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OPTIMAL DYNAMIC INVESTMENT POLICY UNDER FINANCIAL RESTRICTIONS AND ADJUSTMENT COSTS

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Abstract

This paper examines the effects of a convex adjustment cost function on the optimal dynamic investment policy of a firm with financial restrictions. We assume that the management, which operates under decreasing returns to scale, maximizes the shareholder's value of the firm. It turns out that investments are a continuous function of time, that capital never keeps a stationary value and that there exists an unique optimal investment decision rule for the firm.

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Contents

1. Introduction

Surveys by Bensoussan, Kleindorfer & Tapiero (1978) and Feichtinger (1982a, 1982b, 1985) excellently illustrate that many recent papers using optimal control to solve dynamic models analytically, have extended the theory of the firm. Those models provide insight into the economic behaviour of firms over time.

One of the first dynamic models of the firm is the classical model of Jörgenson (1967). The problem with this model is that the resulting optimal solution dictates an instantaneous adjustment of the stock of capital goods to the level of maximum revenue.

In the literature, two ways in particular have been proposed to avoid this unrealistic immediate adjustment. The first way is the introduction of financing limits in the dynamic model of the firm. Examples of such models are those of Leland (1972), Ludwig (1978) and van Loon & Verheyen (1984).

The second way of getting a smoothed adjustment pattern is the introduction of adjustments costs as another aspect governing the dynamics of the firm. Research into this subject has been conducted by e.g. Gould (1968), Lucas (1967) and Treadway (1969). The article by Söderström (1976) contains a good survey of the theory of adjustment costs.

In this contribution we will analyse the impact of coupling financing and adjustment costs on the optimal policy within a really dynamic model of the firm. Section 2 contains a global survey of the theory of adjustment costs; in section 3 we will present our dynamic model of the firm. In this model we have incorporated both financing limits and adjustment costs. Section 4 contains a description and further analysis of the optimal solution, which is proved in Appendix 1. In Appendix 2 we derive the development of investments and capital during two optimal trajectories and in Appendix 3 we give the derivation of an investment decision rule.

2. The theory of adjustment costs

Adjustment costs arise with investment expenditures of the firm. In the literature, a distinction is made between external adjustment costs (investment expenditures) and internal adjustment costs (seize on available productive inputs) (Brechling (1975)).
External adjustment costs apply to a monopsonistic market of capital goods: if the firm wants to increase its rate of growth it will be confronted with increasing prices on the market because of its increased demand of capital goods. Other examples of external adjustment costs are: architects' fees, expenditures on job advertisements and costs of moving new employees.
Internal adjustment costs arise because the acquisition of additional capital and/or labour requires resources which could otherwise be used for the production of output. For instance, a firm may have personnel and training departments which are adequate for regular replacement of quits and retirements. Suppose the firm now wishes to raise its level of employment by hiring more people. In consequence more capital and labour have to be invested in both these departments. With given total inputs the level of output must, therefore, fall.
Installation costs and organisation costs are other examples of internal adjustment costs.

We can consider three different shapes of the adjustment cost function as given in figure 2.1. It is always assumed that the first derivative of the cost of adjustment function is positive. The question is whether there are constant, increasing or decreasing costs compared to the rate of adjustment. In accordance with standard terminology, costs of adjustment in these three cases will be called linear, convex and concave (Söderström, 1976).
The curvature of the adjustment cost function could have an impact on the optimal investment policy of the firm. If the cost of adjustment function is convex, marginal adjustment costs are increasing with investment expenditures. Therefore, the total cost of raising capital stock by a given amount will be larger the faster the growth of capital stock, and hence the firm will tend to adjust it slowly.

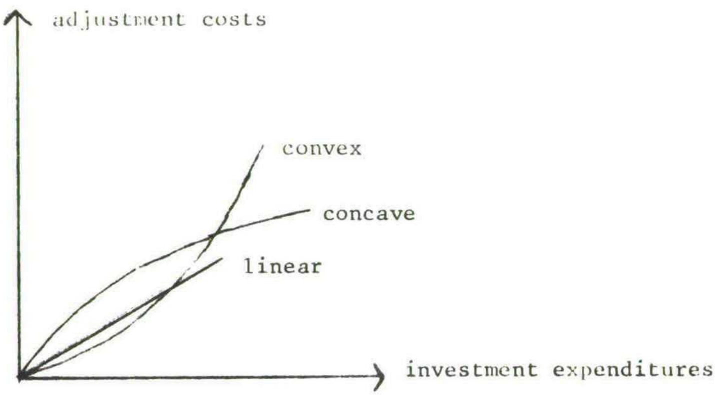


Figure 2.1. The curvature of the adjustment cost function

Concave adjustment cost functions imply declining marginal costs. The firm's policy would be to take advantage of decreasing costs of investment and raise its capital stock instantaneously.

As linear adjustment costs imply only a rising price level of the investments, its impact on the firm's investment policy is obvious.

In the literature, most dynamic models have incorporated a convex cost of adjustment function, but other types of adjustment cost functions can also be considered (Rothschild, 1971).

3. The model

We first assume that the firm behaves as if it maximizes the shareholder's value of the firm. This value consists of the sum of the present value of the dividend stream over the planning period and the present value of final equity at the end of the planning period, so:

$$\text{maximize: } \int_{T=0}^{z} D(T)e^{-iT}dT + X(z)e^{-iz} \tag{1}$$

in which

D(T) = dividend

X(T) = equity

T = time

i = discount rate

z = planning horizon

Assuming that the firm will attract only one kind of money capital: equity and has one production factor: capital goods, we get the balance equation:

$$K(T) = X(T) \tag{2}$$

in which

K(T) = total amount of capital goods

We further assume that earnings after deduction of depreciation and adjustment costs are used to issue dividend or to increase the value of equity through retained earnings.
As far as the adjustment costs are concerned, we assume that they are a convex function of investments. Also, we assume that the firm operates under decreasing returns to scale and that depreciation is proportional to capital goods. The above results in the next state equation of equity:

$$\dot{X} := \frac{dX}{dT} = S(K) - aK(T) - U(I) - D(T) \tag{3}$$

in which

$S(K)$ = earnings, $S(K) > 0,\ \frac{dS}{dK} > 0,\ \frac{d^2S}{dK^2} < 0$

$U(I)$ = adjustment costs, $U(I) \geq 0,\ \frac{dU}{dI} > 0,\ \frac{d^2U}{dI^2} > 0,\ U(0) = 0$

$I(T)$ = gross investments

a = depreciation rate

The impact of investments on the production structure is described by the, now generally used, formulation of net investments:

$$\dot{K} := \frac{dK}{dT} = I(T) - aK(T) \tag{4}$$

As far as its dividend policy is concerned, we assume that the firm is allowed to pay no dividend, so:

$$D(T) \geq 0 \tag{5}$$

Investments are irreversible, so:

$$I(T) \geq 0 \tag{6}$$

At last, we assume a positive value of capital good stock at T = 0:

$$K(0) = K_0 > 0 \tag{7}$$

After some simplifications, we get the following dynamic model of the firm:

$$\max_{I} \int_{T=0}^{z} (S(K)-I(T)-U(I))e^{-iT}dT + K(z)e^{-iz} \tag{8}$$

s.t.

$$\dot{K} = I(T) - aK(T) \tag{9}$$

$$S(K) - I(T) - U(I) \geq 0 \tag{10}$$

$$I(T) \geq 0 \tag{11}$$

$$K(0) = K_0 \tag{12}$$

This model can be solved analytically by using optimal control theory (Kamien & Schwartz (1983)), where the state of the system is described by the amount of capital goods and is controlled by investments. The aim of this control is to reach a maximum value of the objective function.

4. The optimal solution

We obtain necessary and sufficient conditions for an optimal solution using Pontryagin's standard maximum principle. Next, we apply the general solution procedure of van Loon (1983, pp. 115-117) to get the optimal trajectories of the firm (see Appendix 1).

Each trajectory consists of one or more feasible paths, which are characterized by different policies concerning investment expenditures and dividend payments. In our problem the set of feasible paths amounts to three:

path	I	D
1	max	0
2	> 0	> 0
3	0	max

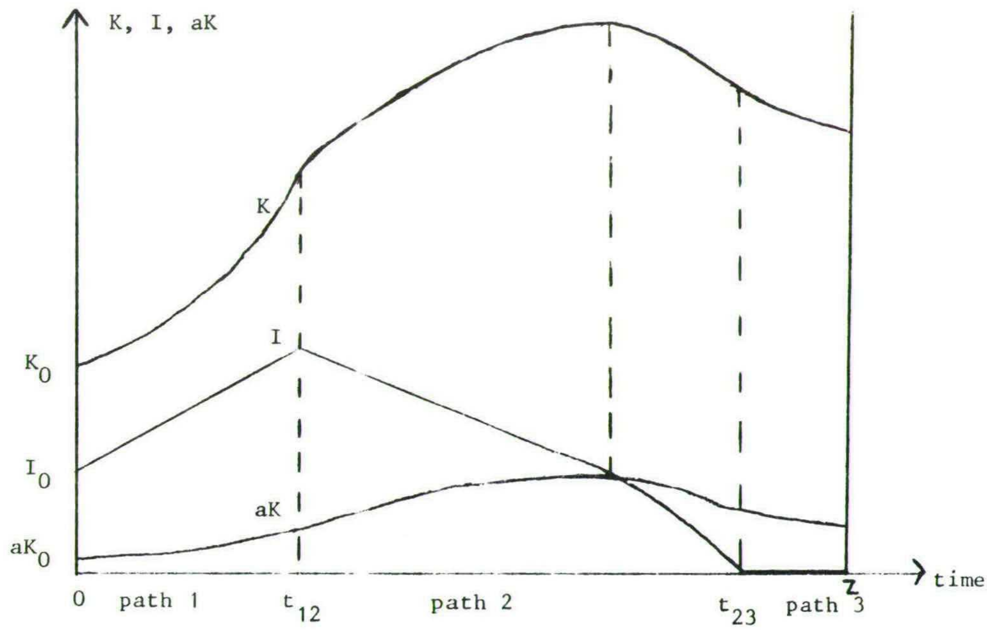
Tabel 4.1. Features of the feasible paths

Depending on the values of K_0 and z, we get different optimal trajectories. Here we will demonstrate two of them, represented in figure 4.1 and 4.2 and derived in Appendix 2. The other patterns are subsections of these two solutions.

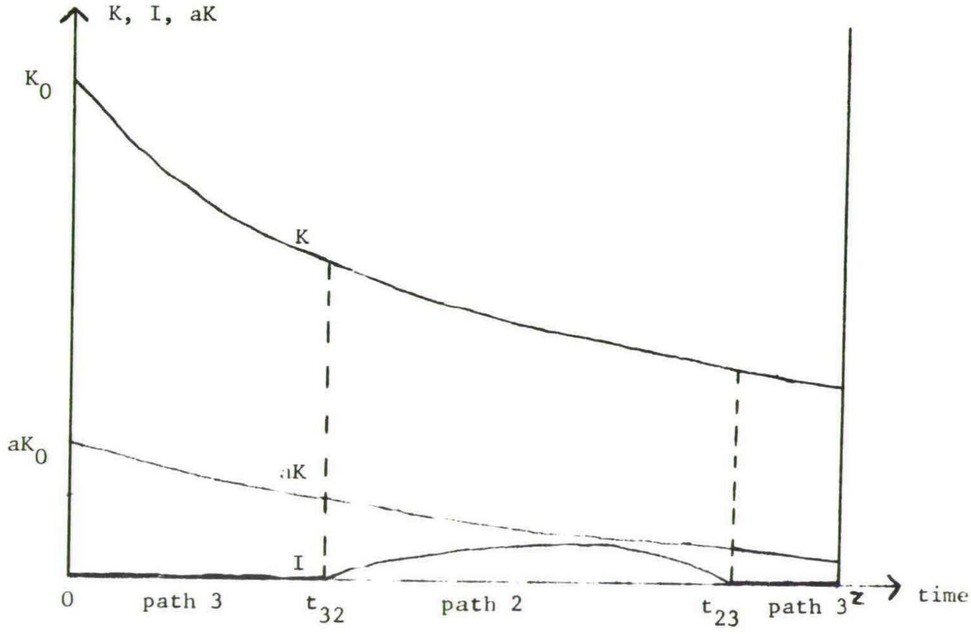
Concerning figure 4.1, we have to remark that the way I diminishes on path 2 depends completely on specific features of S(K) and U(I). So, when S(K) and U(I) are not specified, we do not know whether the slope of I increases, decreases or remains constant on this path. Without these features of S(K) and U(I) we do not have insight into the way I rises on path 1 either.

First, we concentrate on master trajectory 1, which is represented by figure 4.1. On path 1, the next inequality holds (see Appendix 3):

$$1 + \frac{dU}{dI} < e^{-(i+a)(z-T)} + \int_{t=T}^{z} e^{-(i+a)(t-T)} \frac{dS}{dK} \, dT \qquad (13)$$



<u>Figure 4.1</u>. Development of I, K, and aK on master trajectory 1



<u>Figure 4.2</u>. Development of I, K and aK on master trajectory 2

The left-hand side of this expression represents an investment expenditure including adjustment costs, at the right-hand side we find the marginal earnings of investments, consisting of the present value of the remaining new equipment at the end of the planning period (the value of the new equipment decreases with a rate a during the rest of the planning period) plus the present value of additional sales over the whole period due to this new equipment (the production capacity of this equipment decreases with a rate a during the rest of the planning period). Expression (13) means that on path 1, marginal earnings exceed marginal costs of investment expenditures. Therefore, the firm invests at its maximum level, i.e. that level which is feasible considering the financial restrictions, so it does not pay out any dividend and all earnings are spent for investing.

At t_{12} this strategy stops, because marginal earnings become too small ($\frac{dS}{dK}$ decreases when K rises) to finance the rising adjustment costs ($\frac{dU}{dI}$ rises when I rises). Therefore on path 2 investments are kept on such a level that marginal earnings equals marginal costs, so the next equation must hold:

$$1 + \frac{dU}{dI} = e^{-(i+a)(z-T)} + \int_{t=T}^{z} e^{-(i+a)(t-T)} \frac{dS}{dK} dt \qquad (14)$$

This implies decreasing investments and increasing capital stock until I falls below the depreciation level. From this very moment K will also drop. Just when investments become zero, path 2 passes into path 3. This transition is fixed by the moment that the next expression becomes applicable:

$$1 + \frac{dU}{dI} > e^{-(i+a)(z-T)} + \int_{t=T}^{z} e^{-(i+a)(t-T)} \frac{dS}{dK} dt \qquad (15)$$

The inequality shows us that the marginal costs of investments exceed the marginal earnings on path 3. This is caused by the fact that from t_{23} on the remaining time period is too short to defray the adjustment costs of new investments. Therefore, the firm does not invest anymore on path 3.

A noteworthy point is the continuity of I. Larger values of I imply rising marginal adjustment costs because of the convex adjustment cost function. Therefore, adjustment costs are minimized as much as possible if I develops gradually over time.

Another interesting feature is the way in which this pattern will change when the planning period is extended. If z is fixed upon a higher value, then t_{12} as well as t_{23} will be postponed (see Appendix 3). In the case of an infinite time horizon expression (14) continues to hold from t_{23} on, so path 3 disappears completely. This is easy to understand, because now there is always enough time to defray the adjustment costs. On path 2, K will approach a stationary value asymptotically (figure 4.3). Here, the influence of the convex adjustment cost function becomes clear; the optimal value of capital good stock will not be reached within a finite time period, because it is always cheaper to split up the final adjustment into two parts.

On master trajectory 2 (figure 4.2), K_0 is so large (this means that $\frac{dS}{dK}$ is low) that expression (15) holds at T = 0 for all possible values of $\frac{dU}{dI}$. This implies that investments are zero and capital good stock decreases. At t_{32}, $\frac{dS}{dK}$ has risen enough for expression (14) to become applicable. From this very moment I starts to rise, but it never reaches the depreciation level, so K still decreases. At t_{23} the remaining time period is again too short to defray the adjustment costs of new investments. This means that I becomes zero again.

In accordance to master trajectory 1 path 2 will be final path on master trajectory 2 when the time horizon is infinite. In this case capital good stock will approach its stationary value asymptotically from above (figure 4.4).

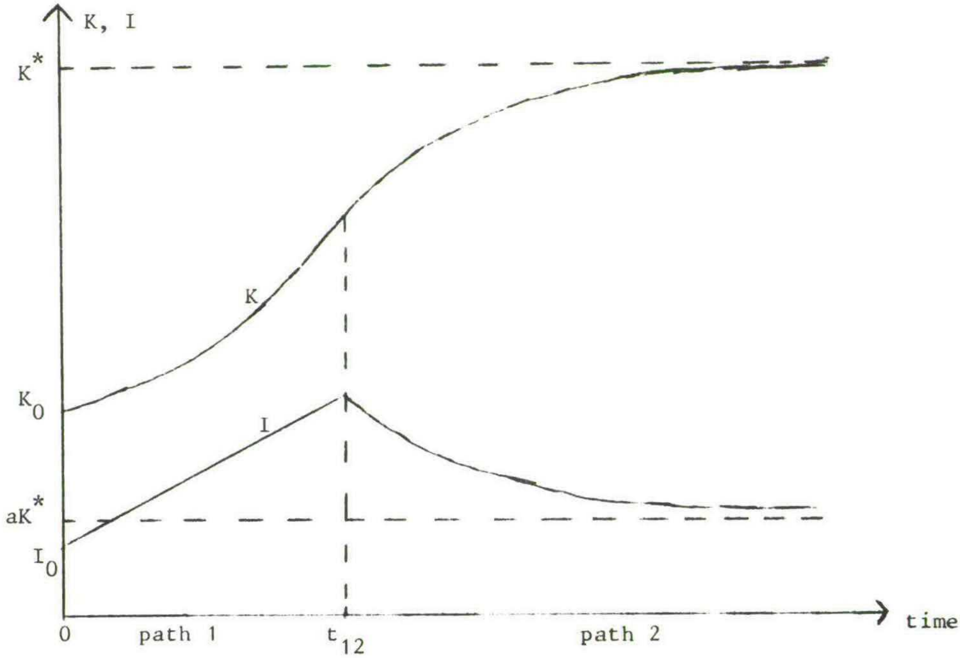


Figure 4.3. Master trajectory 1 in the case of an infinite time horizon

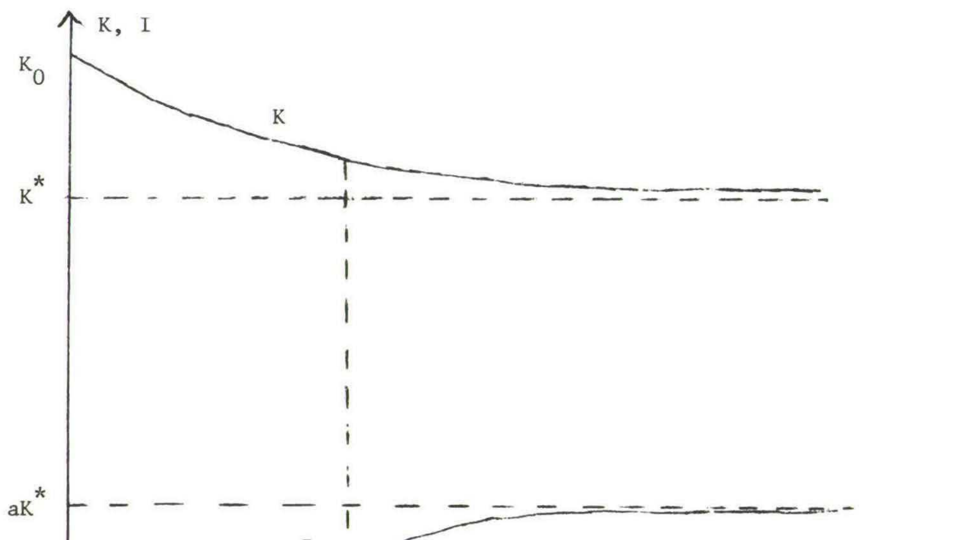


Figure 4.4. Master trajectory 2 in the case of an infinite time horizon

5. Summary

In a dynamic model of the firm we have incorporated financing and a convex cost of adjustment function. We derived the optimal trajectories of the firm by applying Pontryagin's standard maximum principle and the general solution procedure of van Loon. Some striking characteristics of the optimal solution are the continuity of investments during the planning period and the absence of a stationary value of the capital good stock. We have also derived an investment decision rule that explains the optimal policy of the firm by comparing marginal earnings and marginal costs in the succeeding stages of the firm's evolution. With the help of this rule we fix the moments on which the firms policy has to be changed fundamentally and we discuss the relation between these moments and the length of the planning period.

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Appendix 1. Derivation of the optimal solution

We apply Pontryagin's standard maximum principle to obtain the necessary and sufficient conditions.

Let the Hamiltonian be:

$$H = (S(K)-I-U(I))e^{-iT} + \psi(I-aK) \tag{16}$$

and the Lagrangian:

$$L = H + \lambda_1(S(K)-I-U(I)) + \lambda_2 I \tag{17}$$

in which:

- ψ := adjoint variable or co-state variable which denotes the marginal contribution of capital good stock to the performance level
- λ_j := dynamic Lagrange multiplier representing the dynamic 'shadow price' or 'opportunity costs' of the j-th restriction

then the necessary conditions are:

$$\frac{\delta L}{\delta I} = -(1+\frac{dU}{dI})(e^{-iT}+\lambda_1) + \psi + \lambda_2 = 0 \tag{18}$$

$$-\dot{\psi} = \frac{dS}{dK}(e^{-iT}+\lambda_1) - a\psi \tag{19}$$

$$\lambda_1(S(K)-I-U(I)) = 0 \tag{20}$$

$$\lambda_2 I = 0 \tag{21}$$

(20)–(21): complementary slackness conditions

$$\psi(z) = e^{-iz} \quad \text{(transversality condition)} \tag{22}$$

ψ = continuous with piecewise continuous derivatives (23)

$\dot{\psi},\lambda$: continuous on intervals of continuity of I (24)

$$K \quad : \text{continuous} \tag{25}$$

$$I \quad : \text{piecewise continuous} \tag{26}$$

As $H_{optimal}$ is concave in (K,I) and the functions S(K) - I - U(I) and I are quasi-concave in (K,I), these conditions are also sufficient (Van Loon, 1983 (pp. 105)).

Next, we can apply Van Loon's general solution procedure in order to transform these conditions into the optimal trajectories of the firm. These trajectories consist of different paths, which are each of them characterized by the set of active constraints. The properties of these paths are presented by tabel A.1.

path	λ_1	λ_2
1	+	0
2	0	0
3	0	+
4	+	+

<u>Table A.1.</u> The different paths

We will prove that path 4 is infeasible:
From table A.1, (20) and (21) we conclude that on path 4:

$$S(K) - I - U(I) = 0 \tag{27}$$

$$I = 0 \rightarrow U(I) = 0 \tag{28}$$

From (27) and (28) we obtain:

$$S(K) = 0 \rightarrow K = 0 \tag{29}$$

But, (4), (6) and (7) imply that:

$$K > 0 \tag{30}$$

conclusion: path 4 is infeasible

To find the optimal trajectories of the firm, we start at the time horizon z and go backwards in time. According to this strategy we first select all final paths. In order to find these paths, we substitute the transversality condition (22) in (18) for T = z:

$$-(1+\frac{dU}{dI})(e^{-iz}+\lambda_1) + e^{-iz} + \lambda_2 = 0 \tag{31}$$

From this, we can conclude that on a final path it must hold that:

$$\lambda_2 > 0 \tag{32}$$

From (32), we conclude that only path 3 is a feasible final path.

Next, we start a coupling procedure to complete the optimal trajectories. The essence of coupling two paths is to test whether such a coupling will or will not violate the continuity properties of the state variables and the co-state variables. In our model, this means that K and ψ have to be continuous.

As an example of a feasible coupling we will prove that path 1 can precede path 2.

First, we derive the necessary conditions for the continuity of ψ. From (18) and table A.1 we get:

On path 1 it holds that:

$$\psi = (1+\frac{dU}{dI})(e^{-iT}+\lambda_1) \tag{33}$$

On path 2 it holds that:

$$\psi = (1+\frac{dU}{dI})e^{-iT} \tag{34}$$

From (10), (20) and table A.1 we derive:

On path 1 it holds that:

$$S(K) = I + U(I) \tag{35}$$

On path 2 it holds that:

$$S(K) > I + U(I) \tag{36}$$

Because K has to be continuous and S(K) is a continuous function of K, we conclude from (35), (36) and the convexity of U(I):

$$\text{for } t_{12} : \vec{I} > \overset{\leftarrow}{I} \qquad \overrightarrow{(1+\frac{dU}{dI})} > \overleftarrow{(1+\frac{dU}{dI})} \tag{37}$$

(t_{12} is the coupling moment of path 1 and path 2. An arrow to the right (left) indicates the left (right) side limit of the relevant variable at the relevant point of time.)

From (33), (34), (37) and table A.1 we derive the following necessary conditions for the continuity of ψ:

$$I \text{ must be continuous at } t_{12} \tag{38}$$

$$\lambda_1 \text{ must be continuous at } t_{12} \tag{39}$$

Next, we check if K can be continuous when (38) and (39) must hold. When we differentiate (33) to time it holds that on path 1:

$$\dot{\psi} = \frac{d^2U}{dI^2}\dot{I}(e^{-iT}+\lambda_1) - ie^{-iT}(1+\frac{dU}{dI}) + \dot{\lambda}_1(1+\frac{dU}{dI}) \tag{40}$$

From (19) and table A.1 we obtain that on path 1 it holds that:

$$\dot{\psi} = a\psi - \frac{dS}{dK}(e^{-iT}+\lambda_1) \tag{41}$$

After substituting (33) and (40) in (41) we get:

$$((i+a)(1+\frac{dU}{dI}) - \frac{dS}{dK} - \frac{d^2U}{dI^2}\dot{I})e^{-it} =$$

$$(-a(1+\frac{dU}{dI}) + \frac{dS}{dK} + \frac{d^2U}{dI^2}\dot{I})\lambda_1 + \dot{\lambda}_1(1+\frac{dU}{dI}) \tag{42}$$

Analogous to the above, we can derive that on path 2, it holds that:

$$(i+a)(1+\frac{dU}{dI}) - \frac{dS}{dK} - \frac{d^2U}{dI^2}\dot{I} = 0 \tag{43}$$

From (39) and table A.1 we obtain:

$$\overrightarrow{\lambda_1(t_{12})} = 0 \tag{44}$$

$$\overrightarrow{\dot{\lambda}_1(t_{12})} \leq 0 \tag{45}$$

Due to (42), (44) and (45) we get that at the end of path 1, it holds that:

$$(i+a)(1+\frac{dU}{dI}) - \frac{dS}{dK} - \frac{d^2U}{dI^2}\dot{I} \leq 0 \tag{46}$$

From (38), (43) and (46), the continuity in I of $\frac{dU}{dI}$ and $\frac{d^2U}{dI^2}$ and the continuity in K of $\frac{dS}{dK}$ we conclude that a necessary condition of the continuity of K is:

$$\text{for } t_{12} : \overrightarrow{\dot{I}} \geq \overleftarrow{\dot{I}} \tag{47}$$

So, the coupling path 1-path 2 is feasible, if the necessary conditions (38), (39) and (47) hold.

Next, we show an example of an infeasible coupling. We try to couple path 1 to path 3. From (18), (21) and table A.1 we derive that on path 3 it holds that:

$$\psi = (1+\frac{dU}{dI})e^{-iT} - \lambda_2 \tag{48}$$

$$I = 0 \tag{49}$$

Due to (33), (35), (48), (49), table A.1 and the convexity of U(I) we conclude that ψ is continuous if:

$$I,\ \lambda_1 \text{ and } \lambda_2 \text{ are continuous on } t_{13} \tag{50}$$

Due to (35), (49) and (50) we derive that at the end of path 1 it holds that:

$$S(K) = I + U(I) = 0 \rightarrow K = 0 \tag{51}$$

This is in conflict with (30), so the coupling path 1-path 3 is infeasible.

Table A.2 gives us a survey of the feasible and infeasible couplings. From this table we can derive the following possible trajectories:

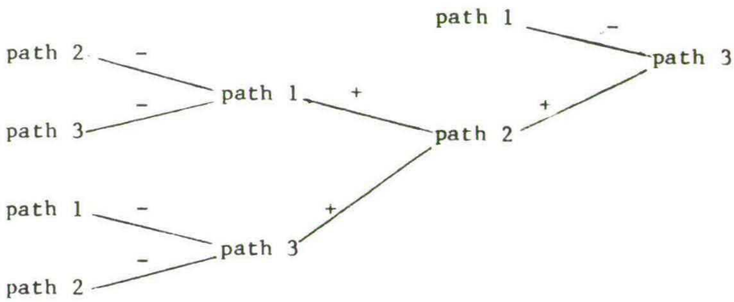
path 1 - path 2 - path 3 (master trajectory 1)

path 3 - path 2 - path 3 (master trajectory 2)

path 2 - path 3

path 3

Depending on the value of K_0 and the length of the planning period one of these trajectories is optimal.



in which

+ : coupling is feasible

- : coupling is infeasible

Table A.2. The feasible and infeasible couplings of the paths

Appendix 2. <u>The development of capital stock and investments on the two master trajectories</u>

First, we concentrate on master trajectory 1.
At T = 0 we make the following assumption:

$$I(0) > aK_0 \qquad (52)$$

Combining this with (4) we get:

$$\dot{K}(0) > 0 \qquad (53)$$

From (35) we obtain that on path 1 it holds:

$$\frac{dS}{dK}\dot{K} = (1+\frac{dU}{dI})\dot{I} \qquad (54)$$

From (53) and (54) we conclude that on path 1 it holds that:

$$\dot{K} > 0 \qquad (55)$$

$$\dot{I} > 0 \qquad (56)$$

Further, from appendix 1 we know that on path 3 it holds that:

$$I = 0 \qquad (57)$$

The above is represented in figure A.1.

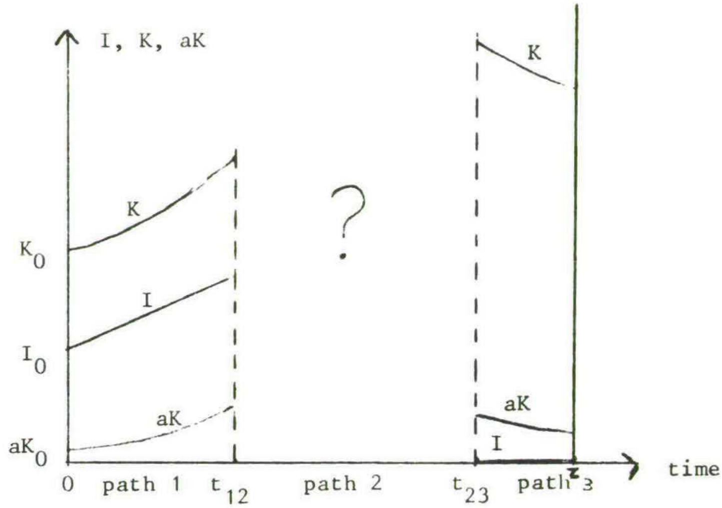


Figure A.1. Development of K, I and aK on path 1 and path 3

Due to (6), (34), (48), (49), table A.1, the convexity of U(I) and the continuity of ψ we can conclude that I must be continuous at t_{23}. So at the end of path 2 (57) becomes applicable.
Let us assume that at the beginning of path 2 it holds that:

$$\dot{I} > 0 \tag{58}$$

From (43) and (58) we derive:

$$(i+a)(1+\frac{dU}{dI}) > \frac{dS}{dK} \tag{59}$$

Due to (58), the convexity of U(I) and the concavity of S(K), we can conclude that $(1+a)(1+\frac{dU}{dI})$ increases and $\frac{dS}{dK}$ decreases (because K increases). As (59) is still satisfied, I continues to rise. This brings us to the conclusion that (58) holds on the entire path 2. Then, after a while restriction (10) will be violated and it is also in conflict with I being zero at the end of path 2, so we have proved that (58) cannot hold at the begin of path 2.

From the above, we conclude that at the beginning of path 2, it holds:

$$(i+a)(1+\frac{dU}{dI}) < \frac{dS}{dK} \tag{60}$$

Now there are four possibilities:

1. $(i+a)(1+\frac{dU}{dI}) < \frac{dS}{dK}$ on the entire path 2

2. $(i+a)(1+\frac{dU}{dI})$ becomes equal to $\frac{dS}{dK}$ when $I > aK$

3. $(i+a)(1+\frac{dU}{dI})$ becomes equal to $\frac{dS}{dK}$ when $I = aK$

4. $(i+a)(1+\frac{dU}{dI})$ becomes equal to $\frac{dS}{dK}$ when $I < aK$

ad 1.

Due to (43) we conclude that $\dot{I} < 0$ on the entire path 2.

ad 2.

Due to (43) we conclude that $\dot{I}$ becomes equal to zero when $I > aK$. Now the level of $(1+a)(1+\frac{dU}{dI})$ does not change and K increases because $I > aK$. This implies that $\frac{dS}{dK}$ diminishes and according to (43) we derive that $(i+a)(1+\frac{dU}{dI}) - \frac{d^2U}{dI^2}\dot{I}$ has to diminish too. So the level of I must change.

I has to diminish because when it starts to rise, it continues to rise at the rest of path 2, which is in conflict with the fact that (57) holds at the end of path 2. Moreover, restriction (10) will be violated after a while.

ad 3.

Due to (43) we conclude that $\dot{I} = 0$ when $I = aK$. This implies that the levels of I and K do not change, so a stationary situation arises which is in conflict with the fact that (57) holds at the end of path 2.

ad 4.

This implies that $(1+a)(1+\frac{dU}{dI}) < \frac{dS}{dK}$ when $I = aK$. Since then I has diminished so $\frac{dU}{dI}$ has diminished, and K has diminished so $\frac{dS}{dK}$ has increased. This brings us to the conclusion that $(i+a)(1+\frac{dU}{dI})$ can't become equal to $\frac{dS}{dK}$ when $I < aK$.

From the above we can conclude that I will diminish on path 2, but an exception is possible: I can remain at the same level during a very short period when I > aK.
Now we have proved figure 4.1.

In the case of an infinite time horizon path 2 will be final path. Then I cannot become equal to zero, so the only satisfactory possibility is the third one (see figure 4.3).
Let us now concentrate on master trajectory 2.
So far, the following is known about the development of I, K and aK on master trajectory 2 (figure A.2).

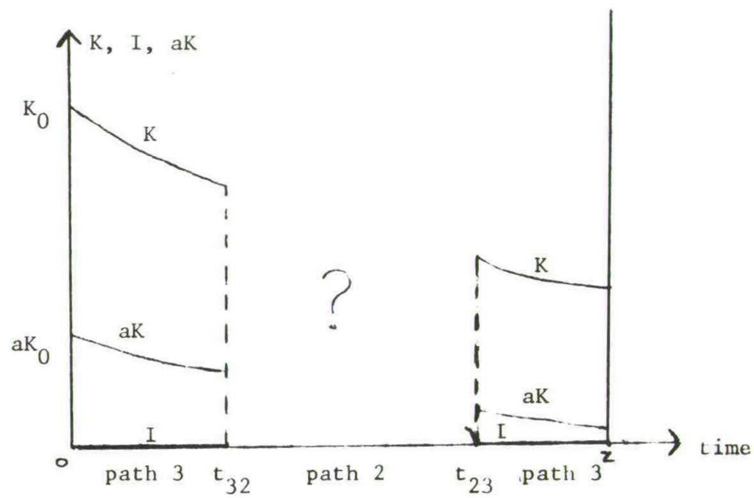


Figure A.2. Development of K, I and aK on path 3

Due to (6), (34), (48), (49), table A.1, the continuity of U(I) and the continuity of ψ we can conclude that I must be continuous at t_{32}. This means that the following must hold:

$$\text{for } t_{32} : \overset{\leftarrow}{\dot{I}} > 0 \tag{61}$$

From figure A.2, we derive:

$$\text{for } t_{23} : \dot{\vec{I}} < 0 \tag{62}$$

Due to (25), (26) and (43) we can conclude that $\dot{I}$ is a continuous function of time on path 2.
Now, there are three possibilities:

1. $\dot{I} = 0$ when $I < aK$

2. $\dot{I} = 0$ when $I = aK$

3. $\dot{I} = 0$ when $I > aK$

ad 1.

This means that depreciations are larger than investments during path 2.

ad 2.

In this case a stationary situation arises, which is in conflict with I being zero at the end of path 2.

ad 3.

This means that $\dot{I} > 0$ when $I = aK$, so (due to (43)) $(i+a)(1+\frac{dU}{dI}) > \frac{dS}{dK}$ when $I = aK$. Since then $\frac{dU}{dI}$ has risen and $\frac{dS}{dK}$ has diminished, so this possibility cannot arise.

From the above, we can conclude that possibility 1 will occur, so we have proved figure 4.2.

In the case of an infinite time horizon path 2 will be final path. Then possibility 2 will arise, because I cannot become equal to zero (figure 4.4).

Appendix 3. Derivation of the investment decision rule

During the whole planning period, it holds that:

$$\psi(T) = \psi(z) - \int_{\tau=t}^{z} \dot{\psi}(\tau)d\tau \tag{63}$$

After substituting (18), (19) and (22) in (63) for $\lambda_1 = \lambda_2 = 0$ it holds that on path 2:

$$(1+\frac{dU}{dI})e^{-iT} = e^{-iz} + \int_{\tau=T}^{z} (\frac{dS}{dK} e^{-i\tau} - a\psi(\tau))d\tau \tag{64}$$

After solving the differential equation (19) on path 2 and path 3 (λ_1=0) and substituting in this solution the transversality condition (22) we get:

$$\psi(\tau) = e^{a\tau}(\int_{t=\tau}^{z} e^{-(i+a)t} \frac{dS}{dK} dt + e^{-(i+a)z}) \tag{65}$$

When we substitute (65) in (64) and differentiate this to time we get:

$$-i(1+\frac{dU}{dI})e^{-iT} + \frac{d^2U}{dI^2} \dot{I}e^{-iT} =$$

$$= ae^{aT}e^{-(i+a)z} - \frac{dS}{dK} e^{-iT} + ae^{aT} \int_{t=T}^{z} e^{-(i+a)t} \frac{dS}{dK} dt \tag{66}$$

After substituting (43) in (66) and deviding this by ae^{-iT} we obtain that on path 2 it holds that:

$$1 + \frac{dU}{dI} = e^{-(i+a)(z-T)} + \int_{t=T}^{z} e^{-(i+a)(t-T)} \frac{dS}{dK} dt \tag{67}$$

In appendix 2 we obtained that at the end of path 2 it holds that:

$$\dot{I} < 0 \tag{68}$$

From (43), (68) and the continuity in t_{23} of K and I we get:

$$\text{for } t_{23} : (i+a)(1+\overleftarrow{\frac{dU}{dI}}) < \overleftarrow{\frac{dS}{dK}} \tag{69}$$

From (57) we derive that on path 3 it holds that:

$$\dot{I} = 0 \tag{70}$$

Due to (69), (70) and the fact that $\frac{dS}{dK}$ rises on path 3 we can conclude that on path 3 it holds that:

$$(i+a)(1+\frac{dU}{dI}) < \frac{dS}{dK} \tag{71}$$

Due to the continuity of I and K on t_{12}, we can conclude that (60) holds at the end of path 1. Also, we know that (56) holds on path 1. Therefore, we can rewrite (46) as follows:

$$-(i+a)(1+\frac{dU}{dI}) + \frac{d^2U}{dI^2}\dot{I} > -\frac{dS}{dK} \tag{72}$$

When we divide (67) by $e^{(i+a)T}$ we get:

$$(1+\frac{dU}{dI})e^{-(i+a)T} = e^{-(i+a)z} + \int_{t=T}^{z} e^{-(i+a)t}\frac{dS}{dK}\,dt \tag{73}$$

After differentiating the left-hand and right-hand side of (73) to time we get:

$$\frac{\partial((1+\frac{dU}{dI})e^{-(i+a)T})}{\partial T} = (-(i+a)(1+\frac{dU}{dI}) + \frac{d^2U}{dI^2}\dot{I})e^{-(i+a)T} \tag{74}$$

$$\frac{\partial(e^{-(i+a)z} + \int_{t=T}^{z} e^{-(i+a)t}\frac{dS}{dK}\,dt)}{\partial T} = -\frac{dS}{dK}e^{-(i+a)T} \tag{75}$$

Concerning master trajectory 1, from (70) through (72), (74) and (75) we can conclude that:

on path 1 it holds that:

$$1 + \frac{dU}{dI} < e^{-(i+a)(z-T)} + \int_{t=T}^{z} e^{-(i+a)(t-T)} \frac{dS}{dK} dt \tag{76}$$

on path 3 it holds that:

$$1 + \frac{dU}{dI} > e^{-(i+a)(z-T)} + \int_{t=T}^{z} e^{-(i+a)(t-T)} \frac{dS}{dK} dt \tag{77}$$

Next, we want to know what happens to t_{12} and t_{23} on master trajectory 1 when the planning period is extended. Therefore we differentiate the left-hand and right-hand side of (67) to z:

$$\frac{\partial(1+\frac{dU}{dI})}{\partial z} = 0 \tag{78}$$

$$\frac{\partial(e^{-(i+a)(z-T)} + \int_{t=T}^{z} e^{-(i+a)(t-T)} \frac{dS}{dK} dt)}{\partial z} = (-(i+a)+\frac{dS}{dK})e^{-(i+a)(z-T)} \tag{79}$$

From (43), the fact that K reaches its maximum level on path 2 and the fact that $\dot{I} \leq 0$ on path 2, we can conclude that during the whole planning period it holds:

$$\frac{dS}{dK} \geq (i+a)(1+\frac{dU}{dI}) \tag{80}$$

Due to (79) and (80) we conclude that the derivative of the right-hand side of (67) to z has a positive value. So, when the planning period is extended (z becomes bigger), the right-hand side of (67) becomes bigger, so I must be on a higher level when path 1 passes into path 2. So, the coupling moment of path 1 and path 2 has to be postponed.
When the coupling of path 2 and path 3 takes place, I must be equal to zero, so the left-hand side of (67) always has the same value at this coupling moment. When z becomes bigger, the right-hand side of (67)

becomes bigger. Due to the fact that the right hand side of (67) diminishes during the time, t_{23} has to be postponed.

Finally, we concentrate on master trajectory 2.
Due to (43), the fact that $\dot{I} > 0$ at the begin of path 2 and the continuity of K and I, it must hold that:

$$\text{for } t_{32} : (i+a)(1+ \overrightarrow{\frac{dU}{dI}}) > \overrightarrow{\frac{dS}{dK}} \tag{81}$$

Due to the fact that $\frac{dS}{dK}$ rises on path 3, we can derive that on path 3 it holds that:

$$(i+a)(1+ \frac{dU}{dI}) > \frac{dS}{dK} \tag{82}$$

From (70), (74), (75) and (82) we can conclude that (77) holds on path 3.

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