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RESEARCH MEMORANDUM

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COORDINATED REPLENISHMENT SYSTEMS WITH DISCOUNT OPPORTUNITIES

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Abstract:
In many practical situations coordination of replenishment orders of a family of items leads to considerable cost savings. A continuous review inventory system with compound Poisson demands and discount opportunities is considered. A well-known class of strategies for the case where cost savings are due to reduced fixed ordering costs is the class of can-order strategies. However, these strategies, which are simply to implement in practice, don't take discount possibilities into account. We propose a method to incorporate discounts in the framework of can-order strategies. When the can-order system triggers a replenishment, an additional decision has to be made whether the can-order replenishment has to be enlarged in such a way that the dollar value of the order exceeds a given discount breakpoint. A class of simple decision rules for this discount evaluation is investigated. For small problems the optimal strategy within this class can be found with a semi-Markov decision model. For large sized problems a one-period look ahead heuristic is proposed. Some numerical examples show that this heuristic performs quite satisfactorily.

1. Introduction

The main part of inventory management literature is devoted to single-item models. These models neglect possible savings which can be achieved by ordering two or more items together. These savings can be caused by reduced ordering costs, reduced freight rates, quantity discounts or improvement of the implementation of stock control. Therefore, joint replenishment models, in which the interaction among different items is explicitly taken into account, are very useful in many practical situations.

The joint replenishment models, which have been investigated in the literature, can be divided into two groups, depending on which of the following problems has to be tackled:

JRP1 : *Joint replenishment problem with savings due to reduced ordering costs*

The problem is to find an ordering strategy for the situation where a fixed ordering cost is incurred for any order that is placed. In addition, an item-specific ordering cost is charged for each particular item included in the replenishment (the fixed cost is shared when two or more items are jointly replenished).

JRP2 : *Joint replenishment problem with savings due to discounts*

In this situation, the problem is to find an ordering strategy for the situation where discounts are available if the total dollar value of an order exceeds a given discount breakpoint. These discounts may take several forms (for example: all units discounts, incremental units discounts or freight rate reduction).

JRP1 is extensively studied in the literature for the case of constant deterministic demand (a good overview is given in the review papers by Aksoy and Erenguc (1988) and Goyal and Satir (1989)).
Less attention has been paid to the stochastic demand case. A well-known class of strategies for the stochastic demand case of **JRP1** are the so-called "can-order" strategies, which are characterized by three parameters (S_i,c_i,s_i) for each item i. Under this type of control the inventory position is continuously reviewed. Whenever the inventory position of any item i drops to or below its must-order point (s_i) a family replenishment is made. All the other items j with an inventory position less than or equal to their can-order point (c_j) are included in this replenishment. The inventory position of each item j in the replenishment is then raised to the order-up-to level (S_j). Silver (1974, 1981), Thompstone and Silver (1975) and Federgruen et al. (1984) have proposed algorithms to find the parameters of the optimal can-order strategy for the case of (compound) Poisson demands.
Periodic replenishment strategies for the stochastic demand case of **JRP1**

have been investigated by Naddor (1975), Chakravarty (1986) and Atkins and Iyogun (1988).

All these strategies have in common that they don't take account of discount opportunities and hence they are not appropriate to deal with **JRP2**.
Miltenburg (1984a, 1984b, 1984c, 1985, 1987) has developed a coordinated replenishment system, based on reorder points, for the situation where quantity discounts provide significant potential savings. Roughly speaking, this system works as follows: if the inventory position of any item in the family drops (to or) below its reorder point, then a replenishment is triggered. Based on the actual inventory positions of all items, the relevant costs, the demand rates and the discount structure, a total order for the entire family is selected. This family order is obtained by aggregating information about all items in one single item. Discounts are evaluated by using a single-item model of Brown (1967). The family order is then allocated among the items in the family such that the expected time until the next replenishment is maximized. In a number of papers , Miltenburg (1985, 1987) and Miltenburg and Silver (1984a,b,c) provide procedures for determining the optimal reorder point of the different items, the size of the family order and the allocation of the family order among the items. Miltenburg assumes that demand occurs according to a Wiener process.
We refer to IBM's IMPACT (1971) for another class of strategies for **JRP2**.

As mentioned before, the can-order systems don't pay attention to discounts. Another shortcoming of can-order systems is that the calculation of the optimal policy is rather complex. On the other hand, however, if the parameters have been determined, the can-order strategy is very easy to implement in practice because of its simple structure.
In this paper, we extend the class of can-order strategies in such a way that discount opportunities can be taken into account. When the can-order system triggers a replenishment, an additional decision has to be made whether the can-order replenishment has to be enlarged so that the total dollar value of the order exceeds the discount breakpoint.

This paper is structured as follows. Section 2 introduces some notations and investigates how to incorporate discounts into the framework of can-order strategies. Further, a simple decision rule is proposed. Using this rule, the optimal strategy is found with a semi-Markov decision model, which is presented in section 3. Since application of this method is prohibitive in real applications, a heuristic method will be considered in section 4. This heuristic is investigated and validated in section 5. Finally, some concluding remarks are given in section 6.

2. Can-order strategies with discount opportunities

We start this section with a detailed description of the model. For a complete list of symbols, we refer to appendix 1. Coordinated replenishment systems for a family of N items with continuous review are considered where demand events are generated by independent compound Poisson processes with rate λ_i for item i. Demand sizes for item i are independent random variables with a probability distribution, which will be denoted by $\{\phi_i(j), 0 \leq j \leq m_i\}$. The expected demand size is denoted by ED_i. Excess demands are backlogged and the lead time is a constant L.
The objective is to minimize the total expected long run average cost per time unit subject to a given service level constraint. The relevant costs consist of ordering, holding and purchase costs.
The ordering costs are divided into two parts: a fixed ordering cost, denoted by K, is charged whenever an order is triggered. A minor ordering cost, k_i, is added if item i is included in the order. When the inventory on hand of item i is H_i, the holding cost of item i is charged at a rate $h_i H_i$ per unit time. Furthermore, the unit purchase cost in dollars of item i is v_i. Unit price discounts (such as all-units discounts) or freight rate discounts may be offered on the total dollar value of a family order.

Ignall (1969) has shown that the overall optimal strategy may have a complex structure for the case that savings by joint replenishments are caused by reduction of ordering costs (**JRP1**). Therefore, in the literature a lot of attention is paid to nearly optimal strategies, like the can-order strategies, which are more easy to implement. The control mechanism of the can-order strategies is already described in the previous section. Note that an individual item i is ordered when $I_i \leq s_i$, or when $I_i \leq c_i$ while $I_j \leq s_j$ for any item j (I_i denotes the inventory position of item i). When an item $j \neq i$ triggers a replenishment, we call this a *"special replenishment opportunity"* for item i.

Although the control mechanism of the can-order strategies is very simple, it is difficult to determine the optimal control parameters (S_i,c_i,s_i) for i=1,..,N. Interaction is caused by the special replenishment opportunities. Several heuristics have been proposed to overcome this difficulty. Most of them are based on a decomposition approach: the N-item problem is decomposed in N single-item problems. Crucial is the assumption that the special replenishment opportunities for item i (the trigger moments of all other items) can be approximated by a Poisson process. The rate of this process, μ_i, is equal to sum of the expected number of trigger events per unit time of the other items. Let β_j denote the expected number of replenishments per unit time triggered by item j, then $\mu_i := \Sigma_{j \neq i}\beta_j$. Given a set of trigger intensities (β_j) the control parameters (S_i,c_i,s_i) are determined from the solution of the single-item problem. An iterative solution procedure is then used since the control parameters of item i influence the rate of special replenishment opportunities of any other item. The iteration process stops when the control parameters are the same in two subsequent iterations. The general solution procedure for computing values of the control parameters of the optimal can-order strategy is given by PROC-CAN:

PROC-CAN :

Step 0 : Choose starting values for β_i (i=1,..N).

Step 1a: Initialize i:=0.

1b: Set i:=i+1, compute $\mu_i := \Sigma_{j \neq i}\beta_j$ (2.1)

1c: Solve the single-item problem **SIP**; i.e. choose the parameters (S_i,c_i,s_i) which minimize the expected long-run average cost of item i per unit time, subject to a given service level constraint, where demands and special replenishment opportunities are generated by independent Poisson processes with rates λ_i and μ_i, respectively. Next compute β_i given the parameters (S_i,c_i,s_i).

Step 2 : Stop when for each item the new control parameters are the same as in the previous iteration and the values of μ_i have sufficiently converged; otherwise go to step 1a if $i=N$ or to step 1b if $i<N$.

Remark 1: Based on this global solution scheme, Silver developed in a series of papers (see Silver (1974, 1981) and Thompstone and Silver (1975)) algorithms to solve the single-item problem in step 1c under different assumptions. In a related paper, Federgruen et al.(1984) carried out step 1c by describing **SIP** with a semi-Markov decision model which is solved by the policy-iteration algorithm. The service level constraint is handled by Lagrangian techniques. In section 4, this algorithm of Federgruen et al. is considered in more detail, because elements of it will be used in our heuristic for the discount case. ■

The can-order strategies so far do not take into account the possibilities for achieving a discount. In the rest of this paper, we investigate the multi-item system with all-units discounts. When the total dollar value of the replenishment (denoted by Q) is greater than or equal to a given discount breakpoint (denoted by Q_d), a discount is achieved which is a given percentage (d) of the total dollar value; so, when $Q \geq Q_d$, the discount is $d \cdot Q$. It will be shown that the methods to be developed are also applicable when other discount structures are given.

It seems reasonable to expect that the control parameters of the can-order strategy will change when discounts are incorporated. In the case of no-discounts, the long-run average purchase cost per unit time is the same under all possible stable strategies. Hence, the purchase cost is not taken into account in the algorithms of Silver and Federgruen et al.. In the case of discounts, however, the long-run average purchase cost depends on the strategy which is used. So, when the decomposition-approach is used to find the control parameters, the purchase cost should be taken into account in **SIP**. However, this cost depends on the total dollar value of the family order.

Since it is very hard to obtain changes in the control parameters due to possible discount possibilities, we propose to use a can-order strategy as if there are no discounts. However, when a can-order replenishment is triggered, a "discount evaluation" is done. When the total dollar value of the can-order replenishment (which will be denoted by Q_{can}) is not enough to achieve the discount, an additional decision has to be made whether Q_{can} has to be enlarged so that the total value of the order exceeds Q_d. A flow-chart of the system, which will be developed, is given in appendix 2.

Several possibilities are available to achieve this enlargement: the order quantities of those items which are already in the can-order replenishment might be enlarged or other items which are not in the can-order replenishment might be added. In this paper we restrict attention to the following simple rule: "include new items into the can-order replenishment and order those items up to their order-up-to level". We emphasize that this choice is only based on the easy structure of the resulting policy; the nice structure of the can-order strategy is preserved when using this rule. Note that the enlargement of the can-order replenishment is restricted under this rule.

To formalize the "discount evaluation" procedure three sets are defined; F_0 denotes the set of items in the can-order replenishment, whereas F_1 consists of the items which are not included. Finally, F_2^* denotes the set of items which are added to the can-order replenishment if one decides to take the discount.
F_2^* is chosen in such a way that the value of the extra order exceeds $(Q_d$-$Q_{can})$ and the extra ordering and holding costs until the next family replenishment are minimized. It is shown in appendix 3 that the problem of determining F_2^* can be modelled as a knapsack problem. This knapsack problem is denoted by **KP**(F_1).
The general procedure, PROC-DE, for "discount evaluation" when the inventory position of the system is $(I_1,..,I_N)$ at a given replenishment epoch, can be formalised as follows:

PROC-DE

Step 1a: Determine $F_0 := \{ i \mid I_i \leq c_i \}$, $F_1 := \{ i \mid I_i > c_i \}$

b: Calculate

$$Q_{can} := \sum_{i \in F_0} (S_i - I_i) \cdot v_i \tag{2.2}$$

c: If $Q_{can} \geq Q_d$, order $i \in F_0$; otherwise go to step 2

Step 2a: Calculate

$$Q_{ex} := \sum_{i \in F_0} (S_i - v_i) \cdot v_i \tag{2.3}$$

b: If $Q_{can} + Q_{ex} < Q_d$, order $i \in F_0$; otherwise go to step 3.

Step 3a: Determine F_2^* by solving the knapsack problem **KP**(F_1).

b: Solve **EP** : decide whether to order:

(i) $i \in F_0$ (discount is not taken), or

(ii) $i \in F_0 \cup F_2^*$ (discount is taken)

Note that Q_{ex} denotes the maximum enlargement of the can-order replenishment if Q_{can} is only enlarged with new items $i \in F_1$ whose inventory position is ordered up to S_i. If this extra order value is not enough to achieve the discount breakpoint, then the discount opportunity is neglected (see step 2b).

The procedure is straightforward except for the case that $Q_{can} < Q_d \leq Q_{can} + Q_{ex}$ (step 3). In this case one has to decide from a cost point of view whether the regular can-order replenishment has to be enlarged or not. The extra items ($i \in F_2^*$) which are ordered when the discount is taken are determined by solving the knapsack problem (step 3a). In section 3 and 4 we will discuss an optimal and a heuristic method to solve the evaluation problem **EP** in step 3b of PROC-DE.

3. An optimal method to solve the discount evaluation problem EP

In the previous section we have proposed the following strategy: periodically (for example, once in every three months), the parameters of the can-order strategy are determined by PROC-CAN (and the algorithm of Federgruen et al.

(1984) for solving **SIP**). Between these periodical reviews the can-order parameters are fixed. Discount opportunies are evaluated at those epochs at which the can-order strategy triggers a replenishment. PROC-DE is used to decide whether the order has to be stretched or not. Only items which are not included in the replenishment are used to enlarge the total value of the order (Q_{can}) if it is lower than the discount breakpoint (Q_d). When it is possible to achieve the discount by adding new items to the order, the evaluation problem **EP** has to be solved. In this section we develop a method to solve this problem **EP** by formulating the proposed replenishment strategy as a semi-Markov decision process.

It seems natural that the decision epochs of the semi-Markov decision proces are given by the moments at which a replenishment is triggered by the can-order system. However, the corresponding one-step transition probabilities turn out to be very complex. Therefore we choose as decision epochs the moments at which a demand occurs for any of the items in the family. Note that this implies that at several decision epochs no replenishment is triggered, and thus, no discount evaluation has to be done. By this choice the decision epochs follow a Poisson process, and the expected time between two subsequent decision epochs equals $\tau := 1/\Sigma_i \lambda_i$ (recall that demand events occur according to independent Poisson processes with rate λ_i for item i).

The state space of the inventory system is described by the vector of inventory positions of all items, $(I_1,..,I_N)$, just after a demand, but before the decision induced by the can-order strategy. Feasible states are those for which the inventory position of all items i is between s_i+1 and S_i, with the exception of at most one item j, whose inventory position is allowed to be equal to or lower than s_j.

There are two possible actions in each state,

- $a=0$: do not change the can-order replenishment,
- $a=1$: order extra items to get the discount.

In several states $(I_1,..,I_N)$ only action $a=0$ is feasible: if $I_i > s_i$ for all items i, if $Q_{can} \geq Q_d$, or if $Q_{can} + Q_{ex} < Q_d$. The set of feasible actions in state $(I_1,..,I_N)$ is denoted by $F(I_1,..,I_N)$.

The relevant costs consist of purchase, ordering and holding costs. Recall that the holding cost is charged at a rate h_i proportional to the inventory on hand of item i. The problem of determining the one-step holding cost is complicated when there is a lead time $L>0$, because the inventory position (I_i) and the inventory on hand (H_i) of item i differ during a time L after a replenishment of that item. Note that the inventory on hand at time $t+L$ depends on the inventory position at t and the demand during $[t,t+L]$. Hence, the inventory position just after the n'th decision epoch t_n determines the distribution of the inventory on hand at time t_n+L. A standard convention to handle positive deterministic lead times, which is also used by Federgruen et al. (1984), is to shift the holding cost in $[t_n+L,t_{n+1}+L]$ towards the time interval $[t_n,t_{n+1}]$.
Denoting the probability distribution of the total demand of item i during de lead time L by $r_i(k)$ and the inventory position of item i after action a by I_i^a, the expected holding cost incurred in $[t_n+L,t_{n+1}+L]$ is given by:

$$h((I_1,..,I_N),a) := \tau \cdot \sum_{i=1}^{N} h_i \cdot \left[\sum_{k=0}^{I_i^a} (I_i^a - k) \cdot r_i(k) \right] \tag{3.1}$$

with $I_i^0 := I_i$ when $i \in F_1$,
$:= S_i$ when $i \in F_0$.
$I_i^1 := I_i$ when $i \notin F_0 \cup F_2^*$,
$:= S_i$ when $i \in F_0 \cup F_2^*$.

The probability distribution of the total demand of item i during L can be computed recursively (see Adelson (1966)).

When choosing action a ($a=0,1$), the one-step costs consist of the immediate ordering and purchase cost together with $h((I_1,..,I_N),a)$, representing the expected holding cost. The one-step cost functions, denoted by $c((I_1,..,I_N),a)$, are calculated as follows:

$c((I_1,..,I_N),0) :=$

$$
\begin{cases}
h((I_1,..,I_N),0) & \text{if } Q_{can} = 0 \\
Q_{can} + K + \sum_{i \in F_0} k_i + h((I_1,..,I_N),0) & \text{if } 0 < Q_{can} < Q_d \\
(1-d) \cdot Q_{can} + K + \sum_{i \in F_0} k_i + h((I_1,..,I_N),0) & \text{if } Q_{can} \geq Q_d
\end{cases}
\tag{3.2}
$$

$c((I_1,..,I_N),1) :=$

$$
(1-d) \cdot [\, Q_{can} + \sum_{i \in F_2^*} (S_i - I_i) \cdot v_i \,] + K + \sum_{i \in F_0 \cup F_2^*} k_i + h((I_1,..,I_N),1) \tag{3.3}
$$

where I_i^a en $h((I_1,..,I_N),a)$ $(i=1,..,N;\ a=0,1)$ are defined as above.

As mentioned before, the methods are also applicable to other discount structures, such as freight rate discounts. It is shown in appendix 4 how the one-step cost functions (3.2) and (3.3) have to be adapted to handle freight rate discounts.

To define the one-step transition probabilities we use the following observation. If we start in state $(I_1,..,I_N)$, then the state at the next decision epoch is $(I_1^a,..,I_i^a\text{-}k,..,I_N^a)$, when the next decision epoch is induced by a demand of k units for item i. For $k=1,..,m_i$, $i=1,..,N$ and $a=0,1$, the one-step transition probabilities of the semi-Markov decision process are given by :

$$
p_{s,t}(a) := \lambda_i \cdot \tau \cdot \phi_i(k) \tag{3.4}
$$

where $s:=(I_1,..,I_i,..,I_N)$ and $t:=(I_1^a,..,I_i^a\text{-}k,..,I_N^a)$, and $p_{s,t}(a)$ is equal to zero elsewhere.

This completes the description of the semi-Markov decision process. Algorithms for computing an average cost optimal strategy for the semi-Markov decision model are the policy-iteration and the value-iteration algorithm (see e.g. Tijms (1986)). Due to the large state space[1] for our problem the latter algorithm is preferable. The value function, $V_n(s)$, follows from formula (3.47) in Tijms (1986):

$$V_n(s) := \min_{a \in F(s)} \left[\, c(s,a) \,/\, \tau + \sum_t p_{s,t}(a) \cdot V_{n-1}(t) \, \right] \qquad (3.5)$$

The algorithm stops after a finite number of iterations with a strategy for which the average cost is within a prespecified region around the minimal average cost per time unit. Note that by choosing action $a=0$ for all states the algorithm gives the appropriate cost expression for a given can-order strategy.

It is clear that this semi-Markov decision approach has not much value for large sized problems. From our numerical investigations it becomes apparent that huge amounts of computer time are necessary to solve only fairly small problems ($N=3$). However, the exact method will be used to validate a heuristic approach, which will be presented in section 4. The numerical comparisons are presented in section 5.

4. A heuristic approach to solve the discount evaluation problem EP

In this section we will describe a fast and simple heuristic for the discount evaluation problem **EP** in step 3b of PROC-DE. It should be noted that the ordering costs, the future holding cost and the purchase cost depend on the decision whether or not to enlarge the can-order replenishment. On the other hand, also the time until the next replenishment and the state at the next decision epoch will be influenced by the selected action.

The heuristic can be characterized as a *"one period" look ahead rule* which compares the sum of the direct costs until the next demand and the future costs for both possible actions. The costs until the next demand arrival are given by the one-step cost functions of the semi-Markov decision model in the

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[1] For example: if $N=3$, $s_i=0$, and $S_i=100$ $\forall$ i, and the demand size is always equal to unity ($\phi_i(1)=1$ $\forall$i), then we distinguish 1.000.000 states.

previous section (see (3.1) up to (3.3)). To approximate the total future costs after the next demand arrival we use the relative values from the policy iteration algorithm for the determination of the can-order system according to Federgruen et al.(1984).

These relative values are defined as the unique solution of a set of linear equations which has to be solved when using the policy-iteration algorithm. In particular, the difference between relative values $v_R(i)$ and $v_R(j)$ denotes for any two states i and j and any policy R the difference in total expected costs over a infinitely long period by starting in state i instead of state j when using strategy R (see e.g. Tijms (1986)). Relative values are used in a Markov decision model to construct from a given strategy R a new strategy R' whose average long run expected cost is less than or equal to that of R.
Federgruen et al. (1984) use this policy-improvement procedure to solve the N decomposed single-item problems **SIP** (see section 2). In their semi-Markov decision model corresponding to the single-item problem for item i, the decision epochs are given by epochs at which a demand or a special replenishment opportunity occurs for item i. The state of the system of this single item model is represented by (x,z), the inventory position x of item i just after a demand (z=0) or after a special replenishment opportunity (z=1). The relative value of state (x,z) of item i using can-order strategy R will be denoted by $v_{i,R}(x,z)$ (the subscript i is deleted in the notation used by Federgruen et al. (1984)).

As mentioned before, these relative values are used as an approximation of the future cost differences after the first demand arrival. However, in the relative values as produced by the algorithm of Federgruen et al. (1984) for the single-item system the purchase cost doesn't play a role, since the long run purchase cost for item i is equal to $v_i \cdot \lambda_i \cdot ED_i$ for every stable policy R. In our model the purchase cost plays a significant role in distinguishing between strategies, due to the discount possibilities. Therefore the purchase cost have to be incorporated into the relative values. The details of this extension are presented in appendix 5. The relative values are computed when the optimal can-order strategy is determined for a given number of periods (see the flow-chart in appendix 2).

The one-period look ahead heuristic is developed to solve the evaluation problem **EP** in step 3b of PROC-DE. It decides for a given state $s := (I_1,..,I_N)$, with $Q_{can} < Q_d \leq Q_{can} + Q_{ex}$, whether the regular can-order replenishment has to be enlarged to achieve the discount. The algorithm for the One-Period-Look ahead Heuristic is outlined below:

PROC-OPLH

Step 1 : Compute $c(s,a)$ for $a=0,1$ from (3.1) up to (3.3).

Step 2 : Compute for $a=0,1$, $k=1,..,m_i$, $i=1,..,N$ for any state $t := (I_1^a,..,I_i^a\text{-}k,..,I_N^a)$

a: $p_{s,t}(a)$ from (3.4),

b:

$$v_R(t) := \sum_{j=1}^{N} v_{j,R}(SIP) \tag{4.1}$$

with:

$v_{j,R}(SIP) := v_{j,R}(I_j^a,1)$ if $I_i^a\text{-}k \leq s_i$ and $i \neq j$,

$:= v_{j,R}(I_j^a\text{-}k,0)$ if $j=i$,

$:= v_{j,R}(I_j^a,0)$ otherwise.

Step 3 : Compute for $a=0,1$:

$$X(s,a) := c(s,a) + \sum_t p_{s,t}(a) \cdot v_R(t) \tag{4.2}$$

If $X(s,0) < X(s,1)$, then order $i \in F_0$ (do not take the discount), otherwise order $i \in F_0 \cup F_2^*$ (take the discount).

The value of state t at the next demand arrival of any item i (which is reached with probability $p_{s,t}(a)$) is computed by the sum of the relative values of the states I_j of the single items j. The relative value of item j (including the purchase cost) is equal to $v_{j,R}(I_j^a,1)$ if there is a special replenishment opportunity for item j and it is equal to $v_{j,R}(I_j^a,0)$ if the demand for item i ($\neq$ j) doesn't trigger a replenishment. Finally, the relative value of item j is equal to $v_{j,R}(I_j^a\text{-}k,0)$ if the next demand arrival is a demand of k units for item j.

We emphasize that the relative values are only approximations for the cost differences, not only because of the underlying decomposition assumption, but also due to the implicit assumption that no discount opportunities will occur in the future.

5. Numerical results

In this section some numerical results are given to validate the one-period look ahead rule for solving the evaluation problem **EP**. The average expected costs per unit time for the heuristic are compared with the optimal value of the average expected costs according to the semi-Markov decision model.
As mentioned in section 3, the applicability of the semi-Markov decision approach is restricted to only small problems ($N=3$). Therefore, simulation is used to validate the heuristic for larger problems ($N=15$). The costs of the strategy which uses the one-period look ahead rule are compared with the costs associated with two extreme heuristics: always take the discount and never take the discount.
Finally, our strategy should be compared with other discount strategies like this of Miltenburg (1987) or IMPACT (1971). However, an appropriate comparison between those strategies is not possible due to the difference in the assumptions of the demand process (e.g. Miltenburg assumes a demand process given by a Brownian motion instead of a Poisson process).

Table 1 lists some numerical data for a family of three items. There are no shortage costs involved, but there is a service level constraint which requires that at least 95% of the demand is satisfied directly from shelf. Further, $\phi_i(1)=1$ for all items i (simple compound demand) and $A=25$, $L=0.25$.
The can-order parameters, which are determined by PROC-CAN (and the algorithm of Federgruen et al. (1984) to solve **SIP**), are also given in table 1. The relative values including purchase costs, which are needed for the one-period look ahead rule, can then be computed by PROC-RV (see appendix 5).

TABLE 1
Data for numerical example 1 (N=3)

item i	v_i	h_i	a_i	λ_i	s_i	c_i	S_i
1	10.00	2.50	5.00	11.00	2	12	19
2	5.00	2.00	10.00	20.00	3	19	37
3	2.00	0.50	15.00	15.00	0	14	44

The dollar value of the discount break, Q_d, and the discount percentage, d, are varied for different experiments. Table 2 summarizes the minimal costs and the costs of the heuristic when using the given can-order strategy and PROC-DE for discount evaluation at replenishment epochs. C_{opt} denotes the exact costs when using the semi-Markov decision model, whereas C_{oplh} denotes the exact costs of the one-period look ahead rule. C_{oplh} is determined by using the value iteration algorithm with the given strategy from PROC-OPLH.

The determination of the optimal strategy takes huge computation times. The average computation time to get the optimal strategy for these small problems is more than four CPU-hours (on a VAX-8700-computer), wheras it takes only one CPU-second to obtain the strategy with the heuristic approach.

TABLE 2
Results for numerical example 1 (N=3)

combination	Q_d	d	C_{opt}	C_{oplh}
1	150	0.03	323.65	323.66
2	150	0.05	318.86	318.88
3	200	0.03	323.80	323.82
4	200	0.05	319.08	319.13
5	250	0.03	323.89	323.91
6	250	0.05	319.21	319.23

From table 2 it follows that the one-period look ahead rule works satisfactorily for the examples considered. However, we want to validate the strategy for larger sized problems. Since the semi-Markov decision approach is not applicable for large problems, simulation is used to find the average expected cost under three different strategies: the one-period look ahead rule for solving

EP is compared with two very crude heuristics: h1 and h2.

Under h1 the discount opportunity is only used when the regular can-order replenishment is enough to qualify for the discount (the replenishment is never enlarged).

Under h2 the discount opportunity is always used if the discount can be reached by adding new items to the regular replenishment (the replenishment is always enlarged with the item(s) $i \in F_2^*$ when it is possible to achieve the discount).

We consider a family of N=15 with A=75 and L=0.25. The items are listed in table 3, along with the values for h_i, v_i, a_i, λ_i and the corresponding can-order parameters s_i, c_i and S_i when the required fraction of demand which is satisfied directly from inventory on hand is 97.5% for all items. It is assumed that for all items the demand size has the same trunctated negative binomial distribution with parameters r=30,p=0.85.

TABLE 3
Data for numerical example 2 (N=15)

item i	h_i	v_i	a_i	λ_i	s_i	c_i	S_i
1	2.50	5.00	15.00	15.00	38	84	118
2	1.50	4.00	15.00	10.00	19	63	99
3	0.75	3.75	15.00	12.00	15	76	131
4	1.25	2.50	15.00	10.00	19	65	104
5	1.75	7.50	15.00	5.00	11	40	63
6	1.50	3.50	30.00	12.00	23	66	122
7	0.25	1.00	30.00	7.00	0	55	158
8	0.75	3.00	30.00	9.00	17	61	129
9	0.50	2.50	30.00	15.00	23	91	197
10	4.25	20.00	30.00	2.00	8	21	35
11	5.50	30.00	45.00	2.00	11	22	37
12	0.25	1.00	25.00	10.00	0	67	180
13	0.75	4.00	25.00	10.00	20	68	133
14	0.50	2.50	25.00	8.00	1	55	126
15	0.25	1.25	45.00	5.00	0	28	135

The discount breakpoint, Q_d, and the discount percentage, d, are varied in the experiments wheras the other parameters are kept fixed. For each combination of Q_d and d the number of simulation runs is determined by the requirement that a 95%-confidence interval has to be obtained with a bandwidth of

four. A single run for a given combination is obtained by simulating the multi-item system until 1000 orders have been triggered. The simulated average costs per time unit of the strategy which uses the one-period look ahead heuristic, h1 and h2 are denoted by C_{oplh}, C_{h1} an C_{h2}, respectively.
It appears that simulation is not a useful alernative for heuristic procedures from a computation time of view. It takes a large amount of computer-time (approximately 6 CPU-hours for the 15-item example) to obtain confidence intervals of acceptable width for the average cost.

TABLE 4
Results for numerical example 2 (N=15)

combination	Q_d	d	C_{h1}	C_{oplh}	C_{h2}
1	1500.0	0.03	4577.33	4573.79	4596.39
2	1500.0	0.05	4524.01	4519.53	4532.49
3	1500.0	0.10	4392.43	4366.41	4370.30
4	1500.0	0.15	4261.08	4207.92	4212.69
5	1500.0	0.20	4129.27	4050.56	4048.92
6	2000.0	0.03	4602.67	4596.34	4639.49
7	2000.0	0.05	4565.37	4548.91	4578.44
8	2000.0	0.10	4475.99	4419.04	4425.55
9	2000.0	0.15	4384.03	4270.81	4271.78
10	2000.0	0.20	4292.31	4116.33	4116.05
11	2500.0	0.03	4630.38	4617.45	4657.10
12	2500.0	0.05	4609.36	4584.80	4602.03
13	2500.0	0.10	4564.87	4470.55	4469.87
14	2500.0	0.15	4518.63	4340.83	4341.56
15	2500.0	0.20	4472.18	4208.27	4210.83

The procentage cost savings of using the one-period look ahead heuristic instead of h1 or h2 are low in most experiments. However, note that the absolute cost savings due to the discount opportunities are bounded. When the can-order strategy is optimal with respect to the ordering and the holding costs, the maximal cost saving per unit time equals:

$$d \cdot \sum_{i=1}^{N} \lambda_i \cdot ED_i \cdot v_i \tag{5.1}$$

Note that the maximal cost savings are only incurred when Q_{can} is always larger than Q_d.

In table 5 we compare the costs of each heuristic with the costs of the best heuristic in the particular situation. In addition, the maximal cost savings (max) due to the discounts are given in the last column.

TABLE 5

Cost deviations, $C_i - \min\{C_{oplh}, C_{h1}, C_{h2}\}$, for i=oplh, h1, h2

comb.	QK	d	h1	oplh	h2	max
1	1500.0	0.03	3.54	0.00	22.60	99.64
2	1500.0	0.05	4.48	0.00	12.96	166.06
3	1500.0	0.10	26.02	0.00	3.89	332.13
4	1500.0	0.15	53.16	0.00	4.77	498.19
5	1500.0	0.20	80.35	1.64	0.00	664.25
6	2000.0	0.03	6.33	0.00	43.15	99.64
7	2000.0	0.05	16.46	0.00	29.53	166.06
8	2000.0	0.10	56.95	0.00	6.51	332.13
9	2000.0	0.15	113.22	0.00	0.97	498.19
10	2000.0	0.20	176.26	0.28	0.00	664.25
11	2500.0	0.03	12.93	0.00	39.65	99.64
12	2500.0	0.05	24.56	0.00	17.23	166.06
13	2500.0	0.10	95.00	0.68	0.00	332.13
14	2500.0	0.15	177.80	0.00	0.73	498.19
15	2500.0	0.20	263.91	0.00	2.56	664.25

The following conclusions can be drawn from the numerical results. In all cases the one-period look ahead heuristic is the best of the three heuristics (note that the cost deviations in combination 5, 10 and 13 are not statistically significant).

The one-period look ahead heuristic is always fairly close to one of the extreme heuristics, although it cannot be predicted in advance to which of one.

Our conclusion is that related to the maximal cost savings obtained by the discount, the one-period look ahead heuristic gives fairly good results.

6. Concluding remarks

A procedure is developed to handle discount opportunities in a can-order system. Control parameters are set at a periodic basis, whereas demands are monitored continuously. The can-order strategy is used to trigger replenishments. At a replenishment epoch, it is possible to enlarge the regular can-order replenishment by using dynamic can-order levels. We proposed to use a one-period look ahead heuristic to evaluate the discount opportunities in those cases in which it is possible to achieve the discount by adding new items in the replenishment. An overview of the system is given in appendix 2. The flow-chart in this appendix shows the relationship among the different procedures which are described in this paper.

For small problems ($N \leq 3$) the performance of the heuristic is evaluated by comparison with the optimal average expected cost per time unit, which is determined by using a semi-Markov decision model. For larger problems the validation is based on simulation. Numerical results show that the proposed heuristic strategy gives fairly good results.

In this paper, we considered all-units discounts, but the method is also appropriate for other discount structures, such as freight cost discounts.

Appendix

Appendix 1 : List of symbols

N : number of items in the family.
λ_i : Poisson arrival rate of customers for item i.
$\phi_i(j)$: probability for item i that the demand size equals j units.
m_i : maximum demand size for item i.
ED_i : expected demand size for item i.
L : lead time.
K : fixed ordering cost per replenishment (independent of the items in the replenishment or the number of items ordered).
k_i : additional ordering cost for item i when it is included in the replenishment (independent of the order-quantity).
h_i : holding cost of item i per unit per unit time.
H_i : inventory on hand of item i.
I_i : inventory position of item i.
v_i : unit purchase price of item i.
S_i : order-up-to level of item i.
c_i : can-order point of item i.
s_i : must-order point of item i.
Q : total dollar value of the family replenishment.
Q_d : discount breakpoint (in dollars).
d : discount percentage.
Q_{can} : total dollar value of the regular can-order replenishment.
Q_{ex} : maximum total dollar value of extra order.
F_0 : set of items in the regular can-order replenishment.
F_1 : set of items which are not included in the regular can-order replenishment.
F_2^* : set of items which are added to the regular can-order replenishment to achieve the discount breakpoint.
τ : expected time between two subsequent demand events for any item in the family.
F(s) : set of feasible actions in state $s:=(I_1,..,I_N)$.
$r_i(k)$: probability for item i that the demand during lead time L equals k units.

I_i^a : inventory position of item i just after decision a has been selected.

$h(s,a)$: one-step holding cost function in state $s:=(I_1,..,I_N)$ when action a is selected.

$c(s,a)$: one-step cost function in state $s:=(I_1,..,I_N)$ when action a is selected.

$p_{s,t}(a)$: the one-step transition probability of going from state $s:=(I_1,..,I_N)$ to $t:=(I_1^a,..I_i^a\text{-}k,..,I_N^a)$ when action a is selected.

$V_n(s)$: value function in state $s:=(I_1,..,I_N)$ after n iterations.

$v_{i,R}(x,z)$: relative value of item i when the inventory position equals x just after a demand event $(z=0)$ or a special replenishment opportunity $(z=1)$ according to the single-item model of Federgruen et al. (1984) (with or without the purchase cost).

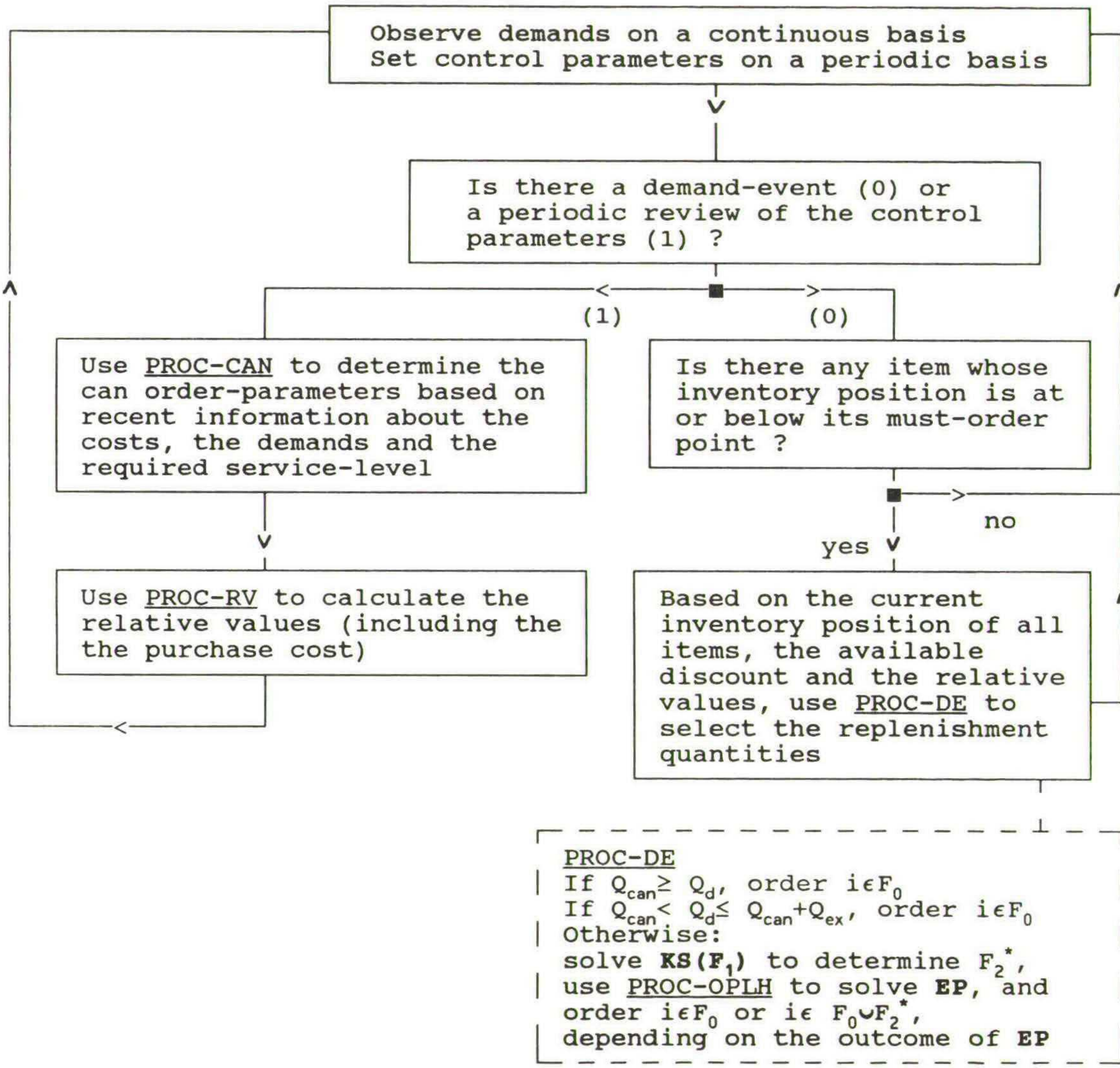
$v_{i,R}(s)$: relative value in state $s:=(I_1,..,I_N)$ of the multi-item model including the purchase cost.

$X(s,a)$: sum of direct costs until the next demand event and the weighted relative value at the next decision epoch, in state $s:=(I_1,..,I_N)$ when action a is selected.

Appendix 2: Flow-chart of the coordinated replenishment system

In figure 1, a flow-chart is given of the coordinated replenishment system with the periodic and daily decisions. Further, it describes the relationship among the different problems and procedures, which are described in this paper.

FIGURE 1
Coordinated replenishment system with discount opportunities



Appendix 3 : Determination of F_2^*

Recall that $F_0 := \{ i \mid I_i \leq c_i \}$ and $F_1 := \{ i \mid I_i > c_i \}$. Let F_2^* denote the set of items belonging to F_1, which are added to the regular can-order replenishment to achieve the discount. We assume that the sequence of can-order replenishments approximately follow a Poisson process with rate ρ, which equals $\Sigma_i\ \beta_i$ (the values of β_i (i=1,..N) follow from PROC-CAN together with the algorithm of Federgruen et al. (1984) for solving the single-item problem **SIP**). Under this assumption, the extra ordering and holding costs until the next replenishment equal:

$$\sum_{i \in F_2^*} \left[a_i + \frac{h_i \cdot (S_i - I_i)}{\rho} \right] \tag{a.1}$$

Note that the extra holding costs are shifted for a fixed period L. The problem of determining F_2^* can now be modelled as follows:

$$\begin{aligned} \min \quad & \sum_{i \in F_1} \left[a_i + \frac{h_i \cdot (S_i - I_i)}{\rho} \right] \cdot x_i \\ s.t. \quad & \sum_{i \in F_1} \left[v_i \cdot (S_i - I_i) \right] \cdot x_i \geq (Q_d - Q_{can}) \\ & x_i = 0,1, \ i \in F_1 \end{aligned} \tag{a.2}$$

Let $y_i = 1 - x_i$, $i \in F_1$. It is easily seen that the above minimization problem transforms to the following standard knapsack problem:

Knapsack problem **KS**(F_1):

$$\begin{aligned} \max \quad & \sum_{i \in F_1} \left[a_i + \frac{h_i \cdot (S_i - I_i)}{\rho} \right] \cdot y_i \\ s.t. \quad & \sum_{i \in F_1} \left[v_i \cdot (S_i - I_i) \right] \cdot y_i \leq (Q_{ex} + Q_{can} - Q_d) \\ & y_i = 0,1, \ i \in F_1 \end{aligned} \tag{a.3}$$

Note that $F_2^* = \{ i \mid y_i = 0, i \in F_1 \}$.

In the literature, several optimal and heuristic procedures have been proposed to solve this classical combinatorial optimization problem (see e.g. Martello and Toth (1990)). The following rule was used in our experiments: when the number of items in the knapsack (the number of items in F_1) is less than five, then the optimal solution for $\mathbf{KS}(F_1)$ is obtained by full enumeration of all possible solutions. Otherwise a heuristic approach is used, which is based on a ranking of the items in decreasing order with respect to the ratio of $\{ a_i + h_i (S_i-I_i)/\rho \}$ and $(S_i-I_i) \cdot v_i$. The knapsack is filled with items from the ordered list until the k'th item doesn't fit. The knapsack is then filled with any other item (>k) from the ordered list that still fits.

Appendix 4: One-step cost functions for freight rate discounts

The traditional quantity discount models analyse unit price discounts. In this paper we investigate one type of unit price discounts, namely all-units discounts. Another type of discount structure, which exist in many practical situations, is a fixed dollar value discount on ordering costs, as described in section 2, or freight costs when the replenishment order exceeds the discount breakpoint. A typical example is the situation where FR:=0 if $Q \geq Q_d$, and FR:=F if $Q < Q_d$, where FR denotes the freight cost per replenishment.

The discount evaluation procedure which is described for the all-units discount structure is also applicable to the freight cost discount structure, when the one-step cost functions (3.2) up to (3.3) are changed into:

$c((I_1,..,I_N),0) :=$

$$\begin{array}{ll} h((I_1,..,I_N),0) & \text{if } Q_{can} = 0 \\ Q_{can} + K + \sum_{i \in F_0} k_i + F + h((I_1,..,I_N),0) & \text{if } 0 < Q_{can} < Q_d \\ Q_{can} + K + \sum_{i \in F_0} k_i + h((I_1,..,I_N),0) & \text{if } Q_{can} \geq Q_d \end{array} \qquad (3.2')$$

$c((I_1,..,I_N),1):=$

$$[\ Q_{can} + \sum_{i \in F_2^*} (S_i - I_i) \cdot v_i\ ] + K + \sum_{i \in F_0 \cup F_2^*} k_i + h((I_1,..,I_N),1) \qquad (3.3')$$

Appendix 5 : Computation of the relative values $v_{i,R}(x,z)$

Recall that $v_{i,R}(x,z)$ denotes the relative value of item i with an inventory position of x units, just after a demand event (z=0) or a special replenishment opportunity (z=1) for a given can-order strategy $R=(S_i,c_i,s_i)$. The algorithm of Federgruen et al. (1984) for the single-item problem **SIP** can be used to compute the relative values. However, purchase costs are then neglected. In this appendix, the same approach as Federgruen et al. (1984) is used to determine the relative values including purchase costs. For convenience, the subscript i will be deleted in the notation.

For a fixed can-order policy R=(S,c,s) the average cost and relative values can be determined by the theory of regenerative processes. The attention is restricted to the cost incurred between two subsequent replenishment orders for the particular item. The regeneration state is the order-up-to level S, the state which is visited just after an order. Now, $t_R(x)$, $q_R(x)$, $h_R(x)$ and $k_R(x)$ can be defined for a can-order system (with control parameters R), which starts in state (x,0) with x>s: $t_R(x)$ is the expected time until the next replenishment order, $q_R(x)$ is the probabilty that this replenishment is triggered by a demand, $h_R(x)$ denotes the sum of the expected holding cost until the next replenishment and the expected purchase cost at that particular replenishment epoch and finally $k_R(x)$ denotes the total holding cost until the next replenishment together with the expected purchase and ordering costs incurred at the replenishment epoch. It follows that:

$$k_R(x):=h_R(x) + (K+k) \cdot q_R(x) + k \cdot (1-q_R(x)) \qquad (a.4)$$

The long run average cost per unit time under the can-order strategy R, denoted by g_R, equals:

$$g_R := k_R(S)/t_R(S) \tag{a.5}$$

$t_R(x)$, $q_R(x)$ and $h_R(x)$ are determined by conditioning on the state of the system after the next decision epoch. The probability that the next decision epoch is induced by a demand is $\lambda \cdot (\lambda+\mu)^{-1}$, wheras $\mu \cdot (\lambda+\mu)^{-1}$ equals the probability that the next decision epoch is induced by a special replenishment opportunity. Hence,

$$\begin{aligned} t_R(x) := {} & (\lambda+\mu)^{-1} + \mu \cdot (\lambda+\mu)^{-1} \cdot t_R(x) \cdot \delta(x-c) \\ & + \lambda \cdot (\lambda+\mu)^{-1} \sum_{j=0}^{x-s-1} t_R(x-j) \cdot \phi(j) \quad x>s \end{aligned} \tag{a.6}$$

and

$$\begin{aligned} q_R(x) := {} & \mu \cdot (\lambda+\mu)^{-1} \cdot q_R(x) \cdot \delta(x-c) \\ & + \lambda \cdot (\lambda+\mu)^{-1} \cdot \Big[\sum_{j=x-s}^{\infty} \phi(j) + \sum_{j=0}^{x-s-1} q_R(x-j) \cdot \phi(j) \Big] \quad x>s \end{aligned} \tag{a.7}$$

with $\delta(i)=1$ if $i>0$ and $\delta(i)=0$ otherwise.

Define $c(x)$ as the holding cost until the next decision epoch (Federgruen et al. (1984) consider also two sorts of penalty costs, but these are disregarded in this appendix). Using the same convention as in section 3 and 4 (the holding cost in $[t_n+L, t_{n+1}+L]$ is assigned to the cost after decision epoch t_n), it follows that

$$c(x) := (\lambda+\mu)^{-1} \cdot h \cdot \sum_{j=0}^{x} (x-j) \cdot r(j) \tag{a.8}$$

where $r(j)$ denotes the distribution function of the demand during the lead time and $(\lambda+\mu)^{-1}$ is the expected time until the next decision epoch.

If the next decision epoch is a demand of j units and $x-j \leq s$, then the item triggers a replenishment and a purchase order for $(S-x+j)$ units is placed. If the first decision epoch is a special replenishment opportunity and $x \leq c$, then the item is also included in the replenishment and a purchase cost of $(S-x) \cdot v$

dollars is incurred. Hence, for $x>s$:

$$
\begin{aligned}
h_R(x) := c(x) &+ \mu\cdot(\lambda+\mu)^{-1}\cdot h_R(x)\cdot\delta(x-c) \\
&+ \lambda\cdot(\lambda+\mu)^{-1}\cdot\sum_{j=0}^{x-s-1} h_R(x-j)\cdot\phi(j) \\
&+ \mu\cdot(\lambda+\mu)^{-1}\cdot(S-x)\cdot v\cdot\delta(c+1-x) \\
&+ \lambda\cdot(\lambda+\mu)^{-1}\sum_{j=x-s}^{\infty}(S-x+j)\cdot v\cdot\phi(j)
\end{aligned}
\tag{a.9}
$$

Finally, the relative values of the given can-order strategy $R=(S,c,s)$ are defined:

$$
\begin{aligned}
v_R(x,0) &:= k_R(x)-g_R\cdot t_R(x) && x>s \\
&:= K+k+(S-x)\cdot v && x\le s \\
v_R(x,1) &:= k_R(x)-g_R\cdot t_R(x) && x>c \\
&:= k+(S-x)\cdot v && x\le c
\end{aligned}
\tag{a.10}
$$

The algorithm for determining $v_{i,R}(x,z)$ is given below:

PROC-RV

Step 1: Compute $t_R(x), q_R(x), h_R(x), k_R(x)$ recursivily from (a.6),(a.7),(a.8),(a.9) and (a.4) for $x=s+1,..,S$.

Step 2: Compute g_R from (a.5).

Step 3: Compute $v_{i,R}(x,z)$ for $x=s_i+1-m_i,..,S_i$ and $z=0,1$ from (a.10).

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