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DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

A MARKOV MODEL FOR OPPORTUNITY MAINTENANCE

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FEW 476

A Markov Model for Opportunity Maintenance

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Abstract

The impact of opportunities on the optimal maintenance policy of a Markov-degrading unit is analyzed. The case where preventive maintenance is restricted to opportunities arising from a Poisson process is compared to the situation that the repair facility is continuously available. For both cases it is shown that the optimal policy is of the control limit type and that the average cost is a unimodal function of the control limit. The embedding technique is then applied to develop an efficient optimization procedure. The analysis extends and unifies existing results.

1 Introduction

A basic model in maintenance optimisation is that of a single unit, which is subject to Markov-degradation and can be replaced, either preventively or correctively, by a new one without time delay. In a discrete-time setting this model includes the standard age-replacement model (see Özekiçi [9]). In this paper we consider two practically important extensions. Firstly, we relax the assumption that the state after performing maintenance is as-good-as-new and allow the state to be inferior, however not depending on the state just before maintenance. This will be called the continuous model. Secondly, the assumption that preventive maintenance can start at any time is replaced by the assumption that this is restricted to opportunities arising from a Poisson process, independently of the degradation process. The continuous model can be considered as a limiting case of this model, which we will refer to as the opportunity model. The main purpose of this paper is to analyze the opportunity model and compare it to the continuous model.

To illustrate the importance of opportunities we mention two practical considerations, relating to both the production as well as the maintenance environment. As argued in Dekker and Dijkstra [1], it is often desired for reasons of cost effectiveness that preventive maintenance is carried out at moments at which the system is not required for service, like the epoch of a major overhaul. Furthermore, in case the repair-crew has to maintain several systems, it will often be unavailable due to other maintenance activities with higher priority.

Another practical observation is that maintenance is often imperfect. E.g. in electricity plants performing maintenance may disturb the system, thereby causing a breakdown instead of preventing it. Therefore we allow for a general state-after-repair distribution.

After stating the model in a Markov decision framework in the next section we obtain optimality results in section 3. We prove that the optimal policy is of the control limit type, and that the average cost is a unimodal function of the control limit. The latter proof is established, using the policy improvement procedure. As we believe, this is a new approach to obtain structural results in Markov decision processes. The connection between the continuous and the opportunity model is explained in section 4. Conditions are given, which guarantee that the optimal control limit in the opportunity case is lower than or equal than the optimal control limit in the continuous model. As illustrated by a counterexample, this inequality does not generally hold, when these conditions are not met. In section 5, we present an efficient algorithm to obtain the optimal policy. This algorithm is based on the embedding technique and the optimality results of section 2. We conclude with a brief discussion of two existing models and their relation with our model so as to indicate its flexibility.

Özekiçi [9] shows that under the increasing failure rate (IFR) assumption the optimal policy for the discounted cost criterion is of the control limit type. Hopp and Wu [5] consider the same problem but they allow for a general type of repair, possibly state-dependent, which includes our extension. However, they do not consider opportunities. Actually, opportunity-models are relatively scarse. For a survey of the literature we refer to Dekker and Dijkstra [1] who discuss the age-replacement model extended with opportunities. Their analysis is however not based on Markov-decision theory, which is in our view a nice and flexible tool for analysing maintenance problems. The connection with our paper will be discussed in more detail in the final section. In this paper the opportunity process is supposed to be independent of the degradation process of the unit under consideration. For an example of a system where the opportunity- and the degradation process are dependent we refer to van der Duyn Schouten and Vanneste [3], who consider a two-component system where the replacement of one component constitutes an opportunity for the other.

2 Model and preliminaries

We start with a discription of the continuous model. Consider a single unit, whose condition is described by a state variable, taking on values from the state space

$$S = \{0,1,...,m+1\}$$

State 0 denotes the good condition, states 1 to m are degraded conditions and $m+1$ is the breakdown state. In the absence of maintenance the unit deteriorates according to a continuous-time Markov chain with transition rates $q_{ij} = \lambda_i \, p_{ij}$ $(i,j \in S)$. Transitions are only possible from state i to $i+1$ or $m+1$, so we can write $p_{i,i+1} = \mathrm{p}_i$ and $p_{i,m+1} = 1\text{-}p_i$ $,0 \leq i \leq m$. (Here $p_m := 0$). The following assumption is made throughout the paper:

Assumption 1. (a) $0 < \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_m$ $(<\infty)$

(b) $0 = p_m < p_{m\text{-}1} \leq ... \leq p_0 < 1$ □

Note that $(1\text{-}p_i)\lambda_i$ is the intensity of jumps from i to the failed state. Assumption 1 implies that this intensity increases with i. When the unit fails, i.e. enters state $m+1$, corrective maintenance (CM) is required. As long as this state is not reached, there is the possibility of starting preventive maintenance (PM). Each type of maintenance has its own characteristics, viz. the state after repair, the amount of time associated with it and the cost involved. Costs may comprise e.g. the purchase costs of a new unit and cost due to production losses. The (nonnegative) expected costs are denoted by c_p and c_f respectively, where the subscript refers to the type of repair (p=PM, f=CM). The repairtime distribution associated with preventive maintenance is denoted by $A(\bullet)$, with expectation α, and similarly we use the notation $B(\bullet)$ and β in case of CM. Let Y_p and Y_f , respectively, be the generic random variable denoting the state-after-maintenance, and define for $i \in S$: $a_i := \mathrm{P}(Y_p = i)$ and $b_i := \mathrm{P}(Y_f = i)$. The time needed for preventive maintenance nor the state-after-maintenance depends on the state in which preventive maintenance is initiated. This reflects the case when components have to be replaced, since then the state-after-repair is determined by the quality of the new, not the old, components. Preventive maintenance can be advantageous with respect to each characteristic, but we do not pose any a priori conditions at this point. Actually, it seems to us that the more interesting cases are the ones where advantages in one respect have to be balanced with disadvantages in the other, like PM being less costly but resulting in a higher state-after-maintenance. We say that the state-after-maintenance is better under PM than under CM if it is stochastically smaller (and vice versa), and we denote $Y_p \preceq Y_f$ (see e.g. Ross [10, p.153] for a precise definition).

The decisions to be taken are, for each possible state of the working unit, whether or not to start PM. In view of the exponential nature of the sojourn time in each state, we notice that PM in state 0 need not be considered, and in addition we may assume that PM starts upon entrance of a certain state. Thus, a policy R prescribes for each intermediate state, whether to start PM upon entrance of that state (action 1) or not (action 0). Denote by $X_R^{(c)}(t)$, $t \geq 0$ the state of the working unit at time t under policy R (the superscript c refers to continuous model). For every stationary policy R, the process $\{X_R^{(c)}(t), t \geq 0\}$ constitutes a semi-Markov process on S. Assumption 1b guarantees that there exists only one recurrent class under every policy R, and that the process is aperiodic. We are interested in that stationary policy R that minimizes the long-run average cost $g(R)$. From Markov decision theory it is known that the average optimal policy R^* can be found as the minimizing action in the average cost optimality equation (see Tijms [11]):

$$v(i) = \min_a \{c(i,a) - g\tau(i,a) + \sum_{j \in S} p_{ij}(a) v(j)\}, \; i \in S \qquad (2.1)$$
$$v(m+1) = 0$$

where the minimization is over actions $a \in A(i)$, the action space of state i, and

$p_{ij}(a)$:= the probability that at the next decision epoch the system will be in state j if action a is chosen in the present state i,

$\tau(i,a)$:= the expected time until the next decision epoch if action a is chosen in the present state i, and

$c(i,a)$:= the expected cost incurred until the next decision epoch if action a is chosen in the present state i.

Decision epochs are the moments at which a transition of state occurs. The set of equations (2.1) uniquely determines the relative values $v(i)$, $i \in S$ and the average cost g of the optimal policy. Applied to the continuous model, this results in the following set of equations:

$$
\begin{aligned}
&v(0)=-g\lambda_0^{-1}+p_0v(1)+(1-p_0)v(m+1)\\
&v(i)=\min\{-g\lambda_i^{-1}+p_iv(i+1)+(1-p_i)v(m+1),c_p-g\alpha+\sum_{j=0}^{m+1}a_jv(j)\},\ 1\le i\le m\\
&v(m+1)=c_f-g\beta+\sum_{j=0}^{m+1}b_jv(j)\\
&v(m+1)=0
\end{aligned}
\tag{2.2}
$$

Let us now turn to the modelling of opportunities. Opportunities for preventive maintenance are supposed to arrive according to a Poisson process with rate μ, independently of the state of the unit. When it is difficult to predict the moment of an opportunity in advance, an exponential time between opportunities might be considered. Indeed, this was the approach followed by Jardine and Hassounah [6], who observed during their research on a vehicle-fleet inspection schedule, that deviations from the scheduled inspection intervals were common practice, and they approximated the time between inspections with the geometric distribution (the discrete counterpart of the exponential distribution). When μ tends to infinity, the continuous model appears as a limiting case.

The decision problem is now related to the question: suppose an opportunity presents itself while the unit is in state i ($1\le i\le m$), should the opportunity to perform PM be taken or not? An appropriate way of modelling this is as follows. For each state i ($1\le i\le m$), we distinguish two actions:

action 0 : do not perform PM during the visit to state i

action 2 : start PM at the next opportunity, provided it occurs before the present state is left.

The probability that an opportunity occurs during the visit to state i is obviously equal to $r_i := \mu(\lambda_i+\mu)^{-1}$. Therefore, we have, e.g.

$$p_{i,i+1}(2)=(1-r_i)\,p_i\,+\,r_i\,a_{i+1}$$

$$\tau(i,2)=\,(\lambda_i+\mu)^{-1}\,+\,\mu(\lambda_i+\mu)^{-1}\,\alpha=\,(1-r_i)\lambda_i^{-1}\,+\,r_i\,\alpha$$

$$c(i,2)=r_i\,c_p$$

In terms of the relative values $w(i)$, $i\in S$ and the minimal average cost h, the average-cost optimality equations now yield:

$$
\begin{aligned}
&w(0)=-h\lambda_0^{-1}+p_0w(1)+(1-p_0)w(m+1)\\
&w(i)=\min\{-h\lambda_i^{-1}+p_iw(i+1)+(1-p_i)w(m+1),\\
&\quad(1-r_i)[-h\lambda_i^{-1}+p_iw(i+1)+(1-p_i)w(m+1)]+r_i[c_p-h\alpha+\sum_{j=0}^{m+1}a_jw(j)]\},\ 1\le i\le m \qquad (2.3)\\
&w(m+1)=c_f-h\beta+\sum_{j=0}^{m+1}b_jw(j)\\
&w(m+1)=0
\end{aligned}
$$

Suppose that in the opportunity model also action 1 (start PM upon entrance of a state) would be allowed, then the term in (2.3) associated with action 2 can be considered as a convex combination of the terms related to action 0 and 1, where the weights are given by r_i and $1\text{-}r_i$ respectively.

It is important to note that, although we presented the model in a continuous-time setting, the analysis equally well applies in a discrete-time framework with the following modifications: the unit deteriorates according to a discrete-time Markov chain (equal time intervals between transitions, i.e. $\lambda_i\equiv 1$, $i\in S$) and the time between opportunities is geometrically distributed with parameter r.

We conclude this section with two definitions and a brief discussion of the policy-improvement theorem.

Definition 1 *(Control limit rule) A policy R_k is a control limit rule (CLR) with control limit k, $1\le k\le m+1$, if we have:* $R_k(i)=0$, $i<k$ and $R_k(i)\neq 0$, $i\ge k$ □

This definition applies to both the continuous as well as the opportunity model.

Definition 2 *(Unimodality) A function $f(\bullet)$ on S is unimodal if:*

(i) if $f(i)\le f(i+1)$ then $f(i)\le f(i+k)$ for all $k\ge 2$

(ii) if $f(i)\le f(i\text{-}1)$ then $f(i)\le f(i\text{-}k)$ for all $k\ge 2$ □

(cf. Federgruen and So [4, p.390])

Let us denote the average cost and relative values associated with a fixed policy R by $g(R)$ and $v_R(i)$, $i\in S$. The following theorem is adopted from Tijms [11], p.208:

Theorem 2.1 *(Policy improvement) Suppose that $g(R)$ and $v_R(i)$, $i\in S$ are the average cost and relative values of a stationary policy R. If the stationary policy $\bar{R}$ is such that, for each state $i\in S$,*

$$c(i,\bar{R}(i))-g(R)\tau(i,\bar{R}(i))+\sum_{j\in S}p_{ij}(\bar{R}(i))v_R(j)\leq v_R(i) \tag{2.4}$$

then $$g(\bar{R})\leq g(R).$$

Moreover, the strict inequality holds if (2.4) *holds for each* $i\in S$ *with strict inequality for at least one state which is recurrent under* $\bar{R}$ □

Remark 2.1 The theorem is also true with the inequality signs reversed.

Remark 2.2 The quantities $g(R)$ and $v_R(i)$, $i\in S$ satisfy eq. (2.1), when the action space is restriced to $A(i)=\{R(i)\}$, $i\in S$. Hence, with each policy, we can associate an adapted version of the equations (2.2) resp. (2.3).

For notational convenience we introduce for every $i\in S$ and $a\in A(i)$ and fixed policy R, the policy-improvement quantity:

$$T_R(i,a):=c(i,a)-g(R)\tau(i,a)+\sum_{j\in S}p_{ij}(a)v_R(j) \tag{2.5}$$

3 Optimality results

3.1 Continuous model

Theorem 3.1 *Any solution of the equations* (2.2) *satisfies:*

$$v(i)\leq v(i+1),\ 1\leq i\leq m \tag{3.1}$$

Proof. By induction to the state variable i. For $i=m$, the inequality immediately follows from (2.2): $v(m)\leq -g\lambda_m^{-1}+v(m+1)\leq v(m+1)$ (the costs are nonnegative). Now, suppose (3.1) holds for $i=k,\ k+1,..,m$. Then it follows that $(v(m+1)-v(k+1))\geq(v(m+1)-v(k+2)\geq 0$, which together with assumption 1 yields:

$$\begin{aligned} v(k)&=\min\{-g\lambda_k^{-1}+v(m+1)-p_k(v(m+1)-v(k+1)),c_p-g\alpha+\sum_{j=0}^{m+1}a_jv(j)\}\\ &\leq\min\{-g\lambda_{k+1}^{-1}+v(m+1)-p_{k+1}(v(m+1)-v(k+2)),c_p-g\alpha+\sum_{j=0}^{m+1}a_jv(j)\}=v(k+1) \end{aligned}$$ □

Corollary 3.1 *There exists an optimal policy which is of the control limit type.*
Proof. A direct consequence of Theorem 3.1.

The corollary implies that, in looking for an optimal policy, we may restrict ourselves to control limit rules. Hence it is important, to know whether the average cost is a unimodal function of the control limit. Before we address this question, we introduce an additional assumption in order to avoid technicalities in the subsequent analysis.

Assumption 2 $(a_0+b_0)>0$ and $b_{m+1}<1$.

Indeed, this can be done without loss of generality. For, suppose that $a_j=b_j=0$, $0\le j\le k$. Then the states 0 to k are all transient states under every stationary policy and are irrelevant for the long-run average cost. Therefore, we might as well leave them out of consideration and renumber the states from $k+1$ onwards. Moreover, if $b_{m+1}=1$ then all states except $m+1$ are transient. Assumption 1 and 2 together ensure that for each control limit policy R_i , $1\le i\le m$, the states 0 to i are recurrent and for the CLR R_{m+1} at least state m is recurrent.

Lemma 3.1 (a) $g(R_i)\le g(R_{i+1})$ *iff* $T_{R_i}(i,0)\ge v_{R_i}(i)$, $1\le i\le m$
(b) $g(R_i)\le g(R_{i-1})$ *iff* $T_{R_i}(i\text{-}1,1)\ge v_{R_i}(i\text{-}1)$, $2\le i\le m+1$

Proof. Part (a). Notice that the policies R_i and R_{i+1} differ only with respect to the action prescribed in state i. We have $R_i(j)=R_{i+1}(j)$ for all $j\in S\backslash\{i\}$ and $R_i(i)=1$, $R_{i+1}(i)=0$. Consequently, $T_{R_i}(j,R_{i+1}(j))=v_{R_i}(j)$, $j\in S\backslash\{i\}$. According to the policy-improvement theorem the inequality $T_{R_i}(i,R_{i+1}(i))=T_{R_i}(i,0)\ge v_{R_i}(i)$ then implies that $g(R_{i+1})\ge g(R_i)$. Also, $T_{R_i}(i,0)<v_{R_i}(i)$ implies that $g(R_{i+1})<g(R_i)$, since state i is recurrent under policy R_{i+1}, due to assumption 2. Together these implications establish the equivalence (a). The same reasoning applies to part (b). We have that $R_i(j)=R_{i-1}(j)$ for all $j\in S\backslash\{i\text{-}1\}$ and $R_i(i\text{-}1)=0$, $R_{i-1}(i)=1$ so that $T_{R_i}(j,R_{i-1}(j))=v_{R_i}(j)$, $j\in S\backslash\{i\text{-}1\}$. Now, $T_{R_i}(i\text{-}1,1)\ge v_{R_i}(i\text{-}1)$ implies that $g(R_{i-1})\ge g(R_i)$ and $T_{R_i}(i\text{-}1,1)<v_{R_i}(i\text{-}1)$ implies $g(R_{i-1})<g(R_i)$ (state i-1 is recurrent under policy R_{i-1}). □

Theorem 3.2 *$g(R_i)$ is a unimodal function of* i, $1\le i\le m+1$

Proof. Referring to the definition of unimodality, we have to prove that:

(a) $g(R_i) \leq g(R_{i+1})$ implies $g(R_i) \leq g(R_{i+k})$ for all $k \geq 2$, $1 \leq i \leq m$-1 and

(b) $g(R_i) \leq g(R_{i-1})$ implies $g(R_i) \leq g(R_{i-k})$ for all $k \geq 2$, $2 \leq i \leq m+1$.

In the proof of each part, we distinguish two cases:

(1) $v_{R_i}(m) \leq v_{R_i}(m+1)$ and

(2) $v_{R_i}(m) > v_{R_i}(m+1)$

(a1) Choose i, $1 \leq i \leq m$-1. According to Lemma 3.1, $g(R_i) \leq g(R_{i+1})$ implies $T_{R_i}(i,0) \geq v_{R_i}(i)$. Using Remark 2.1 it is easily verified that $T_{R_i}(i+l,0) \geq v_{R_i}(i+l)$, $0 \leq l \leq k$ implies $g(R_{i+k}) \geq g(R_i)$. Hence, it is sufficient to prove that

$$T_{R_i}(i+k,0) \geq v_{R_i}(i+k),\ k \geq 1 \tag{3.2}$$

Now it follows from the definition of R_i and the average cost equations for a fixed policy (cf. Remark 2.2) that:

$$v_{R_i}(i) = v_{R_i}(i+1) = \ldots = v_{R_i}(m) \tag{3.3}$$

which together with assumption 1 and the fact that $v_{R_i}(m) \leq v_{R_i}(m+1)$ yields:

$$\begin{aligned} T_{R_i}(j,0) &= -g(R_i)\lambda_j^{-1} + v_{R_i}(m+1) - p_j(v_{R_i}(m+1) - v_{R_i}(j)) \\ &\leq -g(R_i)\lambda_{j+1}^{-1} + v_{R_i}(m+1) - p_{j+1}(v_{R_i}(m+1) - v_{R_i}(j+1)) = T_{R_i}(j+1,0),\ i \leq j \leq m-1 \end{aligned} \tag{3.4}$$

So, the numbers $T_{R_i}(j,0)$ constitute an increasing sequence for $i \leq j \leq m$, whereas the numbers $v_{R_i}(j)$, $i \leq j \leq m$ are constant. Together with the fact that $T_{R_i}(i,0) \geq v_{R_i}(\mathrm{i})$ this yields (3.2).

(a2) The inequality $v_{R_i}(m) > v_{R_i}(m+1)$ together with (3.3) implies that:

$$\begin{aligned} T_{R_i}(i,0) &= -g(R_i)\lambda_i^{-1} + p_i v_{R_i}(i+1) + (1-p_i) v_{R_i}(m+1) \\ &\leq p_i v_{R_i}(m) + (1-p_i) v_{R_i}(m+1) < v_{R_i}(m) = v_{R_i}(i) \end{aligned} \tag{3.5}$$

But this contradicts our assumption $g(R_i) \leq g(R_{i+1})$ in view of Lemma 3.1.

(b1) Choose i, $2 \leq i \leq m+1$ and suppose that $g(R_i) \leq g(R_{i-1})$. This implies, according to Lemma 3.1 that $T_{R_i}(i\text{-}1,1) \geq v_{R_i}(i\text{-}1)$. We will show that

$$v_{R_i}(j) \leq v_{R_i}(j+1),\ 0 \leq j < i\text{-}1 \tag{3.6}$$

That is, the values $v_{R_i}(j)$, $0 \leq j \leq i$-1 constitute an increasing sequence, whereas the values $T_{R_i}(j,1)$ are constant. As a consequence, $T_{R_i}(j,1) \geq v_{R_i}(j)$, $0 \leq j < i$ and using Remark 2.1 the desired result then follows. To prove (3.6) we first note that $v_{R_i}(i\text{-}1) \leq v_{R_i}(i)$. This is immediate from the assumptions in case $i = m+1$, and follows from $T_{R_i}(i\text{-}1,1) = v_{R_i}(i)$ in case $i \leq m$. Starting with this inequality, we obtain (3.6) by induction, using the assumption that

$v_{R_i}(m) \leq v_{R_i}(m+1)$ (cf. the proof of (3.1)).

(b2) Suppose $i \leq m$. In this case, the result immediately follows from:

$$v_{R_i}(j) = -g(R_i)\lambda_j^{-1} + p_j v_{R_i}(j+1) + (1-p_j)v_{R_i}(m+1) \leq v_{R_i}(m) = T_{R_i}(j,1),\ 0 \leq j < i \tag{3.7}$$

The inequality in (3.7) can be proved by induction, using $v_{R_i}(m) > v_{R_i}(m+1)$. For the basis of the induction, ($j=i$-1) we use the fact that $v_{R_i}(i) = v_{R_i}(m)$. If $i=m+1$ case (b2) does not apply, since $v_{R_{m+1}}(m) = -g(R_{m+1})\lambda_m^{-1} + v_{R_{m+1}}(m+1) < v_{R_{m+1}}(\mathrm{m}+1)$. □

3.2 Opportunity model

Theorem 3.3 *Any solution of the equations* (2.3) *satisfies*:

$$w(i) \leq w(i+1),\ 1 \leq i \leq m \tag{3.8}$$

Proof. For $i=m$ the result follows immediately (see the proof of Theorem 3.1), so let us assume that (3.8) holds for $i=k+1, k+2, .., m$. We have to show that $w(k) \leq w(k+1)$. For ease of notation we introduce

$$w(PM) := c_p - h\alpha + \sum_{j=0}^{m+1} a_j w(j) \tag{3.9}$$

and rewrite (2.3) into:

$$\begin{aligned} w(i) = \min\{ & -h\lambda_i^{-1} + w(m+1) - p_i(w(m+1) - w(i+1)), \\ & (1-r_i)[-h\lambda_i^{-1} + w(m+1) - p_i(w(m+1) - w(i+1))] + r_i\ w(PM)\} \end{aligned} \quad 1 \leq i \leq m \tag{3.10}$$

Since, for $p \in [0,1]$ and $x, y \in \mathbb{R}$,

$$px + (1-p)y \geq y \quad iff \quad x \geq y \tag{3.11}$$

the first term in de RHS of (3.10) is greater (smaller) than the second term if and only if the first term is greater (smaller) than $w(PM)$. From the induction hypothesis and assumption 1 we have:

$$\begin{aligned} & (-h\lambda_k^{-1} + w(m+1) - p_k(w(m+1) - w(k+1))) \\ & \leq (-h\lambda_{k+1}^{-1} + w(m+1) - p_{k+1}(w(m+1) - w(k+2)) \end{aligned} \tag{3.12}$$

and

$$1-r_k=1-\frac{\mu}{\lambda_k+\mu}\leq 1-\frac{\mu}{\lambda_{k+1}+\mu}=1-r_{k+1} \tag{3.13}$$

We distinguish two cases. First suppose that

$$(-h\lambda_{k+1}^{-1}+w(m+1)-p_{k+1}(w(m+1)-w(k+2)))\leq w(PM) \tag{3.14}$$

Then (3.10), (3.12) and (3.14) imply:

$$\begin{aligned} w(k)&\leq(-h\lambda_k^{-1}+w(m+1)-p_k(w(m+1)-w(k+1)))\\ &\leq(-h\lambda_{k+1}^{-1}+w(m+1)-p_{k+1}(w(m+1)-w(k+2)))\leq w(PM) \end{aligned} \tag{3.15}$$

From (3.15) we conclude, in view of (3.11):

$$\begin{aligned} w(k)&=(-h\lambda_k^{-1}+w(m+1)-p_k(w(m+1)-w(k+1)))\\ &\leq(-h\lambda_{k+1}^{-1}+w(m+1)-p_{k+1}(w(m+1)-w(k+2))=w(k+1) \end{aligned} \tag{3.16}$$

Next, suppose that the opposite inequality holds in (3.14). Then we conclude from (3.10), (3.11), (3.12) and (3.13):

$$\begin{aligned} w(k)&\leq w(PM)+(1-r_k)\big([-h\lambda_k^{-1}+w(m+1)-p_k(w(m+1)-w(k+1))]-w(PM)\big)\\ &\leq w(PM)+(1-r_{k+1})\big([-h\lambda_{k+1}^{-1}+w(m+1)-p_{k+1}(w(m+1)-w(k+2)]-w(PM)\big)=w(k+1) \end{aligned} \tag{3.17}$$

This completes the induction argument. □

Corollary 3.2 *There exists an optimal policy which is of the control limit type.*
Proof. Immediate from Theorem 3.2.

Remark 3.1 The control limit concept has more meaning in the opportunity model than in the continuous model, since in the presence of opportunities, the decision to do PM at the next opportunity in state i may not be implemented, and it is conceivable that when the opportunity finally occurs, the system has arrived in state $j>i$, and PM might not be optimal in state j. This situation cannot occur in the continuous model, where a PM action is immediately carried out.

Remark 3.2 By adding a state *PM* to the decision process, representing the situation that preventive maintenance is actually being carried out, we can model the start of a preventive maintenance action in state i by a transition from i to state *PM*. The action in *PM* is fixed:

start immediately with maintenance. We then have the relation:

$$w(i)=\min\{-h\lambda_i^{-1}+p_iw(i+1)+(1-p_i)w(m+1),\\ -h\frac{1}{\lambda_i+\mu}+r_i\ w(PM)+(1-r_i)(p_iw(i+1)+(1-p_i)w(m+1))\}\ \ 1\le i\le m \tag{3.18}$$

Rewriting (3.18) yields (3.10). So, we can interpret (3.9) as the relative value corresponding to state PM, when the state space S is augmented with $\{PM\}$. In the continuous model a preventive maintenance action can be started upon entrance of a certain state, so we obtain:

$$v(i)=\min\{-g\lambda_i^{-1}+p_iv(i+1)+(1-p_i)v(m+1),\ v(PM)\}\ \ 1\le i\le m \tag{3.19}$$

with

$$v(PM)=c_p-g\alpha+\sum_{j=0}^{m+1}a_jv(j) \tag{3.20}$$

Clearly, this procedure can also be applied under a fixed policy.

The proof of the following theorem proceeds along the same lines as in the continuous case, but due to the opportunities the arguments are more intricate. Therefore the proof is deferred to appendix A.

Theorem 3.4 $h(R_i)$ is a unimodal function of i, $1\le i\le m+1$.

4 Relation between continuous and opportunity model

First we introduce some additional notation. We will refer to the continuous model as model C, and denote by K_c the optimal control limit for this model (if there are more than one, the smallest). Similarly, the opportunity model is indicated by model O, and the optimal CLR by K_o. Recall that g is the minimal average cost in model C and h in model O.

Theorem 4.1 $g \le h$

Proof. Model C and O only differ with respect to the decision structure. A general model, that comprises both model C and O, is obtained by considering 4 possible actions: 0 (do nothing); 1 (start PM); 2 (start PM at the next opportunity, provided it occurs before the present state is left); 3 (start CM), and the action space $A(0)=\{0\}$, $A(i)=\{0,1,2\}$, $1\leq i\leq m$, and $A(m+1)=\{3\}$. The average cost optimality equation in terms of the average cost f and the relative values $z(i)$, $i\in S$ yields:

$$\begin{aligned}
&z(0)=-f\lambda_0^{-1}+p_0z(1)+(1-p_0)z(m+1)\\
&z(i)=\min\{-f\lambda_i^{-1}+p_iz(i+1)+(1-p_i)z(m+1),\ c_p-f\alpha+\sum_{j=0}^{m+1}a_jz(j),\\
&\quad(1-r_i)[-f\lambda_i^{-1}+p_iz(i+1)+(1-p_i)z(m+1)]+r_i[c_p-f\alpha+\sum_{j=0}^{m+1}a_jz(j)]\},1\leq i\leq m \qquad (4.1)\\
&z(m+1)=c_f-f\beta+\sum_{j=0}^{m+1}b_jz(j)\\
&z(m+1)=0
\end{aligned}$$

This generalized model will be called model G and its policies $R^{(g)}$. By restricting the action space on the intermediate states to $\{0,1\}$ resp. $\{0,2\}$ we obtain the models C and O as special cases. Now, choose a policy $R^{(g)}$ for model G with $R^{(g)}\in\{0,2\}$ such that it is optimal for model O. Then, clearly, $f(R^{(g)})=h(R^{(g)})=h$. Suppose this policy never chooses action 2, then it is also a feasible policy for model C and we obtain: $g\leq g(R^{(g)})=f(R^{(g)})=h$. Next, suppose it chooses action 2 in at least one state i. Because the corresponding term in (4.1), the third one, is a convex combination of the first two terms, it follows that one of these terms is lower than or equal to the third, so the action can be improved. From Theorem 2.1 we conclude that an improved policy can be constructed by replacing all actions of type 2 by either 0 or 1, whichever is the best. This improved policy for model G, which we denote by $\bar{R}^{(g)}$, is now a feasible policy for model C, so we obtain:

$$g\leq g(\bar{R}^{(g)})=f(\bar{R}^{(g)})\leq f(R^{(g)})=h$$

□

Proposition 4.1 $g(R_{m+1})=h(R_{m+1})$ *and* $v_{R_{m+1}}(i)=w_{R_{m+1}}(i)$, $i\in S$

Proof. The policy R_{m+1} assigns action 0 on all intermediate states. Therefore $(g(R_{m+1}),v_{R_{m+1}}(i))$ and $(h(R_{m+1}),w_{R_{m+1}}(i))$ are solutions of the same set of equations.

Proposition 4.2 $K_c=m+1$ *iff* $K_o=m+1$

Proof. Suppose that $K_c=m+1$. This means that $g(R_m)>g(R_{m+1})$, which implies that $T_{R_{m+1}}(m,1)>v_{R_{m+1}}(m)$ (by the same reasoning as in Lemma 3.1(b)). So:

$$(c_p - g(R_{m+1})\alpha + \sum_{j=0}^{m+1} a_j v_{R_{m+1}}(j)) > (-g(R_{m+1})\lambda_m^{-1} + v_{R_{m+1}}(m+1)) \tag{4.2}$$

By Proposition 4.1, (4.2) holds equally well in terms of h and w. Using (3.11), we then obtain the inequality $T_{R_{m+1}}(m,1) > w_{R_{m+1}}(m)$ for the opportunity case, or $h(R_m) > h(R_{m+1})$. In view of the unimodality of $h(R_i)$ we conclude that $K_o = m+1$. The proof of the other implication is similar.

□

Theorem 4.2 *If* *(i)* $a_0 = 1$, *or*

(ii) $a_0 + a_{m+1} = b_0 + b_{m+1} = 1$

Then $K_o \leq K_c$

Proof. Recall that K_c resp. K_o are the *smallest* optimal control limits. In view of Proposition 4.2, there is nothing to prove in case $K_c = m+1$ or $K_c = m$. So we may assume that $K_c \leq m-1$ and also that $K_o \leq m$. It is convenient to consider the decision process, augmented with the state PM (see Remark 3.2), and to set the relative values for this state equal to zero. That is, we put $v(PM) = w(PM) = 0$ instead of $v(m+1) = w(m+1) = 0$, as before. The following two claims show that the assumption $K_o > K_c$ is contradictory to the conditions (i) or (ii). Claim (a): $K_o > K_c$ implies that $w(0) < v(0)$. Claim (b): conditions (i) and (ii) of the theorem imply $w(0) \geq v(0)$. We first prove Claim (a). We have:

$$\begin{aligned} v(i) &= v(PM),\ K_c \leq i \leq m \\ v(PM) &\leq -g\lambda_i^{-1} + p_i v(i+1) + (1-p_i)v(m+1),\ K_c \leq i \leq m \\ w(i) &\geq w(PM),\ K_o \leq i \leq m \\ w(PM) &> -h\lambda_{K_0-1}^{-1} + p_{K_0-1} w(K_0) + (1-p_{K_0-1})w(m+1) \end{aligned} \tag{4.3}$$

The first equation follows from (3.19). The second equation is easily verified for $i = K_c$ (suppose the inequality does not hold, then action 1 in K_c can be improved and K_c is not optimal, according to Theorem 2.1) and the result then follows for $i > K_c$ since the RHS increases with i (cf. the proof of Theorem 3.1). Similarly, $w(K_o) < w(PM)$ would contradict the optimality of K_o, which yields the third equation for $i = K_o$, and the result for $i > K_o$ then follows upon application of Theorem 3.3. Finally, the fourth equation is valid, for if it were not true, K_o would not be the smallest optimal control limit. From (4.3) and $K_o > K_c$ we obtain:

$$-h\lambda_{K_o-1}^{-1}+p_{K_o-1}w(K_o)+(1-p_{K_o-1})w(m+1)<w(PM)= \\ = v(PM)\leq -g\lambda_{K_o-1}^{-1}+p_{K_o-1}v(K_o)+(1-p_{K_o-1})v(m+1) \tag{4.4}$$

or,

$$(h-g)\lambda_{K_o-1}^{-1} > p_{K_o-1}(w(K_o)-v(K_o)) + (1-p_{K_o-1})(w(m+1)-v(m+1)) \tag{4.5}$$

From (4.3) we know that $w(K_o)\geq w(PM)=v(PM)=v(K_o)$. Hence, (4.6) below, holds for $i=K_o$-1. When $w(m+1)\geq v(m+1)$ we obtain (4.6) for all i by using Assumption 1, and in the opposite case (4.6) is immediate, since $(h-g)\geq 0$ (Theorem 4.1):

$$(h-g)>\lambda_i(1-p_i)(w(m+1)-v(m+1)),\ 0\leq i\leq K_o-1 \tag{4.6}$$

We are now able to prove by induction that $w(j)<v(j)$, $0\leq j\leq K_o$-1, which particularly proves (a). To establish the inequality for $j=K_o$-1 we note that $w(K_o\text{-}1)<w(PM)=v(PM)=v(K_o\text{-}1)$ according to (4.3). Next, suppose $w(j)\leq v(j)$ for $j=k+1$ ($<K_o$-1), then the induction hypothesis together with (4.6) yields:

$$w(k)=p_k w(k+1)+(-h\lambda_k^{-1}+(1-p_k)w(m+1)) < p_k v(k+1)+(-g\lambda_k^{-1}+(1-p_k)v(m+1))=v(k) \tag{4.7}$$

which establishes the result for $j=k$. Proof of Claim (b). If $a_0=1$ then we have according to (3.10) and (3.20) that $w(PM)=c_p\text{-}h\alpha+w(0)$ and $v(PM)=c_p\text{-}g\alpha+v(0)$. Since $g\leq h$ and $v(PM)=w(PM)$, we conclude that $w(0)\geq v(0)$. A similar reasoning applies to condition (ii). From $v(PM)=c_p\text{-}g\alpha+a_0v(0)+a_{m+1}v(m+1)$ and $v(m+1)=c_f\text{-}g\beta+b_0v(0)+b_{m+1}v(m+1)$ we obtain

$$v(PM)=(c_p+Hc_f)-g(\alpha+H\beta)+(a_0+Hb_0)v(0)$$
$$where \quad H:=a_{m+1}(1-b_{m+1})^{-1}$$

and similar expressions for $w(PM)$ and $w(m+1)$. Again, this leads to $w(0)\geq v(0)$. □

Notice that Theorem 4.2 holds without any restrictions on the costs c_p, c_f and repair-time distributions $A(\bullet)$, $B(\bullet)$. Numerical experiments support our conjecture that $K_o\leq K_c$ is ensured by the more general condition that $Y_p\leq Y_f$, but we were not able to prove this. That the inequality does not generally hold is illustrated by the following

Counterexample: Choose $m=14$; $a_0=0.4$; $a_{11}=a_{12}=a_{13}=0.2$; $b_0=1$; $c_p=1$, $c_f=20$; $\alpha=\beta=0$; $\mu=0.25$; $\lambda_i=1$, $i\in S$. Then $K_c=2$ and $K_o=5$. To give an impression of the average cost function, we present its value for three different control limits:

$g(R_1)=5.89$ $\quad g=g(R_2)=5.58$ $\quad g(R_{15})=6.18$

$h(R_1)=6.47$ $\quad h=h(R_5)=6.16$ $\quad h(R_{15})=6.18$ □

Indeed, it does not seem unrealistic to assume in practical situations that CM is, although much more costly, better with repect to the state-after-maintenance. The example reveals a counterintuitive property of opportunity maintenance.

5 Optimization algorithm

In contrast with the preceding sections, which primarily focussed on theoretical issues, this section addresses the computational aspects. Exploiting the special structure of the problem, we are able to develop an efficient optimization procedure. This procedure consists of two parts: an iterative search procedure within the space of control limit policies, leading to the optimal control limit rule, and a method to compute the average costs for a fixed policy in each iteration. The latter method is based on the embedding technique, whereas the search procedure relates to the optimality results, obtained in section 2.

Assume for the moment that the average costs and relative values of any CLR can be efficiently calculated. In view of the unimodality, a simple bisection procedure as in Federgruen and So [4, p.391] can now be applied to find the optimal policy. This procedure only requires the average cost of a CLR in each iteration. Alternatively, a search procedure can be based on the policy-improvement quantity (cf. van der Duyn Schouten and Vanneste [3]). This procedure requires in each step the average costs and relative values for the present strategy to construct an improved policy, which forms the initial policy for the next iteration. Thus a sequence of improved policies is constructed, which eventually yields the optimal policy, in view of the unimodality of the average cost function. A step in this adapted version of the general policy-improvement step proceeds as follows. Suppose R_k is the initial policy, for which the relative values and average cost have been determined. By comparing the policy-improvement test quantity with the relative values, it is easily established whether $g(R_{k-1})<g(R_k)$ or $g(R_{k+1})<g(R_k)$ (if the opposite inequality holds in both cases, or equality in either of these cases, then we may conclude from the unimodality results that R_k

already is an optimal policy). Next, suppose that $T_{R_k}(k\text{-}1,1)<v_{R_k}(k\text{-}1)$ (or $g(R_{k-1})<g(R_k)$) then we put $l:=k\text{-}1$ and we continue lowering the control limit l als long as $T_{R_k}(l,1)<v_{R_k}(l)$. This yields an improved policy R_l. An analogous procedure applies for the case $T_{R_k}(k,0)<v_{R_k}(k)$ (or $g(R_{k+1})<g(R_k)$).

Now we turn to the computation of the average cost and relative values for a fixed CLR R_l. The method will be presented for the opportunity case, but the analysis similarly applies to the other case. The quantities $(h(R_l),\ w_{R_l}(i))$ satisfy the equations:

$$\begin{aligned}
&w_{R_l}(i)=-h(R_l)\lambda_i^{-1}+p_i w_{R_l}(i+1)+(1-p_i)w_{R_l}(m+1),\ 0\le i<l\\
&w_{R_l}(i)=(1-r_i)[-h(R_l)\lambda_i^{-1}+p_i w_{R_l}(i+1)+(1-p_i)w_{R_l}(m+1)]+r_i\ w_{R_l}(PM),\ l\le i\le m\\
&w_{R_l}(PM)=c_p-h(R_l)\alpha+\sum_{j=0}^{m+1}a_j w_{R_l}(j)\\
&w_{R_l}(m+1)=c_f-h(R_l)\beta+\sum_{j=0}^{m+1}b_j w_{R_l}(j)\\
&w_{R_l}(m+1)=0
\end{aligned}\tag{5.1}$$

where we use the auxiliary state PM again (see Remark 3.2). In addition, we will identify state $m+1$ with CM. As in section 2, we denote by $X_{R_l}^{(o)}(t)$ the state of the unit at time t. From the semi-Markov process $\{X_{R_l}^{(o)}(t),\ t\ge 0\}$ we derive the embedded process $\{Y_{R_l}(t),\ t\ge 0\}$ where $Y(t):=1$ if the last maintenance activity on or before t was PM and $Y(t):=2$ in case of CM. The process $\{Y_{R_l}(t),\ t\ge 0\}$ is another semi-Markov process on the embedded state space $E=\{1,2\}$. Let us denote the associated relative values by $w^E(i)$ $(=w_{R_l}{}^E(i))$, $i=1,2$, and the average costs by h^E $(=h^E(R_l))$. These quantities satisfy the equations:

$$\begin{aligned}
&w^E(1)=c_1^E-h^E\tau_1^E+p_{11}^E w^E(1)+p_{12}^E w^E(2)\\
&w^E(2)=c_2^E-h^E\tau_2^E+p_{21}^E w^E(1)+p_{22}^E w^E(2)\\
&w^E(2)=0
\end{aligned}\tag{5.2}$$

where c_i^E $(=c_i^E(R_l))$, τ_i^E $(=\tau_i^E(R_l))$, p_{ij}^E $(=p_{ij}^E(R_l))$ represent the expected cost, time and transition probabilities of the embedded process. According to Tijms [11, p.230] we have that $w_{R_l}(PM)=w^E(1)$, $w_{R_l}(CM)=w^E(2)$ and $h(R^l)=h^E$. Therefore, by solving (5.2) we obtain:

$$\begin{aligned}
&h(R_l)=\frac{p_{21}^E c_1^E+p_{12}^E c_2^E}{p_{21}^E\tau_1^E+p_{22}^E\tau_2^E}\\
&w_{R_l}(PM)=\frac{c_1^E\tau_2^E-c_2^E\tau_1^E}{p_{12}^E\tau_2^E+p_{21}^E\tau_1^E}
\end{aligned}\tag{5.3}$$

It is now an easy matter to solve (5.1) from (5.3). Using (5.3) we obtain $w_{R_l}(m)$ directly from

(5.1) and by proceeding downwards with i we find all relative values for $i \in S$ by single-step calculations in a recursive way.

What remains is to find expressions for c_i^E, p_i^E and τ_i^E. To that end, we analyze the absorbing Markov chain $\{Z_{R_l}(t), t \geq 0\}$ obtained from the process $\{X_{R_l}^{(o)}(t), t \geq 0\}$ by converting PM and CM into absorbing states. Let us now define κ_j $(=\kappa_j(R_l))$ and σ_j $(=\sigma_j(R_l))$ for $0 \leq j \leq m+1$ as follows:

κ_j:= probability of absorption into state PM from initial state j

σ_j:= mean time until absorption (either in PM or CM} starting from state j.

Then it can be verified that, for example:

$$c_1^E = c_p$$

$$p_{11}^E = \sum_{j=0}^{m} a_j \kappa_j$$

$$\tau_1^E = \alpha + \sum_{j=0}^{m} a_j \sigma_j$$

and similar expressions for the other quantities. The following theorem gives recursive relations by which the numbers κ_j and σ_j for $0 \leq j \leq m$ can be easily calculated.

Theorem 5.3 *The quantities* κ_j and σ_j, $0 \leq j \leq m$, *satisfy the relations:*

(i) $\kappa_j = r_j + (1-r_j)p_j \kappa_{j+1}$, $l \leq j \leq m$

(ii) $\kappa_j = p_j \kappa_{j+1}$, $0 \leq j < l$

(iii) $\sigma_j = (1-r_j)(\lambda_j^{-1} + p_j \sigma_{j+1})$, $l \leq j \leq m$

(iv) $\sigma_j = \lambda_j^{-1} + p_j \sigma_{j+1}$, $0 \leq j < l$

(Here $\sigma_{m+1} = 0$ and $\kappa_{m+1} = 0$) □

Proof. By conditioning on the epoch of first transition of $\{Z_{R_l}(t), t \geq 0\}$ (cf. Karlin and Taylor [7, p.148]). □

This completes the description of the embedding procedure.

As a final remark, we mention that the analysis of the C-model results in the same expressions as above when $\mu = +\infty$ (or $r_j = 1$) is substituted.

6 Two special cases

The analysis presented in the previous sections extends and unifies existing results from the literature. In this section we will particularly focuss on two known models and relate them to ours. It was already noted in the introduction that the basic model includes the standard age-replacement model as a special case. An age replacement policy with parameter T prescribes to replace a component with lifetime distribution $F(\bullet)$, when it has failed or reached the age T, whichever occurs first. Dekker and Dijkstra [1] consider this model in a continuous-time setting, and extend it with opportunity-based replacements. Using methods from classical analysis, they analyse the average-cost function as a function of T. Although their approach is quite different from ours, the results are very much in agreement. The discretized version of their model is identical to our model with the following specifications: $a_0=b_0=1$; $\alpha=\beta=0$; $\lambda_i\equiv 1$, $p_i=(1\text{-}F(i+1))/(1\text{-}F(i))$, $i\in S$. For numerical results, we refer to their paper. In particular, it is observed that a pretty high cost ratio c_f/c_p is needed to obtain a significant reduction in the value of the optimal control limit when opportunities are taken into account. Our own numerical investigations confirm this conclusion.

By imposing an appropriate cost structure, the model can also be applied to study availability issues. Kawai [8] e.g., considers the availability of a two-unit parallel system with a single repair facility (see also van der Duyn Schouten and Ronner [2]). In short, their system is described as follows. Initially, their are two identical units, one of which is in working condition and the other in (cold) standby position. While functioning, the operating unit gradually deteriorates, whereas the cold standby remains as good as new. When the working unit goes under repair (either PM or CM), the standby takes over its position and we return to the initial situation as soon as the repair is completed. The system is unavailable when the working unit has failed, while the other is still under repair. Our single unit model can be used to describe this system if we introduce a super-unit, which comprises a working unit and a standby unit. If one unit is under repair then we say that the superunit is under repair, otherwise its state is equal to the state of the working unit. Thus, after completion of a repair, the superunit has state i, whenever the unit in working position has state i (possibly $m+1$). The parameters a_i and c_p, e.g., now depend on the repair time distribution. Recall from section 2 that the deterioration of the working unit in the absence of maintenance can be described by a continuous-time Markov chain on S with absorbing state $m+1$. Let us denote this process by $\{D(t),\ t\geq 0\}$ and define:

$$H_{ij}(t):=P(D(t)=j\,|\,D(0)=i) \qquad (6.1)$$

Then it can be verified that:

$$
\begin{aligned}
a_i &= \int_0^\infty H_{0i}(t)dA(t) \\
c_p &= \int_0^\infty H_{0m+1}(t)(1-A(t))dt
\end{aligned}
\qquad (6.2)
$$

The repair cost now represents the expected unavailability during the repair. We note that Kawai allows for a more general transition mechanism, namely from each state to all higher states, but he does not consider opportunities. Refering to the O-model, it will be clear that this is easily incorporated. The equivalence between the C-model and the Kawai model is immediately clear from the optimality equations presented in the paper of Kawai (eq. (11)-(13)) and ours (eq. (2.2)). In contrast with the article of Kawai, we do not need any conditions on the repair time distribution to prove unimodality. Finally, we note that the computation of the quantities as in (6.1) is an easy matter when analytical expressions for the Laplace transforms of $A(\bullet)$, $B(\bullet)$ are available. The procedure is given in the appendix and generalizes the results obtained by van der Duyn Schouten and Ronner [2].

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Appendix A Proof of Theorem 3.4

Proof. The proof of the unimodality of $h(R_i)$ closely follows the line of argument in the proof of Theorem 3.2 and we will therefore concentrate on the parts that deviate from this proof. We will frequently use Lemma 3.1, which equally well applies to the opportunity model, in terms of h and w. The values $(h(R_i), w_{R_i}(j))$ are a solution of the set (5.1) (note that we again use the additional state PM, see Remark 3.2). We distinguish case (a) and (b) as in proof 3.2 and furthermore case (1) and (2) according to:

(1) $w_{R_i}(PM) \leq w_{R_i}(m+1)$

(2) $w_{R_i}(PM) > w_{R_i}(m+1)$

Notice that $w_{R_i}(PM)$ plays the role of $v_{R_i}(m)$ in proof 3.2. Indeed, $v_{R_i}(m) = v_{R_i}(PM)$ if $i \leq m$.

(a1) Choose i, $1 \leq i \leq m$-1. It follows from $h(R_i) \leq h(R_{i+1})$ that $T_{R_i}(i,0) \geq w_{R_i}(i)$ or, by (3.11),

$$w_{R_i}(i) \geq w_{R_i}(PM) \tag{A.1}$$

It suffices to prove that:

$$w_{R_i}(j) \geq w_{R_i}(PM),\ i+1 \leq j \leq m \tag{A.2}$$

We prove (A.2) by induction. First, the case $j=m$. Suppose to the contrary that

$w_{R_i}(m)<w_{R_i}(PM)$, or, in view of (3.11):

$$-h(R_i)\lambda_m^{-1}+w_{R_i}(m+1)<w_{R_i}(PM) \tag{A.3}$$

Rewriting yields (A.4) below for $k=m$ (note that $p_m=0$). By assumption 1 and using the assumption that $w_{R_i}(PM)<w_{R_i}(m+1)$, we obtain

$$h(R_i)>\lambda_k(1-p_k)(w_{R_i}(m+1)-w_{R_i}(PM)),\ i\le k\le m \tag{A.4}$$

or, equivalently,

$$-h(R_i)\lambda_k^{-1}+p_k w_{R_i}(PM)+(1-p_k)w_{R_i}(m+1)\ <\ w_{R_i}(PM),\ i\le k\le m \tag{A.5}$$

Starting with $w_{R_i}(m)<w_{R_i}(PM)$, and using (A.5) it follows by induction that

$$\begin{aligned}&(i)\ \ -h(R_i)\lambda_k^{-1}+p_k w_{R_i}(k+1)+(1-p_k)w_{R_i}(m+1)<\\&\quad<-h(R_i)\lambda_k^{-1}+p_k w_{R_i}(PM)+(1-p_k)w_{R_i}(m+1)\ \ \mathit{and}\qquad (i\le k\le m)\\&(ii)\ \ w_{R_i}(k)<w_{R_i}(PM)\end{aligned} \tag{A.6}$$

In particular, $w_{R_i}(i)<w_{R_i}(PM)$, which contradicts our first conclusion, (A.1). Hence, $w_{R_i}(m)\ge w_{R_i}(PM)$. Next, suppose (A.2) holds for $k=l+1,..,m$ $(l\ge i)$. Then, we show that $w_{R_i}(l)\ge w_{R_i}(PM)$, again by contradiction. For, suppose to the contrary, that $w_{R_i}(l)<w_{R_i}(PM)$. Using the induction hypothesis $w_{R_i}(l+1)\ge w_{R_i}(PM)$, it can be verified that

$$\begin{aligned}&-h(R_i)\lambda_l^{-1}+p_l w_{R_i}(PM)+(1-p_l)w_{R_i}(m+1)\\&\quad\le -h(R_i)\lambda_l^{-1}+p_l w_{R_i}(l+1)+(1-p_l)w_{R_i}(m+1)\le w_{R_i}(PM)\end{aligned} \tag{A.7}$$

where the latter inequality follows from the induction hypothesis and (3.11). Eq. (A.7) yields (A.8) below, for $j=l$. By assumption 1, we obtain :

$$h(R_i)\ge\lambda_j(1-p_j)(w_{R_i}(m+1)-w_{R_i}(PM)),\ j\le l \tag{A.8}$$

Starting with $w_{R_i}(l)<w_{R_i}(PM)$ we arrive at $w_{R_i}(i)<w_{R_i}(PM)$ by the same reasoning as in (A.3)-(A.6), which yields a contradiction with (A.1). Hence, $w_{R_i}(l)\ge w_{R_i}(PM)$ which completes the induction step, and thus the proof of (A.2).

(a2) Analogously to proof 3.2, we can show that the assumption $w_{R_i}(PM)>w_{R_i}(m+1)$ is in contradiction with $h(R_i)\le h(R_{i+1})$. The former assumption implies that $w_{R_i}(j)<w_{R_i}(PM)$, $i\le j\le m$, which can be established by induction: For $j=m$ this inequality is easily verified, and

furthermore $w_{R_i}(j+1)<w_{R_i}(PM)$ implies:

$$w_{R_i}(j)=(1-r_i)[-h(R_i)\lambda_i^{-1}+p_j w_{R_i}(j+1)+(1-p_j)w_{R_i}(m+1)]+r_i\ w_{R_i}(PM)<w_{R_i}(PM) \qquad \text{(A.9)}$$

(b1) The proof of part (b) is based on the relation

$$-h(R_i)\lambda_j^{-1}+p_j w_{R_i}(j+1)+(1-p_j)w_{R_i}(m+1)\le w_{R_i}(PM),\ j\le i-1 \qquad \text{(A.10)}$$

Note that the LHS equals $w_{R_i}(j)$, $j\le i\text{-}1$. From $h(R_i)\le h(R_{i-1})$ we have: $T_{R_i}(i\text{-}1,1)\ge w_{R_i}(i\text{-}1)$ or $w_{R_i}(PM)\ge w_{R_i}(i\text{-}1)$. Eq. (A.10) now easily follows by induction in case $i=m+1$ (cf. (A.3)-(A.6)). Suppose $i\le m$. Now, $w_{R_i}(PM)\le w_{R_i}(m+1)$ implies $w_{R_i}(m)\le w_{R_i}(m+1)$. Proceeding downwards with k $(k\ge i)$ we obtain $w_{R_i}(k)\le w_{R_i}(k+1)$ as long as $w_{R_i}(k+1)\ge w_{R_i}(PM)$ (cf. (3.14) and (3.17)). Should we have $w_{R_i}(k)\le w_{R_i}(PM)$ at a certain stage k (whereas $w_{R_i}(k+1)$-$\ge w_{R_i}(PM)$), then we obtain from

$$\begin{aligned} &h(R_i)\lambda_k^{-1}+p_k w_{R_i}(PM)+(1-p_k)w_{R_i}(m+1) \\ &\le h(R_i)\lambda_k^{-1}+p_k w_{R_i}(k+1)+(1-p_k)w_{R_i}(m+1)\le w_{R_i}(PM) \end{aligned} \qquad \text{(A.11)}$$

that $w_{R_i}(i)\le w_{R_i}(PM)$, for all $i\le k$, (cf. (A.4) and (A.5)), which establishes (A.10). In the other case, i.e. $w_{R_i}(k)\ge w_{R_i}(PM)$ for all $k\ge i$, we particularly have $w_{R_i}(i)\ge w_{R_i}(PM)$, so $w_{R_i}(i\text{-}1)(\le w_{R_i}(PM))\le w_{R_i}(i)$, which provides the start for an inductive proof of $w_{R_i}(j)\le w_{R_i}(j+1)$, $j\le i\text{-}1$, which yields (A.10) (cf. (3.12); use the fact that $w_{R_i}(j)\le w_{R_i}(m+1)$ for all j, which is easily proved).

(b2) Eq. (A.10) now follows directly by induction, starting with $w_{R_i}(i\text{-}1)\le w_{R_i}(PM)$ and using $w_{R_i}(PM)\ge w_{R_i}(m+1)$. □

Appendix B Recursive schemes for the Kawai model

We present an efficient procedure to compute the quantities $\{a_i\}_{i\in S}$ and c_p, as specified for the Kawai-model (see (6.2)). When the Laplace-transform of the repair time distribution $A(\bullet)$ is explicitly known, the recursive schemes given below should quickly yield a solution. The procedure similarly applies to the computation of $\{b_i\}_{i\in S}$ and c_f.

Define:

$$\tilde{A}(s):=\int_0^\infty e^{-sx}A(x)dx$$
$$A_{ij}:=\int_0^\infty H_{ij}(t)(1-A(t))dt \qquad (B.1)$$

(with $H_{ij}(t)$ as defined in (6.1))

Suppose, as in the Kawai model, that the sequence $\{\lambda_i\}_{i=0}^{m}$ is strictly increasing and that transitions to lower states are impossible. It follows from eq.(4) in Kawai (1981) and the definition of $H_{ij}(t)$ that:

$$H_{ii}(t)=e^{-\lambda_i t}$$
$$H_{ij}(t)=(\lambda_j-\lambda_i)^{-1}\left(\sum_{k=i}^{j-1} q_{kj}H_{ik}(t)-\sum_{k=i+1}^{j} q_{ik}H_{kj}(t)\right),\ 0\leq i<j\leq m \qquad (B.2)$$
$$H_{im+1}(t)=1-\sum_{k=i}^{m} H_{ik}(t)$$

From (B.1) and (B.2) we have:

$$A_{ii}=\lambda_i^{-1}-\tilde{A}(\lambda_i)$$
$$A_{ij}=(\lambda_j-\lambda_i)^{-1}\left(\sum_{k=i}^{j-1} q_{kj}A_{ik}-\sum_{k=i+1}^{j} q_{ik}A_{kj}\right),0\leq i<j\leq m \qquad (B.3)$$
$$A_{im+1}(t)=\alpha-\sum_{k=i}^{m} A_{ik}$$

(Note that $A_{ij}=0$ when $j<i$)

From these equations we can recursively solve for A_{ij}, $0\leq i\leq j\leq m+1$. In particular, we obtain A_{0i} , $0\leq i\leq m+1$. Using the relations between a_i and A_{0i}, $0\leq i\leq m+1$, given by Kawai (eq. (14)-(16)) and noting that $c_p=A_{0m+1}$, we obtain the desired quantities.

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