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RESEARCH MEMORANDUM

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OPTIMAL INVESTMENT, FINANCING AND DIVIDENDS: A STACKELBERG DIFFERENTIAL GAME

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FEW 286

OPTIMAL INVESTMENT, FINANCING AND DIVIDENDS:

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ABSTRACT

The paper deals with a problem in the area of 'The Dynamics of the Firm'. Over a finite planning period a firm has two shareholders who may trade shares at a fixed price but external transactions are not allowed. The total amount of shares is fixed and the majority shareholder decides on the rate of dividend payout. Each shareholder wishes to maximize a profit functional comprised of (i) total earnings from share transactions plus dividends and (ii) capital gains at the horizon date. The shareholders are subject to personal taxation on dividends as well as capital gains. Decisions on investments and borrowing/lending are made by the manager who seeks to maximize accumulated profits after corporate taxation.

The problem is modelled as an open-loop Stackelberg differential game such that the manager acts as the leader; the shareholders are followers and play a Nash game. We discuss some conceptual problems related to this formulation. The solution of shareholders' Nash game is derived by standard techniques of optimal control theory. Owing to linearity, the controls become bang-bang. The analysis of the manager's problem is done by using a path-connecting procedure.

1. Introduction

This paper deals with the influence of corporate as well as personal taxation on the optimal investment, financing and dividend policies of a firm. Recent contributions in this area include Ludwig (1978), Ylä-Liedenpohja (1978), Van Loon (1983), and Van Schijndel (1985, 1986ab, 1987). See also the survey article by Lesourne and Leban (1982). These studies, however, assume no separation of ownership and management, i.e. the shareholders are also the managers of the firm. The main topic in this research is the determination of optimal policies for capital investments, dividend and debt. In this connection, the question of corporate taxation is important and was the first to receive attention in the literature. Later works (for instance, van Schijndel (1986ab, 1987)) also considered the impact of different personal tax rates of the shareholders.

The purpose of this paper is to relax the assumption of non-separated ownership and management. Within the framework of a financial model of the firm we consider a company with a manager and (for mathematical convenience) only two shareholders. The latter have different personal tax rates which, in turn, differ from the corporate tax rate. The manager controls the investment rate and is in charge of debt management too. The shareholders control the rate of divident pay-out and can buy and sell shares from each other. No emissions of new stock are undertaken during the time period under consideration and there are no external transactions with shares. Thus, in this respect, the company is viewed as a closed system.

To model a situation with multiple decision makers we apply the theory of differential games. Various conceptual problems arise here, and we shall briefly discuss some of them. The main body of the paper is devoted to the analysis of a non-cooperative game where the manager is a Stackelberg leader, announcing at the start of the game an investment policy and the stockholders respond rationally (as followers) by choosing a dividend policy as well as the amount of internal trade with shares. Because of the complexity of the model, a closed-form solution is apparently not attainable but a number of qualitative results can be stated.

The paper is organized as follows: in Section 2 we establish a differential game model as an open-loop Stackelberg game. This section also contains our reflexions on some conceptual difficulties in the modelling process. In Section 3 the Stackelberg game is analyzed; we characterize the structure of optimal policies and discuss their economic implications. Section 4 concludes the paper with a brief summary of our main results and contains also some suggestions for further research.

2. Model formulation

2.1. Preliminaries

In this section we develop a deterministic dynamic model of a corporate firm with a manager (M) and two shareholders (P_1 and P_2, respectively). Let t denote time and [0,T] a planning period of fixed duration.

To construct the firm's balance equation, we introduce the following variables and constants.

$K = K(t)$ = stock of capital goods.
$K(0) = K_o > 0$ and fixed.
$Y = Y(t)$ = debt ($Y>0$); lending ($Y<0$)
$\bar{Z}$ = common stock at nominal value; $\bar{Z} > 0$ and constant.
$R = R(t)$ = cumulative retained earnings.
$R(0) = R_o > 0$ and fixed.

This yields the balance equation

$$K(t) = Y(t) + \bar{Z} + R(t) \tag{1}$$

such that the shareholders' total equity capital equals $\bar{Z} + R$.
Notice that we have assumed that issue of new shares is not allowed, i.e. $\bar{Z}$ is constant.

To construct the equation for the evolution of the firm's retained earnings we introduce the following assumptions.

- The firm operates under decreasing returns to scale.
- Corporate tax is proportional to profit.
- Depreciation is proportional to the stock of capital goods.
- Borrowing/lending do not carry any transactions costs.

Then the flow of retained earnings is given by

$$E = (1-f)[G(K) - aK - rY] - D \tag{2}$$

where

$D = D(t)$: dividend pay-out rate

$E = E(t)$: retained earnings rate

$G = G(K(t))$: gross revenue; $G > 0$, $G' > 0$, $G'' < 0$

a: depreciation rate; $a > 0$ and constant

f: corporate tax rate; $0 < f < 1$ and constant

r: interest rate on debt = interest rate received from lending; $r > 0$ and constant.[1)]

We assume that dividends are paid in cash, not in shares. Notice that cumulative retained earnings, $R(t)$, are given by

$$R(t) = R_o + \int_0^t E(s)ds. \tag{3}$$

From (2-3) we obtain the state equation for R:

$$\dot{R} = E = (1-f)[G(K) - aK - rY] - D. \tag{4}$$

Define

$I = I(t)$: rate of (gross) investment in the capital stock

Then the state equation for K is given by

$$\dot{K} = I - aK. \tag{5}$$

Next we turn to the division of common stock, $\bar{Z}$, between P_1 and P_2. Let

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1) That is, we assume a perfect capital market.

$Z_i = Z_i(t) = P_i$'s part of $\bar{Z}$, $i = 1,2$

$Z_i(0) = Z_{io} > 0$

such that

$$\bar{Z} = Z_1 + Z_2.$$

Note that if $Z_i > \bar{Z}/2$, then P_i is the majority shareholder. If $Z_j = 0$ then shareholder P_i ($i \neq j$) is the sole owner of the firm. Since $\bar{Z}$ is constant we have

$$\dot{Z}_i = -\dot{Z}_j. \tag{6}$$

In (6), and in the sequel, we have $i = 1,2$; $j = 1,2$; $i \neq j$. Suppose that P_1 and P_2 may trade shares (but not sell to or buy from others) and define

$B_i = B_i(t)$ = purchase rate of P_i

$S_i = S_i(t)$ = selling rate of P_i.

Hence

$$\dot{Z}_1 = B_1 - S_1 = S_2 - S_1. \tag{7a}$$

From (7a), and because of (6), we have

$$\dot{Z}_2 = S_1 - S_2. \tag{7b}$$

Eqs. (7) are state equations for the pair (Z_1, Z_2). But under the assumptions we only need one of these equations and we take (7a) as our third state equation. Henceforth we refer to this equation as (7).

Assume that the shareholders have the following contractual agreement: If P_i wants to sell, then P_j must buy. (Note that P_i can sell to P_j only). Hence a stockholder always has the option to leave the company. On

the other hand, no stockholder can be forced to sell, i.e. nobody can be forced to give up a majority position or to leave the firm. This implies that only the selling rates are controls.
For the sake of simplification, let the price at which share transactions take place be fixed as p = constant > 0.[2)]

To construct the payoffs of the shareholders define the following tax rates.

$$\tau_i = \text{tax rate on personal income for } P_i;\ 0 \leqq \tau_i < 1$$

$$\tau_g = \text{tax rate on capital gains for } P_i;\quad 0 < \tau_g < 1.$$

Suppose that the shareholders are characterized in the following way. P_1 is in a high tax bracket with respect to personal taxation whereas P_2 is in the opposite situation. We shall assume that

$$\tau_1 > \tau_g > \tau_2. \tag{8}$$

Dividends are paid out continuously as a fraction of common stock and total dividends amount to

$$D = C\,\bar{\bar{Z}}.$$

Hence, shareholder P_i receives dividends in amount of

$$D_i = C\,Z_i$$

such that $D_1 + D_2 = D$.

The payoff functionals of the two shareholders can now be specified. Assume that each shareholder wishes to maximize (i) his net income stream

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2) This (rather strong) assumption may be relaxed by letting the share price be determined as the result of a bargaining process between the two shareholders, or by letting the price be dependent on cumulative retained earnings, i.e. $p = p(R)$ such that $p' > 0$.

from share transactions plus dividends and (ii) his share of the company's equity capital at t = T after capital gains taxation.

$$J^1 = \int_0^T [p(S_1-S_2) + (1-\tau_1)C\, Z_1]dt + [(1-\tau_g)R(T) + \bar{Z}]\, \frac{Z_1(T)}{\bar{Z}} \qquad (9a)$$

$$J^2 = \int_0^T [p(S_2-S_1) + (1-\tau_2)C\, Z_2]dt + [(1-\tau_g)R(T) + \bar{Z}]\, \frac{Z_2(T)}{\bar{Z}}. \qquad (9b)$$

Let us also define the payoff functional of the manager. The manager has the following objective functional.

$$J = \int_0^T (1-f)[G(K) - aK - rY]dt. \qquad (10)$$

His objective is simply to maximize total profits after corporate taxation.

We have not incorporate discounting in the objectives (9-10). For the shareholders we can assume a zero discount rate because the possibility of lending money offers the shareholders an alternative investment opportunity with a rate of return equal to r. Hence, a shareholder has a time preference rate of $(1-\tau_i)r$. For the manager it turns out that incorporation of a discount rate in (10) does not change qualitatively our results; this is contrary to what is known from models of the same structure, but where management and ownership are not separated (see e.g. van Loon (1983)). The reason is, briefly stated, that the discount rate in such models (only) influences the dividend policy. However, in the present model where management and ownership are separated the dividend policy is not a control instrument of the manager.

In what is to follow we consider the following scenario.[3] Assume that the manager is the leader in a Stackelberg game where the shareholders are followers playing Nash vis-a-vis each other. For reasons of tractability we suppose that all players employ open-loop strategies.

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3) A motivation for the choice of this particular case can be found in Section 2.2 which also contains some alternative possibilities.

Assume that $Z_{1o} > \bar{Z}/2$ which means that shareholder P_1 initially is in control of the dividend policy, $C(t)$, and suppose that P_1 wishes to maintain this control throughout the game. Hence impose the state constraint

$$Z_1(t) \geqq \bar{Z}/2 \ \forall t \tag{11}$$

for which P_1 is responsible. Shareholder P_2 must guarantee the satisfaction of

$$Z_1 \leqq \bar{Z} \Leftrightarrow Z_2 \geqq \ \forall t. \tag{12}$$

As to the controls C, S_i and I we impose non-negativity conditions as well as upper limits

$$0 \leqq C \leqq C^M = \text{const.} \tag{13a}$$

$$0 \leqq S_i \leqq S^M = \text{const.} \tag{13b}$$

$$0 \leqq I \leqq I^M = \text{const.} \tag{13c}$$

Notice that we have assumed the same upper bound on the S_i's. Hence, by (7), $\dot{Z}_i = 0$ whenever $S_1 = S_2 = 0$ or $S_1 = S_2 = S^M$. The assumption is motivated partly by mathematical convenience, partly by lack of reason for supposing $S_1^M \neq S_2^M$.

The state variable K is constrained by the natural non-negativity condition

$$K \geqq 0 \tag{14}$$

which is satisfied whenever $I \geqq 0$ holds. This is easily seen from (5).

It can be argued (see, for instance, van Loon (1983)) that debts must not exceed a certain fraction of equity capital, that is

$$Y \leqq k(R+\bar{Z}) \qquad k = \text{constant} > 0.$$

This inequality is equivalent to

$$K \leqq (1+k)(R+\bar{Z}). \tag{15}$$

Using (14) yields

$$K = Y + \bar{Z} + R \geqq 0 \Rightarrow -Y \leqq \bar{Z} + R$$

which means that, in case of lending (i.e. $Y < 0$), then the total amount lended cannot exceed total equity capital. Notice that satisfaction of (14) guarantees that this will actually hold. Since debt management is a responsibility of the manager, we assume that he guarantees satisfaction of (15).

In summary, we have posed an open-loop Stackelberg differential game with the following components.

$$\dot{K} = I - aK \qquad K(0) = K_o > 0$$

$$\dot{R} = (1-f)[G(K) - aK - r(K-\bar{Z}-R)] - C\bar{Z} \qquad R(0) = R_o > 0$$

$$\dot{Z}_1 = S_2 - S_1 \qquad Z_1(0) = Z_{1o} > \bar{Z}/2$$

and P_1, P_2 play - for a fixed $I(t)$ - the Nash game

$$P_1: \quad \max_{\substack{0 \leqq C \leqq C^M \\ 0 \leqq S_1 \leqq S^M}} \quad J^1 = \int_0^T [p(S_1-S_2) + (1-\tau_1)CZ_1]dt + [(1-\tau_g)R(T) + \bar{Z}]Z_1(T)/\bar{Z}$$

subject to (4), (5), (7) and

$$Z_1 - \bar{Z}/2 \geqq 0.$$

$$P_2: \quad \max_{0 \leqq S_2 \leqq S^M} \quad J^2 = \int_0^T [p(S_2-S_1) + (1-\tau_2)C(\bar{Z}-Z_1)]dt + [(1-\tau_g)R(T) + \bar{Z}](\bar{Z}-Z_1(T))/\bar{Z}$$

subject to (4), (5), (7) and

$$\bar{Z} - Z_1 \geqq 0.$$

M solves the optimization problem

$$M: \quad \max_{0 \leqq I \leqq I^M} \quad J = \int_0^T (1-f)[G(K) - aK - r(K-\bar{Z}-R)]dt$$

subject to (4), (5), (7) and

$$K \leqq (1+k)(R+\bar{Z})$$

and the followers' rational reactions.

2.2. Discussion of some open issues

Apart from analytical difficulties we encounter in the analysis of above model, there are some conceptual difficulties that deserve some consideration. The model essentially has five control variables; Y (debt/lending), I (investment), C (dividends), S_1,S_2 (share trading), and it is not obvious, for instance, how to 'divide the roles' between the three decision makers.

2.2.1. Hierarchical Relationships

(1) Assume that ownership and management are divided such that the shareholders have delegated to a manager (M) the daily operations of the firm (Aoki (1980, p. 604)). M decides upon the investment plan, I(t). Then we could take M as the leader in a Stackelberg game. Assuming open-loop controls, M announces, at the start of the game, his control I(t). The shareholders, P_1 and P_2, are the followers and we may assume that they play an open-loop Nash game. Hence we solve (first) a Nash game for P_1 and P_2 who must choose C, S_1, and S_2, taking I as given. This results in reaction functions C(I) and $S_i(I)$ where I is considered a time-varying parameter. Next we solve the leader's problem with regard to I.

(2) A reverse case may also be considered (Bagchi (1984)) where we have a hierarchical solution with P_1 and P_2 as the two leaders and, hence, one follower, M. Here, P_1 and P_2 play a Nash game when deciding on C and S_1,S_2. Thus, M decides on the investment rate, I, taking C, S_1 and S_2 as given. This yields M's reaction function $I(C,S_1,S_2)$. Next, the Nash game between P_1 and P_2 is played.

In case (1), the manager is in a stronger position since he is able to impose his strategy upon the shareholders. This scenario could emerge when the manager is authorized to control the firm's investment plan, without interference from the part of the shareholders. The latter behave rather passively by adjusting their decisions (in particular, the dividend policy) to the manager's investment plan. In the reverse case, (2), the shareholders announce their decisions (in particular, the dividend policy) and the manager reacts rationally by choosing an appropriate investment program.

(3) A third case may be envisaged where there are three levels of hierarchy with, for example P_1 at the highest level, P_2 at the middle level, and, consequently, M at the lowest level.

(4) Assuming that the owners also manage the firm, the shareholders control, C,I and their selling rates. The game then becomes an optimal control problem and has been studied by van Schijndel (1987). In that work, however, no share transactions take place.

(5) In a case where $Z_{1o} = Z_{2o}$, no shareholder can unilaterally decide on C and hence, some cooperation must be established in order to reach a decision. There is a rich literature on bargaining solutions in cooperative differential games which possibly could be put into use in the problem at hand.

As already mentioned (cf. Section 2.1) we have chosen in this paper to consider case (1).

2.2.2. Control of Dividend Policy

There are also open questions regarding the determination of the dividend rate, C. If Z_i is greater than $\bar{Z}/2$, then P_i has the voting power to fix C.[4] Hence, in this situation P_i has two control variables: C and S_i while P_j only controls his selling rate, S_j. However, Z_i may change as a result of buying and selling and P_i may lose control of C if, at some instant, $Z_i/\bar{Z}$ goes below one half.[5] Thus, it may happen that C changes from being a control of P_i to being a control of P_j at some instant during the play, and maybe switches back again to P_i at a later instant. Such a situation has not been treated in the differential game literature (to our best knowledge), and it is not quite clear how to handle it in an appropriate way. Let us briefly look at some proposals.

(A) Change the dividend term, $C\bar{Z}$, to $C_i\bar{Z}$ and let C_i $(i = 1,2)$ be a control variable of the majority shareholder. This implies (among other things) that the dynamics for the state variable R will switch as majority switches and calls for the use of control theory with switching dynamics (see Luhmer (1983)). Assume, for instance, that there is only one switch in the control of the dividend policy such that on the interval $[0,t_1)$ shareholder P_1 controls the dividend policy. On $[t_1,T]$, P_2 is in control. Hence

$$\dot{R} = (1-f)[G(K) - aK - rY] - C_1\bar{Z} \qquad [0,t_1)$$

$$\dot{R} = (1-f)[G(K) - aK - rY] - C_2\bar{Z} \qquad [t_1,T]$$

where t_1 is given by $Z_1(t_1) = \bar{Z}/2$.

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4) We assume that all shares have equal voting rights, although in practice some shares may have limited voting rights (or no voting rights at all).

5) But notice that P_i cannot be forced to give up a majority position since he does not have to sell if he does not want to. This follows from the above rules for buying and selling.

(B) An approximative formulation with nice properties is the following.[6] Let dividend payout rate, C, be determined as

$$C = \alpha u_1 + (1-\alpha)u_2$$

where $\alpha = Z_1/\bar{Z}$ and u_1, u_2 are continuous control variables of P_1 and P_2, respectively. This formulation means that each shareholder's possibility of influencing the dividend policy depends on his voting power, expressed by the number of shares in his possession.

(C) We could argue that if P_1 initially controls C, then he would be reluctant to give up his control. Hence, he wants the constraint $Z_1 \geq \bar{Z}/2$ to be satisfied for all t. Formally, we add this constraint to the model and make P_1 responsible for its satisfaction.

The proposal (A) offers an intuitively appealing and flexible way to deal with the problem of switches in majority but is also technically the most complicated. Proposal (B) gives a smooth, but not quite realistic, approximation. In what follows we will use proposal (C). Recall that it was assumed that P_1 has initially the majority and we might suppose that P_1 wants to maintain his control of the dividend policy. If P_2 could decide on dividends then he would certainly pursue a completely different dividend policy than the one of P_1, due to P_2's low personal tax rate.

2.2.3. Information Structure

As already mentioned, we suppose in this paper that M is the leader in a Stackelberg open-loop differential game. At the initial instant, M announces (toward the shareholders) his investment plan, I(t). Taking this plan for given, the followers P_1 and P_2 seek to determine an optimal reaction in terms of their controls C, S_1 and S_2. If the followers choose time-functions C(t), $S_1(t)$ and $S_2(t)$, we have the case of an open-loop Stackelberg game.

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6) We are indebted to Paul van Loon for this suggestion.

Another choice of information structure is the feedback Stackelberg solution where all players know (the time as well as) the current state of the game, summarized in the state vector (Z_1,K,R).[7] Additionally, the followers know the announced strategy $I(Z_1,K,R,t)$ of the leader. Note that in the feedback Stackelberg case, the leader announces a control law based on time and current state.
Thus, the leader tells the followers the rule by which he will adjust his investment rate, contingent on the current state (and time). (It should be noticed that playing feedback strategies could cause problems in the definition of the follower's rational reaction set, since the variable I is not a (simple) time-function. However, in a case where the followers use feedback strategies, but the leader plays open-loop, no such conceptual problems arise).
Obviously, a feedback solution is more satisfying than an open-loop solution; for a large value of T, i.e. a long planning period, it is not very reasonable to believe that whatever happens, each player will stick to his predetermined, fixed plan. Moreover, the open-loop Stackelberg solution is not in general time-consistent.[8]

Playing an open-loop Stackelberg game which constitutes an equilibrium (in the sense that the leader does not wish to deviate from his announced strategy) demands that the leader agrees to stick to his announced time-function, $I(t)$. This agreement is usually viewed as a rule of the game, being enforced by some independent arbitrator, or by reasons of punitive action from the followers. In the problem at hand we may imagine that the manager can be forced to stick to his announced time-function $I(t)$ by threats that are in the hands of the shareholders (the followers);

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7) For recent approaches to the feedback Stackelberg solution, see Papavassiloupoulos and Cruz (1979), Basar and Cruz (1982), Basar et al. (1985).

8) By time-consistency of a solution, say u^*, with associated state x^*, we mean that the restriction of u^* to any subinterval (t,T) also constitutes a solution to the game on (t,T), starting with an initial state x^*.

the latter can ultimately dismiss the manager if het does not adhere to their stipulations.[9)]

To relax (partially) the assumption of no deviations from announced strategies throughout the game, one could consider the possibility that the leader does not commit himself to a fixed policy for the entire interval [0,T] but only for a shorter period. More precisely, let z be the length of time over which the leader makes such a commitment (Reinganum and Stokey (1985)). Then the policy I(t) is chosen as follows: at $t = 0$ the leader chooses $\{I(t),\ 0 \leqq t \leqq z\}$, at $t = z$ he chooses $\{I(t),\ z < t \leqq 2z\}$, and so forth. With K periods we have $T = Kz$. Note that the open-loop case is obtained for $K = 1$.

In this paper we assume that all players use open-loop controls. The manager controls the investment rate and each shareholder controls his selling rate. We study the case where P_1 (the shareholder with the high personal tax rate) initially has more than half of the shares and P_1 does not intend to give up his control of the dividend policy.

3. Analysis of the differential game

3.1. The Shareholders' Problem

In this section we solve the Nash open-loop game for the shareholders, taking the investment policy I(t) as a fixed time function.
Using (1) we eliminate the variable Y.

For P_1 define the Hamiltonian H^1 and the Lagrangian L^1 as follows.[10)]

$$H^1 = \lambda_o^1[p(S_1-S_2) + (1-\tau_1)CZ_1] + \lambda_1^1(S_2-S_1) + \lambda_2^1(I-aK) + \lambda_3^1[(1-f)[G(K) - aK - r(K-\bar{Z}-R)] - C\bar{Z}] \tag{16}$$

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9) In other situations, the leader in a Stackelberg game may try to cheat the follower. This is because the follower is not supposed to know the payoff functional of the leader, and the latter can implement a strategy which differs from the one he announced at the start of the game. See, for example, Hämäläinen (1981).

10) See e.g. Feichtinger and Hartl (1986).

where $\lambda_i^1 = \lambda_i^1(t)$ $(i = 1,3)$ are piecewise continuously differentiable costate variables and $\lambda_o^1 = \text{const.} \geqq 0$.

$$L^1 = H^1 + v^1(Z_1-\bar{Z}/2). \tag{17}$$

We have adjoinded the state variable constraint $Z_1 \geqq \bar{Z}/2$ directly to the Hamiltonian by a piecewise continuous multiplier function $v^1 = v^1(t)$. If an optimal solution exists it satisfies the necessary conditions:

$$(C^*,S_1^*) = \underset{\substack{0 \leqq C \leqq C^M \\ 0 \leqq S_1 \leqq S^M}}{\arg\max}\; H^1(Z_1^*,K^*,R^*,C,S_1,S_2^*,I,\lambda_o^1,\lambda_1^1,\lambda_2^1,\lambda_3^1). \tag{18}$$

Condition (18) yields

$$\lambda_o^1(1-\tau_1)Z_1^* - \lambda_3^1\,\bar{Z} \gtreqless 0 \Rightarrow C^* = \begin{cases} C^M \\ \text{unspecified} \\ 0 \end{cases} \tag{19}$$

and

$$\lambda_o^1 p - \lambda_1^1 \gtreqless 0 \Rightarrow S_1^* = \begin{cases} S^M \\ \text{unspecified.} \\ 0 \end{cases} \tag{20}$$

The costate variables and the multipliers satisfy

$$\dot{\lambda}_1^1 = -\lambda_o^1(1-\tau_1)C^* - v^1 \tag{21a}$$

$$\dot{\lambda}_2^1 = a\,\lambda_2^1 - \lambda_3^1(1-f)[G'(K^*) - a - r] \tag{21b}$$

$$\dot{\lambda}_3^1 = -\lambda_3^1(1-f)r \tag{21c}$$

$$v^1 \geqq 0;\; v^1(Z_1^*-\bar{Z}/2) = 0 \tag{22}$$

$$\lambda_1^1(T) = \lambda_o^1[(1-\tau_g)R^*(T) + \bar{Z}]/\bar{Z} + \gamma^1 \tag{23a}$$

$$\lambda_2^1(T) = 0 \tag{23b}$$

$$\lambda_3^1(T) = \lambda_o^1(1-\tau_g)Z_1^*(T)/\bar{Z} \tag{23c}$$

$$\gamma^1 \geqq 0;\ \gamma^1(Z_1^*(T)-\bar{Z}/2) = 0;\ \gamma^1 = \text{const.} \tag{24}$$

The costate function λ_1^1 is continuous except at junction instants where jumps may occur. The costates λ_2^1 and λ_3^1 are everywhere continuous.

It should be noticed that the (constant) multiplier λ_o^1 cannot automatically be put equal to one. This is because of the state variable constraint $Z_1 \geqq \bar{Z}/2$ which must be satisfied at $t = T$. However, the following lemma can be proved.

<u>Lemma 1</u>. $\lambda_o^1 > 0$, and we may put $\lambda_o^1 = 1$.

<u>Proof</u>. See Appendix 1.

From (21a) we obtain

$$\dot{\lambda}_1^1 \leqq 0 \tag{25}$$

and integration in (21a), using (23a), yields

$$\lambda_1^1 = [(1-\tau_g)R^*(T) + \bar{Z}]/\bar{Z} + \int_t^T [(1-\tau_1)C^*(s) + v^1(s)]ds + \gamma^1 \tag{26}$$

which is positive for all $t \in [0,T]$. Note that λ_1^1 has the interpretation of the shadow price of a unit of Z_1, as assessed by P_1. Hence this shadow price is positive but non-increasing.

From (20) and (25-26) it is obvious that an optimal S_1-policy must be of one of the following types:

(A) $S_1^* \equiv 0$ which occurs if $\lambda_1^1 > p\ \forall t \in [0,T]$

(B) $S_1^* = 0$ on $[0,t_1)$ and $S_1^* = S^M$ on $[t_1,T]$ which occurs if $\lambda_1^1 > p$ for $t \in [0,t_1)$, $\lambda_1^1 = p$ at $t = t_1$ and $\lambda_1^1 < p$ for $t \in (t_1,T]$ (27)

(C) $S_1^* = S^M$ which occurs if $\lambda_1^1 < p\ \forall t \in [0,T]$.

It will suffice to give an economic interpretation of the selling policy of type (C).

Using (26) and (27) yields the result that P_1 should sell shares at the maximal rate for all t, iff

$$[(1-\tau_g)R^*(T) + \bar{Z}]/\bar{Z} + \int_0^T [(1-\tau_1)C^*(t) + v^1(t)]dt + \gamma^1 < p. \tag{28}$$

This amounts to saying that P_1 should sell at the maximal rate if the marginal value, at the initial instant, of keeping a share is less than what could be obtained by selling the unit. (Since $\lambda_1^1(t) < \lambda_1^1(0)\ \forall t \in (0,T]$ the argument applies equally well for all $t \in (0,T]$). Inequality (28) can also be written as

$$[(1-\tau_g)R^*(T) + \bar{Z}]\frac{Z_{1o}}{\bar{Z}} + \int_0^T (1-\tau_1)C^*(t)Z_{1o}dt + \int_0^T v^1(t)Z_{1o}dt$$
$$< Z_{1o}(p-\gamma^1). \tag{29}$$

Recall that $Z_{1o} > \bar{Z}/2$. On the left-hand side of (29) the first term is the capital gain to be collected at t = T if P_1 does not sell shares. The second term is the accumulated dividend in the case of no selling. The last term is non-negative, and identically zero if the constraint $Z_1 \geqq \bar{Z}/2$ never binds. The term on the right-hand side is the (adjusted) sales value of Z_{1o}. Hence, if (29) holds, then P_1 will be better off by selling all his initial stock of shares since the sales value exceeds what can be collected in capital gain and dividends. But notice that he cannot sell his initial amount of shares instantaneously; the best to do is to decrease Z_{1o} as fast as possible by selling at the maximal rate.

Now we turn to a characterization of the dividend policy. Integration in (21c), and using (23c), yields

$$\lambda_3^1 = \exp\{(1-f)r(T-t)\}[(1-\tau_g)Z_1^*(T)/\bar{Z}] \qquad (30)$$

which is positive for all $t \in [0,T]$. The costate λ_3^1 represents the shadow price of a unit of R, as assessed by P_1. Using (21c) yields

$$\dot{\lambda}_3^1 < 0 \ \forall t \in [0,T]. \qquad (31)$$

Hence the shadow price of a unit of R (evaluated by P_1) is positive but strictly decreasing. We get the following types of C-policies:

$$C^* = \begin{cases} C^M \\ C \in [0,C^M] \text{ if } (1-\tau_1)\dfrac{Z_1}{\bar{Z}} \gtreqless \lambda_3^1 \Leftrightarrow \\ 0 \end{cases}$$

$$(1-\tau_1)\frac{Z_1}{\bar{Z}} \gtreqless (1-\tau_g)\frac{Z_1(T)}{\bar{Z}}\exp\{(1-f)r(T-t)\} \qquad (32)$$

where the following lemma can be proved.

<u>Lemma 2</u>. A singular C(t) is infeasible.

<u>Proof</u>. See Appendix 2.

The expression (32) can be economically interpreted in the following way. At any instant the firm has the possibility of using a dollar of its cash-flow to pay out as dividend or, alternatively, to retain the dollar and

- pay back a dollar of debt (or, if debt is already negative, to lend one dollar more)
- to finance a dollar of investment

Notice that payment of interest on debt and corporate tax, i.e. rY and $f[G(K) - aK - rY]$, is mandatory and leaves no choice to the firm. To shareholder P_1 the investment policy is given; hence P_1 can only choose between dividends and/or reducing debt/increasing lending. In (32), the term $(1-\tau_1)Z_1/\bar{Z}$ represents the net amount which P_1 receives if one dollar of dividend is paid out at time t. Recall that λ_3^1 has an interpretation as the marginal contribution to optimal profits, caused by a marginal increase in retained earnings (R). Hence, as long as the net benefit from one present dollar of dividend exceeds the value of retaining the dollar, dividends should be paid out, and vice versa. The second expression in (32) can also be interpreted economically. The right-hand side represents the (net of capital gains tax) amount which P_1 collects at $t = T$ if the dollar at hand is used for decreasing the debt. If debt is decreased by one dollar then the instantaneous interest cost is reduced by $(1-f)r$; the value of this saving over the interval $[t,T]$ equals

$$\int_t^T (1-f)r \exp\{(1-f)rs\}ds = \exp\{(1-f)r(T-t)\}.$$

Hence the term $(1-\tau_g)Z_1 \exp\{\{1-f)r(T-t)\}/\bar{Z}$ represents P_1's share of the interest cost saved by not paying out a dollar of dividend at time t. (If the firm lends money, an additional dollar yields interest income in amount of $\exp\{(1-f)r(T-t)\}$ on $[t,T]$, and similar arguments as for the debt case apply).

In order to characterize in further detail the dividend policy we need to determine S_2 since the pair (S_1,S_2) determines Z_1, cf. (7).

Hence, for P_2, define a piecewise continuous multiplier function

$$v^2 = v^2(t)$$

and the Hamiltonian and the Langrangian, respectively:

$$H^2 = \lambda_o^2[p(S_2-S_1) + (1-\tau_2)C(\bar{Z}-Z_1)] + \lambda_1^2(S_2-S_1) + \lambda_2^2(I-aK) + \lambda_3^2[(1-f)[G(K) - aK - r(K-\bar{Z}-R)] - C\bar{Z}] \tag{33}$$

where λ_i^2 $(i = 1,2,3)$ are piecewise continuously differentiable costate variables; $\lambda_0^2 = \text{const.} \geq 0$.

$$L^2 = H^2 + v^2(\bar{Z}-Z_1). \tag{34}$$

Since the necessary conditions for the problem of P_2 resemble those for P_1 (cf. (18)-(24)), they are not stated here.

As for P_1 we prove that the problem is normal.

<u>Lemma 3</u>. $\lambda_o^2 > 0$.

<u>Proof</u>. See Appendix 3.

H^2-maximization requires that

$$S_2^* = \begin{cases} S^M \\ S \in [0,S^M] \text{ if } \lambda_1^2 \gtreqless -p \\ 0 \end{cases} \tag{35}$$

and it holds that

$$\dot{\lambda}_1^2 \geqq 0. \tag{36}$$

By integration of the costate equation for λ_1^2 we get

$$\lambda_1^2 = -[(1-\tau_g)R^*(T) + \bar{Z}]/\bar{Z} - \gamma^2 - \int_t^T [(1-\tau_2)C^*(s) + v^2(s)]ds \tag{37}$$

which is non-positive for all $t \in [0,T]$. Note that λ_1^2 is the shadow price of a unit of Z_1, as assessed by P_2. Hence, this shadow price is non-positive but non-decreasing. We obtain selling policies of the following forms:

(A) $S_2^* \equiv 0$ which occurs if, for all t, $\lambda_1^2 < -p \Leftrightarrow |\lambda_1^2| > p$

(B) $S_2^* = 0$ on $[0,t_2)$ and $S_2^* = S^M$ on $[t_2,T]$ which occurs if $\lambda_1^2 < -p$ for $t \in [0,t_2)$, $\lambda_1^2(t_2) = -p$ and $\lambda_1^2 > -p$ for $t \in [t_2,T]$ (38)

(C) $S_2^* \equiv S^M$ which occurs if, for all t, $\lambda_1^2 > -p \Leftrightarrow |\lambda_1^2| < p$.

The results so far obtained for the optimal selling policies S_1^* and S_2^* are summarized in Table 1.

Player P_1	Player P_2
(A) $S_1^* \equiv 0$ if $\kappa + \gamma^1 > p$	(A) $S_2^* \equiv 0$ if $\kappa + \gamma^2 > p$
(B) $S_1^* = 0$ on $[0,t_1)$ and $S_1^* = S^M$ on $[t_1,T]$ if $(\kappa+\gamma^1<p) \wedge$ $(\kappa+\gamma^1+ \int_0^T [(1-\tau_1)C^*(s) + v^1(s)]ds>p)$	(B) $S_2^* = 0$ on $[0,t_2)$ and $S_2^* = S^M$ on $[t_2,T]$ if $(\kappa+\gamma^2<p) \wedge$ $(\kappa+\gamma^2+ \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]ds>p)$
(C) $S_1^* \equiv S^m$ if $\kappa+\gamma^1+ \int_0^T [(1-\tau_1)C^*(s) + v^1(s)]ds < p$	(C) $S_2^* \equiv S^m$ if $\kappa+\gamma^2+ \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]ds < p$

$$\kappa := [(1-\tau_g)R^*(T) + \bar{Z}]/\bar{Z}$$

Table 1. Summary of conditions for occurrence of S_i^*-policies

In Table 2 we have listed the possible combinations of S_1^* and S_2^* policies.

P_1 \ P_2	(A) $S_2^* \equiv 0$	(B) $S_2^* = \begin{cases} 0 & \text{on } [0,t_2) \\ S^M & \text{on } [t_2,T] \end{cases}$	(C) $S_2^* \equiv S^M$
(A) $S_1^* \equiv 0$	(1) The zero-selling case ($t_1=t_2=T$)	(2) Infeasible	(3) Infeasible
(B) $S_1^* = \begin{cases} 0 & \text{on } [0,t_1) \\ S^M & \text{on } [t_1,T] \end{cases}$	(4) Infeasible	(5) The general case	(6) Case (5) with $t_2= 0$ ($t_1>t_2$)
(C) $S_1^* \equiv S^M$	(7) Infeasible	(8) Case (5) with $t_1 = 0$ ($t_1<t_2$)	(9) The maximum-selling case ($t_1=t_2=0$)

Tabel 2. Summary of (S_1^*,S_2^*)-regimes

Notice that in the zero-selling case as well as the maximum-selling case we have $Z_1 = Z_{1o}$ for all $t \in [0,T]$. (Whenever $S_1^* = S_2^* = S^M$ on an interval, then there is de facto no trade; the shareholders simply exchange equal amounts of shares which means that the net amount of trade is zero).

We can prove the following lemma.

Lemma 4. The regimes depicted in cells (2), (3), (4), and (7) of Table 2 are infeasible.

Proof. See Appendix 4.

The lemma states that it is never optimal for a player to sell at the maximal rate for all t if the other player does not sell at all, and vice versa. The lemma also states that it is never optimal for a player to use a switching policy against a zero selling policy of the other player, and vice versa.

The remaining regimes (6) and (8) (as well as (1) and (9)) are all subsumed under regime (5). Therefore, we confine our interest to regime (5). The switching instants (t_1 and t_2, respectively) are determined by

$$\lambda_1^i(t_i) = \gamma^i + \kappa + \int_{t_i}^{T} [(1-\tau_i)C^*(s) + v^i(s)]ds = p \qquad i = 1,2 \tag{39}$$

Notice that if $t_1 = t_2$ then $Z_1 \equiv Z_{1o}$ as in regimes (1) and (9).
The following lemma can be established, implying that it suffices to consider regime (5) for the case of $t_1 < t_2$.

Lemma 5. If in regime (5) the switching instants are such that $t_1 > t_2$ then regime (5) reduces to regime (1) or (9).

Proof. See Appendix 5.

Consider regime (5) with $t_1 < t_2$ and the following inequality

$$Z_{1o} - S^M(t_2-t_1) > \bar{Z}/2 \Leftrightarrow S^M(t_2-t_1) < Z_{1o} - \bar{Z}/2 \tag{40}$$

which is satisfied if, for example, Z_{1o} is much larger than $\bar{Z}/2$ (i.e. P_1 has initially a comfortable majority), S^M is relatively small, or t_1 is close to t_2. It turns out that the policy S_1 depends on whether (40) is satisfied or not. Obviously, $Z_1 \equiv Z_{1o}$ on $[0,t_1)$ and no constraints are binding. On $[t_1,t_2)$ we have $\dot{Z}_1 < 0$ implying that (12) cannot be binding; hence $v^2 = 0$. The same holds true on the interval $[t_2,T]$: $v^2 = 0$ on $[t_2,T]$ and $\gamma^2 = 0$. It may happen, however, that (11) becomes tight for some t in the interval $[t_1,t_2)$. But note that if (11) does not become binding in $[t_1,t_2)$ it never does. For $t \in [t_1,t_2)$ we have

$$Z_1 = Z_{1o} - S^M(t-t_1)$$

and suppose that Z_1 hits its lower bound ($\bar{Z}/2$) at $t = \tilde{t}_1$ $(\tilde{t}_1 > t_1)$. There are two subcases to consider.

(I) If $\tilde{t}_1 \geqq t_2$ then $Z_1 > \bar{Z}/2$ $\forall t \in [t_1, t_2]$ and the policy for S_1 is policy (B) given by (27). It is easy to see that this situation occurs if (40) is satisfied.

(II) If $\tilde{t}_1 < t_2$ then $Z_1(\tilde{t}_1) = \bar{Z}/2$. Since we have made P_1 responsible for the satisfaction of the constraint (11), this player must switch from $S_1 = S^M$ to $S_1 = 0$ on the interval $[\tilde{t}_1, t_2)$. This will keep Z_1 equal to its lower bound on $[\tilde{t}_1, t_2)$. When P_2 switches (at $t = t_2$) from $S_2 = 0$ to $S_2 = S^M$, then P_1 resumes his policy $S_1 = S^M$. This situation occurs if (40) is not satisfied.

To summarize: for $t_1 < t_2$ and if (40) does not hold, then the S_1 policy should be modified such that

$$\begin{aligned}
&S_1^* = 0 \Rightarrow Z_1 = Z_{1o} && \text{on } [0, t_1) \\
&S_1^* = S^M \Rightarrow Z_1 = Z_{1o} - S^M(t-t_1) && \text{on } [t_1, \tilde{t}_1) \\
&S_1^* = 0 \Rightarrow Z_1 = \bar{Z}/2 && \text{on } [\tilde{t}_1, t_2) \\
&S_1^* = S^M \Rightarrow Z_1 = \bar{Z}/2 && \text{on } [t_2, T].
\end{aligned} \tag{41}$$

The instant $\tilde{t}_1$ is an entry time; the costate λ_1^1 remains continuous at $t = \tilde{t}_1$.[11)]

Now we are ready to establish what types of dividend policies, C, can occur under the optimal selling policies S_1 and S_2. Using (32) and Table 2 we can conclude the following.

Regime (1) and (9): $C^* \equiv 0$.

Regime (5) ($t_1 < t_2$): Recall that $C^* = 0$ on $[\tilde{t}_1, T]$. The C^*-policy is one of the types given by:

$$
\begin{aligned}
&1. \quad C^* = 0 \quad \text{for } t \in [0, \tilde{t}_1). \\
&2. \quad C^* = 0 \quad \text{for } t \in [0, t_1') \\
&\qquad C^* = C^M \quad \text{for } t \in [t_1', t_1'') \\
&\qquad C^* = 0 \quad \text{for } t \in [t_1'', \tilde{t}_1). \\
&3. \quad C^* = C^M \quad \text{for } t \in [0, \hat{t}_1) \\
&\qquad C^* = 0 \quad \text{for } t \in [\hat{t}_1, \tilde{t}_1).
\end{aligned}
\tag{42}
$$

The policies stated in (42) can be derived from (32); see also Figure 1.

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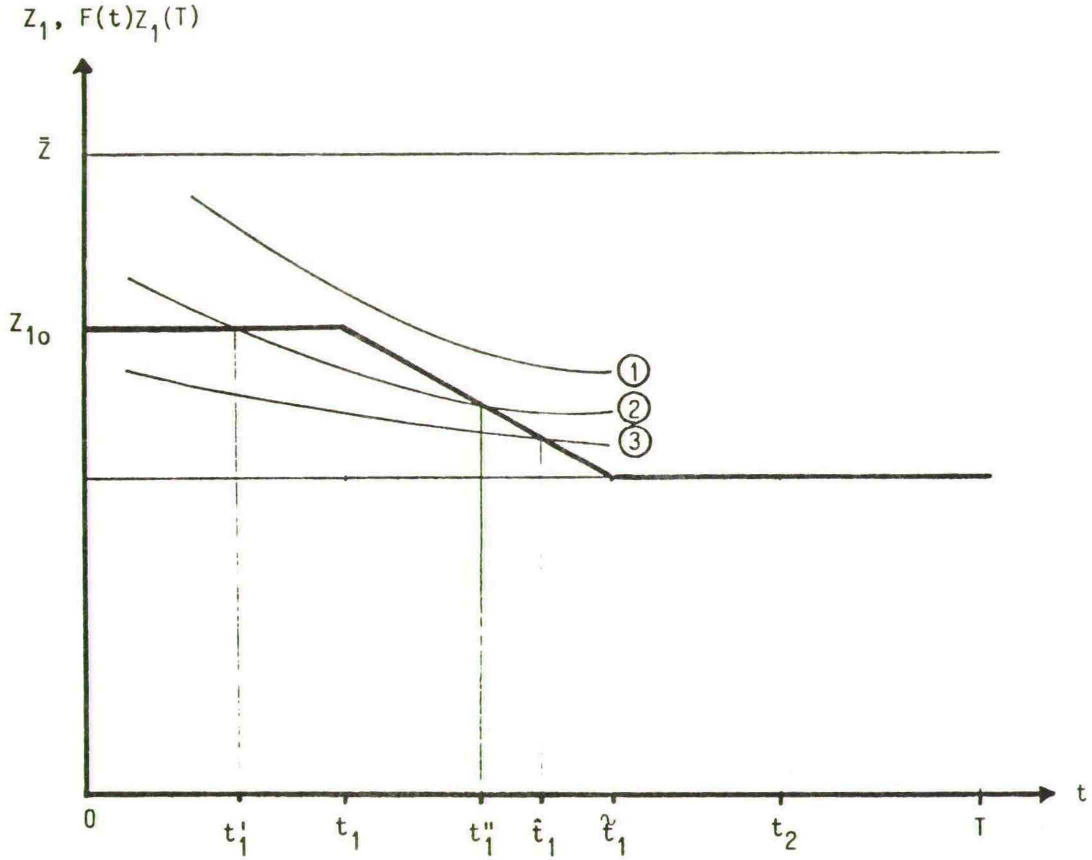
11) This is because the entry to the boundary arc $Z_1 = \bar{Z}/2$ is nontangential, cf. Feichtinger and Hartl (1986).



$$F(t) := \frac{1-\tau_g}{1-\tau_1} \exp\{(1-f)r(T-t)\}$$

Figure 1. Possible C^*-policies on an initial interval.

In Figure 1 we have $\dot{F} < 0$ for all $t \in [0,T]$; $F(t) > (1-\tau_g)/(1-\tau_1) > 1$ for all $t \in [0,T]$. Consider the interval $[0,\tilde{t}_1)$ and refer to Figure 1 where the three curves depict possible paths for $F(t)\bar{Z}/2$; the solid line represents $Z_1(t)$. If, for example, $F(t)\ Z_1(T) > Z_1(t)$ for $t \in [0,\tilde{t}_1)$ then policy $C^* = 0$ is obtained.

It is easy to establish some qualitative conditions for the occurrence of the three dividend policies in (42). In Table 3 such conditions are stated.

C^*-policy type	Tax rates	Initial amount of shares	Net cost of debt
1	$\tau_1 >> \tau_g$	$Z_{1o} \cong \bar{Z}/2$	$(1-f)r$ large
2	$\tau_1 \cong \tau_g$	$Z_{1o} >> \bar{Z}/2$	
3	$\tau_1 \cong \tau_g$	$Z_{1o} >> \bar{Z}/2$	$(1-f)r$ small

Table 3. Qualitative conditions for occurrence of various dividend policies

Under policy 1, no dividends are paid out since the decision is made by the majority shareholder (P_1) who suffers from a high personal tax rate on dividends (τ_1), has only a small majority and the net cost of debt is large. In view of his objective, dividends are discouraged by the high value of τ_1 and the relatively small amount of shares in possession (Z_1). The net cost of debt being large implies that a high value of R(T) (which is desirable) should be achieved by a cautious dividend policy rather than expanding K through debt financed investments. Under policy 3, dividends are initially paid out, motivated by a relatively small personal tax rate of P_1, a comfortable amount of shares (which increases the total amount of dividends received, CZ_1), and a net cost of debt being small. Here, a certain amount of dividends can be defended since taxation on dividends now, and retained earnings later, is approximately the same. Moreover, the loss of retained earnings incurred by the dividend payout can be counterbalanced by attracting debt money (to invest and increase K) since the cost of such funds is relatively low.

3.2. The manager's problem

In this section we turn to the manager's problem. Recall that we consider the manager as the leader in an open-loop Stackelberg game. In Section 3.1 we characterized the solution of the Nash game for the followers P_1 and P_2 whose rational reaction sets are (implicitly) given by

$$C^* = C^*(Z_1) \tag{32}$$

$$S_1^* = S_1^*(\lambda_1^1) \tag{20}$$

where Z_1 and λ_1^1 are determined by (7) and (26), and

$$S_2^* = S_2^*(\lambda_1^2) \tag{35}$$

where λ_1^2 is determined by (37).[12)]

The optimization problem of the manager consists of selecting a piecewise continuous investment rate $I(t)$, such that $0 \leqq I(t) \leqq I^M$, to maximize the payoff functional J given by (10), subject to the original state equations (4), (5), (7), the six costate equations of the followers with their appropriate boundary conditions, and the state variable inequality constraint (15).

Let $\mu(t)$, $\eta_1(t)$ and $\eta_2(t)$ be piecewise continuous multiplier functions and let $\lambda_1(t)$, $\lambda_2(t), \ldots, \lambda_9(t)$ be piecewise continuously differentiable costate variables. Let λ_o be a non-negative constant.

It may be convenient to transform the payoff (10). Using (4) we obtain

$$J = \int_0^T (1-f)[G(K) - aK - r(K-\bar{Z}-R)]dt = \int_0^T \dot{R}\ dt + \bar{Z} \int_0^T C^*\ dt.$$

Integrating on the right-hand side yields

---

12) Notice that these rational reactions do not involve (explicitly) the control, I, of the leader. However, the trajectories C^*, S_1^* and S_2^* do depend on I since I determines K which, together with C^*, determine R. On the other hand, R(T) and C^* determine λ_1^1 and λ_1^2 which, in turn, determine S_1^* and S_2^*. The latter yield the path of Z_1 which determines C^*.

$$J = \int_0^T C^* \bar{Z}\, dt + R(T) - R_o$$

where the term R_o can be disregarded since it is constant. The Hamiltonian becomes

$$\begin{aligned} H = {} & \lambda_o C^* \bar{Z} + \lambda_1 (S_2^* - S_1^*) + \lambda_2 (I - aK) + \\ & \lambda_3 [(1-f)[G(K) - aK - r(K - \bar{Z} - R)] - C^* \bar{Z}] + \\ & \lambda_4 (-(1-\tau_1) C^* - v^1) + \lambda_5 [a\lambda_2^1 - \lambda_3^1 (1-f)(G'(K) - a - r)] + \\ & \lambda_6 (-\lambda_3^1 (1-f) r) + \lambda_7 ((1-\tau_2) C^* + v^2) + \\ & \lambda_8 [a\lambda_2^2 - \lambda_3^2 (1-f)(G'(K) - a - r)] + \lambda_9 (-\lambda_3^2 (1-f) r) \end{aligned} \qquad (43)$$

and the Lagrangian is given by

$$L = H + \mu[(1+k)(R+\bar{Z}) - K] + \eta_1 I + \eta_2 (I^M - I). \qquad (44)$$

The set of necessary conditions is as follows:

$$I^* = \arg\max_{0 \leq I \leq I^M} H \qquad (45)$$

$$\frac{\partial L}{\partial I} = \lambda_2 + \eta_1 - \eta_2 = 0 \qquad (46)$$

$$\dot{\lambda}_1 = \frac{dC^*}{dZ_1} [\bar{Z}(\lambda_3 - \lambda_o) + \lambda_4 (1-\tau_1) - \lambda_7 (1-\tau_2)] \qquad (47a)$$

$$\dot{\lambda}_2 = a\lambda_2 - (1-f)(G'(K) - a - r)\lambda_3 + (1-f)G''(K)(\lambda_5 \lambda_3^1 + \lambda_8 \lambda_3^2) + \mu \qquad (47b)$$

$$\dot{\lambda}_3 = -(1-f) r\, \lambda_3 - \mu(1+k) \qquad (47c)$$

$$\dot{\lambda}_4 = \lambda_1 \frac{dS_1^*}{d\lambda_1^1} \qquad (47d)$$

$$\dot{\lambda}_5 = -a\lambda_5 \tag{47e}$$

$$\dot{\lambda}_6 = (1-f)r\ \lambda_6 + \lambda_5(1-f)(G'(K)-a-r) \tag{47f}$$

$$\dot{\lambda}_7 = -\lambda_1 \frac{dS_2^*}{d\lambda_1^2} \tag{47g}$$

$$\dot{\lambda}_8 = -a\lambda_8 \tag{47h}$$

$$\dot{\lambda}_9 = (1-f)r\ \lambda_9 + \lambda_8(1-f)(G'(K)-a-r) \tag{47i}$$

$$\mu \geqq 0 \qquad \mu[(1+k)(R+\bar{Z}) - K] = 0 \tag{48a}$$

$$\eta_1 I = 0,\ \eta_1 \geqq 0 \qquad \eta_2(I^M-I) = 0,\ \eta_2 \geqq 0 \tag{48b}$$

$$\lambda_1(T) = [\lambda_9(T) - \lambda_6(T)](1-\tau_g)/\bar{Z} \tag{49a}$$

$$\lambda_2(T) = -\alpha \tag{49b}$$

$$\lambda_3(T) = \alpha(1+k) + [\lambda_7(T) - \lambda_4(T)](1-\tau_g)/\bar{Z} + \lambda_o \tag{49c}$$

$$\alpha \geqq 0,\ \alpha[(1+k)(R(T)+\bar{Z}) - K(T)] = 0 \tag{50}$$

$$\lambda_4(0) = \lambda_5(0) = \lambda_6(0) = \lambda_7(0) = \lambda_8(0) = \lambda_9(0) = 0. \tag{51}$$

From (47e,f,h,i) and (51) it appears that

$$\lambda_5 = \lambda_6 = \lambda_8 = \lambda_9 = 0 \text{ for } 0 \leqq t \leqq T \tag{52}$$

which is intuitively reasonable since the followers' costates λ_2^1, λ_3^1, λ_2^2, λ_3^2 do not have direct significance for the manager's problem.

We shall make some assumptions that will facilitate the analysis of the necessary optimality conditions.

<u>First</u>, we assume that, loosely speaking,

I^M is sufficiently large.

This assumption is made for mathematical convenience but seems reasonable in the light of the model's financial structure. The model per se imposes a natural upper bound on the investment rate since the firm cannot finance unlimited investments, neither by borrowing funds nor by retaining profits. The implications of the assumption are the following.

- Whatever control, $I > 0$, we choose, it holds that $I < I^M$.
- The multiplier η_2 is identically zero. Let $\eta_1 = \eta$.

Second, assume that

$$G'(K) > a+r \text{ for } K = K_o. \tag{53}$$

It may be appropriate to make some remarks on the state variable inequality constraint $(1+k)(\bar{Z}+R) \geqq K$. Define

$$h(R,K) \quad (1+k)(\bar{Z}+R) - K$$

which yields

$$dh/dt = (1+k)\dot{R} - \dot{K} = (1+k)[(1-f)[G(K) - aK - r(K-\bar{Z}-R)] - C^{*}\bar{Z}] - (I-aK). \tag{54}$$

This expression shows that $h \geqq 0$ is a first order constraint.
Denote an arc having $h = 0$ as a boundary arc and recall that the adjoint variables associated with K and R are λ_2 and λ_3, respectively. If entry to/exit from a boundary arc occurs in a non-tangential way, then λ_2, λ_3 are continuous at the point of entry/exit (Feichtinger and Hartl (1986)).

Notice that non-tangential entry/exit means that h(R,K) as a function of time has a kink. Hence, at such junction points dh/dt is discontinuous.[13] Also notice that a discontinuity in dh/dt at some t = t' occurs only if a control is discontinuous at t = t'. However, in the problem at hand, control discontinuities are not only those in I but also in C^*, cf. (54). (Obviously, if the controls are continuous then dh/dt is continuous too.) When dh/dt is continuous at a junction point, then the adjoint variables will normally also be continuous.[14] The constant multiplier α (in (49-50)) vanishes whenever the adjoint variables λ_2, λ_3 are continuous at t = T. This follows from the (necessary) jump conditions for the adjoint variables. (Of course, if the state constraint is not effective at t = T, then by (50) we have $\alpha = 0$. Hence the problem concerning α occurs only if the state constraint is binding at t = T).

We start the analysis with some remarks about an optimal investment rate. From (43) and (45) we see that I will be singular whenever $\lambda_2 = 0$. Notice that the costate λ_2 has the interpretation of a shadow price of a unit of K, as assessed by the manager.

Obviously, for $\lambda_2 < 0$ no investment occurs. A necessary condition for optimality of a singular control is given by the generalized Legendre-Clebsch condition which requires that

$$(-1)^k \frac{\partial}{\partial I} \frac{d^{2k}}{dt^{2k}} H_I \leqq 0 \text{ for } k = 0,1,2,...$$

Obviously the condition is satisfied for k = 0. For k = 1 we obtain

$$(1-f)\lambda_3 G''(K) \leqq 0$$

which holds only if $\lambda_3 \geqq 0$.

---

13) Eq. (54) shows that discontinuity of dh/dt implies discontinuity of $\dot{K}$ and/or $\dot{R}$.

14) Seierstad and Sydsæter (1987, p. 318)

On a singular path it holds that $\dot{\lambda}_2 = \ddot{\lambda}_2 = 0$. Hence

$$\dot{\lambda}_2 = -\lambda_3(1-f)(G'(K)-a-r) + \mu = 0 \tag{55}$$

and, whenever μ is differentiable,

$$\ddot{\lambda}_2 = \dot{\mu} - (1-f)\dot{\lambda}_3(G'(K)-a-r) - \lambda_3(1-f)G''(K)(I-aK) = 0. \tag{56}$$

Substituting from (5) and (47c) into (56) yields

$$I^s = aK + \frac{\dot{\mu}}{\lambda_3(1-f)G''(K)} + \frac{\mu}{\lambda_3(1-f)G''(K)}[(1-f)r\lambda_3 + \mu(1+k)] \tag{57}$$

which shows that I^s equals aK (i.e. investment is just at the replacement level) if the state constraint $(1+k)(\bar{Z}+R) \geqq K$ does not bind. For $I^s = aK$, the corresponding value of K, say, K^s, is implicitly given as the (unique) solution of

$$G'(K^s) = a + r \tag{58}$$

which follows from (55).[15] Hence, $I^s = aK^s$ is constant.

Next we characterize the four possible paths by using the complementary slackness conditions.

PATH	1	2	3	4
μ	+	0	0	+
η	0	0	+	+

---

15) Stricktly speaking, $G'(K) = a + r$ does not need to hold in (55) if $\lambda_3 = 0$.

Path 1

This is a boundary path: $K = (1+k)(\bar{Z}+R)$. Moreover $\lambda_2 = 0$ and the control I is singular: I^s is given by (57). The control, say, I^b, which will maintain K equal to $(1+k)(\bar{Z}+R)$ is given by

$$I^b = aK + (1+k)[(1-f)[G(K) - aK - r(K-\bar{Z}-R)] - C^*\bar{Z}]. \tag{59}$$

From $\dot{\lambda}_2 = \lambda_2 = 0$ we obtain

$$\mu = (1-f)(G'(K)-a-r)\lambda_3 > 0 \Rightarrow G'(K) > a+r \tag{60}$$

whenever $\lambda_3 > 0$. (Notice that if $\lambda_3 = 0$ then path 1 cannot occur). Next, observe that

$$I^b \gtreqless aK \Leftrightarrow \dot{K},\dot{R} \gtreqless 0.$$

It is easy to show that if $C^* = 0$ for all t on path 1, then $\dot{R} > 0$. However, $\dot{R} > 0$ may not hold in general, but $\dot{R} > 0$ on a final interval since $C^* = 0$ in such an interval. Path 1 is a feasible final path since the transversality condition $\lambda_2(T) = 0$ is satisfied. However, if $\dot{K} > 0$ then $G'(K)$ decreases and if path 1 is extended on a sufficiently long interval it may happen that $G'(K) > a+r$ is violated.

Path 2

On this path, where $\mu = \eta = 0$, we have $(1+k)(\bar{Z}+R) \geq K$ and $I \geq 0$. Then (57) yields $I^s = aK^s$ and K^s is given by (58) whenever $\lambda_3 > 0$.[16)] To satisfy (46), $\lambda_2 = 0$ must hold which makes path 2 a feasible final path. Notice that $\dot{K} = 0$ and hence $\dot{Y} = \dot{R} = 0$ or $\text{sgn}(\dot{Y}) = -\text{sgn}(\dot{R})$.

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16) If $\lambda_3 = 0$ on an interval then K^s is not uniquely determined by (55), that is, $G'(K) = a+r$ does not necessarily hold. This could cause difficulties in the coupling procedure to follow since the arguments employed require that $G'(K) = a+r$ be satisfied. In Appendix 6 this issue is discussed in more detail.

Path 3

On this path we have $\mu = 0$ and $(1+k)(\bar{Z}+R) \geqq K$. Moreover, $\eta > 0$ implies $I = 0$ and hence $\dot{K} < 0$. It must hold that at least one of $\dot{R}, \dot{Y}$ is negative. Notice that path 3 is infeasible as a final path since $\lambda_2 = -\eta < 0$.

Path 4

This is a boundary path and $I = 0$. Furthermore, $\dot{K}, \dot{R}, \dot{Y} < 0$ and $\lambda_2 = -\eta < 0$ makes path 4 infeasible as a final path. Notice that $\dot{R}$ decreases irrespective of whether $C = 0$ or $C > 0$.

An initial feasible path is a path which satisfies the fixed initial conditions. If we assume that the firm has maximal debt at $t = 0$, then the initial values $K(0)$, $R(0)$ must satisfy[17)]

$$(1+k)(\bar{Z}+R(0)) = K(0). \tag{61}$$

Van Loon (1983) argues that if we in (61) had strict inequality (i.e. debt less than maximal) then the firm would instantaneously attract the missing amount of debt and invest it. After that, the firm starts on a feasible path. A mathematically stringent argumentation for assumption (61) can be found in Feichtinger and Hartl (1986, p. 378).

Recall that paths 3 and 4 are infeasible as final paths; path 2 is a feasible final path and path 1 may be a feasible path. The procedure is now to work backwards from $t = T$ and consider a feasible final path. A first question to answer is the following: is path 1 or path 2 a candidate for an optimal solution for the entire planning period? Path 1 is a candidate for an optimal solution for all t only if $G'(K) > a+r$ holds throughout the interval $[0,T]$. It is, however, questionable if this actually will be satisfied. Path 2, on the other hand, can never be optimal on $[0,T]$ if $\lambda_3(0) > 0$ since (53) is then violated.

The next step is to determine which path can precede a final path. Therefore we test for each feasible final path which paths can precede those final paths. The testing procedure utilizes the properties of paths

---

17) See van Loon (1983), van Schijndel (1987).

1-4 described above as well as continuity properties of state and costate variables. If the set of feasible preceding paths is not empty then we repeat the coupling procedure. The procedure stops when no more paths can precede a feasible string of policies. Note that the initial condition (61) must hold. (A more precise description of the coupling technique can be found in Appendix 7; see also van Loon (1983)). It turns out that there is only one coupling satisfying all necessary conditions. We can prove the following proposition.

Proposition 1: The only policy strings consisting of paths 1-4 that satisfy the necessary optimality conditions are

* PATH 1 → PATH 2

** PATH 1 throughout [0,T]

Proof: See Appendix 7.

In Figures 2 and 3 we have depicted the evolution of some key variables for the case of C-policies of type 1 and 3, (cf. Table 3) for the case of a policy string Path 1 → Path 2.

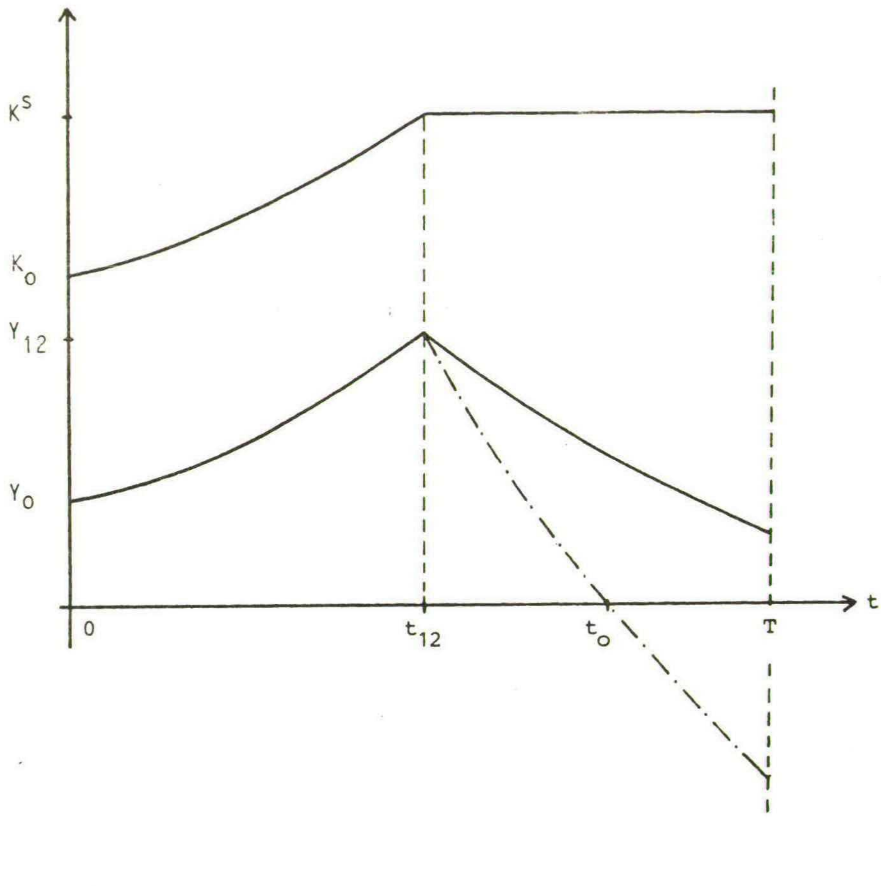

Figure 2. Optimal policies for capital stock (K) and debt/lending (Y) when no dividends are paid out.

In Figure 2 we have the case of zero dividends. It occurs if, among other things, the majority shareholder's personal tax rate is high and/or the net cost of debt is large. Initially (on path 1) K, R and Y are all increasing and debt is maximal. Gross investment, I, is greater than the depreciation, aK, which implies an increasing stock of capital goods. Retained earnings, R, increase since no dividends are paid out and G'(K) >

a+r on path 1. The latter condition means that a marginal unit of investment gives the firm a return, $G'(K) - a$, being greater than the interest rate. This justifies that debt is increased (maximally). However, since K increases, $G'(K)$ decreases and at $t = t_{12}$ $G'(K)$ reaches its stationary level where $G'(K) = a+r$, and we get on path 2. Here, K is constant and R is increasing. Investment is at the replacement level and since retained earnings still increase, the remaining cash is used to pay off debt, i.e. Y decreases. Depending on the parameters of the problem, lending may occur from some instant, say, t_o. On path 2 the evolution of Y is given by

$$\dot{Y} = -(1-f)(G(K^s)-aK^s-rY)$$

which has the solution

$$Y = \frac{G(K^s) - aK^s}{r} + \left[Y_{12} - \frac{G(K^s) - aK^s}{r}\right] \exp\{-(1-f)r(t_{12}-t)\}$$

where Y_{12} is the level of debt at the coupling instant $t = t_{12}$. In Figure 2 the instant t_o (where lending starts) is given by

$$t_o = t_{12} - \frac{1}{(1-f)r} \ln(G(K^s)-aK^s-rY_{12})/(G(K^s)-aK^s)). \tag{62}$$

Notice that if $t_o > T$ then $Y > 0$ for all $t \in [0,T]$ and lending does not occur. Eq. (62) shows that a no-lending case emerges if t_{12} and Y_{12} are large, i.e. the expansion period is long compared to the period of stationary evolution, and the level of debt incurred after the expansion period is large. This seems to be intuitively reasonable.

In Lemma 6 we prove that if the interest rate, r, is sufficiently large (which implies that the net cost of debt, $(1-f)r$, is large) then lending is unlikely to occur. Notice that we still consider the case of zero dividends.

<u>Lemma 6</u>. The instant t_o given by (62) is increasing as a function of the parameter r, i.e. dt_o/dr is positive.

<u>Proof</u>. See Appendix 8.

The lemma states that for increasing values of r, then t_o increases and $t_o > T$ is likely to be the case, implying that Y is positive for all $t \in [0,T]$. Hence, if debt is very costly, no dividends will be paid out (cf. Table 3) and, moreover, lending is unlikely to occur.

In Figure 3 we turn to the case of maximal dividend payment on an initial interval, followed by zero dividends for the rest of the planning period. This case occurs if, for example, the majority shareholder's personal tax rate is comfortable low, see also Table 3. consider Figure 3 where the dividend policy switches before the coupling instant t_{12}.

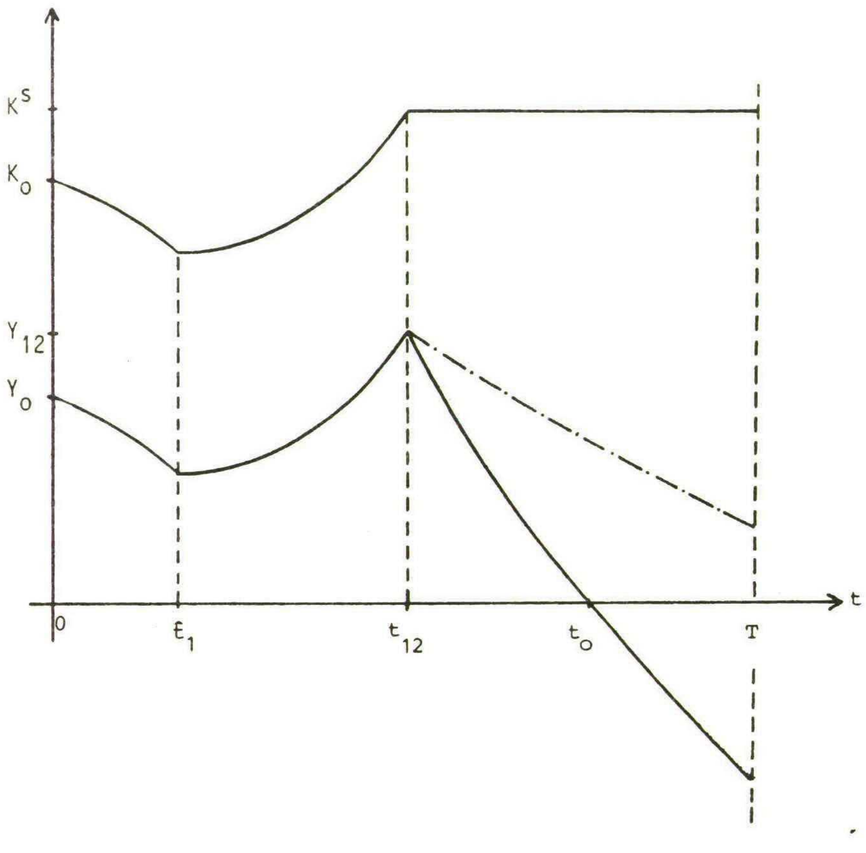

Figure 3. Optimal policies for capital stock (K) and debt/lending (Y) when maximal dividends are paid out on an initial interval ending at $t = \hat{t}_1$ such that $\hat{t}_1 \leq t_{12}$.

On the intervals $[\hat{t}_1, t_{12})$ and $[t_{12}, T]$ the evolution of K, R and Y are the same (qualitatively speaking) as in Figure 2. Notice that if r is very low, lending (on the interval $(t_o, T]$) could occur here. Depending on the actual value of C^M, two situations can be distinguished.

(a) If C^M is sufficiently low, then $\dot{R} = (1-f)(G(K)-aK-rY) - C^M\bar{Z}$ remains positive on $[0, t_{12}]$ and Figure 2 applies.

(b) For C^M sufficiently large, $\dot{R}$ becomes negative and K as well as Y decrease on the interval $[0, \hat{t}_1)$. See Figure 3. In such a situation the firm initially follows a contraction policy. Investment is below the replacement level, implying a decreasing K. Even if investment is low, the cumulative retained earnings decrease since large amounts of dividends are distributed and debt is paid off at the same time.

If the dividend policy switches after the coupling instant, i.e. $\hat{t}_1 > t_{12}$, then the value of C^M again becomes significant.

(α) For C^M sufficiently low, the situation will be as in Figure 2.

(β) For large values of C^M, $\dot{R}$ becomes negative, implying that K decreases. But $\dot{K} > 0$ must hold on some interval before the coupling instant t_{12}, cf. Appendix 7. Hence, for $\hat{t}_1 > t_{12}$ and in case of a large value of C^M, the feasibility of the string "path 1 → path 2" may be lost.

4. Concluding remarks

In this paper we studied a problem in the areas of 'The Dynamics of the Firm' and 'Corporate Finance'. A deterministic, dynamic model was set up with the purpose of characterizing optimal investment, financing and dividend policies of a firm with separation of management and ownership. In the latter respect, the present work differs from, for instance, Van Loon (1983), Van Schijndel (1987) where no such separation exists. To model the possible conflicts between management and shareholders a Stackelberg differential game approach was applied and with a view to tractability we assumed open-loop controls.

More specifically, within the framework of a financial model of the firm we assumed that a manger controls the firm's investment policy over a fixed planning period. With the manager being the Stackelberg leader, the shareholders respond rationally to the announced investment policy by choosing a dividend policy and policies for the internal trade with shares. The dividend policy is decided by the majority shareholder and each shareholder receives dividends in proportion to the fraction of shares he possesses. At the end of the game the owners receive their respective parts of the corporate assets.

An important aspect of the scenario is the presence of taxation. Here, we considered corporate as well as personal taxes; the latter being charged on the streams of income and on the terminal capital gains.

The solution of the Nash game played by the shareholders was obtained by standard methods of optimal control. Due to linearity, the dividend policy as well as the share trading policies turned out to be bang-bang policies. The manager's problem was only solvable in a qualitative way and we applied a path-connecting procedure designed by Van Loon (1983). Here, the optimal string of paths was a simple two-path sequence. At a terminal interval the investment policy is designed to maintain the stock of capital goods at an optimal stationary level and debt is gradually paid off. Depending on the parameters of the problem even lending may occur during this final phase. The initial phase is an expansion phase if no dividends are paid out; for a sufficiently large rate of dividend pay out, an initial time interval of contraction can occur, however.

In order to obtain our results a number of assumptions were made. Those we find most crucial are the following:

(1) The firm is in some respects 'a closed system'. Although debt money can be attracted/paid off, and lending is possible, the amount of common stock is fixed. This means that funds cannot be obtained by emissions of new shares. Moreover, the existing shareholders were not permitted to buy/sell shares from/to investors outside the firm.

(2) The price of a share traded between the shareholders was considered fixed and constant. This assumption obviously deprives the shareholders from a range of interesting options.

(3) Control of the dividend policy cannot switch during the play. This means that the shareholder who initially has the majority of shares will continue to be in this position throughout the planning period. Here we assumed that the high-taxed shareholder had the majority of

shares. Of course, different results would have been obtained if the low-taxed shareholder has had the majority.

(4) The strategies of the manager as well as the shareholders are open-loop, implying that the players are supposed to stick to predetermined plans that are independent of the current state of the game.

An obvious task for future research would be to relax these assumptions. However, one should be prepared to face considerable difficulties in the set-up as well as the analysis of such a model.

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Appendix 1

Proof of Lemma 1: We show that $\lambda_o^1 = 0$ leads to a contradiction. Assuming $\lambda_o^1 = 0$ implies

$$\lambda_1^1 = \gamma^1 + \int_t^T v^1(s)ds \geqq 0, \tag{A.1.1}$$

by using (21a), (23a). Integration in (21c), using $\lambda_3^1(T) = 0$ from (23c), yields $\lambda_3^1 \equiv 0$. To maximize H^1 with respect to (S_1,C) we choose

$$S_1^* = \begin{cases} S^M \\ S_1 \in [0,S^M] \text{ if } \lambda_1^1 \lesseqgtr 0 \\ 0 \end{cases}$$

$$C^* = \begin{cases} C^M \\ C \in [0,C^M] \text{ if } \lambda_3^1 \lesseqgtr 0. \\ 0 \end{cases}$$

Hence S_1^* is either zero or unspecified in the interval $[0,S^M]$; $C^* \in [0,C^M]$. On an interval where $\lambda_1^1 = 0$ H^1 does not depend on S_1 and we can choose any S_1 in $[0,S^M]$, e.g. $S_1^* = 0$. By a similar reasoning, since λ_3^1 vanishes identically, we put $C^* \equiv 0$. In conclusion, $S_1^* = C^* = 0$ $\forall t \in [0,T]$. by (7) we have $\dot{Z}_1 \geqq 0$ which means that $Z_1 > \bar{Z}/2$ for all $t \in [0,T]$. This implies, by (22) and (24), that $v^1 \equiv 0$ and $\gamma^1 = 0$. From (A.1.1) we then obtain $\lambda_1^1 \equiv 0$. Rewrite the payoff in (9a) by using (7):

$$J^1 = -p \int_0^T (S_2-S_1)dt + \frac{1}{\bar{Z}} [(1-\tau_g)R^*(T) + \bar{Z}][Z_{1o} + \int_0^T (S_2-S_1)dt]$$

and calculate

$$\partial J^1/\partial Z_{1o} = [(1-\tau_g)R^*(T) + \bar{Z}]/\bar{Z}. \tag{A.1.2}$$

It is known that $\partial J^1/\partial Z_{1o} = \lambda_1^1(0)$. Since $\lambda_1^1 \equiv 0$ we have $\lambda_1^1(0) = 0$ but in that case (A.1.2) cannot be satisfied since $\bar{Z} + R \geqq 0$ is a constraint which must be satisfied as a part of the necessary conditions. Q.E.D.

Appendix 2

Proof of Lemma 2: For a singular C it must hold that

$$Z_1 = \lambda_3^1 \frac{\bar{Z}}{1-\tau_1} \text{ and } \dot{Z}_1 = \dot{\lambda}_3^1 \frac{\bar{Z}}{1-\tau_1}.$$

Using (21c) yields

$$\dot{Z}_1 = -\lambda_3^1 (1-f)r \frac{\bar{Z}}{1-\tau_1} = -(1-f)rZ_1$$

which implies, by (7), that

$$S_2 - S_1 = -(1-f)rZ_1.$$

Since for C > 0 the S_i's are strictly bang-bang, and $Z_1 > 0$, it must hold that $S_2 = 0$ and $S_1 = S^M$. Thus,

$$S^M = (1-f)rZ_1 \Rightarrow Z_1 = S^M/((1-f)r)$$

which yields $\dot{Z}_1 = 0$ and $S_1 = S_2$. But this contradicts $S_1 = S^M$, $S_2 = 0$ and we conclude that a singular C is not feasible. Q.E.D.

Appendix 3

Proof of Lemma 3: The proof follows the same lines as the proof of Lemma 1 and is omitted.

Appendix 4

Proof of Lemma 4: We shall prove that the regimes (2), (3), (4), and (7) in Table 2 are infeasible.

Consider regime (3): It occurs if

$$\kappa + \gamma^1 > p \tag{A.4.1}$$

and

$$\kappa + \gamma^2 + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]ds < p. \tag{A.4.2}$$

If $\gamma^2 > 0$ then $\gamma^1 = 0$ and the inequalities (A.4.1-2) cannot be satisfied. (This is also true for $\gamma^1 = \gamma^2 = 0$.) We conclude that regime (3) occurs only if $\gamma^1 > 0$. This implies $Z_1(T) = \bar{Z}/2$,

$$\kappa + \gamma^1 > p$$

and

$$\kappa + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]ds < p.$$

By a similar reasoning, <u>regime (7)</u> occurs only if $\gamma^2 > 0$ implying $Z_1(T) = \bar{Z}$,

$$\kappa + \gamma^2 > p$$

and

$$\kappa + \int_0^T [(1-\tau_1)C^*(s) + v^1(s)]ds < p.$$

Consider <u>regime (2)</u> which occurs if

$$\kappa + \gamma^1 > p$$

and

$$\{\kappa + \gamma^2 < p\} \wedge \{\kappa + \gamma^2 + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]ds > p\}.$$

This case occurs only if $\gamma^1 > 0$,

$$\kappa + \gamma^1 > p$$

and

$$\{\kappa < p\} \wedge \{\kappa + \int_0^T [(1-\tau_2)C^*(s) + v^2(s)]ds > p\}.$$

Similarly, <u>regime (4)</u> occurs only if $\gamma^2 > 0$,

$$\kappa + \gamma^2 > p$$

and

$$\{\kappa < p\} \wedge \{\kappa + \int_0^T [(1-\tau_1)C^*(s) + v^1(s)]ds > p\}.$$

(Note that regimes (2) and (3) occur only if $\gamma^1 > 0$, implying $Z_1(T) = \bar{Z}/2$. This makes some economic sense since regimes (2)-(3) both have $S_1 \equiv 0$. Hence, knowing that his majority could ultimately be lost, P_1 prefers not to sell at all. A similar interpretation applies to regimes (4) and (7) where $\gamma^2 > 0$, implying $Z_1(T) = \bar{Z}$, i.e. $Z_2(T) = 0$).

For both regimes (2) and (3) we have $\dot{Z}_1 \geqq 0$, implying $Z_1(T) > Z_{1o} > \bar{Z}/2$ (implying $\gamma^1 = 0$) which contradicts $Z_1(T) = \bar{Z}/2$ (being implied by $\gamma^1 > 0$). Hence these two regimes are infeasible.

For both regimes (4) and (7) we have $\dot{Z}_1 \leqq 0$, implying $Z_1(T) < \bar{Z}$ (implying $\gamma^2 = 0$) which contradicts $Z_1(T) = \bar{Z}$ (being implied by $\gamma^2 > 0$). Hence these regimes are infeasible too. Q.E.D.

<u>Appendix 5</u>

<u>Proof of Lemma 5</u>: First notice that having $t_1 > t_2$ in regime (5) implies that $\dot{Z}_1 \geqq 0$ for all t. When Z_1 is non-decreasing, then C^* is identically equal to zero (cf. (32)). Moreover, the constraint (11) cannot become binding; hence $v^1 = 0$ for all t, and $\gamma^1 = 0$. Using (21a) shows that $\dot{\lambda}_1^1 = 0$ for all t, implying that $\lambda_1^1 \equiv \kappa$.
Using (26)-(27) and Table 1 we observe that S_1^* cannot switch and Table 1 shows that

if $\kappa > p$. then $S_1^* \equiv 0$, and we conclude that $S_2^* \equiv 0$, i.e. $\dot{Z}_1 \equiv 0$. In summary, we have $t_1 = t_2 = T$ as in regime (1)

if $\kappa < p$, then $S_1^* \equiv S^M$, and S_2^* must be identically equal to S^M in order to have $\dot{Z}_1 \geqq 0$. Hence $\dot{Z}_1 \equiv 0$, and $t_1 = t_2 = 0$ as in regime (9).

Q.E.D

Appendix 6

In this appendix we deal with the expression (55) which uniqely determines K^S and I^S on path 2 iff $\lambda_3 > 0$ throughout this path. In relation to the coupling procedure described in Appendix 7 the crucial question is the following. What difference in the results of Appendix 7 does it make if, roughly speaking, $\lambda_3 = 0$ at the instant where path 2 is coupled (before or after) another path? From Appendix 7 it appears that the following cases must be dealt with.

(A) Path 2 → Path 1.

(B) Path 4 → Path 2; Path 3 → Path 2; Path 1 → Path 2.

(A): First notice that on path 2 we have $\lambda_3(G'(K)-a-r) = 0$ from (55). Let t_{21}^- denote an instant just before the coupling instant t_{21} and let $\lambda_3(t_{21}^-) = 0$. Hence $G'(K)-a-r$ is undetermined at $t = t_{21}^-$ and we may have

(1) $G' < a+r$

or

(2) $G' = a+r$

or

(3) $G' > a+r$.

If (1) holds then path 2 cannot be coupled before path 1 since this would require a jump in K. Recall that $G' > a+r$ on path 1.
If (2) holds then the arguments of Appendix 7 (Path 2 → Path 1 → Path 2) show that this coupling is impossible. If (3) holds then consider the costate equation (47c) just before and just after the coupling instant t_{21}. Assume that t_{21} is not a switching point of the dividend policy, C. On path 2 we have

$$\dot{\lambda}_3(t_{21}^-) = \lambda_3(t_{21}^-) = 0$$

whereas on path 1 (t_{21}^+ denoting 'just after' t_{21})

$$\dot{\lambda}_3(t_{21}^+) = -(1-f)r\lambda_3(t_{21}^+) - \mu(1+k).$$

Whenever λ_3 is continuous at $t = t_{21}$ then $\lambda_3(t_{21}^-) = \lambda_3(t_{21}^+)$, implying $\dot{\lambda}_3(t_{21}^+) = -\mu(1+k) < 0$ and $\lambda_3(t_{21}^+) < 0$. This, however, violates the necessary Legendre-Clebsch condition that λ_3 be nonnegative. Hence, if $\lambda_3 = 0$ at $t = t_{21}^-$ then path 2 cannot be coupled before path 1. Following Feichtinger and Hartl (1986, Corollary 6.3) we know that λ_3 is continuous if

(i) I is continuous at t_{21} and the following constraint qualification (CQ) is satisfied:

vectors $(1,I,0)$ and $(-1,0,(1+k)(R+\bar{Z})-K)$

be linearly independent. CQ is satisfied for $I > 0$, but not for $I = 0$. The latter case is dealt with below.

(ii) I is discontinuous at t_{21} and t_{21} is an entry point where entry is non-tangential. Obviously, by (54), entry will be nontangential. If $I(t_{21}) = 0$ then CQ is not satisfied and continuity of λ_3 is not guaranteed. For this case we apply the following argument to prove infeasibility of the coupling path 2 → path 1. We need to distinguish the cases $C(t_{21}) = 0$ and $C(t_{21}) = C^M$.

(a) $C(t_{21}) = 0$. From (59) we have

$$0 = aK + (1+k)(1-f)[G(K) - (a + \frac{rk}{1+k})K]. \qquad \text{(A.6.1)}$$

But $G'(K) > a+r \Rightarrow G(K) > G'(K)K > (a+r)K > (a + \frac{rk}{1+k})K$ which shows that (A.6.1) cannot be satisfied. Hence, with $I(t_{21}) = 0$ the coupling is infeasible.

(b) $C(t_{21}) = C^M$. From (59) we have

$$C^{M}\bar{Z} = aK + (1+k)(1-f)[G(K) - (a + \frac{rk}{1+k})K]. \qquad (A.6.2)$$

We choose to regard (A.6.2) as a borderline case; it would only be by coincidence that $K(t_{21})$ would satisfy (A.6.2). Hence, with $I(t_{21}) = 0$ coupling is infeasible.

(B): Path 2 as final path

From (47c) we obtain that on path 2

$$\lambda_3 = \lambda_3(T)e^{(1-f)r(T-t)}.$$

Recall that $\lambda_3 \geqq 0$ is necessary for a singular path. Hence $\lambda_3 \geqq 0 \Leftarrow \lambda_3(T) \geqq 0$. Eq. (47c) shows that $\dot{\lambda}_3 \leqq 0$ for $\lambda_3 \geqq 0$. If $\lambda_3(T) > 0$ then, at a coupling instant t_{j2} (j = 1,3,4), we have $\lambda_3(t_{j2}) > 0$ and I^S, K^S are well defined on path 2. We choose to disregard the borderline case where $\lambda_3(T) = 0$, cf. (49c).

Appendix 7

Proof of Proposition 1:

First consider path 2 as the final path. Let t_{ij} be the coupling instant between path i and path j (such that path i precedes path j).

Path 4 → Path 2

On path 2, $G'(K) = a+r$; on path 4 $G'(K)$ increases since K decreases. Hence, for a coupling 4 → 2 it must hold that $G'(K) < a+r$ on path 4. This makes path 4 infeasible as an initial path (cf. (61)) and it must preceded by some other path. Path 4 cannot be preceded by neither path 1 not path 2 since on these paths we have $G'(K) > a+r$ and $G'(K) = a+r$, respectively. Could path 4 be preceded by path 3? Only if $G'(K) < a+r$ on path 3. But then the initial condition (61) cannot hold.

Path 3 → Path 2

The same conclusions as for path 4 → path 2 apply.

Path 1 → Path 2

This coupling is feasible. Notice that on path 1 it must hold that $\dot{K} > 0$ since $G'(K) > a+r$ and $G'(K)$ must decrease to $G'(K) = a+r$. Hence, on path 1, $\dot{K} > 0$ at least on some interval before the coupling instant t_{12}.

Now we have to check if path 2, 3 or 4 could precede the string path 1 → path 2. We will do this checking for the dividend policies of type 1, type 2, and type 3 (cf. (42)).

Path 4 → Path 1 → Path 2

Depending on the dividend policy we can have $C^* = 0$ or $C^* = C^M$ at the coupling instant t_{41}.

$C^*(t_{41}) = 0$:

From (5) we have $\dot{R} = (1-f)[G(K) - aK - rY]$. Since $Y = K - \bar{Z} - R$, and the state variables K and R are continuous, we have Y and $\dot{R}$ continuous. On path 4, $\dot{R} < 0 \Rightarrow \dot{R}(t_{41}^-) < 0$. But on path 1, $\dot{R} > 0 \Rightarrow \dot{R}(t_{41}^+) > 0$, which contradicts the continuity of $\dot{R}$. Notice that both paths 1 and 4 are boundary paths. We conclude that the coupling Path 4 → Path 1 is infeasible.

$C^*(t_{41}) = C^M$:

For path 4 it holds that $\dot{R}(t_{41}^-) < 0$. Since $C^* = C^M$ across t_{41}, $\dot{R}$ will be continuous at $t = t_{41}$. Hence $\dot{R}(t_{41}^+) < 0$ must hold on path 1. On path 4 we have $I = 0 \Rightarrow \dot{K} = -aK < 0$, and on path 1 we have $\dot{K} = I^b - aK < 0$ for $t \to t_{41}$ (from the right) since $\dot{R}(t_{41}^+) < 0$. If $I^b > 0$, then $\dot{K}$ will be discontinuous at t_{41}.[18] Assume that $\dot{Y}$ is continuous. Then $\dot{K} = \dot{Y} + \dot{R}$ should be continuous too, which contradicts what has just been stated. Hence $\dot{Y}$ is

---

18) We can safely take $I^b > 0$ since if $I^b = 0$, then path 4 and path 1 coincide and the coupling problem is irrelevant.

discontinuous but this contradicts that $\dot{Y}$ must be continuous since $\dot{Y} = k\dot{R}$. In conclusion, the coupling Path 4 → Path 1 is infeasible.

Path 3 → Path 1 → Path 2

Depending on the dividend policy we can have $C^*(t_{31}) = 0$ or $C^*(t_{31}) = C^M$.

$C^*(t_{31}) = 0$:

On path 3 it holds that $\lambda_2 > 0$ and $\dot{\lambda}_2 = a\lambda_2 - (1-f)[G'(K) - a - r]\lambda_3$, whereas on path 1 we have $\dot{\lambda}_2 = \lambda_2 = 0$. Consider the coupling instant t_{31}. Since path 1 is singular we must have $\lambda_3(t_{31}^+) > 0$ and $\dot{\lambda}_2(t_{31}^+) = \lambda_2(t_{31}^+) = 0$. Hence, on path 3 it must hold that

$$\dot{\lambda}_2(t_{31}^-) = -(1-f)[G'(K)-a-r]\lambda_3(t_{31}^-) \geqq 0.$$

Thus, $G'(K)-a-r \leqq 0$ on path 3 as $K \to K(t_{31}^-)$. However, on path 1 (which is to follow) $G'(K(t_{31}^+)-a-r > 0$ and to retain continuity of K it must hold that $G'(K(t_{31})) = a+r$. But $\dot{K} > 0$ on path 1, implying that $G'(K)$ becomes less than a+r as t increases. This contradicts the requirement (60). Hence, the coupling Path 3 → Path 1 is infeasible.

$C^*(t_{31}) = C^m$:

Consider the coupling instant t_{31} and recall that $(1+k)(\bar{Z}+R) - K \geqq 0$ on path 3 whereas $(1+k)(\bar{Z}+R) - K = 0$ on path 1. If t_{31} is an entry point, then $(1+k)(\bar{Z}+R) - K > 0$ on some interval $(t_{31}-\varepsilon, t_{31})$, $\varepsilon > 0$.[19] But then path 3 violates the initial condition (61).

Hence, if t_{31} is an entry point, then we must have a path which precedes path 3. Such a path can only be path 1. Consider the following policy string.

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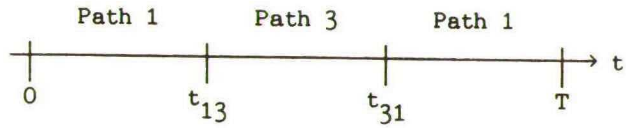
19) If t_{31} is not an entry point, then $(1+k)(\bar{Z}+R) - K = 0$ on some interval $(t_{31}-\delta, t_{31})$. We return to this case later on.



At $t = 0$ it holds that $G'(K) > a+r$ and this can continue to hold on path 1 if K does not increase so much that $G'(K) \leqq a+r$ occurs. To couple path 3 at t_{13} we must have $G'(K(t_{13}^{+})) \geqq a+r$. Working backwards from $t = T$ we must have $G'(K(t_{31}^{-})) \leqq a+r$ by the same argument as stated for the coupling Path 3 → Path 1 in the case where $C^{*}(t_{31}) = 0$. The contradiction now follows. Since $\dot{K} < 0$ on path 3, $G'(K)$ will increase as t increases from t_{13} to t_{31}.

It only remains to consider the policy string 1 → 3 → 1 when t_{31} is not an entry point.
Hence $(1+k)(\bar{Z}+R) - K = 0$ on some interval to the left of t_{31}. But then path 3 as well as path 1 are boundary arcs on an interval containing t_{31} and the arguments stated for the coupling Path 4 → Path 1 → Path 2 ($C^{*}=C^{M}$) apply. In conclusion, the coupling Path 3 → Path 1 is infeasible.

Path 2 → Path 1 → Path 2

$C^{*}(t_{21}) = 0$:

On path 2, $G'(K) = a+r$ and $\dot{K} = 0$. On path 1, $G'(K) > a+r$ and $\dot{K} > 0$, implying that $G'(K)$ decreases. Obviously this coupling is infeasible.

$C^{*}(t_{21}) = C^{M}$:

On paths 1 and 2 we have $\dot{\lambda}_2 = \lambda_2 = 0$. Hence

$$0 = (1-f)(G'(K)-a-r)\lambda_3 \qquad \text{on path 2}$$
$$0 = -(1-f)(G'(K)-a-r)\lambda_3 + \mu \qquad \text{on path 1.} \qquad (A.7.1)$$

Furthermore, $\lambda_3 > 0$ on path 1, $G'(K) = a+r$ on path 2, and $G'(K) > a+r$ on path 1. By continuity of K we must have $G'(K) = a+r$ on path 1 at $t = t_{21}$. But then $\mu(t_{21}^{+}) = 0$ in (A.7.1). Now, $\dot{R}(t_{21}^{-}) = 0$, and since $\dot{R}$ must be continuous across t_{21}, we must have $\dot{R}(t_{21}^{+}) = 0$. This implies $\dot{K}(t_{21}^{+}) = 0$ and

I^b = aK on path 1. Notice that I = aK and $\dot{K}(t_{21}^-) = 0$ on path 2. Hence I = aK is continuous across t_{21}. Extending by continuity of the control, I = aK on an interval $[t_{21}, t_{21} + \varepsilon]$ implies $\dot{K} = 0$ and hence $G'(K) = a+r$. This, however, violates the condition $G'(K) > a+r$ on path 1. In conclusion, the coupling Path 2 → Path 1 is infeasible.

We have now established that no path can precede the string path 1 → path 2. Our analysis also shows that when we take path 1 as the final path, no paths can be coupled before path 1. Hence, we have only two candidates for an optimal policy, namely

* Path 1 → Path 2
** Path 1 throughout [0,T]. Q.E.D.

Appendix 8

Proof of Lemma 6

Define

$$A = -\frac{1}{(1-f)r}\ln\left[\frac{G(K^s) - aK^s - rY_{12}}{G(K^s) - aK^s}\right] \tag{A.8.1}$$

and note that $A > 0$. Eq. (62) can be written as

$$t_o = t_{12} + A$$

which yields

$$\frac{dt_o}{dr} = \frac{dt_{12}}{dr} + \frac{dA}{dr}. \tag{A.8.2}$$

First we note that

$$\frac{dA}{dr} = \frac{1}{(1-f)r^2}\left\{\ln\left[\frac{G(K^s) - aK^s - rY_{12}}{G(K^s) - aK^s}\right] + \frac{rY_{12}}{G(K^s) - aK^s - rY_{12}}\right\} \gtreqless 0$$

for

$$\ln\left[\frac{G(K^s) - aK^s - rY_{12}}{G(K^s) - aK^s}\right] + \frac{rY_{12}}{G(K^s) - aK^s - rY_{12}} \gtreqless 0. \qquad \text{(A.8.3)}$$

To simplify the notation define

$$\alpha = rY_{12} \text{ and } \beta = G(K^s) - aK^s.$$

Hence, (A.8.3) becomes

$$\ln(\tfrac{\beta-\alpha}{\beta}) + \frac{\alpha}{\beta-\alpha} \gtreqless 0 \Leftrightarrow \ln(1 - \tfrac{\alpha}{\beta}) \gtreqless \frac{-\alpha}{\beta-\alpha}. \qquad \text{(A.8.4)}$$

Defining $z = \alpha/\beta$ yields in (A.8.4)

$$\ln(1-z) \gtreqless \frac{-z}{1-z} \Rightarrow \ln(1/(1-z) \lesseqgtr z/(1-z). \qquad \text{(A.8.5)}$$

Define

$$y = 1/(1-z)$$

which yields in (A.8.5)

$$\ln y \lesseqgtr y-1. \qquad \text{(A.8.6)}$$

But $y-1 > \ln y$ for all $y > 0$ except at $y = 1$ where $\ln y = y-1$. However, $y = 1$ cannot occur since $y = 1 \Rightarrow z = 0 \Rightarrow \alpha = 0 \Rightarrow r = 0$ and/or $y_{12} = 0$. Comparing (A.8.3) with (A.8.6) we conclude that $dA/dr > 0$.

The second step is to calculate dt_{12}/dr. On path 1, $\dot{K} > 0$ implies $\dot{\mu} < 0$ and μ decreases from $\mu(0)$ to zero at the start of path 2. The multiplier μ may jump, at $t = t_{12}$, from some positive $\mu(t_{12}^{+})$ to zero. From (56) we know that whenever μ is differentiable then

$$\dot{\mu} = (1-f)\dot{\lambda}_3(G'(K)-a-r) + (1-f)\lambda_3 G''(K)\dot{K}$$

and $\dot{\lambda}_3$ is given by (47c), i.e.

$$\dot{\mu} = (1-f)(G'(K)-a-r)(\mu(1+k)-(1-f)r\lambda_3) + (1-f)\lambda_3 G''(K)\dot{K}. \qquad (A.8.7)$$

In (A.8.7) define

$$\varphi = -(1-f)^2(G'(K)-a-r)r\,\lambda_3 + (1-f)\lambda_3 G''(K)\dot{K}$$

$$\psi = (1-f)(G'(K)-a-r)(1+k)$$

and notice that $\varphi < 0$ and $\psi > 0$ on path 1. Then (A.8.7) can be written

$$\dot{\mu} + \psi\mu = \varphi. \qquad (A.8.8)$$

By integration in (A.8.8) we obtain

$$\mu = \exp(-\int_0^t \psi(s)ds)\left\{\mu(0) + \int_0^t \varphi(s)\exp(\int_0^s \psi(\tau)d\tau)ds\right\}. \qquad (A.8.9)$$

At $t = t_{12}$ we have

$$\mu(t_{12}^+) = \exp(-\int_0^{t_{12}} \psi(s)ds)\left\{\mu(0) + \int_0^{t_{12}} \varphi(s)\exp(\int_0^s \psi(\tau)d\tau)ds\right\} \gtreqqless 0 \qquad (A.8.10)$$

From (A.7.1) (and the remarks below that equation) we know that $\mu(t_{12}^+) = 0$ and (A.8.10) holds with equality. Let

$$F(t_{12},r) = \mu(0) + \int_0^{t_{12}} \varphi(s)\exp(\int_0^s \psi(\tau)d\tau)ds = 0 \qquad (A.8.11)$$

and recall that ψ and φ depend on r. Hence, by (A.8.11), t_{12} is implicitly given as a function of r. By the implicit function theorem we obtain

$$\frac{dt_{12}}{dr} = -\partial F/\partial r/(\partial F/\partial t_{12}) =$$

$$-\frac{\int_0^{t_{12}}\left[\varphi(s)\exp(\int_0^s \psi(\tau)d\tau)\int_0^s \frac{\partial\psi}{dr}d\tau + \exp(\int_0^s \psi(\tau)d\tau)\ \frac{\partial\varphi(s)}{\partial r}\right]ds}{\varphi(t_{12})\exp(\int_0^{t_{12}} \psi(\tau)d\tau)} =$$

$$- 1/(\varphi(t_{12})\exp(\int_0^{t_{12}} \psi(\tau)d\tau)) \times \qquad (A.8.12)$$

$$\int_0^{t_{12}} \exp(\int_0^s \psi(\tau)d\tau)\{-\varphi(s)(1-f)(1+k)s - (1-f)(G'(K)-a-r)(1-f)\lambda_3$$

$$+ (1-f)^2 r\ \lambda_3\}ds.$$

In (A.8.12) we have $\varphi(t_{12}) < 0$ and

$$\varphi(s)(1+k)s + (1-f)(G'(K)-a-r)\lambda_3 - (1-f)r\ \lambda_3 < 0 \qquad (A.8.13)$$

is sufficient for $dt_{12}/dr > 0$. But $G'(K(t_{12}^+)) = a+r$ must hold to guarantee continuity of K (cf. the remarks below (A.7.1)). Then (A.8.13) holds since $\varphi < 0$.

We have shown that $dA/dr > 0$, $dt_{12}/dr > 0$ and using (A.8.2) completes the proof. Q.E.D.

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