UNIVERSITEIT
BRABANT
POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

[redacted]

A (49,16,3,6) STRONGLY REGULAR GRAPH DOES NOT EXIST

F.C. Bussemaker, W.H. Haemers,
R. Mathon and H.A. Wilbrink

FEW 355

A (49,16,3,6) STRONGLY REGULAR GRAPH DOES NOT EXIST*

by

F.C. BUSSEMAKER,
Technological University Eindhoven, The Netherlands,

W.H. HAEMERS,
University of Tilburg, The Netherlands,

R. MATHON,
University of Toronto, Canada,

and

H.A. WILBRINK,
Technological University Eindhoven, The Netherlands.

ABSTRACT.

We prove the non-existence of a strongly regular graph with 49 vertices and degree 16.

*) The research for this paper was done in 1979 at the Technological University of Eindhoven.

1. INTRODUCTION.

A strongly regular graph with 49 vertices and degree 16 has parameters $(v,k,\lambda,\mu) = (49,16,3,6)$. In this paper we show that such a graph cannot exist. Up till now it was the smallest (with respect to the number of vertices) feasible strongly regular graph for which existence was not settled. Our result is the second "ad hoc" non-existence result for strongly regular graphs. Earlier Wilbrink and Brouwer [2] proved that (57,14,1,4) cannot be the parameter set of a strongly regular graph. At the moment the smallest unsettled case is (65,32,15,16). See Brouwer and Van Lint [1] for a survey of recent results on strongly regular graphs.

The present proof involves counting techniques, enumeration, linear algebra, and the use of a computer. Although only little computing time was needed, we could not manage without a computer.

2. COUNTING

Let Γ denote a (49,16,3,6) strongly regular graph, that is, Γ has 49 vertices, each vertex has 16 neighbours, any two adjacent vertices have 3 common neighbours and any two distinct non-adjacent vertices have 6 commom neighbours. For a vertex ∞ of Γ, let Γ_∞ denote the subgraph of Γ induced by the 16 neighbours of ∞. Claerly, Γ_∞ is regular of degree 3, but we have more restrictions for Γ_∞.

LEMMA 1.

i. *The girth of* Γ_∞ *is at least 5.*

ii. *Any two distict pentagons of* Γ_∞ *have at most one edge in common.*

PROOF. For a subgraph H and a vertex α of Γ, let h_i denote the number of vertices outside H adjacent to exactly i vertices of H, and let h_i^α denote the number of vertices outside H adjacent to α and to exactly i vertices of H. Then for $H = K_4$ we have

$$\sum_{i=0}^{4} h_i = 45 \ , \quad \sum_{i=1}^{4} ih_i = 52 \ , \quad \sum_{i=2}^{4} \binom{i}{2} h_i = 6 \ .$$

This yields

$$-1 = \sum_{i=0}^{4} \left(h_i - ih_i + \binom{i}{2}h_i\right) = h_0 + h_3 + 3h_4 \geq 0 ,$$

a contradiction proving that Γ_∞ has no triangles. Next suppose that Γ_∞ contains a 4-gon. Let H be the subgraph of Γ induced by ∞ and the 4-gon (i.e. H is the wheel W_5). Then $h_5 = 0$, since Γ contains no K_4, and we have

$$\sum_{i=0}^{4} h_i = 44 , \quad \sum_{i=1}^{4} ih_i = 64 , \quad \sum_{i=2}^{4} \binom{i}{2}h_i = 18 .$$

This implies

$$-2 = \sum_{i=0}^{4} \left(h_i - ih_i + \binom{i}{2}h_i\right) = h_0 + h_3 + 3h_4 \geq 0 ,$$

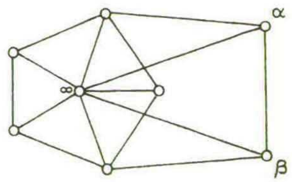
By this contradiction Γ_∞ has no 4-gons. This completes the proof of (i). Suppose Γ_∞ contains a pentagon. Let H be the subgraph of Γ induced by ∞ and the pentagon (i.e. H = W_6). By (i) $h_i = 0$ if $i \geq 4$, hence

$$\sum_{i=0}^{3} h_i = 43 , \quad \sum_{i=1}^{3} ih_i = 76 , \quad \sum_{i=2}^{3} \binom{i}{2}h_i = 35 .$$

This implies that $h_0 + h_3 = 2$. Suppose $h_0 > 0$, that is, there exists a vertex ω adjacent to no vertex of H. Then

$$\sum_{i=0}^{3} h_i^\omega = 16 , \quad \sum_{i=1}^{3} ih_i^\omega = 36 .$$

This gives $2h_0^\omega + h_1^\omega - h_3^\omega = -4$, hence $h_3 \geq h_3^\omega \geq 4$, a contradiction to $h_0 + h_3 = 2$. So $h_0 = 0$, and we find $h_1 = 12$, $h_2 = 29$ and $h_3 = 2$.
We easily have $h_i^\infty = 0$ for $i \geq 3$, $h_0^\infty = 0$, $h_2^\infty = 5$, $h_1^\infty + h_2^\infty = 11$, so $h_1^\infty = 6$. Let X be the set of vertices of Γ not adjacent to ∞ and adjacent to exactly one vertex of H. Then $|X| = h_1 - h_1^\infty = 6$. Next suppose that there exist vertices α and β in Γ_∞, such that together with the vertices of H they induce the following graph H*.



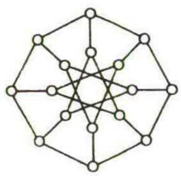
Then

$$\sum_{i=0}^{3} h_i^{\alpha} = 14 \ , \quad \sum_{i=1}^{3} ih_i^{\alpha} = 22 \ .$$

Because $h_0^{\alpha} \leq h_0 = 0$, we have $h_1^{\alpha} = 6 + h_3^{\alpha}$. There is just one vertex outside H^* adjacent to both α and ∞. Therefore, since $h_1^{\alpha} \geq 6$, at least five of the vertices adjacent to α are contained in the set X. The same is true for β. Since $|X| = 6$, at least four vertices of X are adjacent to both α and β. This is a contradiction. Therefore Γ does not contain H^*, and (ii) follows.

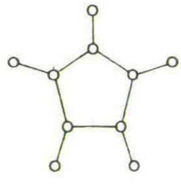
3. ENUMERATION.

The conditions of Lemma 1 are strong enough to enumerate (by hand) all feasible candidates for Γ_∞.
If Γ_∞ contains no pentagon, then the girth is at least six. It is easily seen that the girth cannot be bigger than six and that there is a unique 3-regular graph on 16 vertices with girth six, being:

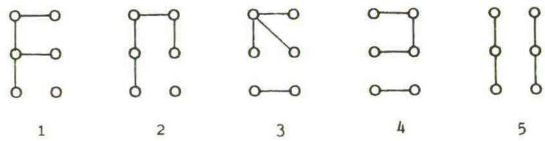


CANDIDATE 1.

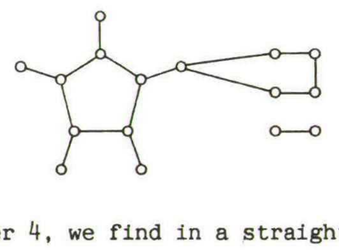
Suppose Γ_∞ contains a pentagon P. Then each vertex of P has just one neighbour outside P. By Lemma 1, these five neighbours are mutually distinct and non-adjacent. Thus Γ_∞ contains the following graph P^* as an induced subgraph.



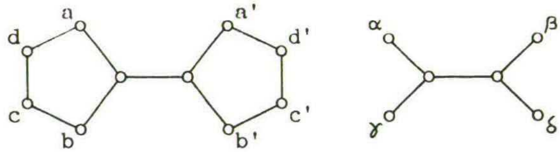
Consider the subgraph F of Γ_∞, induced by the remaining six vertices. One easily sees that F has just four edges and therefore, by Lemma 1, no cycles. Hence F is one of the following graphs:



In the case that F is graph number 1 or graph number 2, the isolated vertex in F is adjacent to three distinct vertices of P*, so two vertices of P* are adjacent to two vertices of the larger component of F. It is easily seen that this cannot be realised without violating Lemma 1. If F is graph number 3 or 4, the two vertices of the isolated edge in F are adjacent to four distinct vertices of P*, so one vertex of P* is adjacent to two vertices of the larger component of F. For number 3 this is clearly impossible, and for number 4 we find that Γ_∞ can be obtained from the following graph by adding eight edges.

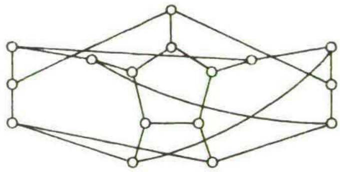


Thus, if F is graph number 4, we find in a straight forward way the following ten feasible structures for Γ_∞, where the adjacencies between the labelled vertices are given in the table below.

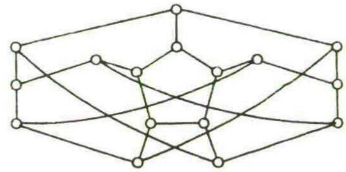


	α	aa'	aa'	aa'	aa'	ad'	ad'	aa'	aa'	aa'	ad'
	β	bb'	bb'	bc'	bd'	bc'	bc'	cb'	cc'	cd'	ca'
	γ	cc'	cd'	cd'	cc'	ca'	cb'	bc'	bd'	bc'	bc'
	δ	dd'	dc'	db'	db'	db'	da'	dd'	db'	db'	db'
CANDIDATE NO.		2	3	4	5	6	7	8	9	10	11

Finally we consider the case when F is graph number 5. Again it is a matter of straigt forward checking that this leads to just two new condidates:



CANDIDATE 12



CANDIDATE 13

This completes the enumeration of the thirteen candidates for Γ_∞.

4. LINEAR ALGEBRA

In terms of the adjacency matrix A the definition of a (49,16,3,6) strongly regular graph reads

$$(*) \qquad A^2 = -3A + 10I + 6J .$$

(We use I for the identity matrix, J for the all-one matrix and j for the all-one vector.) The eigenvalues of A are 16, 2 and -5 with multiplicities 1, 32 and 16, respectively. We say that a graph Γ_1 is *extendable* to a graph Γ, whenever Γ_1 is a subgraph of Γ induced by the neighbours of some vertex of Γ.

LEMMA 2.

Let Γ_1 be a 3-regular graph on 16 vertices with adjacency matrix A_1, and let 2 not be an eigenvalue of A_1. Then Γ_1 is extendable to a (49,16,3,6) strongly regular graph if and only if the vertex set of Γ_1 admits 32 distinct 6-subsets with charateristic vectors $x_1,\ldots,x_{32}$ (say), such that

i. $XX^T + A_1^2 = -3A_1 + 10I + 5J$, *where* $X = [x_1, \ldots, x_{32}]$,

ii. $\tilde{x}_i^T \tilde{A}_1^{-1} \tilde{x}_j = \begin{cases} -16 \text{ if } i = j , \\ -2 \text{ or } 5 \text{ if } i \neq j , \end{cases}$ *for* $i,j = 1,\ldots,32$,

where $\tilde{x}_i = 7x_i - 2j$ $(i = 1,\ldots,32)$ *and* $\tilde{A}_1 = 7A_1 - 14I - 2J$.

PROOF. First suppose we have 32 such subsets. Put $\tilde{X} = 7X - 2J$, $\tilde{A}_2 = \tilde{X}^T \tilde{A}_1^{-1} \tilde{X}$ and

$$\tilde{A} = \begin{bmatrix} -16 & 5j^T & -2j^T \\ 5j & \tilde{A}_1 & \tilde{X} \\ -2j & \tilde{X}^T & \tilde{A}_2 \end{bmatrix} ,$$

then it follows by straight forward verification that $\tilde{A}^2 = 49\tilde{A}$. Hence the (0,1) matrix $A = \frac{1}{7}(\tilde{A} + 14I + 2J)$ satisfies (*) and therefore A is the adjacency matrix of a (49,16,3,6) strongly regular graph.
Next suppose Γ_1 is extendable, that is, there exist matrices X and A_2 such that the matrix

$$A = \begin{bmatrix} 0 & j^T & 0^T \\ j & A_1 & X \\ 0 & X^T & A_2 \end{bmatrix}$$

is the adjacency matrix of a (49,16,3,6) strongly regular graph. Let $x_1, \ldots, x_{32}$ be the columns of X. Then clearly the x_i's consist of 6 ones and 10 zeros, and (i) follows from (*). To prove (ii) define $\tilde{A} = 7A - 14I - 2J$. (In terms of association schemes, $\tilde{A}$ is a multiple of a minimal idempotent in the Bose-Messner algebra.) Since A has eigenvalue 16 with eigenvector j and eigenvalue 2 with multiplicity 32, and since A, I and J have a common full set of eigenvectors, $\tilde{A}$ must have an eigenvalue 0 of multiplicity 33, so $\text{rank}(\tilde{A}) = 16$. Also A_1, I and J have a common full set of eigenvectors and therefore, because A_1 has eigenvalue 3 with eigenvector j and no eigenvalue 2, $\tilde{A}_1$ has no eigenvalue 0, so $\tilde{A}_1$ is non-singular. Thus the following submatrix of $\tilde{A}$ has rank 16.

$$\begin{bmatrix} \tilde{A}_1 & \tilde{x}_j \\ \tilde{x}_i^T & \alpha \end{bmatrix} ,$$

where $\alpha = -16$ if $i = j$, and $\alpha = -2$ or 5 otherwise. Therefore the last column is a linear combination of the other columns, that is, there exists a vector v such that $\tilde{A}_1 v = \tilde{x}_j$ and $\tilde{x}_i^T v = \alpha$. Hence $\alpha = \tilde{x}_i^T \tilde{A}_1^{-1} \tilde{x}_j$. This proves (ii). Finally, if $x_i = x_j$, then (ii) implies $i = j$, so all 32 vectors x_i are distinct.

We remark that the above lemma generalizes to strongly regular graphs for which the multiplicity of one of the eigenvalues equals the degree. These are precisely the so called latin square graphs, negative latin square graphs and conference graphs.

5. COMPUTER RESULTS

For all 13 candidates the eigenvalues of the adjacency matrix have been computed. None has an eigenvalue equal to 2, so Lemma 2 applies in all cases. For each candidate we computed the matrix $\tilde{A}_1^{-1}$, and made a list of all 6-subsets for which the characteristic vector x_i satisfies $\tilde{x}_i^T \tilde{A}_1^{-1} \tilde{x}_i = -16$ (we use the notation of Lemma 2). For each candidate we searched by computer for 32 vectors in the list that also satisfy the other conditions of Lemma 2. For no candidate 32 suitable 6-subsets were found. Hence, we have:

THEOREM.
There exists no (49,16,3,6) *strongly regular graph.*

The mentioned computer search is not necessary. All candidates can be ruled out by ad hoc arguments ones the lists of feasible 6-subsets are computed. For instance, candidates number. 3, 4 and 10 do not work because the list is smaller than 32, and candidates number 5 and 8 do not work because an edge (edge $\{a,d\}$ and $\{a,\alpha\}$ respectively) is contained in just one set of the list, whilst by Lemma 2.i two sets are needed. For the other candidates the ad hoc arguments are more complicated. But it doesn't seem useful to spend a lot of place and effort to treat all these arguments, since a computer search is needed anyway to generate the lists of 6-subsets.

REFERENCES

[1] A.E. Brouwer and J.H. van Lint, *Strongly regular graphs and partial geometries*, pp. 85-122 in: Enumeration and Designs - Proc. Silver Jubilee Conf. on Combinatorics, Waterloo, 1982 (eds. D.M. Jackson & S.A. Vanstone), Academic Press, Toronto, 1984.

[2] H.A. Wilbrink and A.E. Brouwer, *A (57,14,1) strongly regular graph does not exist*, KNAW A 86 (= Indag. Math. 45) (1983) 117-121.

IN 1987 REEDS VERSCHENEN

242 Gerard van den Berg
Nonstationarity in job search theory

243 Annie Cuyt, Brigitte Verdonk
Block-tridiagonal linear systems and branched continued fractions

244 J.C. de Vos, W. Vervaat
Local Times of Bernoulli Walk

245 Arie Kapteyn, Peter Kooreman, Rob Willemse
Some methodological issues in the implementation of subjective poverty definitions

246 J.P.C. Kleijnen, J. Kriens, M.C.H.M. Lafleur, J.H.F. Pardoel
Sampling for Quality Inspection and Correction: AOQL Performance Criteria

247 D.B.J. Schouten
Algemene theorie van de internationale conjuncturele en strukturele afhankelijkheden

248 F.C. Bussemaker, W.H. Haemers, J.J. Seidel, E. Spence
On (v,k,λ) graphs and designs with trivial automorphism group

249 Peter M. Kort
The Influence of a Stochastic Environment on the Firm's Optimal Dynamic Investment Policy

250 R.H.J.M. Gradus
Preliminary version
The reaction of the firm on governmental policy: a game-theoretical approach

251 J.G. de Gooijer, R.M.J. Heuts
Higher order moments of bilinear time series processes with symmetrically distributed errors

252 P.H. Stevers, P.A.M. Versteijne
Evaluatie van marketing-activiteiten

253 H.P.A. Mulders, A.J. van Reeken
DATAAL - een hulpmiddel voor onderhoud van gegevensverzamelingen

254 P. Kooreman, A. Kapteyn
On the identifiability of household production functions with joint products: A comment

255 B. van Riel
Was er een profit-squeeze in de Nederlandse industrie?

256 R.P. Gilles
Economies with coalitional structures and core-like equilibrium concepts

257 P.H.M. Ruys, G. van der Laan
Computation of an industrial equilibrium

258 W.H. Haemers, A.E. Brouwer
Association schemes

259 G.J.M. van den Boom
Some modifications and applications of Rubinstein's perfect equilibrium model of bargaining

260 A.W.A. Boot, A.V. Thakor, G.F. Udell
Competition, Risk Neutrality and Loan Commitments

261 A.W.A. Boot, A.V. Thakor, G.F. Udell
Collateral and Borrower Risk

262 A. Kapteyn, I. Woittiez
Preference Interdependence and Habit Formation in Family Labor Supply

263 B. Bettonvil
A formal description of discrete event dynamic systems including perturbation analysis

264 Sylvester C.W. Eijffinger
A monthly model for the monetary policy in the Netherlands

265 F. van der Ploeg, A.J. de Zeeuw
Conflict over arms accumulation in market and command economies

266 F. van der Ploeg, A.J. de Zeeuw
Perfect equilibrium in a model of competitive arms accumulation

267 Aart de Zeeuw
Inflation and reputation: comment

268 A.J. de Zeeuw, F. van der Ploeg
Difference games and policy evaluation: a conceptual framework

269 Frederick van der Ploeg
Rationing in open economy and dynamic macroeconomics: a survey

270 G. van der Laan and A.J.J. Talman
Computing economic equilibria by variable dimension algorithms: state of the art

271 C.A.J.M. Dirven and A.J.J. Talman
A simplicial algorithm for finding equilibria in economies with linear production technologies

272 Th.E. Nijman and F.C. Palm
Consistent estimation of regression models with incompletely observed exogenous variables

273 Th.E. Nijman and F.C. Palm
Predictive accuracy gain from disaggregate sampling in arima - models

274 Raymond H.J.M. Gradus
The net present value of governmental policy: a possible way to find the Stackelberg solutions

275 Jack P.C. Kleijnen
A DSS for production planning: a case study including simulation and optimization

276 A.M.H. Gerards
A short proof of Tutte's characterization of totally unimodular matrices

277 Th. van de Klundert and F. van der Ploeg
Wage rigidity and capital mobility in an optimizing model of a small open economy

278 Peter M. Kort
The net present value in dynamic models of the firm

279 Th. van de Klundert
A Macroeconomic Two-Country Model with Price-Discriminating Monopolists

280 Arnoud Boot and Anjan V. Thakor
Dynamic equilibrium in a competitive credit market: intertemporal contracting as insurance against rationing

281 Arnoud Boot and Anjan V. Thakor
Appendix: "Dynamic equilibrium in a competitive credit market: intertemporal contracting as insurance against rationing

282 Arnoud Boot, Anjan V. Thakor and Gregory F. Udell
Credible commitments, contract enforcement problems and banks: intermediation as credibility assurance

283 Eduard Ponds
Wage bargaining and business cycles a Goodwin-Nash model

284 Prof.Dr. hab. Stefan Mynarski
The mechanism of restoring equilibrium and stability in polish market

285 P. Meulendijks
An exercise in welfare economics (II)

286 S. Jørgensen, P.M. Kort, G.J.C.Th. van Schijndel
Optimal investment, financing and dividends: a Stackelberg differential game

287 E. Nijssen, W. Reijnders
Privatisering en commercialisering; een oriëntatie ten aanzien van verzelfstandiging

288 C.B. Mulder
Inefficiency of automatically linking unemployment benefits to private sector wage rates

IN 1988 REEDS VERSCHENEN

297 Bert Bettonvil
Factor screening by sequential bifurcation

298 Robert P. Gilles
On perfect competition in an economy with a coalitional structure

299 Willem Selen, Ruud M. Heuts
Capacitated Lot-Size Production Planning in Process Industry

300 J. Kriens, J.Th. van Lieshout
Notes on the Markowitz portfolio selection method

301 Bert Bettonvil, Jack P.C. Kleijnen
Measurement scales and resolution IV designs: a note

302 Theo Nijman, Marno Verbeek
Estimation of time dependent parameters in lineair models using cross sections, panels or both

303 Raymond H.J.M. Gradus
A differential game between government and firms: a non-cooperative approach

304 Leo W.G. Strijbosch, Ronald J.M.M. Does
Comparison of bias-reducing methods for estimating the parameter in dilution series

305 Drs. W.J. Reijnders, Drs. W.F. Verstappen
Strategische bespiegelingen betreffende het Nederlandse kwaliteitsconcept

306 J.P.C. Kleijnen, J. Kriens, H. Timmermans and H. Van den Wildenberg
Regression sampling in statistical auditing

307 Isolde Woittiez, Arie Kapteyn
A Model of Job Choice, Labour Supply and Wages

308 Jack P.C. Kleijnen
Simulation and optimization in production planning: A case study

309 Robert P. Gilles and Pieter H.M. Ruys
Relational constraints in coalition formation

310 Drs. H. Leo Theuns
Determinanten van de vraag naar vakantiereizen: een verkenning van materiële en immateriële factoren

311 Peter M. Kort
Dynamic Firm Behaviour within an Uncertain Environment

312 J.P.C. Blanc
A numerical approach to cyclic-service queueing models