A DYNAMIC MODEL
OF FACTOR DEMAND EQUATIONS

by

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Research memorandum

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1. Introduction

In this paper we will analyse the demand of factor inputs in a dynamic model, assuming profit maximizing firm behaviour, and adjustment costs.

In Section 2 and 3 we will specify the production function, the revenue function and the adjustment costs function. In Section 4 a long-term adjustment model is constructed, using the specifications of Section 2 and 3. The influence of cyclical disturbances on the demand of factor inputs is studied in the context of this long term model.

In Appendix A we will analyse the behaviour of a system of difference equations with begin and endpoint conditions and will study the dependence of the first period decision on the finite time horizon.

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2. The production function and the revenue function

2.1. The production function

We assume a production function of the aggregated type, $Q = F(X)$, where Q is output capacity and X is a vector of aggregated factor inputs, $X = (X_1,\dots,X_n)$. The factor inputs are measured in efficiency units, so that aggregation of different vintages of one factor is possible. We shall not treat in detail the conditions for an aggregated p.f. Instead we assume that for the relevant region of factor inputs, S, the production relations can adequately be described by the function [1].

$$(2.1) \qquad Q = F(X) \qquad\qquad X \in S \;,\; S \subset \mathbb{R}^n_+$$

which satisfies the following properties for $X \in S$

(i) $F(X) > 0$

(ii) $F(X)$ is continuous and twice differentiable for $X \in S$

(iii) $F_i(X) = \dfrac{\partial F}{\partial X_i} > 0 \qquad i = 1,\dots,n$

(iv) $F_{ij}(X) = \dfrac{\partial^2 F}{\partial X_i\, \partial X_j} = \begin{cases} < 0 & i = j \\ > 0 & i \neq j \end{cases} \qquad \begin{matrix} i = 1,\dots,n \\ j = 1,\dots,n \end{matrix}$

(v) $F(\lambda X) = \lambda^v F(X)$

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1) The function (2.1) contains not an explicit technical progress term. For our theoretical analysis the inclusion of (disembodied) technical progress is not essential.

A function which satisfies assumption (i) - (v) and has intuitive appeal is the generalized Cobb-Douglas p.f., which can be derived as follows. The total differential of the function $Q = F(X)$ is

$$(2.2) \qquad dQ = \sum_{i=1}^{n} F_i \, dX_i$$

and after some transformations we find[2)]

$$(2.3) \qquad \frac{dQ}{Q} = \sum_i F_i \cdot \frac{X_i}{Q} \cdot \frac{dX_i}{X_i}$$

or

$$(2.4) \qquad d\ln Q = \sum_i F_i \frac{X_i}{Q} \, d \ln X_i$$

The term $F_i \, X_i/Q_i$ is the production elasticity of factor i; assuming that for $X \in S$ the production elasticities can be reasonable well approximated by constant elasticities α_i we obtain

$$(2.5) \qquad d \ln Q = \sum_i \alpha_i \, d \ln X_i \qquad \alpha_i \geq 0 \quad , i = 1,\ldots,$$

The corresponding production function (p.f.) can then be written as

$$(2.6) \qquad Q = A \prod_i X_i^{\alpha_i}$$

Equation (2.6) satisfies assumptions (i) - (v) and the function is homogenous of degree $v = \Sigma \alpha_i$.

2) Unless stated otherwise all summations are taken over $i = 1,\ldots,n$

2.2. The revenue function

The total net receipts of the firm are $Y = P.Q^s$, where P is the output price per unit, net of costs of materials, and Q^s is the output which can be sold. In general Q^s will depend on P, which can be formally expressed by an output demand curve (o.d.c). The form of the o.d.c. depends on the organisation of the output market. If this market is characterised by perfect competition the o.d.c. is infinitely elastic so that Q^s can freely be changed for a given P which is exogenously determined. In an output market with monopolistic competition Q^s depends on P and P has to be set by the firm [3].

Since in a market with monopolistic competition the uncertainties on firm level are often large it seems preferable to assume a stochastic o.d.c.

(2.7) $$Q^s = G(P) + U$$

where U is a random variable with mean 0. A firm confronted with a stochastic o.d.c. is thus forced to decision making under risk. The price-quantity determination depends the on the attitude of the firm toward risk. For the sake of simplicity we assyme that the risk preferences of the firm are such that he uses an expected o.d.c. [4].

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3) If the o.d.c. has a price elasticity < -1 , the demand (curve) is called elastic and if the price elasticity > -1 the demand (curve) is called in-elastic. The elasticity of demand depends on the saturation level of the market, the position of the firm in the market etc. Note that the elasticity can vary if P changes and can even differ for positive and negative price changes.

4) For a linear cost-function, $C(Q^s) = a + b\ Q^s$, and linear risk preferences the maximation of the expected profit function, $E(\pi) = E[P\ Q^s - C(Q^s)]$, is equivalent to the maximation of the profit function in terms of expected demand, $\pi^* = P\ E(Q^s) - C(E(Q^s))$.

(2.8) $E(Q^s) = G(P)$

In the sequel we will omit the expectation operator E and write Q^s for the expected output demand.

We assume that the function $Q^s = G(P)$ is defined for $P \in S_p$ so that $\forall\, P \in S_p$, $G(P) \in S_Q$ where

(2.9) $S_Q = \{Q | Q = F(X),\ X \in S\}$

and has the following properties [5)]

5) Note that we are only interested in a local approximation of the o.d.c. The o.d.c. in Figure 1 does not satisfy the assymptions (2.10) but can in the region S_p be approximated by a function G(P) which satisfies (2.10).

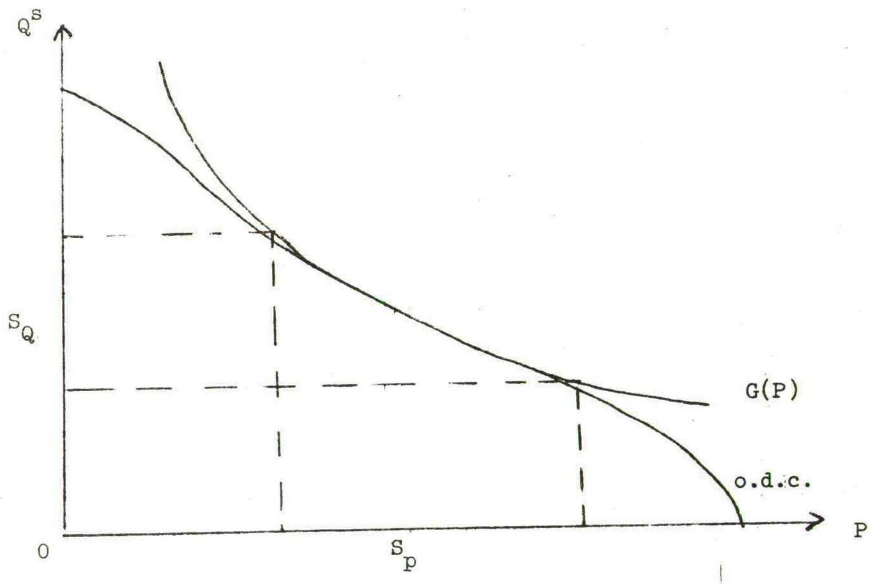

Figura 1

(2.10) i) $G(P)$ is continuous and twice differentiable

ii) $G'(P) < 0$

iii) $G''(P) > 0$

From (2.10) follows that the inverse function $P = H(Q^s)$ exists for all $Q^s \in S_Q$ and that $H(Q^s)$ has the following properties

(2.11) i) $H(Q^s)$ is continuous and twice differentiable

ii) $H'(Q^s) < 0$

iii) $H''(Q^s) > 0$

The revenue function $Y = PQ^s$ can now be written as

$$Y = H(Q^s).Q^s \tag{2.12}$$

The marginal revenue is

$$\frac{\partial Y}{\partial Q^s} = H'Q^s + H = (1 + \frac{1}{\eta})P \tag{2.13}$$

where η is the price elasticity of the o.d.c. $G(P)$. We find that the marginal revenue is positive iff . $\eta < -1$. Further we can express the revenue Y in terms of factor inputs X; if $Q^s < Q$ marginal changes in X do not affect Y so that $\partial Y/\partial X_i = 0$, $i = 1,\ldots,n$. If $Q^s = Q$ we can write

$$Y = H(Q).Q = H(F(X)).F(X) \tag{2.14}$$

and the marginal factor revenue is defined as

$$\frac{\partial Y}{\partial X_i} = (\frac{dH}{dQ}.Q + H).F_i = (1 + \frac{1}{\eta})P.F_i \tag{2.15}$$

From (2.13) and (2.1) follows that $\partial Y/\partial X_i$ is positive iff $\eta < -1$. The case that $Q^s > Q$ is not allowed within our model and will therefore not be analysed. If is of course also possible to obtain expressions for the second derivatives of Y with respect to Q^s and, under the restriction that $Q^s = Q$, with respect to X_i; without additional assumptions on G(P) these expressions are difficult to interpret.

A function which satisfies (2.10) and has other convenient mathematical properties is the constant elasticity demand curve

$$(2.16) \qquad Q^s = a\, P^{\eta} \qquad\qquad \eta < 0$$

We can modify (2.16) so that structural or cyclical changes are explicitly incorporated, e.g. as follows (t is discrete time):

$$(2.17) \qquad Q^s_t = b\, C_t\, (1+g)^t\, P^{\eta}_t$$

where C_t is a cyclical indicator and (1+g) a structural growth factor.

Combining (2.16) with the p.f. (2.6) , and assuming $Q^s = Q$, we find for the revenue function

$$(2.18) \qquad Y = c \prod_i X_i^{\alpha_i (1+ \frac{1}{\eta})}$$

or

$$(2.19) \qquad Y = c \prod_i X_i^{\gamma_i}$$

The γ_i are revenue elasticities of the input factor X_i and are only positive if $\eta < -1$. Let us assume that $0 < \gamma_i < 1$ for i = 1,...,n then the function Y(X) defined in (2.19) has the following properties for $X \in S$

(2.20) i) $Y_i = \frac{\partial Y}{\partial X_i} > 0 \qquad i = 1,\dots,n$

ii) $Y_{ij} = \frac{\partial^2 Y}{\partial X_i\, \partial X_j} = \begin{cases} < 0 & i = j \quad i = 1,\dots,n \\ > 0 & i \neq j \quad j = 1,\dots,n \end{cases}$

iii) $Y(\lambda X) = \lambda^{\Sigma \gamma_i}\, Y(X)$

iv) If $\Sigma\gamma_i < 1$ the function Y(X) is a strictly concave function of X

The Hessian-matrix Γ of the revenue function (2.19) is given by

$$\Gamma = \{\frac{\partial^2 Y}{\partial X_i \; \partial X_j}\} = (\hat{X}^{-1} \; G \; \hat{X}^{-1})Y \tag{2.21}$$

where

$$G = \begin{bmatrix} (\gamma_1-1)\gamma_1 & \cdots & \gamma_1 \gamma_n \\ \gamma_1 \gamma_2 & \cdots & \gamma_2 \gamma_n \\ \vdots & & \\ \gamma_1 \gamma_n & & (\gamma_{n-1})\gamma_n \end{bmatrix}, \quad \hat{X} = \begin{bmatrix} X_1 & & O \\ & \ddots & \\ O & & X_n \end{bmatrix} \tag{2.22}$$

In Section 3 and 4 we will use (2.19), and we will assume that $\eta < -1$, that $0 < \gamma_i < 1$ for $i = 1,\ldots,n$ and that $\Sigma\gamma_i < 1$. Note that the condition $\Sigma\gamma_i = 1$ does not imply that $\Sigma\alpha_i < 1$, since $\Sigma\gamma_i = (1 + \frac{1}{\eta})\Sigma\alpha_i$ and $(1 + \frac{1}{\eta})$ is in general smaller than one.

3. The adjustment process

3.1.1. Introduction

In many neo-classical firm behaviour models the factor inputs (labour and capital) are assumed to be completely variable, i.e. the factor inputs are adjusted immediately to their (long-run) equilibrium position. The production decisions of the firm at each point of time are independent of existing inputs levels; the intertemporal decision process can be decomposed into separate decisions taking place at distinct points of time. This assumption is not very realistic and at variance with the empirical evidence (e.g. the development of factor-shares during the cycle). Quasi-fixity of the capital and labour input can be build in explicitely in the model by introducing external adjustment costs (e.g. by assuming oligopsonistic capital good markets or labour markets) or internal adjustment costs (installation-costs, learning costs) in the form of output forgone. In the profit maximizing model the entrepeneur will, given the presence of adjustment costs, simultaneously determine the equilibrium input and output levels and the adjustment paths of input and output to these equilibrium levels. Pioneering work in this field has be done by Eisner and Strotz; more general models are constructed by R.E. Lucas [3], J.P. Gould [1], R. Schramm [5], A.B. Treadway [7] and D.T, Mortensen [4].

A complication (in neo-classical profit-maximizing models) arises if one allows for changes in the capacity utilization-rates. It is intuitively clear that changes in the capacity utilization-rate more likely if

(i) adjustment costs due to changes in the level of factor inputs are high relative to the costs of changes in the utilization-rate

(ii) the shifts in the output demand curve are transitory (e.g. seasonal variations).

To avoid very complex models, we would suggest a hierarchy of models A first model is a long-run model where long-run equilibrium levels of inputs and output and the adjustment path of inputs and output are jointly determined, assuming positive (internal) adjustment costs and a constant capacity utilization-rate. This model is primarily a structural model where the optimal changes in factor inputs are determined given the expected (structural) development on factor and output markets. In this model the long-run expansion-path of the firm is determined. (See Lucas, Treadway, Mortensen). A second model is a short-run model where the optimal output and the capacity utilization rate is planned given the existing capital stock and labour input.

For econometric purposes the long-run models can be used to specify factor demand equations, which explain determinants of investment and labour-demand of the firm. These equations can be estimated, using annual data. The short-run models are mostly derived to obtain forecasting models for industrial activity, and have to be estimated using monthly or quarterly data. In this study we are mainly interested in the specification of factor demand equations in a long-run model.

3.2. The internal adjustment costs function

Changes in factor inputs bring about adjustment costs. We distinguish external adjustment costs, arising from oligopsony on the factor markets, and output reducing or internal adjustment costs. The specification of the external adjustment costs function depends on the structure of the factor markets. In another paper a model with oligopsony on the labour market will be investigated. In this paper we will investigate the properties of the internal adjustment costs functions. Output reducing adjustment costs may arise as planning costs, installation costs, learning costs and other friction costs internal to the firm. The factor services supplied by the factors labour and capital are used not only to produce the firm's output but also to produce adjustment services, necessary to change the levels of the factors X_i. The existence of internal adjustment costs implies that the (maximum) output produced by the firm depends not only on the factor inputs, X_i, but also on the relative changes in these factor inputs.

Following Treadway and Mortensen we can specify a generalized production function (g.p.f.)

$$(3.1) \qquad Q = f(X,\Delta X) \qquad\qquad X \geq 0$$

We assume that the g.p.f. is continuous and twice differentiable, increasing in X_i and decreasing in ΔX_i, $i = 1,\ldots,n$

$$(3.2) \qquad \frac{\partial f}{\partial X_i} > 0 \quad ; \quad \frac{\partial f}{\partial \Delta X_i} < 0$$

The matrix H of second derivatives can be partitioned in

$$(3.3) \qquad H = \begin{bmatrix} A & C \\ C' & B \end{bmatrix} = \left[\begin{array}{c|c} \dfrac{\partial^2 f}{\partial X_i \, \partial X_j} & \dfrac{\partial^2 f}{\partial X_i \, \partial \Delta X_j} \\ \hline \dfrac{\partial^2 f}{\partial \Delta X_i \, \partial X_j} & \dfrac{\partial^2 f}{\partial \Delta X_i \, \partial \Delta X_j} \end{array}\right]$$

Negative definiteness of the submatrix A corresponds with (strict) concavity of the production function, negative definiteness of the submatrix B implies increasing marginal internal adjustment costs. An important case occurs if C = 0, which implies that the generalized production function can be separated in a standard production function and an internal adjustment cost function.

(3.4) $f(X,\Delta X) = F(X) + A(\Delta X)$

If, in addition, we assume that the matrix B is diagonal a furher separability of the adjustment cost function is possible, $A(\Delta X) = \sum_i A_i(\Delta X_i)$.

In the articles of Lucas, Schram, Treadway and Mortensen different assumptions are made with respect tot the seperability properties. Lucas [3] implicitly assumes that A and B are negative definite, that C is null and B is diagonal. Assuming that the firm maximizes its present value it is possible to derive the multivariate flexible accelerator

(3.5) $\Delta X = M(X - X^{*})$

where X is the vector of actual input levels and X^{*} the vector of stationary or equilibrium levels and M a matrix of adjustment parameters. The long-run equilibrium levels X^{*} can be determined independently of the adjustment process and are, assuming constant price expectations, equivalent to the long-run equilibrium levels derived form traditional static profit maximization models. These results are obtained using a continuous time model; in Schramm [5] analogous results are derived using a discrete time-model.

Mortensen shows for a continuous time model that the results of Lucas depend on the assumptions with respect B and C. Mortensen shows that if C is symmetric, which implies $\partial^2 f/\partial X_i \partial \Delta X_j = \partial^2 f/\partial X_j \partial \Delta X_i$, the results with respect to the adjustment paths are basically the same as the results found by Lucas. If in addition the matrix C is zero in the point $\Delta X = 0$, the stationary point X^{*} is likewise independent of the adjustment process.

3.3. A new specification of the adjustment costs function

Given the g.p.f. we can measure the internal adjustment costs in terms of production volume sacrified for the production of adjustment services. In a perfectly competitive product market the value of the adjustment services is easily measured by multiplying the production volume foregone with the output price P. In the case of imperfect competition on the product market some modifications are necessary.

Let X^A be the part of the factor inputs used for the production of adjustment services

(3.6) $$X^A = g(\Delta X)$$

where g is a vector function. The production volume sacrificed for the production of adjustment services is

(3.7) $$F(X) - F(X - X^A)$$

The "generalized revenue function" can now be written as

(3.8) $$Y = P\,F(X - X^A)$$

where the output price P depends on the production volume. The value of the internal adjustment services follows from

(3.9) $$a(\Delta X) = P(X)F(X) - P(X - X^A)F(X - X^A)$$

so that we can write the revenue function as

(3.10) $$Y = P(X)F(X) - a(\Delta X)$$

which is often a convenient specification in the derivation of the optimal factor demand equations. In this section we will derive an internal adjustment cost function as defined in (3.9).

The adjustment costs-function defined in this section contains both the costs of the learning process complementary to the installation of new capital goods and the introduction of new workers and the installation or re-installation services necessary if the ratio X_i/X_j $(i \neq j)$ changes. As to a reduction in input of factor i, this will not be followed by an instantaneous adjustment of the production technique. The substitution-process is a rather slow one, which implies a temporary under-utilization of all other inputs. This under-utilization is measured, in our approach in the form of adjustment services.

The magnitudee of the adjustment services depends not only on the extent of the changes in individual inputs but also on the direction of these changes. If all factors change in the same direction (expansion or reduction of the firm's activity level) the adjustment services will c.p. be lower than if the changes in the factor inputs show opposite directions (substitution).

A possiple specification of the adjustment services to be produced by factor X_i is

$$(3.11) \quad X_i^A = \sum_{j=1}^{n} \tau_{jj}^{(i)} \left(\frac{\Delta X_j}{X_{j0}}\right)^2 X_{i0} + \sum_{\substack{k=1 \\ k \neq j}}^{n} \sum_{j=1}^{n} \tau_{jk}^{(i)} \left(\frac{\Delta X_j}{X_{j0}}\right)\left(\frac{\Delta X_k}{X_{k0}}\right) . X_{i0} ; \tau_{jk} = \tau_{kj}$$

where X_{i0}, $i = 1,\dots,n$, is a fixed initial factorinput. We can write (3.11) as

$$(3.12) \quad X_i^A = (\Delta X' \, \hat{X}_0^{-1} \, T_i \, \hat{X}_0^{-1} \, \Delta X) X_{i0}$$

where T_i is a n×n symmetric matrix with elements τ_{jk} and

$$(3.13) \quad \hat{X}_0 = \begin{bmatrix} X_{10} & & 0 \\ & \ddots & \\ 0 & & X_{n0} \end{bmatrix}$$

From the discussion on adjustment services follows that T_i is a positive semi definite matrix with main-diagonal elements $\tau_{jj} \geq 0$ and off-diagonal elements $\tau_{jk} \leq 0$ $(j \neq k)$.

The adjustment costs due to internal adjustment services are measured as (See (3.9))

$$(3.14) \quad \mathfrak{a}(\Delta X) = Y(X) - Y(X-X^A) = (\frac{\partial Y}{\partial X}(X_0))' \; X^A = Y_X' \; X^A$$

where $X^A = (X_1^A,\ldots,X_n^A)'$, and the gradient Y_X is measured in $X_0 = (X_{10},\ldots,X_{n0})'$. For the revenue function defined in (2.19) we obtain

$$(3.15) \quad \mathfrak{a}(\Delta X) = \Sigma\gamma_i(\Delta X' \; \hat{X}_0^{-1} \; T_i \; \hat{X}_0^{-1} \; \Delta X)$$

$$= (\Delta X_0' \; \hat{X}_0^{-1}(\Delta\gamma_i \; T_i)\hat{X}_0^{-1} \; \Delta X)Y_0$$

$$= (\Delta X' \; \hat{X}_0^{-1} \; T \; \hat{X}_0^{-1} \; \Delta X)Y_0$$

where $Y_0 = Y(X_0)$ and

$$(3.16) \quad T = \Sigma\gamma_i \; T_i$$

T is a symmetric nxn matrix, which is assumed to be positive definite (so that the adjustment costs are always ≥ 0).

In Section 4 we will need the Hessian matrix

$$(3.17) \quad A = \{\frac{\partial^2 \mathfrak{a}(\Delta X)}{\partial\Delta X_i \; \partial\Delta X_j}\} = (\hat{X}_0^{-1} \; T \; \hat{X}_0^{-1})Y_0 + (\hat{X}_0^{-1} \; T \; \hat{X}_0^{-1})'Y_0$$

Since T is a symmetric positive definite matrix and $\hat{X}$ is a positive definite diagonal matrix, A is a symmetric positive definite matrix. Further we will need the matrix $A^{-1} \; \Gamma$ where Γ is the Hessian matrix of Y, defined in (2.21),

$$(3.18) \quad \Gamma = \{\frac{\partial^2 Y}{\partial X_i \; \partial X_j}\}_{X_0} = (\hat{X}_0^{-1} \; G \; \hat{X}_0^{-1})Y_0$$

exaluated in X_0. If Y is a strictly concave function of X for $X \in S$, Γ is a negative definite matrix. The characteristic values of $A^{-1} \; \Gamma$ can be found from

(3.19) $$|A^{-1}\,\Gamma - \lambda I| = 0$$

which is equivalent with

(3.20) $$|A^{-1}\,\Gamma - \lambda I| = |A^{-1}|\;|\Gamma - \lambda A| = 0$$

From (3.20) follows that all roots λ_i which satisfy $|\Gamma - \lambda A| = 0$ are negative [1]. Further $A^{-1}\,\Gamma$ has n linearly independent characteristic vectors [2].

Finally we define the matrix $\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,\hat{X}_0$ which does not depend on the factor input levels X_0 nor on the output level Y_0 if we use specification (2.19) for the revenue function. We can write

(3.21) $$\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,\hat{X}_0 = \hat{X}_0^{-1}\,\hat{X}_0\,\tfrac{1}{2}\,T^{-1}\,\hat{X}_0.\hat{X}_0^{-1}\,G\,\hat{X}_0^{-1}\,\hat{X}_0.(Y_0^{-1}.Y_0)$$
$$= \tfrac{1}{2}\,T^{-1}\,G$$

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1) Let $|\Gamma - \lambda A| = 0$ since A is positive definite there exists a nonsingular matrix W such that $A = WW'$ or

$$|\Gamma - \lambda A| = |\Gamma - \lambda WW'| = |W|^2 |W^{-1}\,\Gamma\,W'^{-1} - \lambda\,I|$$

where $W^{-1}\,\Gamma\,W'^{-1}$ is a negative definite matrix. From

$$|\Gamma - \lambda A| = 0 \Leftrightarrow |W^{-1}\,\Gamma\,W'^{-1} - \lambda\,I| = 0$$

follows then that all roots λ_i are real and negative..

2) Let $A^{-1}\,\Gamma\,X = \lambda X$ then since $A^{-1} = W'^{-1}\,W^{-1}$ we obtain $W'^{-1}\,W^{-1}\,\Gamma\,X = \lambda X$ or $W^{-1}\,\Gamma\,W'^{-1}\,W'X = \lambda W'X$ or $W^{-1}\,\Gamma\,W'^{-1}\,Y = \lambda Y$ where $Y = W'X$. Since $W\;\;\Gamma\,W'^{-1}$ is a symmetric negative definite matrix, there exist n linearly independent char. vector Y_i and since W' is a non-singular matrix n linearly independent char. vectors X_i .

and $\frac{1}{2} T^{-1} G$ does not depend on X_0 nor on Y_0. 3)

<u>Remark 1</u>. This adjustment cost function is based on internal adjustment services which consist of learning costs and (re)-installation costs. This function is more appropriate to describe expansion or substitution then to describe reduction of the activity level. If the firm reduces its input levels the internal <u>learning</u> costs have to be replaced by external costs as premiums for fired workers or capital losses on sold capital equipment. Since these costs can in general be described by a concave function, we might expect that even in these cases the adjustment costs function described in this section can be seen as a approximation of the true adjustment costs.

<u>Remark 2</u>. If the government takes over part of the wage bill in the case of a temporary shortening of the working-week, this can be seen as a subsidy of the government in the adjustment costs (both internal and external) corresponding to a temporary reduction in the labour-input.

3) In appendix A.2 we will need the characteristic values of the matrix $T^{-1} G$. For the two input case with $\gamma_1 = \gamma_2 = 0.4$ and T is a diagonal matrix with elements γ_1 and γ_2 the char. values of $T^{-1} G$ are - 0.2 and -1. If the matrix T is a diagonal matrix with elements $-\frac{1}{2}\gamma_1$ and $\frac{1}{2}\gamma_2$ the char. values of $T^{-1} G$ are -0.4 and -2. If the matrix T is a diagonal matrix with elements $2\gamma_1$ and $2\gamma_2$ the char. values of $T^{-1} G$ are -0.1 and -0.5

4. The long-term adjustment model

4.1. Introduction and assumptions

In this Section we will derive the adjustment process of the factor inputs to their optimal (equilibrium) values, assuming a profit maximizing firm behaviour. Further assumptions are

(i) the market for investment goods, the labour markets and the capital market are characterized by perfect competition, i.e. the prices on these markets are exogenous variables for the individual firm;

(ii) the product market is characterized by imperfect competition; the (long-run) product demand curve can be described by a constant elasticity demand function;

(iii) the production function and the revenue function are defined in Section 2, eq. (2.6) and (2.19);

(iv) the adjustment costs function is defined in (3.5).

4.2. A profit maximizing model in a stationary situation

We assume that the firm behves as if maximizing the present value of cash-flows over an infinite planning horizon under the condition that for t > T no further adjustments in output or factor inputs will be made. Further we assume constant price expectations for the factor markets and the capital market and a stable long-run product demand curve. Under these conditions the object function can be written as

$$(4.1) \qquad V = \sum_{t=1}^{T} \beta^t (Y_t - \alpha(\Delta X_t) - w'X_t - q'\Delta X_t) + \sum_{t=T+1}^{\infty} \beta^t (Y_T - w'X_T)$$

where $\beta = 1/(1+r)$, r being a constant discount rate, w is a vector of factor rewards [1] and q a vector of purchase prices.

We can formulate the following optimization problem. Maximize

$$(4.2) \qquad V = \sum_{t=1}^{T} \beta^t (Y_t - \mathfrak{a}(\Delta X_t) - w'X_t - q'\Delta X_t) + \frac{\beta^{T+1}}{1-\beta} (Y_t - w'X_T)$$

under the restrictions

$$(4.3) \qquad X_t = X_{t-1} + \Delta X_t \qquad t = 1,\dots,T$$

$$X_t \geq 0$$

Using standard optimization techniques the necessary conditions for a maximum [2], if the maximum lies in the economic relevant region, $X_t > 0$ $(t = 1,\dots,T)$[3], can be written as

$$\frac{\partial Y_t}{\partial X_t} = w + (1-\beta)q + A\Delta X_t - \beta A \,\Delta X_{t+1} \qquad t = 1,\dots,T-1$$

(4.4)

$$\frac{\partial Y_T}{\partial X_T} = w + (1-\beta)q + (1-\beta)A\Delta X_T$$

We will now assume that for all $X_t \in S$ the revenue function Y(X) can be approximated by a quadratic function so that we can linearize $\partial Y_t/\partial X_t$ as follows

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1) The factor rewards consist of wages for the labour inputs and of depreciation allowances and maintenance costs for capital goods.

2) These necessary conditions are in our case also sufficient conditions, since Y(X) is a strictly concave function and $X_t \in S$ for $t = 1,\dots,T$.

3) In fact we assume that $X_t \in S$ $(t = 1,\dots,T)$

(4.5) $$\frac{\partial Y_t}{\partial X_t} \approx \Gamma\,(X_t - X^*) + w + (1-\beta)q$$

where $\frac{\partial Y}{\partial X}(X^*) = w + (1-\beta)q$, $X^* \in S$, and Γ is evaluated in X_0.

Substituting (4.5) into (4.4) we obtain

(4.6)
$$\Gamma(X_t - X^*) = A\,\Delta X_t - \beta\,A\,\Delta X_{t+1} \qquad t = 1,2,\ldots,T-1$$
$$\Gamma(X_T - X^*) = (1-\beta)A\,\Delta X_T$$

or written as a system of difference equations in X_t we obtain

(4.7) $$\beta X_{t+2} + (A^{-1}\,\Gamma - (1+\beta)I)X_{t+1} + X_t = (A^{-1}\,\Gamma)X^* \qquad t = 0,1,2,\ldots$$

with endpoint conditions

(4.8) $$(A^{-1}\,\Gamma - (1-\beta)I)X_T + (1-\beta)X_{T-1} = (A^{-1}\,\Gamma)X^*$$

and beginpoint conditions $X_t = X_0$ for $t = 0$.

The system of difference equations (4.7) - (4.8) can be solved. The result is

(4.9) $$X_t = \sum_{i=1}^{2n} d_i^*\,c_i\,\lambda_i^t + X^* \qquad t = 0,1,2,\ldots$$

where λ_i are the roots of the characteristic equation of the system of difference equations (4.7), c_i are corresponding characteristic vectors and d_i^* are constants to be determined from begin- and endpoint conditions. After some manipulations we find that (see Appendix A.1)

(4.10)
$$0 < \lambda_i < 1 \qquad i = 1,\ldots,n$$
$$\lambda_i > 1 \qquad i = n+1,\ldots,2n$$

Using (4.10) we can write (4.9) as

(4.11) $X_t = (D_1 \Lambda_1^t + D_2 \Lambda_2^t + I)X^*$

where $D_1 = [d_1 c_1, \ldots, d_n c_n]$, $D_2 = [d_{n+1} c_{n+1}, \ldots, d_{2n} c_{2n}]$,

$$\Lambda_1 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} \lambda_{n+1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{2n} \end{bmatrix}$$

and $d_i = (X_i^*)^{-1} d_i^*$, $i = 1, \ldots, 2n$.

If $T \to \infty$ we can prove, see Appendix A.1, that

(4.12) $\lim_{T\to\infty} d_i = 0 \qquad i = n+1, \ldots, 2n$

$\lim_{T\to\infty} D_2 \Lambda_2^t = 0 \qquad t = 1, 2, \ldots, T$

Further $(D_1 \Lambda_1^t + I)X^*$ satisfies the endpoint conditions (4.8) if $T \to \infty$. Thus we conclude that if T is large we can neglect the unstable part $D_2 \Lambda_2^t X^*$ and write the solution of the system of difference equations as

(4.13) $X_t = (D_1 \Lambda_1^t + I)X^*$

The constants $(d_1, \ldots, d_n)$ can be determined from the beginpoint conditions. We find

(4.14) $D_1 X^* = (X_0 - X^*)$

The following results can now be obtained

(4.15) $(X_t - X_{t-1}) = D_1(\Lambda_1^t - \Lambda_1^{t-1})X^*$

or for $t = 1$

$$(4.16)\quad (X_1 - X_0) = D_1(\Lambda_1 - I)X^*$$

Since D is a non-singular matrix, see Appendix A-1, we can write, using (4.14)

$$(4.17)\quad (X_1 - X_0) = (D_1(\Lambda_1 - I)D_1^{-1})(X_0 - X^*)$$

and

$$(4.18)\quad (X_t - X_{t-1} = (D_1(\Lambda_1 - I)\Lambda_1^{t-1}\, D_1^{-1})(X_0 - X^*)$$

Defining $B = D_1(I - \Lambda_1)D_1^{-1}$ we obtain

$$(4.19)\quad \Delta X_1 = B(X^* - X_0)$$

and

$$\Delta X_t = B(I-B)^{t-1}(X^* - X_0) = B(X^* - X_{t-1})$$

which defines a geometric adjustment process.

From (4.19) follows, premultiplying with the matrix $\hat{X}_0^{-1}$, defined in (3.13),

$$(4.20)\quad \hat{X}_0^{-1}\, \Delta X_1 = (\hat{X}_0^{-1}\, B\, \hat{X}_0)(\hat{X}_0^{-1}\, X^* - \hat{X}_0^{-1}\, X_0)$$

and

$$\hat{X}_0^{-1}\, \Delta X_t = (\hat{X}_0^{-1}\, B\, \hat{X}_0)(I - \hat{X}_0^{-1}\, B\, \hat{X}_0)^{t-1}(\hat{X}_0^{-1}\, X^* - \hat{X}_0^{-1}\, X_0)$$

or defining $\tilde{B} = \hat{X}_0^{-1}\, B\, \hat{X}_0, \tilde{X}_t = \hat{X}_0^{-1}\, X_t$, $\Delta\tilde{X}_t = \hat{X}_0^{-1}\, \Delta X_t$, and $\iota = (1,\ldots,1)'$

$$(4.21)\quad \Delta\tilde{X}_1 = \tilde{B}(\tilde{X}^* - \iota)$$

$$\Delta\tilde{X}_t = \tilde{B}(I - \tilde{B})^{t-1}(\tilde{X}^* - \iota)$$

$\Delta\tilde{X}_1$ and $\Delta\tilde{X}_t$ in (4.21) are the solutions of the "rescaled" system of difference equations (4.7)

(4.22) $\beta\,\tilde{X}_{t+2} + (\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,\hat{X}_0 - (1+\beta)I)\tilde{X}_{t+1} + \tilde{X}_t = (\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,\hat{X}_0)\tilde{X}^*$

with endpoint conditions

(4.23) $(\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,\hat{X}_0 - (1-\beta)I)X_T + (1-\beta)\tilde{X}_{T-1} = (\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,X_0)\tilde{X}^*$

and beginpoint conditions

(4.24) $\tilde{X}_t = \iota$ for $t = 0$

The matrix $\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,\hat{X}_0 = \frac{1}{2}\,T^{-1}\,G$ is defined in (3.21). Since $T^{-1}\,G$ does not depend on the (initial) levels of output or factor inputs the matrix $\tilde{B}$, corresponding to the system (4.22) - (4.24), does not depend on X_0 or Y_0 but only on the discount factor β and the elements of $T^{-1}\,G$.

In Appendix A.2 the behaviour of ΔX_1 is analysed as function of the finite time horizon T. As might be expected, the first period adjustment in factor inputs for finite T, $\Delta X_1(T)$, converges quite rapid to the asymptotic solution (for $T \to \infty$) ΔX_1 in (4.19) if the adjustment costs are low. If the adjustment costs are high the convergence is slower. However in most cases a value of $T \geq 10$ will be sufficient to approximate the finite horizon solution $\Delta X_1(T)$ by ΔX_1 in (4.19).

4.3. A profit maximization model in a situation with cyclical disturbance

We assume that the firm behaves as if maximizing the present value of cash-flows over an infinite horizon under the condition that for $t \geq T$ no further adjustments in output or factor inputs will be made. Further we assume constant price expectations for the factor markets and the capital market and a stable long-run product demand curve except for the first period. In the first period we assume a temporary shift in the product demand curve so that the revenue function can be written as

$$(4.25) \quad Y^C = \eta \, Y^S$$

where Y^S is the stable long-run revenue function, Y^C the revenue function in period 1 and η a cyclical indicator.

Further we have to redefine the adjustment costs function in period 1. From (3.9) follows that the internal adjustment costs in period 1, $A^C(\Delta X)$, can be written as

$$(4.26) \quad A^C(\Delta X) = Y^C(X) - Y^C(X - X^A)$$

where X^A is defined in (3.6). Combining (4.25) and (4.26) we obtain

$$(4.27) \quad A^C(\Delta X) = \eta \, A \, (\Delta X)$$

where $A(\Delta X)$ is the internal adjustment costs function corresponding to the stable long-term revenue revenue Y^S.

We will now formulate an optimization problem under the assumption that actual production is equal to the actual production capacity minus production capacity used for the production of adjustment services.

For a stable long-run output demand function this assumption is not very restrictive but if we study the firm behviour with respect to short-run cyclical disturbances this assumption is not always realistic. However for econometric purposes a distinction between firms and periods where $Q = F(X)$ and firms and/or periods where $Q < F(X)$ is troublesome (aggregation of factor demand equations and a suitable specification of dynamic behaviour are then practically impossible). The optimization problem can now be formulated as[4], maximize

$$(4.28) \quad V = \beta(Y_1^C - \mathfrak{a}^C(\Delta X_1) - w'X_1 - q'\Delta X_1) + \sum_{t=2}^{T} \beta^t (Y_t - \mathfrak{a}(\Delta X_t) - w'X_t - q'\Delta X_t)$$

$$+ (\beta^{T+1})/(1-\beta)(Y_T - w'X_T)$$

under the restrictions

$$(4.29) \quad X_t = X_{t-1} + \Delta X_t \qquad t = 1,\ldots,T$$

$$X_t \geq 0$$

Using standard optimization techniques and supposing that the maximum lies in the economic relevant region , $X_t > 0$ $(t = 1,\ldots,T)$, the first order conditions can be written as

$$(4.30) \quad \frac{\partial Y_1}{\partial X_1} = w + (1-\beta)q + A_c\Delta X_1 - \beta A\,\Delta X_2$$

$$\frac{\partial Y_t}{\partial X_t} = w + (1-\beta)q + A\Delta X_t - \beta A\Delta X_{t+1} \qquad t = 2,\ldots,T-1$$

$$\frac{\partial Y_T}{\partial X_T} = w + (1-\beta)q = (1-\beta)A\Delta X_T$$

---

4) In our model the condition that actual production Q equals actual production capacity minus production capacity used for the production of adjustment services corresponds to the condition that the marginal net revenue, $\partial Y/\partial Q$, is positive. (Net revenue is defined as (gross) revenue minus variable costs as costs of materials etc.)

where A_c is evaluated in X_0.
Linearizing $\partial Y_1/\partial X_1$ and $\partial Y_t/\partial X_t$, t = 2,...,T we obtain

$$(4.31)\quad \Gamma_c(X_1 - X_c^*) = A_c\ \Delta X_1 - \beta A\ \Delta X_2$$

$$\Gamma(X_t - X^*) = A\ \Delta X_t - \beta A\ \Delta X_{t+1} \qquad t = 2,\dots,T-1$$

$$\Gamma(X_T - X^*) = (1-\beta)A\ \Delta X_T$$

where Γ_c is evaluated in X_0 and $\frac{\partial X^C}{\partial X}\ (X_c^*) = w + (1-\beta)q$.

Since from period 2 the firm operates in a stationary situation the change in factor inputs ΔX_2 can be found, using the heuristic argument of the "optimality principle", from the results of Section 4.2. So we obtain

$$(4.32)\quad \Delta X_2 = B(X^* - X_1)$$

Substituting (4.32) in (4.31) we obtain for period 1

$$(4.33)\quad \Gamma_c(X_1 - X_c^*) = A_c\ \Delta X_1 - \beta A\ B(X^* - X_1)$$

and for ΔX_1 we find

$$(4.34)\quad [A_c^{-1}\ \Gamma_c - I - \beta A_c^{-1}\ A\ B]\ \Delta X_1 = A_c^{-1}\ \Gamma_c(X_c^* - X_0) - \beta A_c^{-1}\ A\ B(X^* - X_0)$$

Since

$$(4.35)\quad A_c^{-1}\ \Gamma_c = \eta^{-1}\ A^{-1}.\eta\ \Gamma = A^{-1}\ \Gamma$$

$$A_c^{-1}\ A = \eta^{-1}\ A^{-1}\ A = \eta^{-1}$$

we can write for (4.34)

(4.35) $[A^{-1}\,\Gamma - I - \beta\eta^{-1}\,B]\,\Delta X_1 = A^{-1}\,\Gamma(X_c^{*}-X_0) - \beta\eta^{-1}\,B(X^{*}-X_0)$

Since the matrix $[A^{-1}\,\Gamma - I - \beta\eta^{-1}\,B]$ is negative definite we can solve ΔX_1 uniquely from (4.35) and we obtain

(4.26) $\Delta X_1 = B_1(X_c^{*}-X_0) + B_2\,(X^{*}-X_0)$

where

$$B_1 = [A^{-1}\,\Gamma - I - \beta\eta^{-1}\,B]^{-1}\,A^{-1}\,\Gamma$$

(4.37)

$$B_2 = [A^{-1}\,\Gamma - I - \beta\eta^{-1}\,B]^{-1}.(-\beta\eta^{-1}\,B)$$

The matrices B_1 and B_2 are positive definite. Unfortunately they depend on the initial input levels X_0 and on the cyclical indicator η. Analogous to the derivation in (4.20) - (4.24) we can obtain a "rescaled" solution by premultiplying (4.35) with $\hat{X}_0^{-1}$. We obtain

(4.38) $[\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,\hat{X}_0 - I - \beta\eta^{-1}\,\tilde{B}]\,\Delta\tilde{X}_1 = (\hat{X}_0^{-1}A^{-1}\Gamma\,\hat{X}_0)(\tilde{X}_c-\iota)-\beta\eta^{-1}\tilde{B}(\tilde{X}^{*}-\iota)$

where $\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,\hat{X}_0 = \frac{1}{2}\,T^{-1}\,G$ is defined in (3.21), $\tilde{B}$, $\Delta\tilde{X}_1$, $\tilde{X}$ in (4.20).
The matrices $\hat{X}_0^{-1}\,A^{-1}\,\Gamma\,\hat{X}_0$ and $\tilde{B}$ do not depend on the (initial) levels of output or factor inputs.
We can rewrite (4.38) as

(4.39) $\Delta\tilde{X}_1 = \tilde{B}_1(\tilde{X}_c^{*} - \iota) + \tilde{B}_2(\tilde{X}^{*} - \iota)$

where

$$\tilde{B}_1 = \hat{X}_0^{-1}\,B_1\,\hat{X}_0$$

$$\tilde{B}_2 = \hat{X}_0^{-1}\,B_2\,\hat{X}_0$$

The matrices $\tilde{B}_1$ and $\tilde{B}_2$ do not depend on factor input levels or output level but vary with the cyclical indicator η. From (4.38) follows that the elasticity of the elements of $\tilde{B}_1$ with respect to η is positive but always considerable smaller than one and that the elasticity of the elements of $\tilde{B}_2$ with respect to η is negative but always larger than minus one. Thus the behaviour of $\tilde{B}_1$ and $\tilde{B}_2$ is counter-cyclical if $\eta < 1$ but pro-cyclical if $\eta > 1$.

Remark: If the condition $Q = F(X)$ is satisfied in all periods except in period 1 the first order conditions for period 1 can be written as

$$(4.40) \qquad -A_c \Delta X_1 = w - \beta A \, \Delta X_2$$

where ΔX_2 is determined in (4.31). Substituting (4.31) in (4.40) we obtain

$$(4.41) \qquad (-A_c - \beta AB) \, X_1 = w - \beta A \, B(X^* - X_0)$$

or since $(-A_c - \beta A \, B)$ is a negative definite matrix

$$(4.42) \qquad \Delta X_1 = (-A_c - \beta A \, B)^{-1} \, w + (A_c + \beta A \, B)^{-1} \, \beta A \, B(X^* - X_0).$$

APPENDICES

A. Properties of a system of second order difference equations with begin and endpoint conditions.

A.1. The solution of the system of difference equations

Let the system of n difference equations be given by

$$\beta Y_{t+2} + (A-(1+\beta)I)Y_{t+1} + Y_t = AY^* \qquad t = 0,1,2,... \tag{A.1}$$

where $0 < \beta < 1$, A is a non-singular nxn matrix with negative roots and n linearly independent char. vectors. Further we define the begin and endpoint conditions

$$Y_0 = Y_{(0)} \tag{A.2}$$

$$(A-(1-\beta)I)Y_T + (1-\beta)Y_{T-1} = AY^*$$

Firstly we consider the homogenous part of (A.1)

$$\beta Y_{t+2} + (A-(1+\beta)I)Y_{t+1} + Y_t = 0 \tag{A.3}$$

and try a solution of the form $Y_t = \lambda^t c$ where c is a vector and λ a scalar. Since we are only interested in non-trivial solutions the roots λ_i and corresponding vectors c_i can be found from

$$\lambda^t |I + (A-(1+\beta)I)\lambda + \beta\lambda^2 I| = 0 \tag{A.4}$$

or from

$$\lambda^{t+1} | A - \gamma I| = 0 \tag{A.5}$$

where $\gamma = (1+\beta) - \beta\lambda - \lambda^{-1}$. From the fact that A is a non singular matrix with negative roots and n linearly independent char, vectors follows that all γ which satisfy (A.5) are negative and that there exist n linearly independent vectors c_i which are the characteristic vectors of A, corresponding to the roots γ_i of (A.5)

For the function $f(\lambda) = (1+\beta) - \beta\lambda - \lambda^{-1}$ we find

$$f(\lambda) > 0 \qquad \lambda < 0 \; , \quad 1 < \lambda < 1/\beta$$

$$f(\lambda) = 0 \qquad \lambda = 1 \; , \; \lambda = 1/\beta$$

$$f(\lambda) < 0 \qquad 0 < \lambda < 1 \; , \quad \lambda > 1/\beta$$

and the sign of the first derivative in the relevant region of λ is

$$f'(\lambda) > 0 \qquad 0 < \lambda < 1$$

$$f'(\lambda) < 0 \qquad \lambda > 1/\beta$$

Thus for each γ_i ($\gamma_i < 0$, $i = 1,\ldots,n$) we find two roots $(\lambda_i, \lambda_{i+n})$ where

$$\text{(A.6)} \qquad 0 < \lambda_i < 1$$

$$\lambda_{i+n} > 1/\beta$$

Thus we can write the general solution, Y_t^H, of the system of homogenous difference-equations (A.3) as

$$\text{(A.7)} \qquad Y_t^H = \sum_{i=1}^{n} \bar{d}_i \, \lambda_i^t \, c_i + \sum_{i=1}^{n} \bar{d}_{n+i} \, \lambda_{n+i}^t \, c_i = C\Lambda_1^t \, d_1 + C\Lambda_2^t \, d_2$$

where Λ_1 is a diagonal matrix with elements λ_i ($i = 1,\ldots,n$) and Λ_2 is diagonal matrix with elements λ_{n+i} ($i = 1,\ldots,n$). C is the matrix of characteristic vectors c_i ($i = 1,\ldots,n$) and d_1, d_2 are vectors with elements $\bar{d}_i$, $\bar{d}_{n+i}$ ($i = 1,\ldots,n$) respectively which are constants to be determined from the begin and endpoint conditions. A particular solution of the system (A.1) is given by Y^* so that the solution of this system is

$$Y_t = Y_t^H + Y^* \tag{A.8}$$

Substituting (A.7) and (A.8) in the begin and endpoint conditions we find

$$C(d_1 + d_2) = Y_{(0)} - Y^* \tag{A.9}$$

$$[(A-(1-\beta)I)C\Lambda_1^T+(1-\beta)C\Lambda_1^{T-1}]\,d_1+[(A-(1-\beta)I)C\Lambda_2^T+(1-\beta)C\Lambda_2^{T-1}]\,d_2 = 0$$

We can write (A.9) more compactly als

$$\begin{bmatrix} C & C \\ B_1\ \Lambda_1^{T-1} & B_2\ \Lambda_2^{T-1} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} Y_{(0)} - Y^* \\ 0 \end{bmatrix} \tag{A.10}$$

where

$$B_1 = A\ C\ \Lambda_1 - C\ ((1-\beta)\Lambda_1 - (1-\beta)I)$$

(A.11)

$$B_2 = A\ C\ \Lambda_2 - C\ ((1-\beta)\Lambda_2 - (1-\beta)I)$$

Since $A\ C = C\ \Gamma$ where Γ is a diagonal matrix whose elements γ_i are the roots of (A.5) we can write for B_1, B_2

$$\text{(A.12)} \quad B_1 = C[\Gamma\Lambda_1 - (1-\beta)\Lambda_1 + (1-\beta)I]$$

$$B_2 = C[\Gamma\Lambda_2 - (1-\beta)\Lambda_2 + (1-\beta)I]$$

From (A.6) and (A.12) follows that B_1 and B_2 are non-singular matrices:

$$\text{(A.13)} \quad B_1 = -\beta\ C(\Lambda_1^2 - 2\ \Lambda_1 + I) = -\beta\ C(\Lambda_1 - I)^2$$

$$B_2 = -\beta\ C(\Lambda_2^2 - 2\ \Lambda_2 + I) = -\beta\ C(\Lambda_2 - I)^2$$

From the non-singularity of B_1 and B_2 follows that d_1 and d_2 can be solved uniquely from (A.10), since the matrix in the left hand side of (A.10) is non-singular. From the beginpoint conclitions in (A.10) follows

$$\text{(A.14)} \quad d_1 = C^{-1}\ (Y_{(0)} - Y^*) - d_2$$

Substituting (A.14) in the endpoint conditions in (A;10) we find

$$\text{(A.15)} \quad B_1\ \Lambda_1^{T-1}\ C^{-1}\ (Y_{(0)}-Y^*) + (B_2\ \Lambda_2^{T-1} - B_1\ \Lambda_1^{T-1})d_2 = 0$$

Since (A.15) holds for all T and since B_1 and B_2 do not depend on T we find from (A.15) for $T \to \infty$

$$\text{(A.16)} \quad \lim_{T\to\infty} B_2\ \Lambda_2^{T-1}\ d_2 = 0$$

and thus

$$\text{(A.17)} \quad \lim_{T\to\infty} d_2 = 0$$

Combining (A.17) with (A.14) we find

$$\text{(A.20)} \quad \lim_{T\to\infty} d_1 = C^{-1}\ (Y_{(0)} - Y^*)$$

Finally it follows from (A.17) that for all $t \leq T$

(A.21) $$\lim_{T\to\infty} C\, \Lambda_2^t\, d_2 = 0$$

so that

(A.22) $$\lim_{T\to\infty} Y_t^H = C\, \Lambda_1^t\, d_1$$

where d_1 is determined in (A.20).

Solution (A.8) with d_1 and d_2 determined from (A.14) and (A;15) is an uniquely determined solution of the system of difference equations (A.1) with boundary conditions (A.2). Thus, using a constructive method, we have shown that the system (A.1) with boundary conditions (A.2) has an unique solution[1)]. Further we have shown that the solution depends on T and that for $T \to \infty$ only the stable part of the homogenous solution Y_t^H is left over.

Remark 1: If the endpoint conditions are given by

(A.23) $$(A-(1+\beta)I)Y_T + (1-\beta)Y_{T-1} = A\, Y^* + b$$

we can determine the vectors of constants d_1 and d_2 from

(A.24) $$\begin{bmatrix} C & C \\ B_1\, \Lambda_1^{T-1} & B_2\, \Lambda_2^{T-1} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} Y_{(0)} - Y^* \\ b \end{bmatrix}$$

which implies that d_1 and d_2 can be uniquely solved.

The behaviour of Y_t^H for $T \to \infty$ follows from the analogon of (A.16):

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1) In fact it is not true that every system of difference equations with corresponding boundary equations has an (unique) solution.

(A.25) $$\lim_{T\to\infty} B_2 \Lambda_2^{T-1} d_2 = b$$

which implies

(A.26) $$\lim_{T\to\infty} \Lambda_2^{T-1} d_2 = B_2^{-1} b$$

$$\lim_{T\to\infty} d_2 = 0$$

$$\lim_{T\to\infty} d_1 = C^{-1}(Y_{(0)} - Y^*)$$

For the homogenous solution Y_t^H we can write

(A.27) $$Y_t^H = C \Lambda_1^t d_1 + C \Lambda_2^{t-T} \Lambda_2^T d_2$$

so that if $t \to \infty$ and $T \to \infty$ such that $(t-T)$ is a fixed number we find

(A.28) $$\lim_{t,T\to\infty} Y_t^H = \lim_{t\to\infty} C \Lambda_1^t d_1 + C \Lambda_2^{t-T} \lim_{T\to\infty} \Lambda_2^T d_2$$

$$= C \Lambda_2^{t-T} B_2^{-1} b$$

Further we find that if for fixed t, $T \to \infty$

(A.29) $$\lim_{T\to\infty} Y_t^H = C \Lambda_1^t d_1 + \lim_{T\to\infty} C \Lambda_2^{t-T} \lim_{T\to\infty} \Lambda_2^T d_2$$

$$= C \Lambda_1^t d_1$$

Thus if T is large we can for small values of t approximate the homogenous solution Y_t^H by its stable part.

Remark 2. A slightly different system of difference equations is given by

(A.1.a) $$\beta Y_{t+2} + (A+B-(1+\beta)I)Y_{t+1} + Y_t = (A+B)Y^*$$

where A is as defined in (A.1) and B is a metrix with char. roots ≤ 0. Further we have as boundary restrictions

(A.2.a) $Y_0 = Y_{(0)}$

$$(A+\alpha B-(1-\beta)I)Y_T + (1-\beta)Y_{T-1} = (A+B)Y^* + b$$

where $0 < \alpha < 1$.

The general solution of (A.1.a) is completely analogous to the solution (A.7):

(A.7.a) $Y_t = C_a \Lambda_{1a}^t d_{1a} + C_a \Lambda_{2a}^t d_{2a} + Y^*$

where C_a is the matrix of char. vectors of the matrix A + B, Λ_{1a} and Λ_{2a} are the matrices of char. roots of the diff.eq. (A.1.a) and d_{1a} and d_{2a} are the corresponding vectors of constants to be determined form the begin and endpoint-conditions. The properties of Λ_{1a} and Λ_{2a} are identical to the properties of Λ_1 and Λ_2 in (A.7) and it follows from the assumptions on A and B that C_a is a non-singular matrix.

The analogon of (A.11) is

(A.11.a) $B_{1a} = (A + \alpha B)C_a \Lambda_{1a} - (1-\beta)C_a(\Lambda_{1a} - I)$

$$B_{2a} = (A + \alpha B)C_a \Lambda_{2a} - (1-\beta)C_a(\Lambda_{2a} - I)$$

Since α and β vary independently from each other the matrices B_{1a} and B_{2a} are in general non singular so that the vector $(d_{1a}, d_{2a})'$ can be solved uniquely from the analogon of (A.24). Further we find as analogon of (A.25)

(A.25.a) $\lim_{T\to\infty} B_{2a} \Lambda_{2a}^{T-1} d_{2a} = b + (1-\alpha)BY^*$

which implies

$$(A.26.a.)\quad \lim_{T\to\infty} \Lambda_2^{T-1}\, d_2 = B_{2a}^{-1}\, b + B_{2a}^{-1}\,(1-\alpha)BY^*$$

$$\lim_{T\to\infty} d_2 \qquad = 0$$

and together with the analogon of (A.14)

$$\lim_{T\to\infty} d_1 = C_a^{-1}\,(Y_{(0)} - Y^*)$$

For Y_t^H we obtain results completely analogous to (A.28) and (A.29) so that for large values of T and small values of t the homogenous solution Y_t^H can be approximated by its stable part: $C_a\ \Lambda_{1a}^t\ d_{1a}$.

A.2. The dependence of the first period decision on the finite time horizon.

The system of difference equations (A.1) with boundary conditions (A.2) can be rewritten in a cumulative fashion as

$$\text{(A.30)}\quad \begin{bmatrix} I-A & -\beta I & 0 & 0 & \dots & 0 & 0 \\ -A & I-A & -\beta I & & & & \\ -A & -A & I-A & -\beta I & & & \\ \vdots & \vdots & \vdots & & & & \\ -A & -A & -A & -A & & I-A & -\beta I \\ -A & -A & -A & -A & & -A & (1-\beta)I-A \end{bmatrix} \begin{bmatrix} \Delta Y_1 \\ \Delta Y_2 \\ \Delta Y_3 \\ \vdots \\ \Delta Y_{T-1} \\ \Delta Y_T \end{bmatrix} = \begin{bmatrix} A(Y_{(0)}-Y^*) \\ A(Y_{(0)}-Y^*) \\ A(Y_{(0)}-Y^*) \\ \vdots \\ A(Y_{(0)}-Y^*) \\ A(Y_{(0)}-Y^*) \end{bmatrix}$$

where A is the matrix defined in A.1. Since the matrix in the left hand side of (A.30) is a non-singular matrix for any T the vector $(\Delta Y_1, \dots, \Delta Y_T)$ can be solved from (A.30) uniquely.

A simple algorithm to solve ΔY_1 from (A.30) consists of combining the rows of the matrix in (A.30) so that all elements in the last row vanish except the first element:

$$\text{(A.31)}\quad D_{1T}\,\Delta Y_1 = D_{2T}(Y_0 - Y^*)$$

where D_{1T} and D_{2T} depend on T. For $\Delta Y_2, \dots, \Delta Y_T$ a solution can be obtained in a similar way.

To analyse the behaviour of D_{1T} and D_{2T} if T varies we formulate the following lemma which can be proved by using the complete induction theorem.

Lemma

Let $k = T-2$, then ΔY_1 can be solved from the following matrix expression

$$\text{(A.32)}\quad D_{2k+2}\,\Delta Y_1 = D_{2k+1}\,(Y_0 - Y^*)$$

where

$$D_{2k+1} = D_{2k-1} + 1/\beta\ D_{2k}.A \qquad D_{2k+2} = 1/\beta\ D_{2k} - D_{2k+1}$$

with starting matrices

$$D_1 = A + 1/\beta\ ((1-\beta)I-A)A \qquad D_2 = 1/\beta\ ((1-\beta)I-A)-D_1$$

Further, since $A = C\ \Gamma\ C^{-1}$ where Γ is a diagonal matrix with negative elements, the following results can be obtained

(A.33) $$D_{2k+1} = C\ L_{2k+1}(\Gamma)C^{-1}$$

$$D_{2k+2} = C\ L_{2k+2}(\Gamma)C^{-1}$$

where $L_{2k+1}(\Gamma)$ is a polynomial expression in the diagonal matrix Γ and is a diagonal matrix with negative elements and $L_{2k+2}(\Gamma)$ is also a polynomial expression in the diagonal matrix Γ and is a diagonal matrix with positive elements.

From the lemma and (A.33) follows that ΔY_1 can be obtained from

(A.34) $$\Delta Y_1 = C(L_{2k+2}(\Gamma))^{-1}\ L_{2k+1}(\Gamma)C^{-1}\ (Y_0 - Y^*)$$

Since L_{2k+2} and L_{2k+1} are both diagonal matrices we can restrict our investigation of the behaviour of the matrix product in (A.34) as function of k to an investigation of the behaviour of the elements of the product $L_{2k+2}^{-1}\ L_{2k+1}$ as function of k.

Let V_k be a diagonal element of L_{2k+1} corresponding to the root γ of A (element γ of Γ) and let X_k be the corresponding element of L_{2k+2}, for $k = 0,1,\ldots,T-2$. From (A.32) and (A.33) we then obtain

(A.35) $$V_{k+1} = V_k + 1/\beta\ \gamma\ X_k \qquad k = 0,1,2,\ldots$$

$$X_{k+1} = 1/\beta\ X_k - V_{k+1}$$

with initial conditions

(A.36) $V_0 = \gamma + \gamma/\beta\ (1-\beta-\gamma)$

$X_0 = 1/\beta\ (1-\gamma)^2 - 1$

In (A.35) and (A.36) we have defined a system of first order difference equations with initial conditions, This system can be analysed using standard techniques.

We can rewrite (A.35) to

$$\begin{bmatrix} V_{k+1} \\ X_{k+1} \end{bmatrix} - \begin{bmatrix} 1 & \gamma/\beta \\ -1 & 1-\gamma/\beta \end{bmatrix} \begin{bmatrix} V_k \\ X_k \end{bmatrix} = 0$$

The roots of this system can be found from solving the characteristic equation

(A.37) $$\begin{bmatrix} 1-\alpha & \gamma/\beta \\ -1 & \frac{1-\gamma}{\beta} - \alpha \end{bmatrix} = 0$$

or

(A.38) $(1-\alpha)(\frac{1-\gamma}{\beta} - \alpha) + \frac{\gamma}{\beta} = 0$

or

$\alpha^2 - (1 + \frac{1-\gamma}{\beta})\alpha + \frac{1}{\beta} = 0$

so that

(A.40) $\alpha_{1,2} = \frac{1}{2}(1 + \frac{1-\gamma}{\beta}) \pm \frac{1}{2}\sqrt{(1 + \frac{1-\gamma}{\beta})^2 - \frac{4}{\beta}}$

Since $(\frac{\beta+(1-\gamma)}{\beta})^2 - \frac{4}{\beta} > 0$ for all $\gamma < 0$ and $0 < \beta < 1$, both roots are real and further we find

(A.41) $0 < \alpha_2 < \alpha_1$ and $\alpha_1 > 1$

The solution of the system (A.36) can be written as

$$\text{(A.41)}\qquad \begin{bmatrix} V_k \\ X_k \end{bmatrix} = a_1\, \alpha_1^k\, Z_1 + a_2\, \alpha_2^k\, Z_2 \qquad\qquad k = 0,1,2,\ldots$$

where Z_1, Z_2 are the characteristic vectors corresponding to α_1 and α_2. These characteristic vectors can be solved from

$$\begin{bmatrix} 1-\alpha_i & \gamma/\beta \\ -1 & \frac{1-\gamma}{\beta} - \alpha_i \end{bmatrix} \begin{bmatrix} Z_{i1} \\ Z_{i2} \end{bmatrix} = 0 \qquad\qquad i = 1,2,\ldots$$

which yields

$$\text{(A.43)}\qquad Z_{i1} = 1 \qquad\qquad i = 1,2,\ldots$$

$$Z_{i2} = -\frac{\beta}{\gamma}\,(1-\alpha_i)$$

The constants a_1 and a_2 can be obtained from the initial conditions (A.36).

For the analysis of $L_{2k+2}^{-1}\, L_{2k+1}$ we are interested inthe behaviour of

$$\text{(A.44)}\qquad \frac{V_k}{X_k} = \frac{a_1\, \alpha_1^k + a_2\, \alpha_2^k}{a_1(1-\alpha_1)\alpha_1^k + a_2(1-\alpha_2)\alpha_2^k} \cdot \frac{-\gamma}{\beta}$$

or

$$\text{(A.45)}\qquad \frac{V_k}{X_k} = \frac{a_1 + a_2\left(\frac{\alpha_2}{\alpha_1}\right)^k}{a_1(1-\alpha_1) + a_2(1-\alpha_2)\left(\frac{\alpha_2}{\alpha_1}\right)^k} \cdot \frac{-\gamma}{\beta}$$

If $T \rightarrow \infty$ and thus $k \rightarrow \infty$ we obtain for the quotient V_k/X_k

$$\text{(A.46)}\qquad \lim_{k\to\infty} \frac{V_k}{X_k} = \frac{-\gamma}{\beta(1-\alpha_1)}$$

Using (A.46) we can obtain a solution for ΔY_1 if $T \to \infty$. Combining (A.34) and (A.46) we obtain

(A.47) $$\lim_{T \to \infty} \Delta Y_1 = C\ F\ C^{-1}\ (Y_0 - Y^*)$$

and F is a diagonal matrix with element f_{ii}:

(A.48) $$f_{ii} = \frac{-\gamma_i}{\beta(1-\alpha_{i1})}$$

where γ_i is the i-th root of A and α_{i1} is defined in (A.40).

An analytic analysis of the behaviour of Y_k/X_k is extremely difficult and not very promising so that we will confine ourselves to a numerical analysis for $\gamma = -0.5$, $\beta = 0.9$. For $\gamma = -0.5$ and $\beta = 0.9$ we find for α_1 and α_2, Z_1, Z_2 and a_1, a_2:

$$\alpha_1 = 2.15 \ ; \ Z_{11} = 1 \ ; \ Z_{21} = 1 \ ; \ a_1 = -1.48\ \frac{0.83}{1.67}$$

$$\alpha_2 = 0.52 \ ; \ Z_{12} = -2.07 \ ; \ Z_{22} = 0.864 \ ; \ a_2 = -0.15\ \frac{0.83}{1.67}$$

and for k = 0 we find [2)]

$$\frac{V_0}{X_0} = \frac{a_1 + a_2}{a_1(1-\alpha_1)+a_2(1-\alpha_2)} \cdot \frac{-\gamma}{\beta} = -0.5556$$

$$\frac{V_1}{X_1} = \frac{a_1\ \alpha_1 + a_2\ \alpha_2}{a_1(1-\alpha_1)\alpha_1+a_2(1-\alpha_2)\alpha_2} \cdot \frac{-\gamma}{\beta} = -0.50$$

$$\frac{V_2}{X_2} = \frac{a_1\ \alpha_1^2 + a_2\ \alpha_2^2}{a_1(1-\alpha_1)\alpha_1^2+a_2(1-\alpha_2)\alpha_2^2} \cdot \frac{-\gamma}{\beta} = -.487$$

---

2) In fact the same results can be obtained by solving the algorithm (A.32), (A.33) directly for k = 0,1,2,3,...

$$\frac{V_3}{X_3} = -0.484$$

$$\lim_{k\to\infty} \frac{V_k}{X_k} = -0.4783$$

In this example the convergence of V_k/X_k to its limit is quite rapid.

Though this approach is more adequate to analyse numerically the behaviour of ΔY_1 if T varies then the approach in Section A.1, we have not been able to obtain general statements based on an analytical analysis of expression (A.45). In Table 1 we will give additional numerical results for several γ. Finally we will show in Section A.3 that the approach in this Section is basically the same as the approach in Section A.1.

Table 1 shows the results for $\gamma = -0.1$ and $\beta = 0.9$ (corresponding to very high adjustment costs and thus to a low adjustment speed of Y_t to Y^*) and $\gamma = -2$ (corresponding to low adjustment costs and thus to a high adjustment speed of Y_t to Y^*). [3)]

3) See footnote 3 in Section 3.3.

Table 1

$\gamma = -0.1$ $\beta = 0.9$ $\alpha_1 = 1.46$ $\alpha_2 = 0.76$ $Z_{11} = 1$ $Z_{21} = 1$ $Z_{12} = -4.14$ $Z_{22} = 2.16$ $a_1 = 0.38$ $a_2 = -0.54$	$\gamma = -2$ $\beta = 0.9$ $\alpha_1 = 4.055$ $\alpha_2 = 0.275$ $Z_{11} = 1$ $Z_{21} = 1$ $Z_{12} = -1.375$ $Z_{22} = 0.326$ $a_1 = -6.76$ $a_2 = -0.098$
$\frac{V_0}{X_0} = -2.67$ $V_1/X_1 = -0.684$ $V_2/X_2 = -0.418$ $V_3/X_3 = -0.323$ $V_4/X_4 = -0.282$ $V_5/X_5 = -0.262$ $\lim_{k\to\infty} V_k/X_k = -0.24$	$\frac{V_0}{X_0} = -0.74$ $\frac{V_1}{X_1} = -0.728$ $\lim \frac{V_k}{X_k} = -0.7274$

A.3. <u>A comparison of the results of Section A.1 and A.2.</u>

In Section A.1. we obtained an expression for Y_1 for $T \to \infty$, see A.22.

(A.49) $$Y_1 = C\ \Lambda_1\ d_1 + Y^*$$

where $d_1 = C^{-1}(Y_{(0)} - Y^*)$ so that since $Y_0 = Y_{(0)}$

(A.50) $$Y_1 - Y_0 = C\ \Lambda_1\ C^{-1}\ (Y_0 - Y^*) - (Y_0 - Y^*)$$

or

(A.51) $$\Delta Y_1 = C(\Lambda_1 - I)C^{-1}\ (Y_0 - Y^*)$$

where Λ_1 is the matrix of stable roots of the homogenous system of difference equations (A.3) and C is the matrix of corresponding characteristic vectors.

In Section A.2 we obtained for ΔY_1 and $T \to \infty$ the expression, See (A.47)

(A.52) $$\Delta Y_1 = C\ F\ C^{-1}(Y_0 - Y^*)$$

where F is defined in (A.48).

Since both methods are equivalent the following result must hold

(A.53) $$C(\Lambda_1 - I)C^{-1} = C\ F\ C^{-1}$$

or

(A.54) $$\Lambda_1 - I = F$$

Since both matrices are diagonal matrices we can redefine (A.54) in terms of its diagonal elements as

(A.55) $\lambda_i - 1 = f_{ii}$ $\qquad i = 1,\ldots,n$

or

(A.56) $\lambda_i - 1 = \frac{-\gamma_i}{\beta(1-\alpha_{i1})}$ $\qquad i = 1,\ldots,n$

where λ_i is defined in (A.6); γ_i is a characteristic root of A and α_{i1} is defined in (A.40).

Expressing γ_i and α_{i1} in terms of λ_i we find, dropping the suffix i

$$\alpha_1 = \frac{1}{2}\left(\lambda + \frac{1}{\lambda\beta}\right) + \frac{1}{2}\left|\lambda - \frac{1}{\lambda\beta}\right|$$

For $\left|\lambda - \frac{1}{\lambda\beta}\right|$ we can write, since $0 < \lambda < 1$ and $0 < \beta < 1$,

$$\left|\lambda - \frac{1}{\lambda\beta}\right| = \frac{1}{\lambda\beta} - \lambda$$

so that

$$\alpha_1 = \frac{1}{\lambda\beta}$$

For (A.56) we find

$$\lambda - 1 = -\frac{(1+\beta) - \beta\lambda - \lambda^{-1}}{\beta - \lambda^{-1}}$$

or

$$(\lambda-1)(\beta-\lambda^{-1}) = \beta\lambda - \beta + \lambda^{-1} - 1$$

or

$$(\lambda-1)\ (\beta-\lambda^{-1}) = \beta(\lambda-1) - \lambda^{-1}\ (\lambda-1)$$

Since this derivation holds for every $i = 1,\ldots,n$ we have shown that (A.54) holds.

A.4. The existence of an optimal solution for an infinite horizon model

In Section 4 and in Section A.1, A.2 and A.3 we have analysed the behaviour of the adjustment process if $T \to \infty$. Implicitly it was assumed that the optimization problem defined in (4.2) is well defined for $T \to \infty$. In this Section we will show that this assumption is satisfied.

Define the optimization problem (4.2) as

Maximize

$$(A.57) \qquad \sum_{t=1}^{T} \beta^{t} \, NR\,(X_t) + \frac{\beta^{T+1}}{1-\beta} NR_T(X_T)$$

under the restrictions

$$X_t = X_{t-1} + \Delta X_t$$

$$X_t \in S$$

where S is a compact subset [1] of $\mathbb{R}^n$ and $NR_t(X_t)$ is defined as

$$(A.58) \qquad NR(X_t) = Y_t - \mathfrak{a}(\Delta X_t) - w'X_t - q'\Delta X_t \qquad t = 1,\ldots,T$$

$$NR_T(X_T) = Y_T - w'X_T$$

From (A.58) and the definitions of the function $Y, \mathfrak{a}(\Delta X_t)$ given in Secton 3 follows that the (net revenue) functions NR and NR_T are uniform continuous differentiable functions for $X_t \in S$. This implies that NR and NR_T are bounded for all $X_t \in S$, and that the discounted net revenue

$$(A.59) \qquad \sum_{t=1}^{T} \beta^{t} \, NR(X_t) + \frac{\beta^{T+1}}{1-\beta} NR_T(X_T) \qquad X_t \in S$$

---

1) The restriction $X_t \in S$ is not very restrictive, given the strict concavity of Y_t the set of X_t's which yield a non-negative discounted net revenu is for all T a compact subset $\subset \mathbb{R}^n$.

is bounded for all $T > 0$.

We now define the vector (X) as the vector $(X_1, X_2, \ldots, X_t, \ldots)$ so that $X_t \in S$ for all $t \geq 1$. An optimal solution for the infinite horizon problem is defined as the vector $(\hat{X})$ such that

(A.60) $\forall (X), \forall \varepsilon > 0, \forall T > 0, \exists \tau \geq T$:

$$\sum_{t=1}^{\tau} \beta^t NR(\hat{X}_t) + \frac{\beta^{\tau+1}}{1-\beta} NR_\tau(\hat{X}_\tau) \geq \sum_{t=1}^{\tau} \beta^t NR(X_t) + \frac{\beta^{\tau+1}}{1-\beta} NR_\tau(X_\tau) - \varepsilon$$

For a similar definition for a continuous time model see Halkin [9, p. 269].

Further we define the sequence of vectors $(\hat{X})_T = (\hat{X}_{1T}; \hat{X}_{2,T}; \ldots; \hat{X}_{t,T}; \ldots)$ as the sequence of optimal solutions of the finite horizon problems. Now suppose that there exists a vector $(\hat{X})$ such that

(A.61) $$\lim_{T \to \infty} (\hat{X})_T = (\hat{X})$$

in the sence that

(A.62) $\forall \eta > 0, \exists T > 0$:

$$\forall t \leq T \quad |\hat{X}_t - \hat{X}_{t,T}| < \eta$$

Let $(\hat{X})$ satisfy (A.61) and (A.62) then follows from the uniform continuity of NR and NR_T

(A.63) $\forall \varepsilon_1 > 0, \forall \varepsilon_2 > 0 \ \exists T > 0$:

$$\forall t \leq T \quad |NR(\hat{X}_t) - NR(\hat{X}_{t,T})| < \varepsilon_1$$

$$\text{and} \quad |NR_T(\hat{X}_T) - NR_T(\hat{X}_{T,T})| < \varepsilon_2$$

From (A.63) follows that $(\hat{X})$ defined in (A.61) and (A.62) satisfies the definition (A.60) so that $(\hat{X})$ is an optimal solution of the infinite horizon problem.

That $(\hat{X})$ defined in (A.61) and (A.62) satisfies (A.60) follows from: choose an (X), then for each $\tau > 0$

$$\text{(A.64)} \qquad \sum_{t=1}^{\tau} \beta^t \, NR(\hat{X}_{t,\tau}) + \frac{\beta^{\tau+1}}{1-\beta} NR_\tau(\hat{X}_{\tau,\tau}) \geq \sum_{t=1}^{\tau} \beta^t \, NR(X_t) + \frac{\beta^{\tau+1}}{1-\beta} NR_\tau(X_\tau)$$

where $(\hat{X})_\tau = (\hat{X}_{1,\tau}, \hat{X}_{2,\tau}, \ldots, \hat{X}_{t,\tau}, \ldots)$ is the optimal solution for the problem with horizon τ. Further follows from (A.61) and (A.63) that for all $\varepsilon > 0$ and for all $T > 0$, $\exists \, \tau \geq T$:

$$\text{(A.65)} \qquad \left| \left(\Sigma\beta^t NR(\hat{X}_t) + \frac{\beta^{\tau+1}}{1-\beta} NR_\tau(\hat{X}_\tau)\right) - \left(\Sigma\beta^t NR(\hat{X}_{t,\tau}) + \frac{\beta^{\tau+1}}{1-\beta} NR_\tau(\hat{X}_{\tau,\tau})\right) \right| < \varepsilon$$

Combining (A.64) and (A.65) we conclude that $(\hat{X}) = (\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_t, \ldots)$ satisfies (A.60).

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