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RESEARCH MEMORANDUM

# COST ALLOCATION AND COMMUNICATION

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# COST ALLOCATION AND COMMUNICATION

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*Abstract.* This paper considers a special class of cost allocation problems, where the communication possibilities among the agents are restricted. Integral formulas are derived for two allocation rules: the Myerson value and the position value.

*Key words.* Cost allocation, communication, Myerson value, position value.

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## 1. INTRODUCTION

Consider a group of economic agents who all want to make use of some machines or facilities. If the agents cooperate they can possibly share some facilities and thus save costs. In section 2 it is shown that this kind of cost allocation problems give rise to concave cost games which lie in the cone generated by all dual unanimity games.

Subsequently, we investigate the consequences of a restriction in communication possibilities between the agents on this kind of cost allocation problems. Throughout this paper we assume that the communication possibilities can be modelled by means of a (communication) graph. Based on the Shapley value, two solution concepts for communication situations were introduced: the Myerson value and the position value. An axiomatic characterization of the Myerson value was given by *Myerson* (1977) and *Borm et al.* (1990) gave an axiomatic characterization of the position value in case of cycle-free communication graphs. *Van den Nouweland* and *Borm* (1990) proved that if a communication graph is cycle-complete (cycle-free) and the underlying cost game is concave, then the Myerson value (position value) is in the core of the corresponding graph-restricted game.

*Owen* (1986) and *Borm et al.* (1990) provided integral formulas to compute the Myerson value and the position value in situations where the communication graph is cycle-free and the underlying game is a quadratic measure game. In section 3 we derive integral formulas for the Myerson value and the position value in situations where the underlying cost game is a dual unanimity game.

## 2. THE MODEL

Let $N := \{1, \ldots, n\}$ and $2^N := \{S \mid S \subseteq N\}$. By $TU^N$ we denote the class of all *transferable utility games* $(N, v)$ with player set $N$ and *characteristic function* $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. There are basically two ways to interpret a $TU$-game $(N, v)$, the amount $v(S)$ can represent the revenue/gains for a *coalition* $S \subseteq N$ or it can represent the costs for this coalition. We prefer to denote a $TU$-game by $(N, c)$ if it is to be interpreted as a cost game.

In this paper we consider cost games that are generated by cost allocation problems of the form $\langle N, F, p, d\rangle$. Here, $N$ is the set of players, $F$ is a finite set of facilities, $p : F \to \mathbb{R}_+$ is a function that assigns to every facility its non-negative price, and $d : N \to 2^F$ is a function that assigns to every player the subset of facilities demanded by this player.

Let $\langle N, F, p, d\rangle$ be such a cost allocation problem. Then the players in a coalition $S \subseteq N$ have to purchase each facility that at least one of them demands. On the other hand, if two (or more) players demand the same facility, then they only have to purchase it once. Hence, this allocation problem leads to a cost game $(N, c)$ with

$$c(S) := \sum_{r \in F} p(r)\, u^*_{N_r}(S),$$

where, for all $r \in F$, $N_r := \{i \in N \mid r \in d(i)\}$ is the set of players who demand facility $r$, and $u^*_{N_r}$ is the *dual unanimity game* on $N_r$, defined by $u^*_{N_r}(S) = \begin{cases} 1 & \text{if } S \cap N_r \neq \emptyset \\ 0 & \text{else} \end{cases}$.
The proof of theorem 1 is straightforward and therefore omitted.

THEOREM 1. The class of all cost games corresponding to cost allocation problems of the form $\langle N, F, p, d \rangle$ is the convex cone generated by all dual unanimity games $u^*_M$ with $M \in 2^N \backslash \{\emptyset\}$.

A cost game $(N, c)$ is called *concave* if it is more advantageous to join a larger coalition or, in formula, if

$$c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$$

for all $i \in N$ and all $S \subseteq T \subseteq N \backslash \{i\}$. The class of concave games with player set $N$ is a convex cone and $(N, u^*_M)$ is a concave game for every $M \subseteq N$. Therefore, a direct consequence of theorem 1 is that all cost games corresponding to cost allocation problems of the form $\langle N, F, p, d \rangle$ are concave.
It may be noted that the *Shapley value* $\Phi(N, c) \in \mathbb{R}^N$ (cf. *Shapley* (1953)) of the cost game $(N, c)$ corresponding to the cost allocation problem $\langle N, F, p, d \rangle$ has a nice interpretation. As is easily seen the Shapley value of a dual unanimity game $(N, u^*_M)$ equals 0 for $i \notin M$ and $\frac{1}{|M|}$ for $i \in M$. Since the Shapley value is linear, it follows that

$$\Phi_i(N, c) = \sum_{r \in d(i)} \frac{p(r)}{|N_r|}$$

for all $i \in N$, which implies that the costs of each facility are equally divided among the players that make use of it.

## 3. INTEGRAL FORMULAS FOR THE MYERSON VALUE AND THE POSITION VALUE

So far, we implicitly assumed that all players can freely communicate with one another. Now suppose that communication between the players is restricted and that the communication possibilities are determined by an undirected *(communication) graph*

$(N, A)$ in which the points are the players and the arcs correspond to pairs of players who can communicate directly. A triple $(N, c, A)$, where $(N, c)$ is a cost game and $(N, A)$ is a communication graph, is called a *communication situation.*
Let $(N, c, A)$ be a communication situation. Then the players in a coalition $S \subseteq N$ can effect communication through all communication links of

$$A(S) := \{\{i, j\} \in A \mid \{i, j\} \subseteq S\}.$$

Hence a coalition $S$ splits up into *(communication) components* in the following way: $T \subseteq S$ is a component within $S$ if and only if the graph $(T, A(T))$ is connected and there is no set $\overline{T}$ such that $T \subsetneq \overline{T} \subseteq S$ and $(\overline{T}, A(\overline{T}))$ is connected. We denote the resulting partition of $S$ by $S/A$.
Taking into account these communication restrictions, the costs $c_A(S)$ for a coalition $S \in 2^N$ can be defined as

$$c_A(S) := \sum_{T \in S/A} c(T).$$

$(N, c_A)$ is called the *graph-restricted* game.
One can also focus on the communication links. The cost savings for the grand coalition induced by the presence of the communication links in $L \subseteq A$ are defined as

$$r_N^c(L) := \sum_{i \in N} c(\{i\}) - \sum_{T \in N/L} c(T).$$

$(A, r_N^c)$ is called the *arc (cost savings) game.*
Now we are ready to formulate the definitions of the Myerson value and the position value.
The *Myerson value* $\mu(N, c, A) \in \mathbb{R}^N$ (cf. *Myerson* (1977)) is defined as the Shapley value of the corresponding graph-restricted game, i.e.

$$\mu(N, c, A) := \Phi(N, c_A).$$

The *position value* of a communication situation (cf. *Borm et al.* (1990)) is based upon the Shapley value of the corresponding arc game: the corresponding cost savings of each arc are equally divided among the players it connects. With the aid of these cost savings the corresponding cost allocation rule $\pi(N, c, A) \in \mathbb{R}^N$, defined by

$$\pi_i(N, c, A) := c(\{i\}) - \sum_{a \in A_i} \tfrac{1}{2} \Phi_a(A, r_N^c)$$

for all $i \in N$, is called the position value of the communication situation $(N, c, A)$. Here, $A_i := \{\{i, j\} \in A \mid j \in N\}$ is the set of all communication links of which player $i$ is an end point.

Both the Myerson value and the position value are linear with respect to the underlying cost game. Therefore, in deriving integral formulas for both values we can restrict our attention to dual unanimity games, because all cost games we consider are positive combinations of dual unanimity games (cf. theorem 1).

First we derive an integral formula for the Myerson value of a communication situation $(N, u_M^*, A)$, $M \subseteq N$, if the communication graph $(N, A)$ is a tree.
Let $(N, c, A)$ be a communication situation. As is well known, the game $(N, c_A)$ can be written as

$$c_A = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta_{c_A}(S)\, u_S,$$

where $u_S$ is the *unanimity game* on $S$, defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{else} \end{cases}$$

and the *dividends* $\Delta_{c_A}(S)$ (cf. *Harsanyi* (1959)) are given by

$$\Delta_{c_A}(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|}\, c_A(T)$$

for all $S \in 2^N \setminus \{\emptyset\}$.
In the following we consider a fixed communication situation $(N, u_M^*, A)$ where $|N| \geq 2$ and $(N, A)$ is a tree.
In this case the dividends $\Delta_{(u_M^*)_A}(S)$ are given by (cf. *Owen* (1986))

$$\Delta_{(u_M^*)_A}(S) = \begin{cases} (-1)^{|S \cap M|+1} & \text{if } (S, A(S)) \text{ is connected and } \mathrm{Ext}(S, A(S)) = S \cap M \\ 0 & \text{else} \end{cases}$$

for all $S \in 2^N \setminus \{\emptyset\}$. Here, for a connected graph $(S, A(S))$

$$\mathrm{Ext}(S, A(S)) := \{i \in S \mid |A_i \cap A(S)| \leq 1\}$$

denotes the set of *extreme points*.
Consequently,

$$\mu_i(N, u_M^*, A) = \Phi_i(N, (u_M^*)_A) = \sum_{S \in \Sigma(A): i \in S} \frac{(-1)^{|S \cap M|+1}}{|S|} \tag{1}$$

for all $i \in N$, where

$$\Sigma(A) := \{S \subseteq N \mid (S, A(S)) \text{ is connected and } \mathrm{Ext}(S, A(S)) = S \cap M\}.$$

In order to apply (1) we have to find all elements in $\Sigma(A)$ containing a player $i$. This can be done in the following way.
For every $S \subseteq C$ with $C \in N/A$, we can define the *connected hull* of $S$ (cf. *Owen* (1986)) by

$$H(S) := \bigcap\{T \mid S \subseteq T \subseteq C, (T, A(T)) \text{ is a connected graph}\}.$$

Let $i \in N$ and let $A_i := \{\{i, i(1)\}, \ldots, \{i, i(t)\}\}$. Clearly, since $(N, A)$ does not contain a cycle, the partition $N/(A \backslash A_i)$ contains $t$ distinct components $C(1), \ldots, C(t)$ with $i(k) \in C(k)$ for all $k \in \{1, \ldots, t\}$. For each $k \in \{1, \ldots, t\}$ we define a connected subgraph $(T(k), A(T(k)))$ of $(C(k), A(C(k)))$ by

$$T(k) := \bigcup_{j \in M \cap C(k)} \{H(\{i(k), j\}) \mid |H(\{i(k), j\}) \cap M| = 1\}. \tag{2}$$

So, in particular, $T(k) = \emptyset$ if $C(k) \cap M = \emptyset$, $T(k) = \{i(k)\}$ if $i(k) \in M$ and in all other cases $\text{Ext}(T(k), A(T(k))) \backslash \{i(k)\} = T(k) \cap M$.
In deriving a generating function (cf. *Owen* (1972)) for the Myerson value we suppose that each player has a probability $x$ to be 'active' or 'operational' and we compute for each subgraph $(T(k), A(T(k)))$ the probability $P_k(x)$ that at least one of the players in $M \cap T(k)$ can actually interact with player $i$. Using the inclusion-exclusion principle this probability $P_k(x)$ is given by

$$P_k(x) = \sum_{S \subseteq (T(k) \cap M), S \neq \emptyset} (-1)^{|S|+1} x^{|H(S \cup \{i(k)\})|} \tag{3}$$

for all $k \in \{1, \ldots, t\}$. Note that $P_k(x) = 0$ if $T(k) \cap M = \emptyset$. The expected costs player $i$ generates by linking up the components $C(1), \ldots, C(t)$ are described by the generating function $\theta_i(x)$, where

$$\theta_i(x) := \begin{cases} 1 - \sum_{k=1}^{t} P_k(x) & \text{if } i \in M \\ -\sum_{r=2}^{t} \sum_{K \subseteq \{1,\ldots,t\}, |K|=r} (-1)^r \prod_{k \in K} P_k(x) & \text{if } i \notin M. \end{cases} \tag{4}$$

In particular we have that $\theta_i(x) = 0$ if $i \notin M$ and $t = 1$.
Now we can formulate

**THEOREM 2.** Let $(N, A)$ be a tree with $|N| \geq 2$. Then for all $M \in 2^N \backslash \{\emptyset\}$,

$$\mu_i(N, u_M^*, A) = \int_0^1 \theta_i(x) dx \tag{5}$$

for all $i \in N$, where $\theta_i(x)$ is defined as in (4).

*Proof.* Let $i \in M$. Then

$$\int_0^1 \theta_i(x)dx = 1 - \sum_{k=1}^{t}(\int_0^1 P_k(x)dx)$$

$$= 1 - \sum_{k=1}^{t}(\int_0^1 \sum_{S \subseteq (T(k) \cap M), S \neq \emptyset} (-1)^{|S|+1} x^{|H(S \cup \{i(k)\})|} \, dx)$$

$$= 1 - \sum_{k=1}^{t} \sum_{S \subseteq (T(k) \cap M), S \neq \emptyset} \frac{(-1)^{|S|+1}}{|H(S \cup \{i(k)\})| + 1}. \tag{6}$$

Note that, since $i \in M$, for each $k \in \{1, \ldots, t\}$ a coalition $S \subseteq T(k) \cap M$ with $S \neq \emptyset$ uniquely determines a set $T = H(S \cup \{i\}) \in \Sigma(A)$, which satisfies $\text{Ext}(T, A(T)) \setminus \{i\} = S$. Hence, (6) equals

$$1 - \sum_{k=1}^{t} \sum_{S \in \Sigma_k(A): i \in S, |S| \geq 2} \frac{(-1)^{|S \cap M|}}{|S|}, \tag{7}$$

where $\Sigma_k(A) := \{S \in \Sigma(A) \mid S \subseteq (T(k) \cup \{i\})\}$ for all $k \in \{1, \ldots, t\}$.
The fact that $i \in M$ implies that for all $S \in \Sigma(A)$ with $i \in S$ and $|S| \geq 2$ there is a $k \in \{1, \ldots, t\}$ such that $S \subseteq T(k) \cup \{i\}$. Therefore, (7) equals

$$1 - \sum_{S \in \Sigma(A): i \in S, |S| \geq 2} \frac{(-1)^{|S \cap M|}}{|S|}. \tag{8}$$

Since $i \in M$, we have $\{i\} \in \Sigma(A)$, so (8) equals

$$\sum_{S \in \Sigma(A): i \in S} \frac{(-1)^{|S \cap M|+1}}{|S|} = \mu_i(N, u_M^*, A).$$

Let $i \in N \setminus M$. First note that we may assume that $|\{k \in \{1, \ldots, t\}| \, T(k) \neq \emptyset\}| \geq 2$, for otherwise $\theta_i(x) = 0$ and $\{S \in \Sigma(A) \mid i \in S\} = \emptyset$, so trivially (5) is satisfied.
Since for all $K \subseteq \{1, \ldots, t\}$ with $|K| \geq 2$

$$\prod_{k \in K} P_k(x) = \prod_{k \in K}( \sum_{S \subseteq (T(k) \cap M): S \neq \emptyset} (-1)^{|S|+1} x^{|H(S \cup \{i(k)\})|})$$

$$= \sum_{(S(k))_{k \in K} \in \Gamma(K)} (\prod_{k \in K} (-1)^{|S(k)|+1} x^{|H(S(k) \cup \{i(k)\})|}), \tag{9}$$

where $\Gamma(K) := \{(S(k))_{k \in K} \mid S(k) \subseteq (T(k) \cap M), S(k) \neq \emptyset \text{ for all } k \in K\}$, it follows that

$$\int_0^1 \theta_i(x)dx = \sum_{r=2}^{t} \sum_{K\subseteq\{1,\ldots,t\},|K|=r} (-1)^{r+1} \sum_{(S(k))_{k\in K}\in\Gamma(K)} \frac{(-1)^{\sum_{k\in K}(|S(k)|+1)}}{1+\sum_{k\in K}|H(S(k)\cup\{i(k)\})|}$$

$$= \sum_{r=2}^{t} \sum_{K\subseteq\{1,\ldots,t\},|K|=r} \sum_{(S(k))_{k\in K}\in\Gamma(k)} \frac{(-1)^{1+\sum_{k\in K}|S(k)|}}{|H(\bigcup_{k\in K} S(k))|}. \qquad (10)$$

Since $i \notin M$, for each $K \subseteq \{1,\ldots,t\}$ with $|K| \geq 2$ a set $(S(k))_{k\in K} \in \Gamma(K)$ uniquely determines a set $T = H(\bigcup_{k\in K} S(k)) \in \Sigma(A)$, which satisfies $\text{Ext}(T, A(T)) = \bigcup_{k\in K} S(k)$ and $i \in T$. Hence, (10) equals

$$\sum_{S\in\Sigma(A):i\in S} \frac{(-1)^{|S\cap M|+1}}{|S|} = \mu_i(N, u_M^*, A). \qquad \square$$

We illustrate the actual computation of the Myerson value in

**EXAMPLE 1.** Let $N = \{1,\ldots,10\}$ and $M = \{1,3,4,7,10\}$. The graph $(N, A)$ is represented in figure 1.

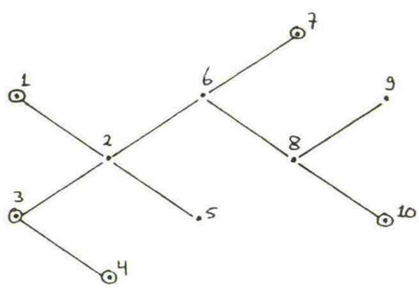


figure 1

Consider player $2 \notin M$. Following (2) we obtain four subgraphs corresponding to $T(1) = \{1\}$, $T(2) = \{3\}$, $T(3) = \emptyset$ and $T(4) = \{6,7,8,10\}$. The corresponding polynomials (cf. (3)) are $P_1(x) = P_2(x) = x$, $P_3(x) = 0$ and $P_4(x) = x^2 + x^3 - x^4$. Hence, according to (4),
$\theta_2(x) = -P_1(x)P_2(x) - P_1(x)P_4(x) - P_2(x)P_4(x) + P_1(x)P_2(x)P_4(x) =$
$-x^2 - 2x^3 - x^4 + 3x^5 - x^6$. So, $\mu_2(N, u_M^*, A) = \int_0^1 \theta_2(x)dx = -\frac{71}{105}$.

Now consider player $3 \in M$. We obtain two subgraphs corresponding to $T(1) = \{4\}$ and $T(2) = \{1, 2, 6, 7, 8, 10\}$. The corresponding polynomials are $P_1(x) = x$ and $P_2(x) = x^2 + x^3 + x^4 - x^4 - x^5 - x^5 + x^6 = x^2 + x^3 - 2x^5 + x^6$. So, according to (4), $\theta_3(x) = 1 - P_1(x) - P_2(x) = 1 - x - x^2 - x^3 + 2x^5 - x^6$ and $\mu_3(N, u_M^*, A) = \int_0^1 \theta_3(x)dx = \frac{3}{28}$.
For the sake of completeness we note that

$$\mu(N, u_M^*, A) = \frac{1}{420}(255, -284, 45, 210, 0, -249, 269, -123, 0, 297).$$

It may seem that the calculation of the polynomials $P_k(x)$ (cf. (3)) may be quite lengthy, especially if the sets $T(k) \cap M$ have a large number of elements. An alternative way to obtain these polynomials is described below.
Let $i \in N$ and let $\big(T(k), A(T(k))\big)$ be one of the connected subgraphs corresponding to player $i$ as described in (2). Suppose each player $p$ has a probability $x_p$ to be 'active'. Consider the polynomial

$$1 - \prod_{j \in T(k) \cap M} (1 - \prod_{p \in H(\{i(k), j\})} x_p) \tag{11}$$

and expand it. Now reduce the obtained polynomial to a multilinear polynomial by the simple recourse of reducing each higher exponent to a 1. Finally, by replacing the probabilities $x_p$ by the probability $x \in [0, 1]$ we obtain the polynomial $P_k(x)$.
In this, we are effectively using the multilinear extension (cf. *Owen* (1972)). As we know, the partial derivative $\frac{\partial f}{\partial x_i}$ of this extension corresponds to the expectation that other players will collaborate with player $i$. Typically, for $S \subseteq N \backslash \{i\}$, the term

$$u(S) := \prod_{j \in S} x_j \tag{12}$$

corresponds to the probability that all members of $S$ collaborate, given that each $j \in S$ has probability $x_j$ of collaboration and assuming independence. Now, if $S$ and $T$ are disjoint, then

$$u(S \cup T) = u(S) \cdot u(T), \tag{13}$$

since the players in $S$ and $T$ are independent. When $S \cap T \neq \emptyset$, however, (13) is not quite correct. Rather,

$$u(S \cup T) = \overline{u(S) \cdot u(T)}, \tag{14}$$

where the bar corresponds to a reduction operation: each exponent larger than 1 is reduced to 1. If, for example, $S = \{1, 2\}$ and $T = \{2, 3\}$, then $u(S) = x_1 x_2, u(T) = x_2 x_3$ and $u(S \cup T) = \overline{x_1 x_2^2 x_3} = x_1 x_2 x_3$.

EXAMPLE 2. Let $(N, u_M^*, A)$ be the communication situation as described in example 1. Again consider player 2. For the subgraph corresponding to $T(1) = \{1\}$, expression (11) yields

$$1 - (1 - x_1),$$

which results in the polynomial $P_1(x) = x$.
In the same way we obtain $P_2(x) = x$ for the polynomial corresponding to $T(2) = \{3\}$.
Defining the empty product to be 1, we easily see $P_3(x) = 0$.
Finally, for the subgraph corresponding to $T(4) = \{6, 7, 8, 10\}$, expression (11) yields

$$1 - (1 - x_6 x_7)(1 - x_6 x_8 x_{10}).$$

Expanding this, we obtain

$$x_6 x_7 + x_6 x_8 x_{10} - x_6^2 x_7 x_8 x_{10}.$$

Reducing this polynomial and then replacing all $x_p$'s by $x$, we obtain

$$x^2 + x^3 - x^4,$$

which is exactly the desired polynomial $P_4(x)$.

We now concentrate on the position value. The arc game corresponding to the communication situation $(N, u_M^*, A)$ can be expressed as

$$r_N^{u_M^*} = \sum_{L \in 2^A \setminus \{\emptyset\}} \Delta_{r_N^{u_M^*}}(L)\, u_L,$$

where, for all $L \in 2^A \setminus \{\emptyset\}$, $(A, u_L)$ is the (arc) unanimity game on $L$. The results of *Borm et al.* (1990) imply that for all $L \in 2^A \setminus \{\emptyset\}$

$$\Delta_{r_N^{u_M^*}}(L) = \begin{cases} (-1)^{|N(L) \cap M|} & \text{if } (N(L), L) \text{ is a tree and Ext } (N(L), L) = N(L) \cap M \\ 0 & \text{else,} \end{cases}$$

where $N(L)$ is the set of players who are end points of an arc in $L$. Hence, for all $a \in A$

$$\Phi_a(A, r_N^{u_M^*}) = \sum_{L \in \Lambda(A): a \in L} \frac{(-1)^{|N(L) \cap M|}}{|L|}, \tag{15}$$

where $\Lambda(A) := \{L \subseteq A \mid (N(L), L) \text{ is a tree, Ext}(N(L), L) = N(L) \cap M\}$.
In order to apply (15) we have to find all elements in $\Lambda(A)$ containing an arc $a$. This can be done in the following way.

Let $a = \{i(1), i(2)\} \in A$. Clearly, since $(N, A)$ is a tree, the partition $N/(A \backslash \{a\})$ consists of two distinct components $C(1)$ and $C(2)$ with $i(1) \in C(1)$ and $i(2) \in C(2)$. For each $k \in \{1, 2\}$ we define the connected subgraph $(T(k), A(T(k)))$ of $(C(k), A(C(k)))$ as in (2). In deriving a generating function for $\Phi(A, r_N^{u_M^*})$ we suppose that each arc has a probability $x$ to be 'available' and we compute for each subgraph $(T(k), A(T(k)))$ the probability $P_k(x)$ that at least one of the players in $M \cap T(k)$ can actually interact with player $i(k)$. Using the inclusion-exclusion principle this probability $P_k(x)$ is given by

$$P_k(x) = \sum_{S \subseteq (T(k) \cap M), S \neq \emptyset} (-1)^{|S|+1} x^{|A(H(S \cup \{i(k)\}))|} \tag{16}$$

for $k \in \{1, 2\}$. The expected cost savings an arc generates by linking up the components $C(1)$ and $C(2)$ are described by the generating function $\theta_a(x)$,

$$\theta_a(x) := P_1(x) \cdot P_2(x). \tag{17}$$

Now we can formulate

**THEOREM 3.** Let $(N, A)$ be a tree with $|N| \geq 2$. Then for all $M \in 2^N \backslash \{\emptyset\}$,

$$\Phi_a(A, r_N^{u_M^*}) = \int_0^1 \theta_a(x) dx \tag{18}$$

for all $a \in A$, where $\theta_a(x)$ is defined as in (17).

*Proof.* Let $a \in A$. We may assume that $T(1)$ and $T(2)$ are both non-empty, for otherwise $\theta_a(x) = 0$ and $\{L \in \Lambda(A) \mid a \in L\} = \emptyset$, so, trivially (18) is satisfied. Now

$$\begin{aligned} \int_0^1 \theta_a(x) dx &= \int_0^1 P_1(x) P_2(x) dx \\ &= \int_0^1 \prod_{k=1}^2 \Big( \sum_{S \subseteq (T(k) \cap M): S \neq \emptyset} (-1)^{|S|+1} x^{|A(H(S \cup \{i(k)\}))|} \Big) dx \\ &= \int_0^1 \sum_{(S(1), S(2)) \in \Gamma(\{1,2\})} \Big( \prod_{k=1}^2 (-1)^{|S(k)|+1} x^{|A(H(S(k) \cup \{i(k)\}))|} \Big) dx, \end{aligned} \tag{19}$$

where $\Gamma(\{1, 2\}) := \{(S(1), S(2)) \mid S(k) \subseteq (T(k) \cap M), S(k) \neq \emptyset \text{ for } k \in \{1, 2\}\}$.

Carrying out the integration in expression (19) we obtain

$$\sum_{(S(1),S(2))\in\Gamma(\{1,2\})} \frac{(-1)^{\sum_{k=1}^{2}(|S(k)|+1)}}{1+\sum_{k=1}^{2}|A(H(S(k)\cup\{i(k)\}))|}$$

$$= \sum_{(S(1),S(2))\in\Gamma(\{1,2\})} \frac{(-1)^{|S(1)|+|S(2)|}}{\left|A\Big(H(S(1)\cup S(2))\Big)\right|}. \tag{20}$$

Note that each pair $(S(1),S(2))\in\Gamma(\{1,2\})$ uniquely determines a set $L=A\big(H(S(1)\cup S(2))\big)\in\Lambda(A)$, which satisfies $a\in L$ and $\mathrm{Ext}(N(L),L)=S(1)\cup S(2)$. Hence, (20) equals

$$\sum_{L\in\Lambda(A):a\in L}\frac{(-1)^{|N(L)\cap M|}}{|L|}=\Phi_a(A,r_N^{u_M^*}). \qquad \square$$

We illustrate the actual computation of the position value in

**EXAMPLE 3.** Let $(N,u_M^*,A)$ be the communication situation as described in example 1. Consider player 6. To obtain the position value of player 6 we have to compute $\Phi_{\{2,6\}}(A,r_N^{u_M^*})$, $\Phi_{\{6,7\}}(A,r_N^{u_M^*})$ and $\Phi_{\{6,8\}}(A,r_N^{u_M^*})$.
For $a:=\{2,6\}$ we obtain (cf. (2)) the subgraphs corresponding to $T(1)=\{1,2,3\}$ and $T(2)=\{6,7,8,10\}$ with corresponding polynomials (cf. (16)) $P_1(x)=2x-x^2$ and $P_2(x)=x+x^2-x^3$. Hence, according to (17), $\theta_a(x)=2x^2+x^3-3x^4+x^5$ and $\Phi_a(A,r_N^{u_M^*})=\int_0^1\theta_a(x)dx=\frac{29}{60}$.
For $b:=\{6,7\}$ we obtain $T(1)=\{1,2,3,6,8,10\}$ and $T(2)=\{7\}$ and the polynomials $P_1(x)=3x^2-x^3-2x^4+x^5$ and $P_2(x)=1$. Hence, $\theta_b(x)=3x^2-x^3-2x^4+x^5$ and $\Phi_b(A,r_N^{u_M^*})=\int_0^1\theta_b(x)dx=\frac{31}{60}$.
Finally, for $c:=\{6,8\}$ we obtain $T(1)=\{1,2,3,6,7\}$ and $T(2)=\{8,10\}$ with corresponding polynomials $P_1(x)=x+2x^2-3x^3+x^4$ and $P_2(x)=x$. Hence, we have $\theta_c(x)=x^2+2x^3-3x^4+x^5$ and $\Phi_c(A,r_N^{u_M^*})=\int_0^1\theta_c(x)dx=\frac{2}{5}$.
We now compute

$$\pi_6(N,u_M^*,A)=u_M^*(\{6\})-\sum_{\ell\in A_6}\tfrac{1}{2}\Phi_\ell(A,r_N^{u_M^*})=0-\tfrac{1}{2}\left(\tfrac{29}{60}+\tfrac{31}{60}+\tfrac{24}{60}\right)=-\tfrac{7}{10}.$$

For the sake of completeness we note that

$$\pi(N,u_M^*,A)=\frac{1}{120}(84,-101,24,60,0,-84,89,-48,0,96).$$

Similarly to the method described for the polynomials with respect to the Myerson value, there is an alternative way to compute the polynomials $P_k(x)$ as described in (16). The desired polynomial is obtained by considering the polynomial

$$1 - \prod_{j \in T(k) \cap M} (1 - \prod_{a \in A\left(H(\{i(k), j\})\right)} x_a)$$

and then following a reduction procedure similar to the one described before.

In deriving integral formulas for the Myerson value and the position value we restricted our attention to communication graphs that are trees. However, since for all communication situations $(N, c, A)$ and all components $T \in N/A$ we have that

$$\mu_i(N, c, A) = \mu_i(T, c\,|_T, A(T))$$

and

$$\pi_i(N, c, A) = \pi_i(T, c\,|_T, A(T))$$

for all $i \in T$ ( where $c\,|_T$ denotes the restriction of $c$ to $T$), the integral formulas can be used to compute both values for communication situations with cycle-free communication graphs.

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