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RESEARCH MEMORANDUM

A NEW ALGORITHM FOR THE LINEAR COMPLEMENTARITY PROBLEM ALLOWING FOR AN ARBITRARY STARTING POINT

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Abstract

The linear complementarity problem is to find vectors $s \in \Re^n$ and $z \in \Re^n$ such that $s = Mz + q$, $s^T z = 0$, $s \geq 0$, $z \geq 0$, where M and q are the data of the problem. In this paper we propose a new complementary pivoting algorithm for solving the LCP as a more efficient alternative to the algorithms proposed by Lemke and by Talman and Van der Heyden. The algorithm can start at an arbitrary vector z^0 in $\Re^n_+$ and converges under the same conditions as the other two methods.

1 Introduction

The linear complementarity problem (LCP) is to find vectors $s \in \Re^n$ and $z \in \Re^n$ satisfying

$$s = Mz + q, \qquad s^T z = 0, \qquad s \geq 0, \qquad z \geq 0, \tag{1.1}$$

where M is a given $n{*}n$-matrix and q a given n-vector. The LCP is quite common in mathematical programming because the problem is frequently met in different areas of scientific research where optimization plays an important role. Often these optimization problems lead to Karush-Kuhn-Tucker conditions which take the form of an LCP.

The popularity of the LCP in mathematical programming has led to a variety of algorithms attempting to solve the problem. Among this variety of algorithms the Lemke complementary pivot algorithm [3] is undoubtedly one of the most renowned algorithms. The Lemke algorithm is a path-following algorithm starting in $z = 0$ and generating a piecewise linear path of so-called almost complementary solutions either towards a solution to the LCP or towards infinity.

The major drawback of the Lemke complementary pivot algorithm is that one is stuck to the fixed starting point $z = 0$. This feature causes inefficiencies when one has some idea concerning the possible location of a solution to the LCP. For example, when one tries to solve a nonlinear complementarity problem by a sequence of linear complementarity problems (see Mathiesen [4]) one cannot start Lemke's algorithm at the solution to the previous LCP in order to solve the present LCP in the sequence. This inefficiency in processing the information makes it worthwile to adapt Lemke's algorithm for an arbitrarily chosen starting point. In [5] Talman and Van der Heyden present a whole class of algorithms generalizing the Lemke complementary pivot algorithm to an arbitrarily chosen starting point. All the algorithms in this class however use a pivot system of at least $n+1$ equations in order to guarantee possible convergence of the algorithm where n is the dimension in problem (1.1). Moreover none of these algorithms seem to be very natural in solving the LCP. To get rid of these inefficiencies we propose a new pivoting algorithm to solve the LCP allowing for an arbitrarily chosen starting point. This algorithm has a natural interpretation as a path-following algorithm and it does not need more than n equations in the pivot system.

The algorithm leaves the starting point in one out of $n+1$ possible directions. There are n rays that connect the starting point with each of the n axes of $\Re^n_+$ and one ray connects the starting point with the origin. This allows the algorithm to leave the starting point z^0 in such a way that, with $s^0 = Mz^0 + q$, it will raise z_i from z_i^0 by moving into the direction of the i-th axis when s_i^0 is negative and smaller than all other components of s^0, while the algorithm will lower z proportionally from z^0 towards the origin if all the components of s^0 are positive. In particular this latter feature endows the algorithm with a very natural interpretation. For example, the algorithm will stop with a solution to the LCP if it reaches the origin. This is contrary to the algorithm in the Talman and Van der Heyden class of algorithms having also $n+1$ rays to leave the starting point. In that algorithm there are n rays that leave the starting point parallel to each of the axes and there is one ray connecting the starting point with the origin but the algorithm must continue along one of the axes of $\Re^n_+$ when reaching the origin.

In our algorithm the intersection of the rays with each of the axes can arbitrarily be chosen. In Section 4 of this paper we suggest a particular choice of these intersections such that it is possible to see in advance whether the algorithm might not solve the problem.

The paper is divided as follows. First we describe the algorithm. The algorithm follows a path of points through different subsets of $\Re^n_+$. In Section 2 these subsets are defined as well as the way to generate a path through these subsets. The steps of the algorithm are enumerated in Section 3 while Section 4 is dedicated to convergency issues.

2 The algorithm

Let S be a set in $\Re^n$ and let $f : S \to \Re^n$ be a function. A point $\bar{x}$ in S is defined to be a stationary point of f on S if $\bar{x}^T f(\bar{x}) \geq x^T f(\bar{x})$ for all x in S. The stationary point problem (SPP) on S with respect to f is to find a stationary point of f on S.

The LCP is equivalent to the SPP on $\Re^n_+$ with respect to the affine function g defined by $g(z) = -Mz - q$ on $\Re^n_+$, as can easily be shown. Taking notice of this interpretation of the LCP we propose an algorithm to solve the LCP which follows a piecewise linear path of points in $\Re^n_+$ starting in some arbitrarily chosen point $z^0 \in \Re^n_+ \backslash \{0\}$. Each point z on the path is such that it is a stationary point of the function g on the set $H(t) \cap \Re^n_+$ for some $t \geq 0$ where

$$H(t) := \{z^0 + \textstyle\sum_{j=1}^{n+1} \lambda_j q(j) \mid \lambda_j \geq 0 \text{ for } j = 1, \ldots, n+1, \text{ and } \sum_{j=1}^{n+1} \lambda_j = t\}.$$

For $j = 1, \ldots, n$, $q(j) = ae(j) - z^0$, with $e(j)$ denoting the j-th unit vector in $\Re^n_+$, and $q(n+1) = -z^0$. The number a is an arbitrarily chosen number satisfying $a > \sum_{h=1}^n z_h^0$ in order to assure that $z^0 \in H(t)$ for all $t \geq 0$.[1] The number t can be considered as a homotopy parameter running from zero to infinity. For $t = 0$ the set $H(0)$ only consists of the starting point z^0. Hence z^0 is a stationary point on $H(0)$ of g. For $t = 1$ the set $H(1)$ is the convex hull of the origin and the points $ae(j)$, $j = 1, \ldots, n$, on the axes of $\Re^n_+$. If the algorithm generates a stationary point z of g on $H(1) \cap bd\Re^n_+$ with $\lambda_{n+1} > 0$ then z is also a stationary point of g on $\Re^n_+$ and thereby a solution to the LCP. For $t \geq 1$ the set $H(t) \cap \Re^n_+$ is equal to the convex hull of the origin and the points $[(1-t)\sum_{h=1}^n z_h^0 + ta]e(j)$, $j = 1, \ldots, n$, on the axes of $\Re^n_+$. In this way the algorithm follows a path of points starting in some arbitrarily chosen point $z^0 \in \Re^n_+ \setminus \{0\}$ and, barring degeneracy, terminates either on a ray or at a solution.

In order to characterize a stationary point of g on $H(t) \cap \Re^n_+$ let $z \in H(t) \cap \Re^n_+$ for some $t \geq 0$. By definition of $H(t)$ it follows that z is a convex combination of $z^0 + tq(j)$, $j = 1, \ldots, n+1$. Then there exists a subset T of $\{1, \ldots, n+1\}$ such that $z - z^0$ is a nonnegative linear independent combination of $q(j)$, $j \in T$. So, for subsets T of $\{1, \ldots, n+1\}$ one can define a subset of $\Re^n_+$, denoted $A(T)$, which is spanned by $q(j)$, $j \in T$.

[1] In Section 4 we will make use of this liberty in choosing a by letting the choice of a depend on the matrix M and vector q (see Lemma 4.2).

Definition 2.1 *For* $T \subset \{1, \ldots, n+1\}$

$$A(T) = \emptyset \text{ if } n+1 \in T \text{ and } z_h^0 = 0 \text{ for all } h \notin T$$

and otherwise

$$A(T) = (\{z^0\} + cone\{q(j) \mid j \in T\}) \cap \Re_+^n.$$

If z lies in the boundary of $H(t) \cap \Re_+^n$ such that $t \geq 1$, $\lambda_{n+1} = 0$, and $z_h = 0$ for some index h for which $z_h^0 > 0$, then z is a nonnegative linear combination with sum at least a of the unit vectors $e(j)$, $j \in T$, for some subset T of $\{1, \ldots, n\}$. So, for some specific subsets T of $\{1, \ldots, n\}$ one can define a subset of $bd\Re_+^n$, denoted $A^0(T)$, which is spanned by $e(j)$, $j \in T$.

Definition 2.2 *For* $T \subset \{1, \ldots, n+1\}$

$$A^0(T) = \emptyset \text{ if } n+1 \in T \text{ or } z_h^0 = 0 \text{ for all } h \notin T$$

and otherwise

$$A^0(T) = \{\sum_{j \in T} \lambda_j a e(j) \mid \lambda_j \geq 0 \text{ for } j \in T \text{ and } \sum_{j \in T} \lambda_j \geq 1\}.$$

The algorithm is such that it generates, starting at z^0, a piecewise linear path of points through subsequent subsets $A(T)$ and $A^0(T)$, for varying subsets T of $\{1, \ldots, n+1\}$, by maintaining so-called *T-completeness* in each point on the path.

Definition 2.3 *For* $T \subset \{1, \ldots, n+1\}$ *a point* $z \in \Re_+^n$ *is* T-complete *if* $j \in T$ *when* $s_j = \min_h s_h \leq 0$ *and* $n+1 \in T$ *when* $\min_h s_h \geq 0$, *where* $s = Mz + q$.

When the algorithm generates a piecewise linear path of T-complete points through subsequent subsets $A(T)$ and $A^0(T)$ for varying subsets T of $\{1, \ldots, n+1\}$ then each point on this path is a stationary point of g on $H(t) \cap \Re^n_+$ for some $t \geq 0$. This is proved in Theorem 2.1.

Lemma 2.1 *Let $P := \{x \in \Re^n \mid Ax \leq b\}$ be a nonempty polytope and let $\overline{x} \in P$. If c is a nonnegative linear combination of the binding constraints of P in $\overline{x}$ then $c^T x \leq c^T \overline{x}$ for all $x \in P$.*

Proof:

Let A contain m rows $a_1^T, \ldots, a_m^T$. Let the set $X(x)$ be defined as $X(x) := \{j \mid a_j^T x = b_j\}$ for all $x \in P$. Let $c = \sum_{j=1}^m \lambda_j a_j$ where $\lambda_j = 0$ for all $j \notin X(\overline{x})$ and $\lambda_j \geq 0$ for all $j \in X(\overline{x})$. Hence c is a nonnegative linear combination of the binding constraints of P in $\overline{x}$. Then, for all $x \in P$ it holds that

$$c^T x = \sum_{j \in X(\overline{x})} \lambda_j a_j^T x_j \leq \sum_{j \in X(\overline{x})} \lambda_j b_j = \sum_{j \in X(\overline{x})} \lambda_j a_j^T \overline{x}_j = c^T \overline{x}.$$

$\square$

Theorem 2.1 *If z is a T-complete point in $A(T)$ or $A^0(T)$ for some $T \subset \{1, \ldots, n+1\}$ then z is a stationary point of g on $H(t) \cap \Re^n_+$ for some $t \geq 0$.*[2]

Proof:

Let $\overline{z}$ be a T-complete point in $A(T)$ or in $A^0(T)$ for some $T \subset \{1, \ldots, n+1\}$. Let $\overline{t}$ be such that $\overline{z} \in bd(H(\overline{t}) \cap \Re^n_+)$. Then $\overline{z}$ being a T-complete point

[2]Notice that the reverse is also valid. The proof is left to the reader.

implies that $g(\overline{z}) = \beta e - \mu$ where $\beta = \max\{0, -\min_h g_h(\overline{z})\}$, $\mu_j = g_j(\overline{z}) + \beta$ for all $j \notin T \cup \{n+1\}$, and $\mu_j = 0$ for $j \in T$, and the vector e is such that all components of e are equal to one. Combining this with $z \in A(T)$ or $z \in A^0(T)$ and the fact that $H(\overline{t}) \cap \Re^n_+$ can be rewritten as $\{z \in \Re^n_+ \mid z \geq (1-\overline{t})z^0$ and $\sum_{j=1}^n z_j \leq \sum_{j=1}^n z_j^0 + \overline{t}(a - \sum_{j=1}^n z_j^0)\}$ it is easy to see that $g(\overline{z})$ is a nonnegative combination of the binding constraints of $H(\overline{t}) \cap \Re^n_+$ in $\overline{z}$. Then with Lemma 2.1 it follows that $g(\overline{z})^T z \leq g(\overline{z})^T \overline{z}$ for all $z \in H(\overline{t}) \cap \Re^n_+$. Hence $\overline{z}$ is a stationary point of g on $H(\overline{t}) \cap \Re^n_+$. □

In the next section we describe how to follow the piecewise linear path of T-complete points in $A(T)$ and $A^0(T)$ for varying T, which starts at z^0, by complementary pivot steps.

3 The steps of the algorithm.

Definition 2.3 leads to a pivot system in each point z on the path generated by the algorithm from the starting point z^0 either towards a solution of the LCP or towards infinity. To make this clear let us denote $-\min_h s_h$ by β if $\min_h s_h \leq 0$. Then the T-completeness condition at a point z is equivalent to the existence of $\mu_j \geq 0$ $(j \notin T \cup \{n+1\})$, $\beta \geq 0$ if $n+1 \notin T$ and $\beta = 0$ if $n+1 \in T$ such that

$$Mz + q = -\beta e + \sum_{j \notin T \cup \{n+1\}} \mu_j e(j).$$

Combined with $z \in A(T)$ or $z \in A^0(T)$ the appropriate pivot system for T-completeness at a point z in $A(T)$ or $A^0(T)$ is given in the next two lemma's where $M_{.j}$ denotes the j-th column of the matrix M.

Lemma 3.1 *A point $z \in A(T)$ is T-complete for some $T \subset \{1, \ldots, n+1\}$ if and only if the system of equations*

$$\sum_{j \in T} \lambda_j M q(j) - \sum_{j \notin T \cup \{n+1\}} \mu_j e(j) + \beta e = -q - M z^0 \tag{3.1}$$

has a solution $\lambda_j \geq 0$ $(j \in T)$, $\mu_j \geq 0$ $(j \notin T \cup \{n+1\})$, $\beta \geq 0$ if $n+1 \notin T$ and $\beta = 0$ if $n+1 \in T$, such that $z = z^0 + \sum_{j \in T} \lambda_j q(j)$.

Lemma 3.2 *A point $z \in A(T)$ for some $T \subseteq \{1, \ldots, n\}$ with $z_i^0 = 0$ for all $i \notin T$ or a point $z \in A^0(T)$ for some $T \subseteq \{1, \ldots, n\}$ is T-complete if and only if the system of equations*

$$\sum_{j \in T} \lambda_j a M_{.j} - \sum_{j \notin T \cup \{n+1\}} \mu_j e(j) + \beta e = -q \tag{3.2}$$

has a solution $\lambda_j \geq 0$ $(j \in T)$, $\mu_j \geq 0$ $(j \notin T \cup \{n+1\})$, and $\beta \geq 0$ such that $z = \sum_{j \in T} \lambda_j a_j e(j)$.

Notice that the pivot systems in (3.1) and (3.2) all contain n equations in $n+1$ variables leaving us with one degree of freedom. If nonempty, the solution set of each system forms a line segment, assuming nondegeneracy. This line segment corresponds to a linear piece of T-complete points in $A(T)$ or in $A^0(T)$ with either one or two end points. As we will show below each end point of a line segment of solutions to a system of equations for some $T \subset \{1, \ldots, n+1\}$ either corresponds to the starting point z^0 or to a solution to the LCP or is an end point of a line segment of solutions to exactly one other system of equations possibly for a different set T. The point z^0 will correspond to an end point of only one line segment of solutions.

These properties make the path of points generated by the algorithm from z^0 a piecewise linear path through subsequent subsets $A(T)$ and $A^0(T)$ for varying $T \subset \{1,\ldots,n+1\}$. Each linear piece can be followed by making a linear programming pivot step in the appropriate pivot system with the variable being zero (or making a binding constraint) at an end point.

A linear piece of T-complete points in $A(T)$ for some subset $T \subseteq \{1,\ldots,n\}$ for which $z_i^0 = 0$ for all $i \notin T$ can be generated by making a pivot step in system (3.1) or in system (3.2). Which one of these systems will be appropriate depends on in which system the previous pivoting step was made. This feature causes the algorithm to generate the path through different subsets of $\Re^n_+$ in an efficient way. Changing from one pivot system to the other one at an end point of a line segment requires a redefinition of the variables λ_j, $j \in T$, in the new pivot system. The setup in Lemma 3.1 and Lemma 3.2 allows us to make as few of these changes of variables as possible.

Suppose the algorithm is following a linear piece of T-complete points in $A(T)$ or in $A^0(T)$ for some $T \subset \{1,\ldots,n+1\}$, i.e., a pivot step is made in one of the systems of equations (3.1) or (3.2) with a variable being zero at an end point of the line segment of solutions. When the linear piece has another end point, say z, then, assuming nondegeneracy, exactly one of the following cases will occur for the solution at this end point:

Case 1: λ_p is zero for some $p \in T\backslash\{n+1\}$, while $T\backslash\{p\} \neq \emptyset$. Then z is an end point lying in $A(T\backslash\{p\})$ or in $A^0(T\backslash\{p\})$ depending on whether $z \in A(T)$ or $z \in A^0(T)$ respectively. The algorithm proceeds in $A(T\backslash\{p\})$ or $A^0(T\backslash\{p\})$ by pivoting the column $e(p)$ into the appropriate system of equations thereby raising μ_p from zero and maintaining $T\backslash\{p\}$-completeness.

Case 2: In system (3.1), λ_p is equal to

$$(\sum_{j\in T\backslash\{p\}} \lambda_j - 1)\left(\frac{z_p^0}{a - z_p^0}\right) \text{ for some } p \in T.$$

Then z is an end point lying in $A^0(T\setminus\{p\})$. Let

$$\overline{\lambda}_j = \lambda_j + (1 - \sum_{h\in T}\lambda_h)\left(\frac{z_j^0}{a}\right), \text{ for } j \in T\backslash\{p\}.$$

Then $\overline{\lambda}_j \geq 0$ $(j \in T\setminus\{p\})$, $\mu_h \geq 0$ $(h \notin T \cup \{n+1\})$, $\mu_p = 0$, and $\beta \geq 0$ is a solution to system (3.2) and z is an end point of a linear piece of $T\backslash\{p\}$-complete points in $A^0(T\backslash\{p\})$. The algorithm proceeds in $A^0(T\backslash\{p\})$ by changing system (3.1) into system (3.2) and pivoting the column $e(p)$ into the new system (3.2) thereby raising μ_p from zero in order to maintain $T\backslash\{p\}$-completeness.

Case 3: $\sum_{j\in T}\lambda_j$ is equal to 1 in system (3.1) while $n+1 \in T$ or $z_h^0 > 0$ for some $h \notin T$, or in system (3.2) while $z_h^0 > 0$ for some $h \notin T$. If $n+1 \in T$ then $s_j = 0$ and $z_j = \lambda_j a \geq 0$ for $j \in T$ while $s_j = \mu_j \geq 0$ and $z_j = (1 - \sum_{j\in T}\lambda_j)z_j^0 = 0$ for $j \notin T$, leaving us with a solution to the LCP in z. Otherwise, suppose $n+1 \notin T$. Then z is an end point of a linear piece of T-complete points in $A(T)$ as well as in $A^0(T)$. So, if z were the end point of a linear piece of T-complete points in $A(T)$ then the algorithm proceeds by generating a linear piece of T-complete points in $A^0(T)$. This linear piece of T-complete points in $A^0(T)$ is generated by changing system (3.1) into system (3.2) and pivoting the column $aM_{.k}$ or $e(k)$ into the new system (3.2), depending on whether $Mq(k)$ or $e(k)$ was the last pivot column in (3.1). Notice that $\sum_{j\in T}\lambda_j$ is then raised from 1. Conversely, if z were the end point of a linear piece of T-complete points in $A^0(T)$ then the algorithm proceeds by generating a linear piece of T-complete points in $A(T)$. This linear piece of T-complete points in $A(T)$ is generated by changing system (3.2) into system (3.1) and pivoting the column $Mq(k)$ or $e(k)$ into the new system (3.1), depending on whether $aM_{.k}$ or $e(k)$ was the last pivot column in the system (3.2). Hence $\sum_{j\in T}\lambda_j$ is lowered from 1.

Case 4: In system (3.2) it holds that for some $p \in T$

$$\sum_{i\in T\setminus\{p\}} \lambda_i z_p^0 + \lambda_p \left(a - \sum_{i\in T} z_i^0\right) = z_p^0 \text{ while } z_p^0 > 0.$$

Then z is an end point lying in $A(T\setminus\{p\})$. Let

$$\overline{\lambda}_h = \lambda_h + z_h^0 \left(\frac{1-\sum_{j\in T\setminus\{p\}}\lambda_j}{a-\sum_{j\in T} z_j^0}\right) \text{ for } h \in T\setminus\{p\}.$$

Then $\overline{\lambda}_h \geq 0$ $(h \in T\setminus\{p\})$, $\mu_h \geq 0$ $(h \notin T \cup \{n+1\})$, $\mu_p = 0$, and $\beta \geq 0$ is a solution to (3.1) and z is an end point of a linear piece of $T\setminus\{p\}$-complete points in $A(T\setminus\{p\})$. The algorithm proceeds by changing system (3.2) into system (3.1) and pivoting the column $e(p)$ into system (3.1) thereby raising μ_p from zero in order to maintain $T\setminus\{p\}$-completeness.

Case 5: μ_k is zero for some $k \notin T \cup \{n+1\}$. Suppose $z \in A(T)$. If $z_h^0 = 0$ for all $h \notin T \cup \{k\}$ while $n+1 \in T$ or if $T \cup \{k\} = \{1,\ldots,n+1\}$ then z is a solution to the LCP. Otherwise z is an end point of a linear piece of $T\cup\{k\}$-complete points in $A(T\cup\{k\})$. The algorithm proceeds by pivoting the column $Mq(k)$ into the system (3.1) or $aM_{.k}$ into the system (3.2) thereby raising λ_k from zero in order to maintain $T\cup\{k\}$-completeness.

Suppose $z \in A^0(T)$. If $z_h^0 = 0$ for all $h \notin T \cup \{k\}$ then z is an end point of a linear piece of $T \cup \{k\}$-complete points in $A(T \cup \{k\})$, otherwise z is an end point of a linear piece of $T \cup \{k\}$-complete points in $A^0(T \cup \{k\})$. The algorithm proceeds in both cases by pivoting the column $aM_{.k}$ into the system (3.2) thereby raising λ_k from zero in order to maintain $T \cup \{k\}$-completeness.

Case 6: β is zero. Then z is a solution to the LCP if $z \in A^0(T)$ or if $z \in A(T)$ and $z_h^0 = 0$ for all $h \notin T$. Otherwise, z is an end point of a linear piece of $T \cup \{n+1\}$-complete points in $A(T \cup \{n+1\})$. The algorithm proceeds by pivoting the column $-Mz^0$ into (3.1) thereby raising λ_{n+1} from zero in order to maintain $T \cup \{n+1\}$-completeness.

Case 7: λ_{n+1} is zero while $T \setminus \{n+1\} \neq \emptyset$. Then z is an end point of a linear piece of $T \setminus \{n+1\}$-complete points in $A(T \setminus \{n+1\})$. The algorithm proceeds by pivoting the column e into system (3.1) thereby raising β from zero in order to maintain $T \setminus \{n+1\}$-completeness.

The cases 1 to 7 describe the performance of the algorithm at the end points of all possible line segments generated by the algorithm except at z^0 where the algorithm is initiated. To show that z^0 is an end point of a (unique) linear piece of T-complete points in $A(T)$ for some $T \subset \{1, \ldots, n+1\}$, let us denote $Mz^0 + q$ by s^0. If $\min_h s_h^0 < 0$ let k be such that $s_k^0 = \min_h s_h^0$. Then the starting point z^0 is T^0-complete with $T^0 = \{k\}$ and the system of equations

$$-\sum_{j \neq k, n+1} \mu_j e(j) + \beta e = -q - Mz^0 \tag{3.3}$$

has a unique solution $\mu_j = s_j^0 - s_k^0 \geq 0$ $(j \neq k, n+1)$, and $\beta = -s_k^0 > 0$. So, assuming nondegeneracy, z^0 is an end point of a linear piece of $\{k\}$-complete points in $A(\{k\})$. In order to follow this linear piece the algorithm starts by pivoting the column $Mq(k)$ into (3.3) thereby raising λ_k from zero.

If $\min_h s_h^0 \geq 0$ then the starting point z^0 is T^0-complete with $T^0 = \{n+1\}$ and the system of equations

$$-\sum_{j=1}^{n} \mu_j e(j) = -q - Mz^0 \tag{3.4}$$

has a unique solution $\mu_j = s_j^0 \geq 0$ $(j = 1, \ldots, n)$. Assuming nondegeneracy, z^0 is the end point of a linear piece of $\{n+1\}$-complete points in $A(\{n+1\})$. In order to follow this linear piece the algorithm starts by pivoting the column $-Mz^0$ into (3.4) thereby raising λ_{n+1} from zero.

4 Convergence issues

Starting in some arbitrarily chosen point $z^0 \in \Re^n_+ \backslash \{0\}$ the algorithm generates a piecewise linear path of T-complete points through adjacent subsets $A(T)$ or $A^0(T)$, for varying $T \subset \{1, \ldots, n+1\}$, as described in Section 3. This path either ends up with a solution to the LCP as defined in (1.1) or it ends up with a ray towards infinity. The end points of the path giving rise to a solution to the LCP have already been described during the enumeration of the cases in Section 3. Lemma 4.1 summarizes all the cases in which the algorithm ends up with a solution.

Lemma 4.1 *Let z be an end point of a linear piece of T-complete points on the path generated by the algorithm in $A(T)$ or in $A^0(T)$ for some $T \subset$*

$\{1, \ldots, n+1\}$. *Then z is a solution to the LCP if one of the following cases holds:*

i) $z \in A(T)$, $n+1 \in T$, $\mu_k = 0$ *for some* $k \notin T$, *and* $z_h^0 = 0$ *for all* $h \notin T \cup \{k\}$ *or* $T \cup \{k\} = \{1, \ldots, n+1\}$;

ii) $z \in A^0(T)$ *and* $\beta = 0$;

iii) $z \in A(T)$, $z_h^0 = 0$ *for all* $h \notin T$, *and* $\beta = 0$;

iv) $z \in A(T)$, $n+1 \in T$, *and* $\sum_{j \in T} \lambda_j = 1$;

where the variables $\lambda_j \geq 0$ $(j \in T)$, $\mu_j \geq 0$ $(j \notin T \cup \{n+1\})$, *and* $\beta \geq 0$ *are the solution to the appropriate pivot system at* z.

The possibility of divergence urges us to impose a convergence condition on the problem. Notice that divergence can only occur when the algorithm is generating a path of points in $A^0(T)$ or in $A(T)$ with $z_i^0 = 0$ for all $i \notin T$ and $n+1 \notin T$, i.e., when Lemma 3.2 is valid. Therefore we can restrict our attention to the possible occurence of a ray to system (3.2) for some $T \subseteq \{1, \ldots, n\}$. System (3.2) however is equivalent to the system used in Lemke [3] to solve the LCP. So, a convergence theorem on Lemke's algorithm can be used for our algorithm.

The convergence theorem we impose is the most general one so far. It is given in Jones [2] and the theorem is a slight generalization of the result in Evers [1]. Before giving the convergence theorem we remark that a square matrix C is said to be *copositive* if $x^T C x \geq 0$ whenever x is nonnegative, and a square matrix P is said to be *copositive-plus* if P is copositive and if, in addition, $(P + P^T)z = 0$ whenever $z^T P z = 0$, $z \geq 0$.

Theorem 4.1 *Suppose M can be written as $P+C$ where P is copositive-plus and symmetric and C is copositive. If the system $q + Px - C^T y \geq 0,\ y \geq 0$ is feasible, then the algorithm terminates at a solution.*

Proof: See Jones[2].

The statement of the convergence theorem ends our description of the algorithm. What remains is a more precise delimitation of the possibilities of choosing the number a. We already put one limitation on a which is independent of the problem as defined in (1.1) but guarantees that each $A(T)$, $T \subset \{1, \ldots, n+1\}$, is convex. To make the choice of a dependent on the data of the problem, i.e., on M and q, we suggest to choose a such that for all $j \in \{1, \ldots, n\}$ no $\{j\}$-complete points in $A^0(\{j\})$ can be found. For every j, let a_j be such that no j-complete points in $A^0(\{j\})$ exist. This implies that the system (3.2) for T equal to $\{j\}$,

$$\lambda_j a_j M_{.j} - \sum_{h \neq j, n+1} \mu_h e(h) + \beta e = -q, \tag{4.1}$$

does not have a solution $\lambda_j \geq 1,\ \mu_h \geq 0\ (h \neq j, n+1),\ \beta \geq 0$. This is the case if for all $\lambda_j \geq 1$, it holds that $\beta < 0$ or $\mu_h < 0$ for some $h \neq j, n+1$. It can easily be seen that the following condition on a_j assures that for all $\lambda_j > 1$ it holds that $\beta < 0$ or $\mu_h < 0$ for some $h \neq j, n+1$:

$$\text{if } M_{jj} > 0 \text{ then } a_j > \min\left\{\frac{-q_j}{M_{jj}},\ \min_{h: M_{hj} < M_{jj}} \left\{\frac{q_h - q_j}{M_{jj} - M_{hj}}\right\}\right\}, \tag{4.2}$$

$$\text{if } M_{jj} = 0 \text{ then } a_j > \min_{h:M_{hj}<M_{jj}} \left\{ \frac{q_j - q_h}{M_{hj}} \right\}.$$

Of course we assume M to fulfil the conditions imposed by the convergence theorem, Theorem 4.1. Then $M_{jj} \geq 0$ for all $j \in \{1, \ldots, n\}$. If these conditions do not hold it is possible that for all $j \in \{1, \ldots, n\}$ a_j can not be calculated according to (4.2). Then one knows in advance that the algorithm could diverge and that the LCP might not even have a solution.

Condition (4.2) suggests how to determine a.

Corollary 4.1 *Suppose M can be written as $P+C$ where P is copositive-plus and symmetric and C is copositive, and the system $q + Px - C^T y \geq 0,\ y \geq 0$ is feasible. Let a be chosen such that $a > \max\{\sum_{h=1}^{n} z_h^0, a_1, \ldots, a_n\}$ where a_j is such that*

if $M_{jj} > 0$ then

$$a_j \geq \min \left\{ \frac{-q_j}{M_{jj}}, \min_{h:M_{hj}<M_{jj}} \left\{ \frac{q_h - q_j}{M_{jj} - M_{hj}} \right\} \right\};$$

if $M_{jj} = 0$ then

$$a_j \geq \min_{h:M_{hj}<M_{jj}} \left\{ \frac{q_j - q_h}{M_{hj}} \right\}$$

for all $j \in \{1, \ldots, n\}$. Then the algorithm always converges and can not generate $\{j\}$-complete points in $A^0(\{j\})$, $j \in \{1, \ldots, n+1\}$.

A final remark is dedicated to the choice of the starting point z^0. Notice that $z^0 = 0$ was excluded. This was meant to be a simplification of the presentation of the algorithm. But it can easily be seen that for $z^0 = 0$ the algorithm in fact reduces to the Lemke algorithm. To see this take $q(j) = e(j)$ instead of $ae(j)$ for all $j \in \{1, \ldots, n\}$ in Definition 2.1.

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