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NON-UNIFORMITIES IN SPATIAL LOCATION MODELS

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Abstract

In this paper we study the impact of demand dispersals on the equilibrium outcome in a two-dimensional spatial location model with two firms. The corresponding location-then-price game is solved by backwards induction. Standard optimization techniques however are not applicable due to the non quasi-concavity of the profit functions. Candidate equilibria for the 'overall game' (referred to as global equilibria) can be found however by restricting the firms' strategies as to end up in a specific demand region. From the set of local equilibria it is possible then to determine the (unique) global equilibrium. In equilibrium firms' locations and prices are such that both firms face equal demands and have equal profits.

Keywords: location theory, demand dispersals, equilibrium.

Introduction

Spatial competition theory, based on Hotelling's (1929) well-known paper, in some sense has not evolved much since then. Although it is possible to list many papers on this subject very little has been written on the distribution of consumers. The main assumption is that consumers are uniformely distributed along a line segment. The only paper that we are aware of that has dealt with non-uniformly distributed consumers and a two-dimensional space analytically is the one of Lederer and Hurter (1986).

The main criticism to their model is that their approach is based on the assumption of discriminatory pricing. There is no clear reason for doing this, but the results will be quite different if we make the assumption of mill pricing (i.e. transportation costs are not included) as was shown by Gabszewicz and Thisse (1986). Eisselt (1991) moreover studies the situation where one of the firms uses discriminatory pricing and the other firm uses mill pricing.

Although there is not much evidence of which pricing policy to be used, we would argue that a mill pricing policy is much more of practical interest than a discriminatory pricing policy. An important reason for firms not being able to price discriminate is that they are unable (or perhaps unwilling) to customise the product to the individual consumer's desires (see al-Nowaihi and Norman (1992)). Mill pricing consequently leads to sales maximization, whereas discriminatory pricing leads to minimization of transportation cost (see Gabszewicz and Thisse (1986)).

Our approach to model non-uniformly distributed consumers will therefore be based on the assumption of mill-pricing. The main problem with competitive location models is that they are often too complex to deal with analytically. In this paper we restrict ourselves to a situation where two firms are located along a single line. The situation with m firms in a two-dimensional space remains future research.

In this paper we study the situation in which consumers are distributed piecewise uniformly. Essentially there are two reasons for representing demand dispersals in this way. The first reason is that we want to model a broad class of demand functions, without requiring concavity, and the second reason is that the data available for practical applications usually is of this kind (think of density per metres square for example).

The determination of all possible location-price equilibria is done in the spirit of Hotelling's approach. First the firms choose their locations and after that the firms choose their prices, given the locations. The corresponding two-stage game is solved then by backwards induction[1].

The basic problem in general is that the profit functions of the firms are not differentiable everywhere due to the fact that demand is kinked. It seems however that there is little difference with the situation where demand is defined as a uniform distribution, in which case it is obvious that there is a unique indifferent consumer. Intuitively we expect the same for a situation where demand is defined as a piecewise uniform distribution. Because the exact location of the indifferent consumer is determined completely by the location choices and the price policies of the firm, it is a hard task to maximize firms' profits directly. The main reason for this is the (in general) non-quasi-concavity of the firm's profit function (see Gabszewicz and Thisse (1986)). In a companion paper we study the profit function more rigorously (see Webers (1993a)).

In this paper we try to overcome the difficulties arising from the non-quasi-concavity of the profit functions by using the following two-step analysis. First we determine a solution for the situation where the location of the indifferent consumer is restricted to one of the subintervals of the line segment. This is possible with standard first order techniques because the profit function is piecewise quasi-concave. A solution to the overall maximization problem can be found then from the solutions for these subintervals. We will show that the firms' location and price choices are such that both firms face the same demand in equilibrium. Each firm's location and price choice is in other words determined completely by the demand distribution. We thereby assume that both firms are in the market.

The paper is organized as follows. In Section 1 we present the model. In Section 2 we define the equilibrium concept. In Section 3 and Section 4 we characterize the equilibrium and give some equilibrium properties. In Section 5 we present a slightly modified model in which the locations of the firms are restricted and in Section 6 we conclude.

[1] Anderson, de Palma and Hong (1992) compared the simultaneous price and location game with the two-stage location-then-price game and suggest that equilibrium locations are further apart under the second game and that profits are higher then, since firms internalize the harmful price competition effect of moving close to each other. However they note that if prices are less costly to adjust than locations, then a two-stage location-then-price game is deemed the relevant equilibrium concept, as was also suggested by Tirole (1988) for example.

1 The model

We consider a market region described by the interval $M = [0, n]$ in $\mathcal{R}^1$, for some positive integer n. In the economy there are two types of agents, sellers and buyers. We assume that there are two sellers in the market. These sellers, which we will also refer to as firms, produce the same, single good with unit production costs c. The buyers, the consumers, are characterized by their location along the line segment M. The consumers have identical preferences and wish to purchase a single unit of the good. The implicit assumption made here is that the consumers have an infinite willingness to pay.

In contrast to most of the existing literature on spatial competition we allow for geographical demand dispersals, i.e. consumers are not necessarily distributed uniformly along the line segment. The distribution of the consumers along the line segment is modeled in the following way. On each interval $[i, i+1]$ with $i \in \{0, \ldots, n-1\}$ consumers are located with density $d_{i+1} > 0$ such that

$$\sum_{i=0}^{n-1} d_{i+1} = 1.$$

The demand distribution will be denoted by the tuple $< d_1, d_2, \ldots, d_n >$ with $n \in \mathcal{N}$. Note that we have the standard uniform case when $n = 1$ or $d_{i+1} = 1/n$ for each $i \in \{0, \ldots, n-1\}$. The requirement that $d_{i+1} > 0$ for each i is needed to side step the difficulties arising when firm 1 strives to get the consumer located at i and firm 2 strives to get the consumer located at $i+1$. The locations that firm 1 and firm 2 choose will be denoted by a and b, respectively, and their prices by p_1 and p_2. We will allow for location choices outside the interval M and refer to this situation as 'locations outside the city'. As Tirole (1988) already noted the possibilities for horizontal differentiation will be greater then. Furthermore this assumption avoids technical 'corner' difficulties as in the original Hotelling model, as was noted by Salop (1979) among others.

The optimal locations of the firms are determined by two effects, a direct effect and a strategic effect. With the direct effect we mean that a location that is more close to the centre leads to a greater market share and therefore higher profits for a fixed price. With the strategic effect we mean that the price competition is higher when the firms are closer to each other. The outcome of the trade-off between the positive and negative effect of clustering is not clear in general.

The transportation cost the consumers will have to pay are assumed to be quadratic. As d'Aspremont, Gabszewicz and Thisse (1979) show, this assumption ensures the existence of a price equilibrium (when demand is distributed uniformly), whatever the locations a and b may be, because this assumption prevents discontinuities in the profit function[2]. Anderson (1988) studies the existence of equilibria for the more general case that transportation costs are 'linear-quadratic', i.e. a quadratic function form representation (with a constant term equal to 0). It is shown that the parameters of the quadratic and the linear term have to satisfy very stringent conditions in order to sustain an equilibrium.

The final assumption we make is that $a < b$, i.e. firm 1 locates to the left of firm 2. We thus ignore the difficulties arising from the coordination problem the firms face. For a detailed discussion on this problem see Bester, de Palma, Leininger, von Thadden and Thomas (1991). There it is shown that there will be less differentiation in such a case, due to the fact that each firm has a positive probability of ending up "left to the middle".

The indifferent consumer, i.e. the consumer indifferent between buying from firm 1 and buying from firm 2, is located at x equal to

$$x(a, b, p_1, p_2) = \frac{a+b}{2} + \frac{p_2 - p_1}{2(b-a)}, \tag{1}$$

given the location a of firm 1, the location b of firm 2, the price p_1 of firm 1 and the price p_2 of firm 2. We assume that $0 \leq x(a, b, p_1, p_2) \leq n$ and that in equilibrium both firms are in the market.

If both firms anticipate that the indifferent consumer will be located in the interval $[i, i+1]$ (due to their strategies) then the (anticipated) demand of firm 1 can be written as

$$D_1^i(a, b, p_1, p_2) = (d_0 + \cdots + d_i) + d_{i+1}(x(a, b, p_1, p_2) - i)$$

[2]Another way to restore equilibrium (see for example Lerner and Singer (1937)) is to assume that firms are unable to cut their rivals' prices (frequently called the no-mill-price undercutting restriction). The shortcoming of this procedure however is that one of the basic (and probably most important) ingredients of price competition is eliminated.

and the (anticipated) demand of firm 2 can be written as

$$D_2^i(a, b, p_1, p_2) = 1 - D_1^i(a, b, p_1, p_2)$$

with $d_0 \equiv 0$. For simplicity we denote $\mathcal{D}_i = d_0 + \cdots + d_i$. The costs c are normalized to zero. Given a, b, p_1, p_2 and i the (anticipated) profits of firm k with $k \in \{1, 2\}$ are $\Pi_k^i(a, b, p_1, p_2) = p_k D_k^i(a, b, p_1, p_2)$.

2 The equilibrium concept

To study the behaviour of the firms with respect to their location choices and their price policies we consider the following two-stage game with complete information. In the first stage firms choose locations, afterwards they become aware of these locations. In the second stage firms choose prices (and receive profits). As in Shaked and Sutton (1982) and Lederer and Hurter (1986) the set of Nash equilibria for this game is very large and very difficult to characterize. We use the perfect equilibrium notion of Selten (1975). In our setting a tuple of strategies is a perfect equilibrium if the price strategies in the second stage of the game form a Nash equilibrium and the location strategies in the first stage of the game form a Nash equilibrium, given these equilibrium price schemes. The concept of perfect equilibrium captures the idea that, when firms choose their locations, they explicitly take the impact of their location decisions on prices into account.

It can be shown that there will exist only a perfect equilibrium when both firms coordinate on the same interval. This is caused by the fact that in equilibrium both firms face the same indifferent consumer. When both firms do not coordinate on the same interval of M, then either no price equilibrium exists or one of the firms anticipates the indifferent consumer to lie in one interval and the other firm anticipates the indifferent consumer to lie in the next interval. As a consequence the indifferent consumer will be located at the common endpoint. Then a price equilibrium exists but one of the firms can increase (or keep) its profits by coordinating on the same interval as its opponent.

Therefore we assume that both firms anticipate that the indifferent consumer is located in say the i^{th} interval of M, i.e. $i \leq x(a, b, p_1, p_2) \leq i+1$. We refer to the competetive game where the indifferent consumer is assumed to be located in the i^{th} interval as Γ^i. A meaningful noncooperative solution for the location-then-price game is the local location-price equilibrium concept, which in fact is the perfect equilibrium concept for the (restricted) game Γ^i. Recall that $a < b$ by assumption.

Definition 2.1 *A local location-price equilibrium (LLPE) for the game Γ^i is a quadruple $< a^*, b^*, p_1(a,b), p_2(a,b) >$ such that it is a perfect equilibrium for Γ^i, i.e.*

(i) *$p_1(a,b)$ and $p_2(a,b)$ is a Nash equilibrium in the second stage of the game for all a and b*

(ii) *a^* and b^* is a Nash equilibrium in the first stage of the game given the price schemes in the second stage.*

Our purpose now is to determine from the set of local equilibria a global equilibrium for the 'overall' game which we will refer to as Γ. Note that the local equilibrium concept is very strong, in the sense that optimal behaviour with respect to location choices and price policies is possibly restricted by the assumed location of the indifferent consumer. In a global equilibrium however, location choices and price policies will be reached without restricting either of the two firms. As we will see the perfectness condition (ii) requires the price policies to be unrestricted. For a global equilibrium we therefore need that the prices and the locations of an LLPE are chosen freely, i.e. not restricted by the boundary conditions, in one of the intervals. We call this an unconstrained LLPE. With this notion we can define a noncooperative solution for the location-then-price game Γ.

Definition 2.2 *A global location-price equilibrium (GLPE) for the game Γ is a quadruple $< a^*, b^*, p_1(a,b), p_2(a,b) >$ such that*

(i) *this quadruple is an unconstrained LLPE with respect to some Γ^i with $i \in \{0, \dots, n-1\}$*

(ii) *for both firms profits are maximized over all unconstrained LLPEs.*

It will be shown that there is at least one and at most two unconstrained LLPEs. In case of two unconstrained LLPEs these equilibria are paired, i.e. the resulting indifferent consumer is the same. For both firms profits are highest if they coordinate on the interval with the lowest demand density, which means that there will be a unique GLPE.

3 Characterization of local equilibria

In maximizing profits firms choose locations in the first stage and price policies in the second stage that are optimal in the sense of Nash, given their expectations with respect to demand. Let $i \in \{0, \dots, n-1\}$ be fixed. The two-stage game Γ^i is solved then by backwards induction. In the second stage of the game both firms maximize profits with respect to their own price (given the price of the other firm and given the locations a and b) subject to the condition that the indifferent consumer lies in the i^{th} interval, i.e. $i \leq x(a, b, p_1, p_2) \leq i + 1$. For firm 1 this yields price $p_1(a, b, p_2)$ given by

$$p_1(a,b,p_2) = \begin{cases} p_2 + (b-a)(a+b-2i) & \text{if } p_2 < \alpha \\ \frac{p_2}{2} + (b-a)(\frac{a+b-2i}{2} + \frac{\mathcal{D}_i}{d_{i+1}}) & \text{if } \alpha \leq p_2 \leq \alpha + 4(b-a) \\ p_2 + (b-a)(a+b-2i-2) & \text{if } p_2 > \alpha + 4(b-a) \end{cases} \quad (2)$$

with $\alpha = 2(b-a)(i - \frac{a+b}{2} + \frac{\mathcal{D}_i}{d_{i+1}})$, and for firm 2 this yields price $p_2(a, b, p_1)$ given by

$$p_2(a,b,p_1) = \begin{cases} p_1 + (b-a)(2i+2-a-b) & \text{if } p_1 < \beta \\ \frac{p_1}{2} + (b-a)(\frac{2i-a-b}{2} + \frac{1-\mathcal{D}_i}{d_{i+1}}) & \text{if } \beta \leq p_1 \leq \beta + 4(b-a) \\ p_1 + (b-a)(2i-a-b) & \text{if } p_1 > \beta + 4(b-a) \end{cases} \quad (3)$$

with $\beta = 2(b-a)(-i-2+\frac{a+b}{2} + \frac{1-\mathcal{D}_i}{d_{i+1}})$.

From the price schemes given by equations (2) and (3) we can determine the equilibrium price schemes for every a and b. There only exists a Nash equilibrium in locations however when prices are given by the interior solutions, i.e.

$$p_1(a,b,p_2) = \frac{p_2}{2} + (b-a)(\frac{a+b-2i}{2} + \frac{\mathcal{D}_i}{d_{i+1}})$$

$$p_2(a,b,p_1) = \frac{p_1}{2} + (b-a)(\frac{2i-a-b}{2} + \frac{1-\mathcal{D}_i}{d_{i+1}}).$$

Therefore, given a and b the equilibrium price schemes are given by

$$\begin{aligned} p_1(a,b) &= (\tfrac{b-a}{3})(a+b-2i+\tfrac{2+2\mathcal{D}_i}{d_{i+1}}) \\ p_2(a,b) &= (\tfrac{b-a}{3})(2i-a-b+\tfrac{4-2\mathcal{D}_i}{d_{i+1}}) \end{aligned} \quad (4)$$

under the restriction that $\alpha \leq p_2(a,b) \leq \alpha + 4(b-a)$ and $\beta \leq p_1(a,b) \leq \beta + 4(b-a)$, which is the same as requiring that $i \leq x(a,b,p_1(a,b),p_2(a,b)) \leq i+1$.

When a and b are such that $i < x(a,b,p_1(a,b),p_2(a,b)) < i+1$ the second order conditions guarantee a maximum but for $x(a,b,p_1(a,b),p_2(a,b)) = i$ and $x(a,b,p_1(a,b),p_2(a,b)) = i+1$ the second derivative is equal to zero. We come back to this later. From (1) and (4) we see that $x(a,b,p_1(a,b),p_2(a,b))$ equals

$$x(a,b) = \frac{1}{6}\left\{a + b + 4i + \frac{2 - 4\mathcal{D}_i}{d_{i+1}}\right\}. \tag{5}$$

The condition $i \leq x(a,b) \leq i+1$ can be written then as

$$2i + \frac{4\mathcal{D}_i - 2}{d_{i+1}} \leq a + b \leq 2i + 6 + \frac{4\mathcal{D}_i - 2}{d_{i+1}}. \tag{6}$$

The next step is to determine the optimal locations, given the equilibrium price schemes from equation (4) and given the location of the other firm. Firm 1 thus maximizes $\Pi_1^i(a,b) = \Pi_1^i(a,b,p_1(a,b),p_2(a,b))$ with respect to a subject to condition (6). From

$$\Pi_1^i(a,b) = \left\{(\frac{b-a}{3})(a + b - 2i + \frac{2 + 2\mathcal{D}_i}{d_{i+1}})\right\}\{\mathcal{D}_i + d_{i+1}(x(a,b) - i)\}$$

we get

$$\frac{\partial \Pi_1^i(a,b)}{\partial a} = \frac{-d_{i+1}}{18}\left\{a + b - 2i + \frac{2 + 2\mathcal{D}_i}{d_{i+1}}\right\}\left\{3a - b - 2i + \frac{2 + 2\mathcal{D}_i}{d_{i+1}}\right\}. \tag{7}$$

The first term of the righthand side of (7) is of course strictly negative. The second term is strictly positive because according to (6) it holds that $a + b - 2i + (2 + 2\mathcal{D}_i)/d_{i+1} = a + b - 2i + (2 - 4\mathcal{D}_i)/d_{i+1} + 6\mathcal{D}_i/d_{i+1} \geq 6\mathcal{D}_i/d_{i+1} \geq 0$, with both equality signs for $x(a,b) = i = 0$. However a and b must be such that $x(a,b)$ will be greater than zero in equilibrium because both firms have to be in the market. The third term is zero for

$$a = \frac{1}{3}\left\{b + 2i - \frac{2 + 2\mathcal{D}_i}{d_{i+1}}\right\}. \tag{8}$$

It can be verified that the second order conditions for a maximum are satisfied. With condition (6) we get that firm 1's profit maximizing location $a(b)$ is given by

$$a(b) = \begin{cases} 2i + \frac{4\mathcal{D}_i - 2}{d_{i+1}} - b & \text{if } b \le \gamma \\ \frac{1}{3}(b + 2i - \frac{2\mathcal{D}_i + 2}{d_{i+1}}) & \text{if } \gamma \le b \le \gamma + \frac{9}{2} \\ 2i + \frac{4\mathcal{D}_i - 2}{d_{i+1}} - b + 6 & \text{if } b \ge \gamma + \frac{9}{2} \end{cases} \quad (9)$$

with $\gamma = i + \frac{14\mathcal{D}_i - 4}{4d_{i+1}}$.

Firm 2 maximizes $\Pi_2^i(a,b) = \Pi_2^i(a,b,p_1(a,b),p_2(a,b))$ with respect to b subject to condition (6). From

$$\Pi_2^i(a,b) = \left\{ (\frac{b-a}{3})(2i - a - b + \frac{4 - 2\mathcal{D}_i}{d_{i+1}}) \right\} \{1 - \mathcal{D}_i - d_{i+1}(x(a,b) - i)\}$$

we get

$$\frac{\partial \Pi_2^i(a,b)}{\partial b} = \frac{-d_{i+1}}{18} \left\{ a + b - 2i - \frac{4 - 2\mathcal{D}_i}{d_{i+1}} \right\} \left\{ a - 3b + 2i + \frac{4 - 2\mathcal{D}_i}{d_{i+1}} \right\}. \quad (10)$$

The first term of the righthand side of (10) is strictly negative. The second term is also strictly negative because according to condition (6) it holds that $a + b - 2i - (4 - 2\mathcal{D}_i)/d_{i+1} = a + b - 2i + (2 - 4\mathcal{D}_i)/d_{i+1} + (6\mathcal{D}_i - 6)/d_{i+1} \le 6 + (6\mathcal{D}_i - 6)/d_{i+1} \le 0$, with both equality signs for $x(a,b) = i + 1 = n$. However, $x(a,b)$ will be smaller than n in equilibrium because both firms have to be in the market. The third term is zero for

$$b = \frac{1}{3}\left\{ a + 2i + \frac{4 - 2\mathcal{D}_i}{d_{i+1}} \right\}. \quad (11)$$

Again the second order conditions guarantee a maximum. Combination with condition (6) yields that firm 2's profit maximizing location $b(a)$ is given by

$$b(a) = \begin{cases} 2i + \frac{4\mathcal{D}_i - 2}{d_{i+1}} - a & \text{if } a \le \delta \\ \frac{1}{3}(a + 2i + \frac{4\mathcal{D}_i - 2}{d_{i+1}}) & \text{if } \delta \le a \le \delta + \frac{9}{2} \\ 2i + \frac{4\mathcal{D}_i - 2}{d_{i+1}} - a + 6 & \text{if } a \ge \delta + \frac{9}{2} \end{cases} \quad (12)$$

with $\delta = i + \frac{14\mathcal{D}_i - 10}{4d_{i+1}}$.

From the reaction functions (9) and (12) it can be shown that there are different types of location equilibria, determined by the demand distribution. If a^* and b^* are such that $a^* = a(b^*)$, $b^* = b(a^*)$ and letting $x^* = x(a^*, b^*)$ we get the following. Whenever $\mathcal{D}_i > \frac{1}{2}$ or $\mathcal{D}_{i+1} < \frac{1}{2}$ then either firm 1's location or firm 2's location is restricted. If $\mathcal{D}_i > \frac{1}{2}$ then $a^* = i + \frac{2\mathcal{D}_i - 4}{4d_{i+1}}$, $b^* = i + \frac{14\mathcal{D}_i - 4}{4d_{i+1}}$ and $x^* = i$. If $\mathcal{D}_{i+1} < \frac{1}{2}$ then $a^* = i + \frac{9}{2} + \frac{14\mathcal{D}_i - 10}{4d_{i+1}}$, $b^* = i + \frac{3}{2} + \frac{2\mathcal{D}_i + 2}{4d_{i+1}}$ and $x^* = i + 1$. In both cases we get corner solutions and the corresponding equilibria are referred to as restricted equilibria. It is obvious that these equilibria cannot be a global equilibrium for the overall game.

If i is such that both $\mathcal{D}_i \leq \frac{1}{2}$ and $\mathcal{D}_{i+1} \geq \frac{1}{2}$ then we get an unrestricted equilibrium given by equations (8) and (11). This yields

$$\begin{aligned} a^* &= i - \tfrac{1 + 4\mathcal{D}_i}{4d_{i+1}} \\ b^* &= i + \tfrac{5 - 4\mathcal{D}_i}{4d_{i+1}}. \end{aligned} \tag{13}$$

Substitution of (13) into (4) gives the equilibrium prices

$$p_1^* = p_2^* = \frac{3}{2d_{i+1}^2} \tag{14}$$

with $p_1^* = p_1(a^*, b^*)$ and $p_2^* = p_2(a^*, b^*)$, and substitution of (13) into (5) gives the equilibrium location of the indifferent consumer

$$x^* = i + \frac{1 - 2\mathcal{D}_i}{2d_{i+1}}. \tag{15}$$

Note that condition (6) indeed is satisfied for a^* and b^* given by (13) because $\mathcal{D}_i \leq \frac{1}{2}$ and $\mathcal{D}_{i+1} \geq \frac{1}{2}$.

Corollary 3.1 *Let i be such that $0 \leq (1 - 2\mathcal{D}_i)/2d_{i+1} \leq 1$. Then the equilibrium outcome for the game Γ^i consists of location choices $a^* = i - (1 + 4\mathcal{D}_i)/4d_{i+1}$ and $b^* = i + (5 - 4\mathcal{D}_i)/4d_{i+1}$ and price choices $p_1^* = p_2^* = 3/2d_{i+1}^2$.*

Proof This follows immediately from Definition 2.1 and equations (13) and (14).

□

The following lemma states what the equilibrium profits are.

Lemma 3.2 *Let i be such that $0 \leq (1 - 2\mathcal{D}_i)/2d_{i+1} \leq 1$. The equilibrium locations and prices in the game Γ^i are such that both firms face the same demand. Furthermore both firms use the same price policies and the profit of firm k is given by $\Pi_k^{i*} = \Pi_k^i(a^*, b^*, p_1^*, p_2^*) = 3/(4d_{i+1}^2) \;\; \forall k \in \{1, 2\}$.*

Proof Using (15) we can write $D_1(a^*, b^*, p_1^*, p_2^*) = \mathcal{D}_i + d_{i+1}(x^* - i) = \mathcal{D}_i + (1/2 - \mathcal{D}_i) = 1/2$. From (14) we know furthermore that the firms use the same price policies, $p_1^* = p_2^* = 3/2d_{i+1}^2$. Profits are given then by $\Pi_1^{i*} = \Pi_2^{i*} = 3/4d_{i+1}^2$ with $\Pi_k^{i*} = \Pi_k^i(a^*, b^*, p_1^*, p_2^*) \; \forall k \in \{1, 2\}$.

□

Thus the two firms always charge the same price in equilibrium and furthermore both firms will have different locations in equilibrium. This means that the principle of minimum differentiation no longer holds. Note that these results are equivalent to the results Lederer and Hurter (1986) found for the situation of discriminatory pricing.

Next we need to verify what happens when $x^* = i$ or $x^* = i + 1$. To check whether profits are indeed maximized at the corner solutions $x^* = i$ or $x^* = i + 1$ we need to compare the profits achievable in the interval at the left to the corner solution and the profits achievable in the interval at the right to the corner solution. Suppose that $x^* = i$. Then x^* is either in the $(i-1)^{th}$ or in the i^{th} interval, i.e. $i - 1 \leq x^* \leq i$ or $i \leq x^* \leq i + 1$. There is a maximum for x^* in the i^{th} interval if and only if for both firms profits are higher than for x^* in the $(i-1)^{th}$ interval and there is a maximum for x^* in the $(i-1)^{th}$ interval if the opposite holds. From Lemma 3.2 we see that there cannot be a situation where one of the firms has maximum profits for x^* in the $(i-1)^{th}$ interval whereas the other firm has maximum profits for x^* in the i^{th} interval. It is easy to see that $\Pi_k^{i*} \geq \Pi_k^{i-1*}$ is equivalent to $d_{i+1} \leq d_i$ and $\Pi_k^{i*} \leq \Pi_k^{(i-1)*}$ is equivalent to $d_{i+1} \geq d_i \; \forall k \in \{1, 2\}$. So if $x^* = i$ the first order conditions give maximum profits, no matter what d_i and d_{i+1} exactly are. For $x^* = i + 1$ there are again two possibilities and the result is the same.

The existence of an unrestricted LLPE for the game Γ^i is determined completely by the demand distribution. The condition $0 \leq (1-2\mathcal{D}_i)/2d_{i+1} \leq 1$ can be rewritten as $\mathcal{D}_i \leq 1/2$ and $\mathcal{D}_{i+1} \geq 1/2$. Moreover we know that $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$ cannot both be equal to 1/2 because of $d_{i+1} > 0 \; \forall i \in \{0, \ldots, n-1\}$. It is obvious that there will only be an unrestricted LLPE for Γ^i if i satisfies very stringent conditions.

In the following lemma we prove that there exists an unrestricted LLPE for at least one $i \in \{0, \ldots, n-1\}$.

Lemma 3.3 *For each demand distribution* $< d_1, d_2, \ldots, d_n >$ *with* $n \in \mathcal{N}$ *there is at least one* Γ^i *with* $i \in \{0, \ldots, n-1\}$ *for which an unrestricted LLPE exists.*

Proof We prove this by contradiction using an induction argument. Let $\mathcal{V}_i$ be defined as $\{0, \ldots, i\}$ with $i \leq n-1$ and $i \in \mathcal{N}$. Suppose that there is no unrestricted LLPE for Γ^i with $i \in \mathcal{V}_0$. From Corollary 3.1 we know that the condition $0 \leq (1 - 2\mathcal{D}_i)/2d_{i+1} \leq 1$ doesn't hold for $i = 0$ then. With $\mathcal{D}_0 = 0$ this can be rewritten as $0 \leq 1/2d_1 \leq 1$. Clearly $0 \leq 1/2d_1$ always holds, therefore it must be that $1/2d_1 > 1$ or equivalently $d_1 < 1/2$. Next we suppose that there is no unrestricted LLPE for any Γ^i with $i \in \mathcal{V}_1$. If there is no unrestricted LLPE for Γ^1 then the condition $0 \leq (1 - 2\mathcal{D}_1)/2d_2 \leq 1$ doesn't hold. Because there is no unrestricted LLPE for Γ^0 we furthermore know that $d_1 < 1/2$. But then the condition simplifies to $1 - 2d_1 \leq 2d_2$. For this condition not to hold it must be that $d_1 + d_2 < 1/2$. By induction we see that there is no unrestricted LLPE for any Γ^i with $i \in \mathcal{V}_{n-1}$ if and only if $d_1 + d_2 + \cdots + d_n < 1/2$ which contradicts $d_1 + d_2 + \cdots + d_n = 1$. Therefore there exists an unrestricted LLPE for some Γ^i with $i \in \{0, \ldots, n-1\}$.

□

As was noted before the local location-price equilibrium concept is very strong and the question arises if there is possibly a unique unrestricted LLPE, i.e. whether or not there is only one value of i for which there exists an unrestricted LLPE for Γ^i. If there is a unique unrestricted LLPE for Γ^i this is necessarily the unique GLPE for Γ.

Lemma 3.4 *For each demand distribution* $< d_1, d_2, \ldots, d_n >$ *with* $n \in \mathcal{N}$ *there are at most two* Γ^i *with* $i \in \{0, \ldots, n-1\}$ *for which an unrestricted LLPE exists. If there exists an unrestricted LLPE for* Γ^i *and* Γ^j *with* $i, j \in \{0, \ldots, n-1\}$ *and* $i < j$ *then necessarily* $j = i + 1$.

Proof In Lemma 3.3 we proved that there always exists an unrestricted LLPE. So there exists an i such that there is an unrestricted LLPE for some Γ^i. As we have seen this is equivalent to requiring that $\mathcal{D}_i \leq 1/2$ and $\mathcal{D}_{i+1} \geq 1/2$. Now suppose that there also exists a $j > i$ (without loss of generality) such that there exists an unrestricted LLPE for Γ^j, thus $\mathcal{D}_j \leq 1/2$ and $\mathcal{D}_{j+1} \geq 1/2$. But then $\mathcal{D}_{j+1} \geq 1/2 \geq \mathcal{D}_j \geq \mathcal{D}_{i+1} \geq 1/2$ which means that $\mathcal{D}_{i+1}$ must be equal to $1/2$ and j be equal to $i + 1$. It is easy to see that there are at most two equilibria. Suppose to the contrary that there exists a third equilibrium for say Γ^m with $m > j$ then we get $\mathcal{D}_m \geq \mathcal{D}_{j+1} = \mathcal{D}_j + d_{j+1} \geq 1/2 + d_{j+1} > 1/2$ while $\mathcal{D}_m \leq 1/2$ is required.

□

From this lemma it is clear that there can be only two unrestricted LLPEs when the demand distribution is such that for some i it holds that $\mathcal{D}_i = 1/2$. But above we have seen that when x^* is a corner solution then on one of the intervals of which x^* is an endpoint, the equilibrium outcome is dominating unless the demand is equally distributed on both intervals. This leads to the following corollary.

Corollary 3.5 *For each demand distribution* $< d_1, d_2, \ldots, d_n >$ *with* $n \in \mathcal{N}$ *there exists a unique unrestricted LLPE, i.e. there exists exactly one* $i \in \{0, \ldots, n-1\}$ *for which an unrestricted LLPE for* Γ^i *exists, whenever* $\mathcal{D}_i \neq 1/2$. *Otherwise there is an unrestricted LLPE for both* Γ^{i-1} *and* Γ^i.

If there are two unrestricted LLPEs, one for Γ^{i-1} and one for Γ^i, then Lemma 3.2 says that profits are equal whenever $d_i = d_{i+1}$. In case $d_i < d_{i+1}$ then the unrestricted LLPE for Γ^{i-1} yields higher profits for both firms and in case $d_i > d_{i+1}$ then the unrestricted LLPE for Γ^i yields higher profits for both firms. This leads to our main result.

Theorem 3.6 *For each demand distribution* $< d_1, d_2, \ldots, d_n >$ *with* $n \in \mathcal{N}$ *there exists a unique GLPE.*

Proof First we consider the case for which there is a unique unrestricted LLPE. Let $i^* \in \{0, \ldots, n-1\}$ be the value of i for which the unrestricted LLPE for Γ^i exists and denote this equilibrium by $< a^*, b^*, p_1(a,b), p_2(a,b) >$. Of course the conditions for $< a^*, b^*, p_1(a,b), p_2(a,b) >$ to be a GLPE are satisfied then. Next we consider the case for which there is not a unique unrestricted LLPE. In this situation there are two subsequent values of i such that there exists an unrestricted LLPE for Γ^i, say for $i = i^* - 1$ and for $i = i^*$. If $d_{i^*} \neq d_{i^*+1}$ then there is a unique value of i for which profits for both firms are maximized. Otherwise for both values of i profits are maximized. We have to show now that the location and price choices are identical for $i = i^* - 1$ and for $i = i^*$. We will only do this for firm 1, for firm 2 the procedure is analogous. For $i = i^*$ we get from (13) that $a^* = i^* - (1 + 4\mathcal{D}_{i^*})/4d_{i^*+1} = i^* - (1 + 4\mathcal{D}_{i^*-1} + 4d_{i^*})/4d_{i^*} = i^* - 1 - (1 + 4\mathcal{D}_{i^*-1})/4d_{i^*}$, which is also the optimal location of firm 1 for $i = i^* - 1$. Furthermore we see that $p_1(a,b) = 1/3(b-a)(a+b-2i^* + (2 + 2\mathcal{D}_{i^*}/d_{i^*+1}) = 1/3(b-a)(a + b - 2(i^*-1) + (2 + 2\mathcal{D}_{i^*-1}/d_{i^*})$ and analoguosly for $p_2(a,b)$.

□

4 Properties of location-price equilibria

In this section we will discuss some basic properties of location-price equilibria. These are mainly properties related to the degree of differentiation (the distance between the two firms). First we relate the degree of differentiation in a situation of uniformly distributed consumers to a situation of non-uniformly distributed consumers.

Lemma 4.1 *Let $< a^*, b^*, p_1(a,b), p_2(a,b) >$ be an unrestricted LLPE for Γ^i. If $d_{i+1} \geq 1/n$ then $b^* - a^* \leq 3n/2$ and if $d_{i+1} \leq 1/n$ then $b^* - a^* \geq 3n/2$. In particular if the consumers are distributed uniformly (i.e. $d_i = 1/n$ $\forall i \in \{0, \ldots, n-1\}$) then $b^* - a^* = 3n/2$. Furthermore in the uniform case $a^* = -n/4$ and $b^* = 5n/4$.*

Proof Because $< a^*, b^*, p_1(a,b), p_2(a,b) >$ is an unrestricted LLPE for Γ^i we know that a^* and b^* are given by equation (13). But then we can write $b^* - a^* = i + (5 - 4\mathcal{D}_i)/4d_{i+1} - (i - (1 + 4\mathcal{D}_i)/4d_{i+1}) = 3/2d_{i+1}$. If $d_{i+1} \geq 1/n$

then $b^* - a^* \leq 3/2n$ and if $d_{i+1} \leq 1/n$ then $b^* - a^* \geq 3/2n$. So for $d_{i+1} = 1/n$ the equality sign holds. In the uniform case $\mathcal{D}_i$ can be written as $\mathcal{D}_i = i/n$. Substitution in (13) yields $a^* = i - (1+4\mathcal{D}_i)/4d_{i+1} = i - (1+4(i/n))/(4/n) = -n/4$ and $b^* = i + (5 - 4\mathcal{D}_i)/4d_{i+1} = i + (5 - 4(i/n))/(4/n) = 5n/4$.

□

This lemma says that the degree of differentiation only depends on the demand distribution. If demand is concentrated in the centre for example both firms will locate nearer to the centre. If demand is concentrated at the endpoints both firms will locate further away from the centre. In the uniform case we see that it is optimal for both firms to locate outside the city, more precisely at a distance of $n/4$ from the endpoints.

Lemma 4.2 *Let $< a^*, b^*, p_1(a,b), p_2(a,b) >$ be an unrestricted LLPE for Γ^i. Then it holds that $a^* < i \leq x^* \leq i+1 < b^*$, i.e. the indifferent consumer is located to the right of firm 1 and to the left of firm 2.*

Proof From Corollary 3.1 it follows immediately that $a^* < i$ because both $\mathcal{D}_i$ and d_{i+1} are non-negative. Remains to be proved that $i + 1 < b^*$. The other inequalities hold per definition. Rewriting $b^* = i + (5 - 4\mathcal{D}_i)/4d_{i+1}$ as $b^* = i + 1 + (5 - 4\mathcal{D}_i - 4d_{i+1})/4d_{i+1} = i + 1 + (5 - 4\mathcal{D}_{i+1})/4d_{i+1}$ shows that $b^* > i + 1$ because $5 - 4\mathcal{D}_{i+1}$ is non-negative.

□

This lemma formalizes Smithies' (1941) notions of "competetive region" for the region $[a^*, b^*]$ and of "hinterlands" for the region $(-\infty, a^*]$ and $[b^*, \infty)$ with respect to either of the two firms. It is obvious that the size of the competetive region depends crucially on the demand distribution. Nevertheless the demand dispersal is irrelevant for the size of the market areas of the firms as we saw in Lemma 3.2. A natural consequence is that their market areas are equal if the consumers are distributed symmetrically along the line.

Corollary 4.3 *The search for the GLPE for Γ, is equivalent to searching an i for which $\mathcal{D}_i \leq 1/2 \leq \mathcal{D}_{i+1}$. Searching iteratively from $i = 0$ to $i = n-1$ leads to a unique value i^* for i whenever $d_{i^*} \neq d_{i^*+1}$ in case $\mathcal{D}_{i^*} = 1/2$. Otherwise there is an i^* such that both $i = i^* - 1$ and $i = i^*$ satisfy the* *condition $\mathcal{D}_i \leq 1/2 \leq \mathcal{D}_{i+1}$. For both $i = i^* - 1$ and $i = i^*$ the equilibrium locations are $a^* = i - (1 + 4\mathcal{D}_i)/4d_{i+1}$ and $b^* = i + (5 - 4\mathcal{D}_i)/4d_{i+1}$ and the equilibrium prices are $p_1^* = p_2^* = 3/2d_{i+1}^2$.*

Proof This follows immediately from Corollary 3.1 and Corollary 3.5.

□

This corollary is very helpful in determining the equilibria for the two stage location-then-price game. Once we have found an i, starting from zero, for which $\mathcal{D}_i \leq 1/2 \leq \mathcal{D}_{i+1}$ we know the GLPE.

Example 4.4

We will illustrate the search for the equilibria of the two-stage location-then-price game for the situation where $n = 3$. The demand distribution is given then by $< d_1, d_2, d_3 >$ with $d_i > 0 \ \forall i \in \{1, 2, 3\}$. Corollary 3.1 says that there is an unrestricted LLPE for the game Γ^i whenever the condition $0 \leq (1 - 2\mathcal{D}_i)/2d_{i+1} \leq 1$ holds. Writing out this condition for $i = 0$ yields $\mathcal{D}_0 \leq 1/2$, $d_1 > 0$ and $d_1 \geq (1/2)(1 - 2\mathcal{D}_0)$. For $i = 1$ this yields $\mathcal{D}_1 \leq 1/2$, $d_2 > 0$ and $d_2 \geq (1/2)(1 - 2\mathcal{D}_1)$. For $i = 2$ this yields $\mathcal{D}_2 \leq 1/2$, $d_3 > 0$ and $d_3 \geq (1/2)(1 - 2\mathcal{D}_2)$. Substituting $\mathcal{D}_0 = 0$, $d_3 = 1 - d_1 - d_2$, $\mathcal{D}_1 = d_1$ and $\mathcal{D}_2 = d_1 + d_2$ simplifies the conditions to $d_1 \geq 1/2$ for $i = 0$, $d_1 \leq 1/2$ and $d_1 + d_2 \geq 1/2$ for $i = 1$ and $d_1 + d_2 \leq 1/2$ for $i = 2$. In figure 1 we see that there are at most two equilibria. It is furthermore obvious that there is a unique equilibrium for all demand distributions $< d_1, d_2, d_3 >$ with $d_1 \neq 1/2$ and $d_1 + d_2 \neq 1/2$.

First we consider the situation where demand is distributed uniformly, i.e. $< d_1, d_2, d_3 >$ equals $< 1/3, 1/3, 1/3 >$. Then there is a unique unrestricted LLPE for $i = 1$. In equilibrium it holds that $a^* = -3/4$, $b^* = 15/4$ and $p_1^* = p_2^* = 27/2$ according to Corollary 3.1. From (15) we see that x^* equals $3/2$ so equilibrium profits are $\Pi_1^{1*} = \Pi_2^{1*} = 27/4$.

Next we consider a situation where demand is concentrated near the centre. Let $< d_1, d_2, d_3 >$ be equal to $< 1/5, 3/5, 1/5 >$ for example. Again there is a unique unrestricted LLPE for $i = 1$. In equilibrium it holds that $a^* = 1/4$ and $b^* = 11/4$ and $p_1^* = p_2^* = 25/6$. Equilibrium profits are $\Pi_1^{1*} = \Pi_2^{1*} = 25/12$.

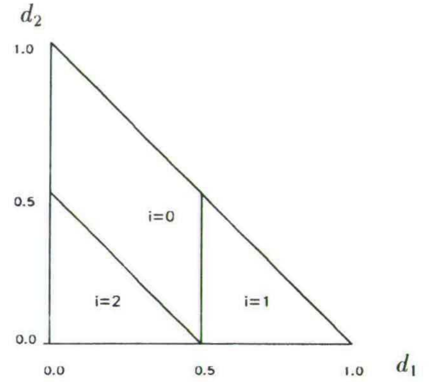

Figure 1: Equilibria for $n = 3$

5 Equilibrium solutions for restricted firms

As Horstmann and Slivinsky (1985) argued the most important feature of the line segment representation is not its linearity but the fact that it posseses endpoints. Due to these endpoints the set of possible solutions is smaller and the search for equilibria should be easier. Nevertheless, as noted before, the equilibrium characterization becomes more difficult because the best responses of the firms now also depend on the restrictions on their locations.

The idea of restricting firms' locations comes from geograhic considerations at the one hand and from the anology with restrictions on the consumers' willingness to pay on the other hand. In a situation of inelastic demand firms have a smaller incentive to differentiate (horizontally) because the probability that demand declines increases. Due to this fact the firms will not choose locations that are further than some arbitrary lower bound or upper bound.

In this section we study the same problem as before, but now with the additional constraints that $a \geq \underline{a}$ and $b \leq \overline{b}$ with $\underline{a}$ a lower bound on firm

1's location choice and $\bar{b}$ an upper bound on firm 2's location choice. We assume that $\underline{a} < \bar{b}$. In particular we are interested in the effects on the equilibrium locations that we have found before. Tirole (1988) shows that the outcome for the uniform case is given by $a^* = 0$ and $b^* = n$ when $\underline{a} = 0$ and $\bar{b} = n$. Recall from Lemma 4.1 that the result for the unrestricted case was $a^* = -n/4$ and $b^* = 5n/4$.

For the restricted case we see that the third term in (10) is positive for $a + 2i + (4 - 2\mathcal{D}_i)/d_{i+1} > 3\bar{b}$ and that the third term in (7) is negative for $b + 2i - (2 + 2\mathcal{D}_i)/d_{i+1} < 3\underline{a}$. We denote $T_i = 2i - (2 + 2\mathcal{D}_i)/d_{i+1}$ and $S_i = 2i + (4 - 2\mathcal{D}_i)/d_{i+1}$. The optimal location choices $a(b)$ and $b(a)$ can be given then by

$$a(b) = \begin{cases} \underline{a} & \text{if } b + T_i \leq 3\underline{a} \\ (b + T_i)/3 & \text{if } b + T_i \geq 3\underline{a} \end{cases} \tag{16}$$

and

$$b(a) = \begin{cases} (a + S_i)/3 & \text{if } a + S_i \leq 3\bar{b} \\ \bar{b} & \text{if } a + S_i \geq 3\bar{b}. \end{cases} \tag{17}$$

From equations (16) and (17) we can determine immediately the Nash equilibrium locations, which will be denoted by a^{**} and b^{**}.

Corollary 5.1 *Let i be such that $\mathcal{D}_i \leq 1/2 \leq \mathcal{D}_{i+1}$ and let $\underline{a}$ and $\bar{b}$ be given. The Nash equilibrium in locations is given then by*

1. $a^{**} = \underline{a}$ *and* $b^{**} = \bar{b}$ *if* $T_i \leq 3\underline{a} - \bar{b}$ *and* $S_i \geq 3\bar{b} - \underline{a}$;

2. $a^{**} = \underline{a}$ *and* $b^{**} = (\underline{a} + S_i)/3$ *if* $S_i + 3T_i \leq 8\underline{a}$ *and* $S_i \leq 3\bar{b} - \underline{a}$;

3. $a^{**} = (\bar{b} + T_i)/3$ *and* $b^{**} = \bar{b}$ *if* $T_i \geq 3\underline{a} - \bar{b}$ *and* $3S_i + T_i \geq 8\bar{b}$;

4. $a^{**} = (S_i + 3T_i)/8$ *and* $b^{**} = (3S_i + T_i)/8$ *if* $S_i + 3T_i \geq 8\underline{a}$ *and* $3S_i + T_i \leq 8\bar{b}$.

The equilibrium characterization is analogous to the unrestricted case and we do not discuss that in detail. For each situation in Corollary 5.1 the equilibrium prices $p_1^{**} = p_1(a^{**}, b^{**})$ and $p_2^{**} = p_2(a^{**}, b^{**})$ can be calculated from equation (4) and the indifferent consumer $x^{**} = x(a^{**}, b^{**})$ can be calculated from (5).

As an illustration we briefly look at the situation where $n = 2$, $\underline{a} = 0$ and $\bar{b} = 2$. This means that firms are not allowed to locate outside the city. First we look at the game Γ^0. With $T_0 = -2/d_1$ and $S_0 = 4/d_1$ we see after some calculation that $a^{**} = 0$ and $b^{**} = 2$ if $1/2 \leq d_1 \leq 2/3$ and $a^{**} = 0$ and $b^{**} = 4/3d_1$ if $2/3 \leq d_1 \leq 1$. Next we look at the game Γ^1. With $T_1 = -4d_1/(1-d_1)$ and $S_1 = (6-4d_1)/(1-d_1)$ we see that $a^{**} = 0$ and $b^{**} = 2$ if $1/3 \leq d_1 \leq 1/2$ and $a^{**} = 2 - 4/3(1-d_1)$ and $b^{**} = 2$ if $0 \leq d_1 \leq 1/3$.

In figure 2 the optimal location choices as a function of the parameter d_1 are depicted for both the restricted case with $\underline{a} = 0$ and $\bar{b} = 2$ and the unrestricted case.

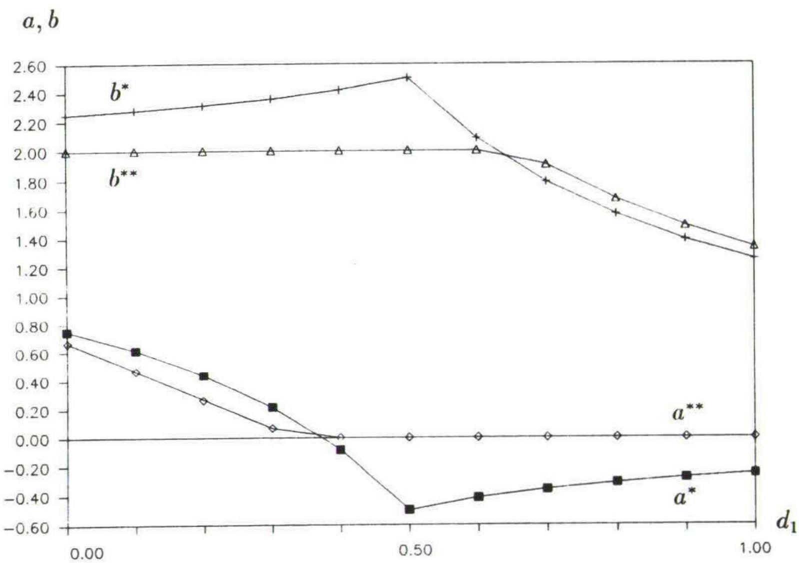

Figure 2: Equilibrium locations of the firms

We see that both firms adjust their strategy when there are restrictions on their locations. If one of the firms is restricted the other firm moves further away in order to avoid strong price competition. Nevertheless restrictions on the firms' locations will always result in a lower degree of differentiation.

Although you might argue that firms being restricted in their location choice will have lower profits because they are more restrained to competition and therefore have smaller "hinterlands", the above results show that the opposite effect, i.e. the other firm moves further away in order to avoid competition, will be dominating if the demand dispersal is large enough.

6 Some concluding remarks

This paper describes how the problem of maximizing non-quasi concave profit functions in two-stage spatial location models can be solved analytically. The advantage of this is that we do not need to make (very) strong assumptions with respect to the demand function and we need not to turn to simulative techniques, which ensures that the impact of demand dispersals can be traced explicitly.

Equilibrium behavior seems to be determined completely by the demand distribution. This results to a 'fair' solution, i.e. both firms set equal prices and face the same demand. In other words, none of the firms is worse of by being labeled firm 1 or firm 2.

To test our analytical results we did some numerical calculations using the well known Newton-Raphson method (see Webers and Webers (1993)). The results confirmed our thoughts that our (simple) analytical results are very useful for people who have to do the same type of calculations, because this numerical method is very time consuming in a situation of strong demand dispersals (and is strongly dependent on the starting values).

The fields of interests that come from this analysis are the following. An important step is to extend the analysis to a two-dimensional space while allowing for more than two firms. Furthermore we feel the need to weaken the assumption of an infinite willingness to pay. In Webers (1993b) a first attempt in this direction is made by showing the effects of a finite willingness to pay in a situation of uniformly distributed consumers. These results need to be generalized for the case of non-uniformly distributed consumers, before being implemented in the two-dimensional framework.

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