[redacted]

THE EQUAL GAIN SPLITTING RULE FOR SEQUENCING SITUATIONS AND THE GENERAL NUCLEOLUS

Imma Curiel, Herbert Hamers,
Jos Potters, Stef Tijs

Research Memorandum FEW 629

Communicated by Dr. P.E.M. Borm

THE EQUAL GAIN SPLITTING RULE FOR SEQUENCING SITUATIONS AND THE GENERAL NUCLEOLUS

IMMA CURIEL[1] HERBERT HAMERS [2] JOS POTTERS [3]

STEF TIJS[2]

December 1, 1993

Abstract

The Equal Gain Splitting (EGS) rule is introduced by *Curiel, Pederzoli* and *Tijs (1989)* in the context of sequencing situations. This paper gives an alternative characterization of the EGS rule. For this characterization we consider the efficiency property and a fairness property on the set $\mathcal{H}$, consisting of the coalitions of the form $\{1, 2, ..., i\}, 1 \leq i \leq n-1$ and their complements. The set $\mathcal{H}$ also yields a generalized core for sequencing games. It is shown that this core is a convex hull of 2^{n-1} vectors and that the EGS rule is the average of these vectors. Moreover, it is shown that the EGS rule coincides with the general nucleolus with respect to $\mathcal{H}$ as introduced in *Maschler, Potters* and *Tijs (1992)*. Finally, necessary and sufficient conditions for the EGS rule to coincide with the nucleolus are given.

[1]Department of Mathematics and Statistics, University of Maryland Baltimore County, MD 21228, USA.

[2]Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

[3]Department of Mathematics, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands. Herbert Hamers is financially supported by the Dutch Organization for Scientific Research(NWO).

1 Introduction

In one-machine sequencing situations each agent (player) has one job that has to be processed on a single machine. Each job is specified by its processing time, the time the machine takes to handle the job. We assume that the cost of a player depend linearly on the completion time of his job. Furthermore, there is an initial order on the jobs of the agents before the processing of the machine starts.

Each group of agents (coalition) is allowed to obtain cost savings by rearranging their jobs in a way that is admissible with respect to the initial order. By defining the worth of a coalition as the maximum cost savings a coalition can make by admissible rearrangements, we obtain a cooperative game called sequencing game, related to the one machine sequencing situation. This game theoretic approach has been taken in *Curiel, Pederzoli* and *Tijs (1989)*. They also defined the EGS (equal gain splitting) rule on the class of sequencing situations to obtain a division of the value of the grand coalition. It has been shown that the EGS rule is in the core of the corresponding sequencing game. *Curiel, Potters, Rajendra Prasad, Tijs* and *Veldman (1993)* considered the class of component additive games and introduced the β-rule as the average of two marginal vectors. They showed that the class of sequencing games is contained in the class of component additive games. Moreover, they showed that the β-rule coincides with the EGS rule in the class of sequencing games. Hence, the EGS rule can be regarded as an alternative to the Shapley value. It takes the average of two marginal vectors, while the Shapley value takes the average of all marginal vectors.

In this paper an alternative characterization for the EGS rule is presented. For this purpose we introduce the class $\mathcal{H}$ of coalitions of the form $\{1,...,i\}$ or $\{i+1,...,n\}, 1 \leq i \leq n-1$. The EGS rule will be characterized by efficiency and the fair head-tail split property. A division rule satisfies the fair head-tail property if the marginal benefits of a union of each head and its complement is equally shared. Further, the set $\mathcal{H}$ yields a generalized core. It is shown that this core is the convex hull of 2^{n-1} vectors and that the EGS rule is the average of these vectors. It is shown that the general nucleolus with respect to $\mathcal{H}$, as introduced by *Maschler, Potters* and *Tijs (1992)*, coincides with the

EGS rule. Finally, necessary and sufficient conditions for the EGS rule to coincide with the nucleolus are given.

2 Sequencing situations and the EGS rule

This section describes sequencing situations and the corresponding sequencing games. Further, we recall the definition of the EGS rule and some well known facts from game theory. For the EGS rule we give a new characterization by efficiency and the fair head-tail split property.

In a one machine sequencing situation there is a queue of agents, each with one job, to be processed by one machine. The finite set of agents is denoted by $N = \{1, ..., n\}$. The position of the agents in the queue is described by a bijection $\sigma : N \rightarrow \{1, ..., n\}$. Specifically, $\sigma(i) = j$ means that player i is in position j. We assume that there is an initial order $\sigma_0 : N \rightarrow \{1, ..., n\}$ on the jobs of the players before the processing of the machine starts. We number the agents in such way that the initial queue corresponds with σ_0 is defined by $\sigma_0(i) = i$ for all $i \in N$. The processing time p_i of the job of player i is the time the machine takes to handle this job. Further, it is assumed that every agent has an affine cost function $c_i : [0, \infty) \rightarrow \mathbf{R}$ defined by $c_i(t) = \alpha_i t + \beta_i$ with $\alpha_i > 0, \beta_i \in \mathbf{R}$. So $c_i(t)$ is the cost for agent i if he is t units of time in the system.

A sequencing situation as described above is denoted by (N, p, α), where $N = \{1, ..., n\}, p = (p_i)_{i \in N}$ and $\alpha = (\alpha_i)_{i \in N}$. The vector $\beta = (\beta_i)_{i \in N}$ is omitted in the description of the sequencing situation since the fixed costs it represents are independent of the positions of the players in the queue.

For player $i \in \{1, ..., n-1\}$ we define the following sets with respect to the initial order σ_0. The head of player i is the set $\{1, 2, ..., i\}$ and the tail of player i is the set $\{i, i+1, ..., n\}$. Note that the head (tail) of player i contains the set of players that precedes (follows) i in the initial order σ_0. The collection of coalitions consisting of all heads and all tails is denoted by $\mathcal{H}$.

If the processing order is given by $\sigma : N \rightarrow \{1, ..., n\}$ then the completion time of player i is equal to $C(\sigma, i) := \sum_{j:\sigma(j) \leq \sigma(i)} p_j$. The total costs of $c_\sigma(S)$ of a coalition $S \subset N$, is given by $c_\sigma(S) := \sum_{i\in S} \alpha_i(C(\sigma, i)) + \beta_i$.

The (maximal) cost savings of a coalitions S depend on the set of admissible rearrangements of this coalition. A bijection $\sigma : N \to \{1, ..., n\}$ is called admissible for S if it satisfies the following two conditions:

(i) The completion time of each agent outside the coalition S is equal to his completion time in the initial order: $C(\sigma_0, i) = C(\sigma, i)$ for all $i \in N\backslash S$.

(ii) the players of S are not allowed to jump over players outside S:

$$\{1, ..., i\} \cap N\backslash S = \{j \mid \sigma(j) \leq \sigma(i)\} \cap N\backslash S \text{ for all } i \in S.$$

The set of admissible rearrangements for a coalition S is denoted by Σ_S.

Before the cooperative sequencing game is given, we recall some well known facts concerning cooperative games. A *cooperative game* is a pair (N, v) where N is a finite set of players and v is a mapping $v : 2^N \to \mathbf{R}$ with $v(\emptyset) = 0$ and where 2^N is the collection of all subsets of N.

A game (N, v) is called *superadditive* if for all coalitions $S, T \in 2^N$ with $S \cap T = \emptyset$ we have

$$v(S \cup T) \geq v(S) + v(T)$$

Cooperative game theory focuses on 'fair' and/or 'stable' division rules for the worth $v(N)$ of the grand coalition. A core element $x = (x_i)_{i\in N} \in \mathbf{R^N}$ is such that no coalition has an incentive to split off, i.e.

$$\sum_{i\in N} x_i = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \in 2^N.$$

where $x(S) = \sum_{i\in S} x_i$. The core $C(v)$ consists of all core elements. A game is called balanced if its core is non-empty.

For the nucleolus of a game, introduced by *Schmeidler (1969)*, we need the following notation. Let $F := (F_T)_{T\in 2^N}$, where F_T is the excess function corresponding to T defined by $F_T(x) := v(S) - x(S)$. The function $\Theta : \mathbf{R^{2^n}} \to \mathbf{R^{2^n}}$ is the map that orders the coordinates in a weakly decreasing order. Then the nucleolus of a game (N, v) is defined by

$$\eta(N, v) = \{x \in I(v) \mid \Theta \circ F(x) \preceq_L \Theta \circ F(y) \text{ for all } y \in I(v)\}$$

where $I(v) := \{x \mid x(N) = v(N), x(i) \geq v(i) \text{ for all } i \in N\}$ is the imputation set of (N, v).

Given a sequencing situation (N, p, α) the worth of a coalition S of the corresponding sequencing game*(Curiel et al.(1989))* is defined as the maximal cost savings the coalition can achieve by means of an admissible rearrangement. Formally,

$$v(S) = \max_{\sigma \in \Sigma_S} \{ \sum_{i \in S} (\alpha_i C(\sigma_0, i) + \beta_i) - \sum_{i \in S} (\alpha_i C(\sigma, i) + \beta_i) \} \tag{1}$$

Curiel et al. (1993) showed that (1) is equivalent to the following expression:

$$v(S) = \sum_{i<j} g_{ij} u_{\{i,i+1,\ldots,j\}}(S)$$

Here $g_{ij} := \max\{\alpha_j p_i - \alpha_i p_j, 0\}$ represents the gain attainable for player i and j in case player i is directly in front of player j. The game $u_{\{i,i+1,\ldots,j\}}$ is the simple game defined by

$$u_{\{i,i+1,\ldots,j\}}(S) := \begin{cases} 1 & \text{if } \{i, i+1, \ldots, j\} \subset S \\ 0 & otherwise \end{cases}$$

The Equal Gain Splitting (EGS) rule in a sequencing situation (N, p, α) is defined for all $i \in N$ by

$$EGS_i(N, p, \alpha) := \frac{1}{2} \sum_{j<i} g_{ji} + \frac{1}{2} \sum_{j>i} g_{ij}$$

Note that the optimal order of a queue can be obtained from the initial order by switching neighbours only. In the EGS rule a player obtains half of the gains of all neighbour switches he is actually involved in, to obtain the optimal order.

Example 1 Let $N = \{1, 2, 3\}, p = (2, 2, 1)$ and $\alpha = (4, 6, 5)$. Then $g_{12} = 4, g_{23} = 4$ and $g_{13} = 6$. This implies that $EGS_1(N, p, \alpha) = \frac{1}{2}(4 + 6) = 5, EGS_2(N, p, \alpha) = \frac{1}{2}(4 + 4) = 4, EGS_3(N, p, \alpha) = \frac{1}{2}(6 + 4) = 5$. □

Curiel et al. (1989) showed that a sequencing game is superadditive (even convex) and that the EGS rule is in the core of the sequencing game. Moreover, they provided a characterization of the EGS rule by using efficiency, dummy property, equivalence property and switch property. We will give an other characterization in terms of efficiency and the fair head-tail split property.

Definition 1 *Let (N, v) be a sequencing game. Then a vector $x \in R^n$ satisfies the fair head-tail split property if*

$$x(S) - v(S) = x(S^c) - v(S^c) = \frac{1}{2} \Delta_S \text{ for all } S \in \mathcal{H}$$

where $\Delta_S = v(N) - v(S) - v(S^c)$.

Note that from the fair head-tail property it follows that the excesses $F_S(x)$ and $F_{S^c}(x)$ are equal for all $S \in \mathcal{H}$. Before the characterization of the EGS rule is given we recall the definition of a σ_0 component additive game and of the β-rule, both introduced by *Curiel, Potters, Rajendra Prasad, Tijs* and *Veltman (1993)*. A coalition S is called connected with respect to σ_0 if for al $i, j \in S$ and $k \in N$, $\sigma_0(i) < \sigma_0(k) < \sigma_0(j)$ implies $k \in S$. Let T be a coalition that is not connected. A coalition S is a component of T if $S \subset T$, S is connected and for every $i \in T \backslash S, T \cup \{i\}$ is not connected. The components of T form a partition of T which we denote by $T \backslash \sigma_0$. A game (N, v) is called a σ_0 component addivitive games if it satisfies the following three conditions:

(i) $v(i) = 0$ for all $i \in N$.

(ii) (N, v) is superadditive.

(iii) $v(T) = \sum_{S \in T \backslash \sigma_0} v(S)$

Let (N, v) be a σ_0 component additive game then

$$\beta_i(v) = \frac{1}{2}\{v(\{1, ..., i\}) - v(\{1, ..., i-1\}) + v(\{i, ..., n\}) - v(\{i+1, ..., n\})\} \tag{2}$$

Curiel et al. (1993) showed that the class of sequencing games is contained in the class of σ_0 component additive games and that the β rule restricted to the class of sequencing games coincides with the EGS rule. It is immediately clear that the β-rule satisfies efficiency and fair head-tail split property.

Theorem 1 *Let (N, v) be the sequencing game corresponding to the sequencing situation (N, p, α). Then $EGS(N, p, \alpha)$ is the unique rule that satisfies efficiency and the fair head-tail split property.*

PROOF: From (2) and the fact that the EGS rule coincides with the β-rule on the class of sequencing games it follows that the EGS rule satisfies the two properties. Conversely, assume that a vector x has both properties. Then it follows from the fair head-tail split property that for $k \in N \backslash \{n\}$

$$x(\{1, ..., k\}) - v(\{1, ..., k\}) = x(\{k+1, ..., n\}) - v(\{k+1, ..., n\}) \tag{3}$$

and

$$x(\{1, ..., k-1\}) - v(\{1, ..., k-1\}) = x(\{k, ..., n\}) - v(\{k, ..., n\}) \tag{4}$$

From (3) and (4) it follows by subtracting that

$$x(\{k\}) - v(\{1, ..., k\}) + v(\{1, ..., k-1\}) = -x(\{k\}) - v(\{k+1, ..., n\}) + v(\{k, ..., n\})$$

Hence, $x(\{k\}) = \beta_k(v) = EGS_k(N,p,\alpha)$ for all $k \in N\backslash\{n\}$. From the efficiency it follows immediately that $x(\{n\}) = EGS_n(N,p,\alpha)$. □

3 The head tail core

In this section the head tail core is introduced. We characterize the extreme points of this set and show that the EGS rule is the average of these extreme points.

The head tail core $C^{\mathcal{H}}(v)$ of a sequencing game (N,v) is defined by

$$C^{\mathcal{H}}(v) = \{x \in \mathbf{R}^n \mid x(N) = v(N), x(S) \geq v(S) \text{ for all } S \in \mathcal{H}\}. \tag{5}$$

Obviously, we have that the core $C(v)$ of the game (N,v) is contained in the head tail core $C^{\mathcal{H}}(v)$. The following example shows that the core of a game can be a strict subset of the head tail core.

Example 2 Consider the sequencing situation of example 1. Then the corresponding sequencing game is given by $v(\{1,2\}) = v(\{2,3\}) = 4$, $v(\{1,2,3\}) = 14$ and $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = 0$. An extreme point of $C^{\mathcal{H}}(v)$ is the vector $(10,-6,10)$. Obviously, this vector is not an element of $C(v)$ (see figure 1). □

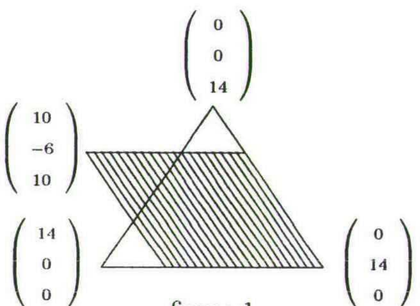

figure 1

The dual game (N,v^*) of (N,v) is defined by $v^*(S) = v(N) - v(S^c)$ for all $S \subset N$. Note that $v(N) = v^*(N)$ and that $v(S) \leq v^*(S)$ for all S by the superadditivity of (N,v).

Now (5) is equivalent to

$$C^{\mathcal{H}}(v) = \{x \in R^n \mid v(\{1,...,i\}) \leq \sum_{k=1}^{i} x_k \leq v^*(\{1,...,i\}) \text{ for all } i \in N\}$$

$$= \{x \in R^n \mid x = L^{-1}y, v(\{1,...,i\}) \leq y_i \leq v^*(\{1,...,i\}) \text{ for all } i \in N\}$$

where L is the $n \times n$ non-singular lower triangular matrix with ones on and below the diagonal, and zeros above the diagonal. It is easy to see that the linear map L gives a 1-1 correspondence between $C^{\mathcal{H}}$ and the set D defined by

$$D := \{y \in R^n \mid v(\{1,...,i\}) \leq y_i \leq v^*(\{1,...,i\}) \text{ for all } i \in N\}$$

Consequently, there is a 1-1 correspondence between the extreme points of both sets. Clearly, the extreme points of D correspond to a system of n equations of the form

$$y_i = a_i,\ i = 1,...,n \quad \text{where } a_i \in \{v(\{1,...,i\}), v^*(\{1,...,i\})\} \quad \text{for all } i \in N$$

Let $J \subset N$. We put $h^J = L^{-1}(y^J)$ where y^J is the extreme point of D corresponding to the system of equations given by

$$y_i^J = \begin{cases} v(\{1,...,i\}) & i \in N \backslash J \\ v^*(\{1,...,i\}) & i \in J \end{cases}$$

Then $E = \{h^J \mid J \subset N\}$ is the set of extreme points of $C^{\mathcal{H}}(v)$. Consequently, we have

Theorem 2 $\quad C^{\mathcal{H}}(v) = conv\{h^J \mid h^J \in E\}$.

where h^J corresponds to the solutions of the set of equations

$$\begin{aligned} &\textstyle\sum_{k=1}^{i} x_k = v^*(\{1,...,i\}) \quad i \in J \\ &\textstyle\sum_{k=1}^{i} x_k = v(\{1,...,i\}) \quad i \in N \backslash J \end{aligned} \tag{6}$$

The number of extreme points is $2^{|I|}$ where $I = \{i \in N \mid v(\{1,...,i\}) \neq v^*(\{1,...,i\})\}$. Hence, in the generic case we have 2^{n-1} different extreme points since $v^*(N) = v(N)$ and consequently $n \notin I$.

The following theorem shows that the average of the extreme points of the head tail core generated by (6) is the EGS rule.

Theorem 3 *Let (N, p, α) be a sequencing situation and (N, v) the corresponding sequencing game. Then $EGS(N, p, \alpha) = \frac{1}{2^{n-1}} \sum_{J \subset N} h^J$.*

PROOF: Note that for all $i \in N$ and all $J \subset N$ we have $y_i^J + y_i^{N \backslash J} = v(\{1,...,i\}) + v^*(\{1,...,i\})$. Consequently, for all $J \subset N$ it follows that $L(h^J + h^{N \backslash J}) = y^J + y^{N \backslash J} = y^{\emptyset} + y^N = L(h^{\emptyset} + h^N)$. Since $h^{\emptyset} + h^N = 2EGS(N, p, \alpha)$ we have

$$\frac{1}{2^{n-1}} \sum_{J \subset N} h^J = \frac{1}{2^n} \sum_{J \subset N} (h^J + h^{N \backslash J}) = \frac{1}{2^n} 2^n EGS(N, p, \alpha) = EGS(N, p, \alpha).$$

□

4 The EGS rule and the general nucleolus

In this section it is shown that the EGS rule has some similarity with the nucleolus in case the set of coalitions is restricted to the set $\mathcal{H}$. In particular, it is shown that the general nucleolus, introduced by *Maschler et al.(1993)*, coincides with the EGS rule.

The following example shows that the EGS rule is not necessarily equal to the nucleolus of a sequencing game.

Example 3 Let $N = \{1,2,3\}, p = (2,2,1)$ and $\alpha = (4,6,5)$. Then $v(\{1,2\}) = v(\{2,3\}) = 4$, $v(\{1,2,3\}) = 14$ and $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{1,3\}) = 0$. The nucleolus of the game is $(4\frac{2}{3}, 4\frac{2}{3}, 4\frac{2}{3})$, whereas the EGS rule equals $(5,4,5)$. □

In *Maschler et al.(1993)* a general nucleolus is defined on the class of truncated games, an extension of the class of cooperative games. A truncated game with respect to the cooperative game (N, v) is a quadruplet $(N, \mathcal{S}, v, \Pi)$, where N is the player set, $\mathcal{S}$ is a subset of $2^N\backslash\{\emptyset, N\}$ called the set of permissible coalitions, $v : \mathcal{S} \to \mathbf{R}$ is the characteristic function and Π a set of permissible pre-imputations.

Let $F := (F_T)_{T\in\mathcal{S}}$, where F_T are the excess functions defined by $F_T(x) := v(T) - x(T)$. The function $\Theta : \mathbf{R}^{|\mathcal{S}|} \to \mathbf{R}^{|\mathcal{S}|}$ is the map that orders the coordinates in a weakly decreasing order. Then the general nucleolus of the truncated game $(N, \mathcal{S}, v, \Pi)$ is defined by

$$\mathcal{N}(N, \mathcal{S}, v, \Pi) = \{x \in \Pi \mid \Theta \circ F(x) \preceq_L \Theta \circ F(y) \text{ for all } y \in \Pi\}.$$

If (N, p, α) is a sequencing situation and (N, v) the corresponding sequencing game, we introduce the truncated game $(N, \mathcal{H}, \overline{v}, \Pi)$ where $\overline{v}(S) = v(S)$ for all $S \in \mathcal{H}$ and $\Pi := \{x \in \mathbf{R^n} \mid x(N) = v(N)\}$, the pre-imputation set of (N, v). The following theorem states that the EGS rule is the general nucleolus of the truncated game $(N, \mathcal{H}, \overline{v}, \Pi)$.

Theorem 4 *Let $(N, \mathcal{H}, v, \Pi)$ be the truncated game of the sequencing game (N, v) corresponding to (N, p, α). Then $EGS(N, p, \alpha)$ coincides with $\mathcal{N}(N, \mathcal{H}, v, \Pi)$.*

PROOF: First it is shown that $\mathcal{N}(N, \mathcal{S}, v, \Pi)$ is a non-empty set.

Let $y \in \Pi\backslash C^{\mathcal{H}}(v)$, then there exists an $S \in \mathcal{H}$ such that $y(S) < v(S)$. Since for any $x \in C^{\mathcal{H}}(v)$ it holds that $x(T) \geq v(T)$ for all $T \in \mathcal{H}$ we have that $\Theta \circ F(x) \preceq_L \Theta \circ F(y)$. Hence, $\mathcal{N}(N, \mathcal{S}, v, \Pi) = \{x \in C^{\mathcal{H}} \mid \Theta \circ F(x) \preceq_L \Theta \circ F(y) \text{ for all } y \in C^{\mathcal{H}}(v)\}$. Since $C^{\mathcal{H}}(v)$ is a compact set and $\Theta \circ F$ is a continuous map it follows that $\mathcal{N}(N, \mathcal{S}, v, \Pi) \neq \emptyset$.

Let $x \in \mathcal{N}(N, \mathcal{S}, v, \Pi)$. Suppose there exists a $k \in N$ and an $\epsilon > 0$ such that

$$[v(\{1,...,k\}) - x(\{1,...,k\})] - [v(\{k+1,...,n\}) - x(\{k+1,...,n\})] > \epsilon$$

Take $y \in \Pi$ such that $y_i = x_i$ for all $i \in N \backslash \{k, k+1\}$, $y_k = x_k + \epsilon$ and $y_{k+1} = x_{k+1} - \epsilon$. Then

$$v(T) - x(T) = v(T) - y(T) \quad \text{for all } T \in \mathcal{H} \backslash \{\{1,...,k\}, \{k+1,...,n\}\},$$

$$v(\{1,...,k\}) - x(\{1,...,k\}) = v(\{1,...,k\}) - y(\{1,...,k\}) + \epsilon$$

and

$$v(\{k+1,...,n\}) - x(\{k+1,...,n\}) = v(\{k+1,...,n\}) - y(\{k+1,...,n\}) - \epsilon.$$

Hence, $\Theta \circ F(y) \prec_L \Theta \circ F(x)$. This is in contradiction with the definition of $\mathcal{N}$. In a similar way we can show that there exists no $k \in N$ such that

$$[v(\{1,...,k\}) - x(\{1,...,k\})] - [v(\{k+1,...,n\}) - x(\{k+1,...,n\})] < 0.$$

This implies that for all $S \in \mathcal{H}$ we have $F_S(x) = F_{S^c}(x)$. Then it follows from theorem 1 that $x = EGS(N, p, \alpha)$. □

5 Final remarks

In section 2 we stated that the class of σ_0 component additive games contains the class of sequencing games and that the β-rule restricted on the class of sequencing games coincides with the EGS rule. In the proofs of the previous results of this paper we only used the properties of a σ_0 component additive game and the expression of the β value. This implies that all results can be extended to the β rule with respect to σ_0 component additive games.

Finally, a necessary and sufficient condition on a sequencing situation is given such that the EGS rule and the nucleolus of the corresponding sequencing game coincide.

Theorem 5 *Let (N, p, α) be a sequencing situation and (N, v) be the corresponding sequencing game. Then $EGS(N, p, \alpha) = \eta(N, v)$ if and only if*

$$\sum_{a=1}^{p-1} \sum_{b=i+1}^{n} g_{ab} \leq \sum_{a=1}^{p-1} \sum_{b=p}^{i} g_{ab} \quad \textit{for all } 1 \leq p \leq i < n$$

$$\sum_{a=1}^{p-1} \sum_{b=i+1}^{n} g_{ab} \leq \sum_{a=p}^{i} \sum_{b=i+1}^{n} g_{ab} \quad \textit{for all } 1 < p \leq i \leq n.$$

PROOF: From *Potters* and *Reijnierse (1992)* it follows that the nucleolus of a sequencing game is the unique point x satisfying efficiency and $\bar{s}_{ii+1}(x) = \bar{s}_{i+1i}(x)$ for all $i \in N \setminus \{n\}$. Here $\bar{s}_{ij}$ is defined as follows

$$\bar{s}_{ij}(x) := \max\{v(T) - x(T) \mid i \in T \subset N \setminus \{j\},\ T \text{ is connected}\}.$$

From this and the fact that the EGS-rule is the unique rule that satisfies efficiency and the fair head-tail property, it follows that the nucleolus is equal to the allocation given by the EGS-rule when for all $i \in N \setminus \{n\}$ the maximum in the definition of $\bar{s}_{ii+1}$ is achieved at $T = \{1, ..., i\}$ and the maximum in the definition of $\bar{s}_{i+1i}$ is achieved at $T = \{i+1, ..., n\}$ when we take x to be equal to the allocation given by the EGS-rule. Since only connected coalitions have to be considered for these maxima we obtain the following inequalities.

$$F_{\{1,\dots,i\}}(x) \geq F_{\{p,\dots,i\}}(x) \text{ for all } 1 \leq p \leq i < n$$

$$F_{\{i+1,\dots,n\}}(x) \geq F_{\{i+1,\dots,q\}}(x) \text{ for all } i+1 \leq q \leq n.$$

where F is the excess function. This set of inequalities is equivalent to

$$F_{\{1,\dots,i\}}(x) \geq F_{\{p,\dots,i\}}(x) \text{ for all } 1 \leq p \leq i < n$$

$$F_{\{p,\dots,n\}}(x) \geq F_{\{p,\dots,i\}}(x) \text{ for all } 1 < p \leq i \leq n. \tag{7}$$

Since

$$v(\{p, ..., i\}) = \sum_{a=p}^{i-1} \sum_{b=a+1}^{i} g_{ab} = \frac{1}{2}\sum_{a=p}^{i-1} \sum_{b=a+1}^{i} g_{ab} + \frac{1}{2}\sum_{b=p+1}^{i} \sum_{a=p}^{b-1} g_{ab}$$

and

$$x(\{p, ..., i\}) = \frac{1}{2}\sum_{a=p}^{i} \sum_{b=a+1}^{n} g_{ab} + \frac{1}{2}\sum_{b=p}^{i} \sum_{a=1}^{b-1} g_{ab}$$

we have for any $p, i \in N, p < i$ that

$$v(\{p, ..., i\}) - x(\{p, ..., i\}) = -\frac{1}{2}\left(\sum_{a=p}^{i-1} \sum_{b=i+1}^{n} g_{ab} + \sum_{b=i+1}^{n} g_{ib} + \sum_{b=p+1}^{i} \sum_{a=1}^{p-1} g_{ab} + \sum_{a=1}^{p-1} g_{ap}\right)$$

$$= -\frac{1}{2}\left(\sum_{a=p}^{i} \sum_{b=i+1}^{n} g_{ab} + \sum_{b=p}^{i} \sum_{a=1}^{p-1} g_{ab}\right).$$

Define $G_{I\times J} = \sum_{a\in I}\sum_{b\in J} g_{ab}$ for any connected $I, J \subset N$ with $g_{ab} = 0$ if $a \geq b$. Then (7) is equivalent to

$$G_{\{1,\dots,i\}\times\{i+1,\dots,n\}} \leq G_{\{p,\dots,i\}\times\{i+1,\dots,n\}} + G_{\{1,\dots,p-1\}\times\{p,\dots,i\}} \text{ for all } 1 \leq p \leq i < n$$

$$G_{\{1,\dots,p-1\}\times\{p,\dots,n\}} \leq G_{\{p,\dots,i\}\times\{i+1,\dots,n\}} + G_{\{1,\dots,p-1\}\times\{p,\dots,i\}} \text{ for all } 1 < p \leq i \leq n.$$

These reduce to

$$G_{\{1,\dots,p-1\}\times\{i+1,\dots,n\}} \leq G_{\{1,\dots,p-1\}\times\{p,\dots,i\}} \text{ for all } 1 \leq p \leq i < n$$

$$G_{\{1,\ldots,p-1\}\times\{i+1,\ldots,n\}} \leq G_{\{p,\ldots,i\}\times\{i+1,\ldots,n\}} \text{ for all } 1 < p \leq i \leq n.$$

which proves the if part.

In the only if part we have that $EGS(N,p,\alpha) = \eta(N,v)$. The efficiency and the head-tail fairness implies that the maximum in the definition of s_{ii+1} is achieved at $T' = \{1,...,i\}$ and the maximum in the definition of $\bar{s}_{i+1i}$ is achieved at $T = \{i+1,...,n\}$. From the if part follows the result immediately. $\square$

References:

CURIEL I., PEDERZOLI G., TIJS S. (1989), Sequencing games, *European Journal of Operational Research*, **40**, 344-351.

CURIEL I., POTTERS J., RAJENDRA PRASAD V., TIJS S., VELTMAN B. (1993), Sequencing and Cooperation, *To appear in: Operations Research.*

CURIEL I., POTTERS J., RAJENDRA PRASAD V., TIJS S., VELTMAN B.(1993), Cooperation in one machine scheduling, *Zeitschrift für Operations Research*, no. 38,113-129.

MASCHLER M., POTTERS J. , TIJS S. (1992), The general nucleolus and the reduced game property, *International Journal of Game Theory*, **21**, 85-106.

POTTERS J., REIJNIERSE H. (1992), Γ component additive games, *Department of Mathematics, University Nijmegen (KUN)*, no. 9228.

SCHMEIDLER D. (1969), The nucleolus of a characteristic function game, *Siam Journal of Applied Mathematics*, **17**,1163-1170.

IN 1992 REEDS VERSCHENEN

532 F.G. van den Heuvel en M.R.M. Turlings
Privatisering van arbeidsongeschiktheidsregelingen
Refereed by Prof.Dr. H. Verbon

533 J.C. Engwerda, L.G. van Willigenburg
LQ-control of sampled continuous-time systems
Refereed by Prof.dr. J.M. Schumacher

534 J.C. Engwerda, A.C.M. Ran & A.L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^*X \; A = Q$.
Refereed by Prof.dr. J.M. Schumacher

535 Jacob C. Engwerda
The indefinite LQ-problem: the finite planning horizon case
Refereed by Prof.dr. J.M. Schumacher

536 Gert-Jan Otten, Peter Borm, Ton Storcken, Stef Tijs
Effectivity functions and associated claim game correspondences
Refereed by Prof.dr. P.H.M. Ruys

537 Jack P.C. Kleijnen, Gustav A. Alink
Validation of simulation models: mine-hunting case-study
Refereed by Prof.dr.ir. C.A.T. Takkenberg

538 V. Feltkamp and A. van den Nouweland
Controlled Communication Networks
Refereed by Prof.dr. S.H. Tijs

539 A. van Schaik
Productivity, Labour Force Participation and the Solow Growth Model
Refereed by Prof.dr. Th.C.M.J. van de Klundert

540 J.J.G. Lemmen and S.C.W. Eijffinger
The Degree of Financial Integration in the European Community
Refereed by Prof.dr. A.B.T.M. van Schaik

541 J. Bell, P.K. Jagersma
Internationale Joint Ventures
Refereed by Prof.dr. H.G. Barkema

542 Jack P.C. Kleijnen
Verification and validation of simulation models
Refereed by Prof.dr.ir. C.A.T. Takkenberg

543 Gert Nieuwenhuis
Uniform Approximations of the Stationary and Palm Distributions of Marked Point Processes
Refereed by Prof.dr. B.B. van der Genugten

544 R. Heuts, P. Nederstigt, W. Roebroek, W. Selen
Multi-Product Cycling with Packaging in the Process Industry
Refereed by Prof.dr. F.A. van der Duyn Schouten

545 J.C. Engwerda
Calculation of an approximate solution of the infinite time-varying LQ-problem
Refereed by Prof.dr. J.M. Schumacher

546 Raymond H.J.M. Gradus and Peter M. Kort
On time-inconsistency and pollution control: a macroeconomic approach
Refereed by Prof.dr. A.J. de Zeeuw

547 Drs. Dolph Cantrijn en Dr. Rezaul Kabir
De Invloed van de Invoering van Preferente Beschermingsaandelen op Aandelenkoersen van Nederlandse Beursgenoteerde Ondernemingen
Refereed by Prof.dr. P.W. Moerland

548 Sylvester Eijffinger and Eric Schaling
Central bank independence: criteria and indices
Refereed by Prof.dr. J.J. Sijben

549 Drs. A. Schmeits
Geïntegreerde investerings- en financieringsbeslissingen; Implicaties voor Capital Budgeting
Refereed by Prof.dr. P.W. Moerland

550 Peter M. Kort
Standards versus standards: the effects of different pollution restrictions on the firm's dynamic investment policy
Refereed by Prof.dr. F.A. van der Duyn Schouten

551 Niels G. Noorderhaven, Bart Nooteboom and Johannes Berger
Temporal, cognitive and behavioral dimensions of transaction costs; to an understanding of hybrid vertical inter-firm relations
Refereed by Prof.dr. S.W. Douma

552 Ton Storcken and Harrie de Swart
Towards an axiomatization of orderings
Refereed by Prof.dr. P.H.M. Ruys

553 J.H.J. Roemen
The derivation of a long term milk supply model from an optimization model
Refereed by Prof.dr. F.A. van der Duyn Schouten

554 Geert J. Almekinders and Sylvester C.W. Eijffinger
Daily Bundesbank and Federal Reserve Intervention and the Conditional Variance Tale in DM/$-Returns
Refereed by Prof.dr. A.B.T.M. van Schaik

555 Dr. M. Hetebrij, Drs. B.F.L. Jonker, Prof.dr. W.H.J. de Freytas
"Tussen achterstand en voorsprong" de scholings- en personeelsvoorzieningsproblematiek van bedrijven in de procesindustrie
Refereed by Prof.dr. Th.M.M. Verhallen

556 Ton Geerts
Regularity and singularity in linear-quadratic control subject to implicit continuous-time systems
Communicated by Prof.dr. J. Schumacher

557 Ton Geerts
Invariant subspaces and invertibility properties for singular systems: the general case
Communicated by Prof.dr. J. Schumacher

558 Ton Geerts
Solvability conditions, consistency and weak consistency for linear differential-algebraic equations and time-invariant singular systems: the general case
Communicated by Prof.dr. J. Schumacher

559 C. Fricker and M.R. Jaïbi
Monotonicity and stability of periodic polling models
Communicated by Prof.dr.ir. O.J. Boxma

560 Ton Geerts
Free end-point linear-quadratic control subject to implicit continuous-time systems: necessary and sufficient conditions for solvability
Communicated by Prof.dr. J. Schumacher

561 Paul G.H. Mulder and Anton L. Hempenius
Expected Utility of Life Time in the Presence of a Chronic Noncommunicable Disease State
Communicated by Prof.dr. B.B. van der Genugten

562 Jan van der Leeuw
The covariance matrix of ARMA-errors in closed form
Communicated by Dr. H.H. Tigelaar

563 J.P.C. Blanc and R.D. van der Mei
Optimization of polling systems with Bernoulli schedules
Communicated by Prof.dr.ir. O.J. Boxma

564 B.B. van der Genugten
Density of the least squares estimator in the multivariate linear model with arbitrarily normal variables
Communicated by Prof.dr. M.H.C. Paardekooper

565 René van den Brink, Robert P. Gilles
Measuring Domination in Directed Graphs
Communicated by Prof.dr. P.H.M. Ruys

566 Harry G. Barkema
The significance of work incentives from bonuses: some new evidence
Communicated by Dr. Th.E. Nijman

567 Rob de Groof and Martin van Tuijl
Commercial integration and fiscal policy in interdependent, financially integrated two-sector economies with real and nominal wage rigidity.
Communicated by Prof.dr. A.L. Bovenberg

568 F.A. van der Duyn Schouten, M.J.G. van Eijs, R.M.J. Heuts
The value of information in a fixed order quantity inventory system
Communicated by Prof.dr. A.J.J. Talman

569 E.N. Kertzman
Begrotingsnormering en EMU
Communicated by Prof.dr. J.W. van der Dussen

570 A. van den Elzen, D. Talman
Finding a Nash-equilibrium in noncooperative N-person games by solving a sequence of linear stationary point problems
Communicated by Prof.dr. S.H. Tijs

571 Jack P.C. Kleijnen
Verification and validation of models
Communicated by Prof.dr. F.A. van der Duyn Schouten

572 Jack P.C. Kleijnen and Willem van Groenendaal
Two-stage versus sequential sample-size determination in regression analysis of simulation experiments
Communicated by Prof.Dr. F.A. van der Duyn Schouten

573 Pieter K. Jagersma
Het management van multinationale ondernemingen: de concernstructuur
Communicated by Prof.Dr. S.W. Douma

574 A.L. Hempenius
Explaining Changes in External Funds. Part One: Theory
Communicated by Prof.Dr.Ir. A. Kapteyn

575 J.P.C. Blanc, R.D. van der Mei
Optimization of Polling Systems by Means of Gradient Methods and the Power-Series Algorithm
Communicated by Prof.dr.ir. O.J. Boxma

576 Herbert Hamers
A silent duel over a cake
Communicated by Prof.dr. S.H. Tijs

577 Gerard van der Laan, Dolf Talman, Hans Kremers
On the existence and computation of an equilibrium in an economy with constant returns to scale production
Communicated by Prof.dr. P.H.M. Ruys

578 R.Th.A. Wagemakers, J.J.A. Moors, M.J.B.T. Janssens
Characterizing distributions by quantile measures
Communicated by Dr. R.M.J. Heuts

579 J. Ashayeri, W.H.L. van Esch, R.M.J. Heuts
Amendment of Heuts-Selen's Lotsizing and Sequencing Heuristic for Single Stage Process Manufacturing Systems
Communicated by Prof.dr. F.A. van der Duyn Schouten

580 H.G. Barkema
The Impact of Top Management Compensation Structure on Strategy
Communicated by Prof.dr. S.W. Douma

581 Jos Benders en Freek Aertsen
Aan de lijn of aan het lijntje: wordt slank produceren de mode?
Communicated by Prof.dr. S.W. Douma

582 Willem Haemers
Distance Regularity and the Spectrum of Graphs
Communicated by Prof.dr. M.H.C. Paardekooper

583 Jalal Ashayeri, Behnam Pourbabai, Luk van Wassenhove
Strategic Marketing, Production, and Distribution Planning of an Integrated Manufacturing System
Communicated by Prof.dr. F.A. van der Duyn Schouten

584 J. Ashayeri, F.H.P. Driessen
Integration of Demand Management and Production Planning in a Batch Process Manufacturing System: Case Study
Communicated by Prof.dr. F.A. van der Duyn Schouten

585 J. Ashayeri, A.G.M. van Eijs, P. Nederstigt
Blending Modelling in a Process Manufacturing System
Communicated by Prof.dr. F.A. van der Duyn Schouten

586 J. Ashayeri, A.J. Westerhof, P.H.E.L. van Alst
Application of Mixed Integer Programming to A Large Scale Logistics Problem
Communicated by Prof.dr. F.A. van der Duyn Schouten

587 P. Jean-Jacques Herings
On the Structure of Constrained Equilibria
Communicated by Prof.dr. A.J.J. Talman

IN 1993 REEDS VERSCHENEN

588 Rob de Groof and Martin van Tuijl
The Twin-Debt Problem in an Interdependent World
Communicated by Prof.dr. Th. van de Klundert

589 Harry H. Tigelaar
A useful fourth moment matrix of a random vector
Communicated by Prof.dr. B.B. van der Genugten

590 Niels G. Noorderhaven
Trust and transactions; transaction cost analysis with a differential behavioral assumption
Communicated by Prof.dr. S.W. Douma

591 Henk Roest and Kitty Koelemeijer
Framing perceived service quality and related constructs A multilevel approach
Communicated by Prof.dr. Th.M.M. Verhallen

592 Jacob C. Engwerda
The Square Indefinite LQ-Problem: Existence of a Unique Solution
Communicated by Prof.dr. J. Schumacher

593 Jacob C. Engwerda
Output Deadbeat Control of Discrete-Time Multivariable Systems
Communicated by Prof.dr. J. Schumacher

594 Chris Veld and Adri Verboven
An Empirical Analysis of Warrant Prices versus Long Term Call Option Prices
Communicated by Prof.dr. P.W. Moerland

595 A.A. Jeunink en M.R. Kabir
De relatie tussen aandeelhoudersstructuur en beschermingsconstructies
Communicated by Prof.dr. P.W. Moerland

596 M.J. Coster and W.H. Haemers
Quasi-symmetric designs related to the triangular graph
Communicated by Prof.dr. M.H.C. Paardekooper

597 Noud Gruijters
De liberalisering van het internationale kapitaalverkeer in historisch-institutioneel perspectief
Communicated by Dr. H.G. van Gemert

598 John Görtzen en Remco Zwetheul
Weekend-effect en dag-van-de-week-effect op de Amsterdamse effectenbeurs?
Communicated by Prof.dr. P.W. Moerland

599 Philip Hans Franses and H. Peter Boswijk
Temporal aggregration in a periodically integrated autoregressive process
Communicated by Prof.dr. Th.E. Nijman

600 René Peeters
On the p-ranks of Latin Square Graphs
Communicated by Prof.dr. M.H.C. Paardekooper

601 Peter E.M. Borm, Ricardo Cao, Ignacio García-Jurado
Maximum Likelihood Equilibria of Random Games
Communicated by Prof.dr. B.B. van der Genugten

602 Prof.dr. Robert Bannink
Size and timing of profits for insurance companies. Cost assignment for products with multiple deliveries.
Communicated by Prof.dr. W. van Hulst

603 M.J. Coster
An Algorithm on Addition Chains with Restricted Memory
Communicated by Prof.dr. M.H.C. Paardekooper

604 Ton Geerts
Coordinate-free interpretations of the optimal costs for LQ-problems subject to implicit systems
Communicated by Prof.dr. J.M. Schumacher

605 B.B. van der Genugten
Beat the Dealer in Holland Casino's Black Jack
Communicated by Dr. P.E.M. Borm

606 Gert Nieuwenhuis
Uniform Limit Theorems for Marked Point Processes
Communicated by Dr. M.R. Jaïbi

607 Dr. G.P.L. van Roij
Effectisering op internationale financiële markten en enkele gevolgen voor banken
Communicated by Prof.dr. J. Sijben

608 R.A.M.G. Joosten, A.J.J. Talman
A simplicial variable dimension restart algorithm to find economic equilibria on the unit simplex using n(n+1) rays
Communicated by Prof.Dr. P.H.M. Ruys

609 Dr. A.J.W. van de Gevel
The Elimination of Technical Barriers to Trade in the European Community
Communicated by Prof.dr. H. Huizinga

610 Dr. A.J.W. van de Gevel
Effective Protection: a Survey
Communicated by Prof.dr. H. Huizinga

611 Jan van der Leeuw
First order conditions for the maximum likelihood estimation of an exact ARMA model
Communicated by Prof.dr. B.B. van der Genugten

612 Tom P. Faith
Bertrand-Edgeworth Competition with Sequential Capacity Choice
Communicated by Prof.Dr. S.W. Douma

613 Ton Geerts
The algebraic Riccati equation and singular optimal control: The discrete-time case
Communicated by Prof.dr. J.M. Schumacher

614 Ton Geerts
Output consistency and weak output consistency for continuous-time implicit systems
Communicated by Prof.dr. J.M. Schumacher

615 Stef Tijs, Gert-Jan Otten
Compromise Values in Cooperative Game Theory
Communicated by Dr. P.E.M. Borm

616 Dr. Pieter J.F.G. Meulendijks and Prof.Dr. Dick B.J. Schouten
Exchange Rates and the European Business Cycle: an application of a 'quasi-empirical' two-country model
Communicated by Prof.Dr. A.H.J.J. Kolnaar

617 Niels G. Noorderhaven
The argumentational texture of transaction cost economics
Communicated by Prof.Dr. S.W. Douma

618 Dr. M.R. Jaïbi
Frequent Sampling in Discrete Choice
Communicated by Dr. M.H. ten Raa

619 Dr. M.R. Jaïbi
A Qualification of the Dependence in the Generalized Extreme Value Choice Model
Communicated by Dr. M.H. ten Raa

620 Dr. J.J.A. Moors
Limiting distributions of moment- and quantile-based measures for skewness and kurtosis
Communicated by Prof.Dr. B.B. van der Genugten

621 Job de Haan, Jos Benders, David Bennett
Symbiotic approaches to work and technology
Communicated by Prof.dr. S.W. Douma

622 René Peeters
Orthogonal representations over finite fields and the chromatic number of graphs
Communicated by Dr.ir. W.H. Haemers

623 W.H. Haemers, E. Spence
Graphs Cospectral with Distance-Regular Graphs
Communicated by Prof.dr. M.H.C. Paardekooper