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THE ALGEBRAIC RICCATI EQUATION AND SINGULAR OPTIMAL CONTROL: THE DISCRETE-TIME CASE

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THE ALGEBRAIC RICCATI EQUATION AND SINGULAR OPTIMAL CONTROL: THE DISCRETE-TIME CASE*

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Abstract We consider a *general* infinite-horizon linear-quadratic control problem, with *arbitrary* stability constraints, subject to a standard *discrete-time* system. In particular, we derive *necessary and sufficient* conditions for the *existence of the optimal cost*, and we characterize this optimal cost as a certain solution of the associated *linear matrix inequality*. This solution turns out to satisfy the corresponding (possibly *singular*) *algebraic Riccati equation*, and thus we can establish a *map* from all possible stability constraints to the set of positive semidefinite solutions of this equation. As a by-result, we present a *necessary and sufficient* condition for the *existence* of a *positive semidefinite* solution of the *general* Riccati equation. Finally, we derive *necessary and sufficient* conditions for the *existence* of *optimal controls* if the underlying discrete-time system is *left invertible*, and these optimal controls turn out to be implementable by a *unique feedback law*.

Keywords Infinite-horizon linear-quadratic control, discrete-time system, arbitrary stability constraints, regularity and singularity, linear matrix inequality, algebraic Riccati equation, left invertibility, feedback law.

1. Introduction and preliminaries.

Consider the standard discrete-time system Σ

$$x(i+1) = Ax(i) + Bu(i), y(i) = Cx(i) + Du(i), \tag{1}$$

together with the additional output variable

$$z(i) = Sx(i) + Tu(i), \tag{2}$$

where, for all $i \geq 0, u(i) \in \mathbf{R}^m, x(i) \in \mathbf{R}^n$ and $x(0) = x_0, y(i) \in \mathbf{R}^r$ and $z(i) \in \mathbf{R}^s$. All matrices involved are real and constant. Also, for every x_0 and every control sequence $u = \{u(i)\}_{i=0}^{\infty}$, we define the function

$$J(x_0, u) := \Sigma_{i=0}^{\infty} y'(i)y(i). \tag{3}$$

Then we are interested in the *Linear-Quadratic Control Problem with z-stability* $(LQCP)_z$: For all x_0, determine

$$J_z(x_0) := \inf \{J(x_0, u) | u \text{ is such that } lim_{i\to\infty} z(i) = 0\}, \tag{4}$$

and if, for every $x_0, J_z(x_0) < \infty$, then compute (if possible) for every x_0 a control sequence $\bar{u}$ such that $J_z(x_0) = J(x_0, \bar{u})$ and $lim_{i\to\infty} z(i) = 0$. The problem is called *regular* if $\ker (D) = 0$ and *singular* if $\ker (D) \neq 0$. If $\ker (S) = 0$ and $T = 0$, then $(\text{LQCP})_z$ will be called the LQCP *with* (state) *stability*, and if $s = 0$, then we will speak of the LQCP *without stability*. Regular as well as singular cases for these two problems are discussed in the lengthy [1], by means of Silverman's structure algorithm. In the present paper, we will treat $(\text{LQCP})_z$ with S and T *arbitrary, regardless* whether D is left invertible or not, by a more direct algebraic approach. In Section 2 we will derive *necessary and sufficient* conditions for *existence* of the *optimal cost* (1.4). Finally, in Section 3 we will present *necessary and sufficient* conditions for *existence* of *optimal controls* if the underlying system Σ is *left invertible* [1, p. 351], i.e., if $\rho :=$ normal rank $(T(z)) = m$, with $T(z) := D + C(zI - A)^{-1}B$ [1, p. 350].

We will need a few well-known observations, as well as some new statements. Assume that there exists a matrix $M_z \geq 0$ such that, for all $x_0 \in \mathbf{R}^n, J_z(x_0) \leq x_0' M_z x_0$. Then there *exists* a unique $K_z \in \mathbf{R}^{n\times n}, K_z \geq 0$, such that, for all $x_0, J_z(x_0) = x_0' K_z x_0$ [2, Lemma 5]. Next, let

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$i \geq 1$, and assume that $u(j)$ $(j = 0, 1, ..., (i-1))$ are given. Then the function $J_z : \mathbf{R}^n \to \mathbf{R}^+$ satisfies the *Dissipation Inequality* (e.g. [2, Lemma 1])

$$x_0' K_z x_0 \leq \Sigma_{j=0}^{i-1} y'(j)y(j) + x'(i)K_z x(i), \tag{5}$$

$$\text{and } x_0' K_z x_0 = \inf\{\Sigma_{j=0}^{i-1} y'(j)y(j) + x'(i)K_z x(i) | u(j), j = 0, ..., (i-1)\} \tag{6}$$

$$\text{If, moreover, } P(K) := \begin{bmatrix} C'C + A'KA - K & C'D + A'KB \\ D'C + B'KA & D'D + B'KB \end{bmatrix}, \text{ with } K = K' \in \mathbf{R}^{n\times n}, \tag{7}$$

$$\text{then } \Sigma_{j=0}^{i-1} y'(j)y(j) + x'(i)Kx(i) - x_0' K x_0 = \Sigma_{j=0}^{i-1} [x'(j)\ u'(j)] P(K) \begin{bmatrix} x(j) \\ u(j) \end{bmatrix}. \tag{8}$$

(Sketchy proof for (1.8): Take $i = 1$. Then (1.8) is clear form (1.1) and (1.7). Next, take $i = 2$. Then the left hand-side of (1.8) is equal to $\{y'(1)y(1) + x'(2)Kx(2) - x'(1)Kx(1)\} + \{y'(0)y(0) + x'(1)Kx(1) - x_0' K x_0\} = [x'(1)\ u'(1)]P(K)] \begin{bmatrix} x(1) \\ u(1) \end{bmatrix} + [x'(0)\ u'(0)]P(K) \begin{bmatrix} x(0) \\ u(0) \end{bmatrix}$, etc.) Combination of (1.5) with (1.8) now yields that the right hand-side of (1.8) for $K = K_z$ is positive semidefinite, and thus, by taking $i = 1$ and realizing that x_0 as well as $u(0)$ are arbitrary, we find that

$$P(K_z) \geq 0, \tag{9}$$

$$\text{and hence } K_z \in \Gamma := \{K \in \mathbf{R}^{n\times n} | K = K', P(K) \geq 0\}, \tag{10}$$

the solution set of the *Linear Matrix Inequality* (LMI). If, in addition,

$$\phi(K) := C'C + A'KA - K - (C'D + A'KB)(D'D + B'KB)^+(D'C + B'KA) \tag{11}$$

(where N^+ denotes the Moore-Penrose inverse of the matrix N), and

$$F(K) := -(D'D + B'KB)^+(D'C + B'KA)\ (K = K' \in \mathbf{R}^{n\times n}), \tag{12}$$

then $K \in \Gamma \Leftrightarrow \phi(K) \geq 0, D'D + B'KB \geq 0, -(D'D + B'KB)F(K) = D'C + B'KA$ (Schur's Lemma), and rank $(P(K))$ = rank $(\phi(K))$+ rank $(D'D + B'KB)$. Hence, for every $\bar{x} \in \mathbf{R}^n, \bar{u} \in \mathbf{R}^m$ and every $K \in \Gamma$,

$$[\bar{x}'\bar{u}']P(K) \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \bar{x}'\phi(K)\bar{x} + [\bar{u}' - \bar{x}'F'(K)][D'D + B'KB][\bar{u} - F(K)\bar{x}] \tag{13}$$

and thus, for every x_0 and every $i \geq 1$, (1.6) transforms by (1.8) into

$$\begin{aligned} \inf\{\Sigma_{j=0}^{i-1}[x'(j)\phi(K_z)x(j) + [u'(j)\ \ & -x'(j)F'(K_z)][D'D + B'K_zB][u(j) - \\ & F(K_z)x(j)]] | u(j), j = 0, ..., (i-1)\} = 0. \end{aligned} \tag{14}$$

We establish that Γ contains the matrix that represents J_z, the optimal cost for $(\text{LQCP})_z$, provided that J_z is bounded from above by a quadratic form. Furthermore

Proposition 1.1.

$\phi(K_z) = 0$, rank $(P(K_z))$ = rank $(D'D + B'K_zB)$.

Proof. Take $i = 1$ in (1.14). Then the infimum is attained for $u(0) = F(K_z)x_0$. It follows that $x_0'\phi(K_z)x_0 = 0$ for every x_0.

Lemma 1.2.

Assume that $K \in \Gamma$. Then rank $(P(K)) \geq \rho$. If $P(K) = \begin{bmatrix} C_K' \\ D_K' \end{bmatrix} [C_K\ D_K]$, with $[C_K\ D_K]$ right invertible and $T_K(z) := D_K + C_K(zI - A)^{-1}B$, then rank $(P(K)) = \rho$ if and only if $T_K(z)$ is right invertible as a rational matrix.

Proof. Follows directly by appropriately rewriting some proofs in [3].

Corollary 1.3.

Let $K \in \Gamma$ and $\phi(K) = 0$. Then rank $(P(K))$ = rank $(D'D + B'KB) = \rho$.

Proof. It is clear that $C_K'(I - D_K(D_K'D_K)^+D_K')C_K = \phi(K)$ in Lemma 1.2 and since $\phi(K) = 0$, it follows that D_K is right invertible. Hence, by Lemma 1.2, rank $(D'D + B'KB)$ = rank

$(P(K)) = \rho$.

Corollary 1.4.

If $\rho = m$, then $D'D + B'K_z B > 0$.

Proposition 1.5.

Let $x_0 \in \mathbf{R}^n, u = \{u(i)\}_{i=1}^{\infty}$ be such that $z(i) \to 0 (i \to \infty)$, and set $v(i) := u(i) - F(K_z)x(i), i \geq 0$. Then $\Sigma_{i=0}^{\infty} v'(i)[D'D + B'K_z B]v(i) < \infty \Leftrightarrow J(x_0, u) < \infty$. In addition, if $J(x_0, u) < \infty$, then

$$J(x_0, u) = \Sigma_{i=0}^{\infty} v'(i)[D'D + B'K_z B]v(i) + x_0' K_z x_0. \tag{15}$$

Proof. If $J(x_0, u) < \infty$, then $x'(i)K_z x(i) \leq \Sigma_{j=i}^{\infty} y'(j)y(j)$, by definition and time-invariancy (!) and hence $x'(i)K_z x(i) \to 0 (i \to \infty)$. Thus, by (1.8), (1.13) and Proposition 1.1, we get (1.15). Conversely, if $\Sigma_{i=0}^{\infty} v'(i)[D'D + B'K_z B]v(i) < \infty$, then, again by (1.8), (1.13) and Proposition 1.1, $J(x_0, u)$ cannot be infinite, as $x'(i)K_z x(i) \geq 0$ for all i. Thus, $J(x_0, u) < \infty$ and hence, as $z(i) \to 0, x'(i)K_z x(i) \to 0 (i \to \infty)$, and we have (1.15).

Corollary 1.6.

Assume in Proposition 1.5 that $\rho = m$. Then $J(x_0, u) = x_0' K_z x_0 \Leftrightarrow u(i) = F(K_z)x(i)$ for every $i \geq 0$.

Proof. By Corollary 1.4, $D'D + B'K_z B > 0$. Now apply Proposition 1.5.

It follows from Corollary 1.6 that if for every x_0 an *optimal* control sequence for $(\text{LQCP})_z$ exists, then this sequence can be given in terms of a feedback law, and this feedback law is *unique*, if $\rho = m$. For more details, see Section 3.

We will close this preliminary Section with the following, partly new, algebraic results. Let $\Gamma_0 \subset \Gamma$ denote the set of solutions of the algebraic Riccati equation (ARE):

$$\Gamma_0 := \{K \in \Gamma | \phi(K) = 0\}. \tag{16}$$

Let, further, for any $G \in \mathbf{R}^{m \times n}$ and any $K = K' \in \mathbf{R}^{n \times n}$,

$$A_G := A + BG, C_G := C + DG, S_G := S + TG, \tag{17}$$

$$\phi_G(K) := C_G' C_G + A_G' K A_G - K - (A_G' K B + C_G' D)(D'D + B'KB)^+ (B'KA_G + D'C_G). \tag{18}$$

Proposition 1.7.

(a) If $K \geq 0$, then $\phi(K) \geq 0 \Leftrightarrow \phi_G(K) \geq 0$ and $\phi(K) = 0 \Leftrightarrow \phi_G(K) = 0$.

(b) If $K \in \Gamma_0$, then $K = A_{F(K)}' K A_{F(K)} + C_{F(K)}' C_{F(K)}$.

(c) If $D'D + B'KB$ is invertible, then $A_{F(K)} = A_G - B(D'D + B'KB)^{-1}(D'C_G + B'KA_G)$. If, moreover, $0 \leq K \in \Gamma_0$, then $D'C_{F(K)} + B'KA_{F(K)} = 0$.

Proof. If $K \in \Gamma$ and $K \geq 0$, then $P(K) \geq 0 \Leftrightarrow \phi(K) \geq 0$ ad $\phi(K) = 0 \Leftrightarrow$ rank $(P(K)) =$ rank $(D'D + B'KB)$. If $P_G(K)$ stands for (1.7) with A_G and C_G instead of A and C, then, obviously, $P_G(K) \geq 0 \Leftrightarrow P(K) \geq 0$, and thus we establish (a). For (b), see [1, (14)]. Next, if $D'D + B'KB$ is invertible, then $(D'D + B'KB)^{-1}(D'C_G + B'KA_G) + F(K) = G$. Finally, by (a), $\phi(K) = 0 \Leftrightarrow \phi_{F(K)}(K) = 0$, and the proof of (c) is then completed by applying (b).

2. Necessary and sufficient conditions for the existence of the optimal cost.

In the sequel, $< A|im(B) >= im(B) + Aim(B) + ... + A^{n-1}im(B), <$ ker $(C)|A >=$ ker $(C) \cap$ ker $(CA) \cap ... \cap$ ker $(CA^{n-1}), \mathcal{X}_s(A)$ denotes the *stable* subspace of A [6, Ch. VI] and $\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}$ is the *system matrix* of Σ. If $A(\mathcal{L}_1) \subset \mathcal{L}_1, A(\mathcal{L}_2) \subset \mathcal{L}_2$ and $\mathcal{L}_1 \subset \mathcal{L}_2$, then $\sigma(A|\mathcal{L}_2/\mathcal{L}_1)$ denotes the spectrum of the quotient map induced by A on $\mathcal{L}_2/\mathcal{L}_1$, the quotient space of $\mathcal{L}_2$ over $\mathcal{L}_1$ (for maps and matrices we use the same symbols).

Definition 2.1 [1, Section III].

Let $\mathcal{V} = \mathcal{V}(\Sigma)$ denotes the space of points $x_0 \in \mathbf{R}^n$ for which there exist $u(i)$ $(i \geq 0)$ such that, for all $i \geq 0, y(i) = 0$. Let, moreover, $\mathcal{V}_z = \mathcal{V}_z(\Sigma)$ denote the space of points $x_0 \in \mathbf{R}^n$ for which there exist $u(i)$ $(i \geq 0)$ such that, for all $i \geq 0, y(i) = 0$ and $z(i) = 0$.

Proposition 2.2.

$\mathcal{V}$ is the largest subspace $\mathcal{L}$ for which there exists a map $G : \mathbf{R}^n \to \mathbf{R}^m$ such that $(A+BG)\mathcal{L} \subset \mathcal{L}, (C + DG)\mathcal{L} = 0$. If $G \in \mathcal{G}(\Sigma) := \{H : \mathbf{R}^n \to \mathbf{R}^m | (A_H)\mathcal{V} \subset \mathcal{V}, (C_H)\mathcal{V} = 0\}$, then $\mathcal{V} =<$

ker $(C+DG)|A+BG>.\mathcal{V}_z$ is the largest subspace $\mathcal{L}$ for which there exists a map $G:\mathbf{R}^n\to\mathbf{R}^m$ such that $(A+BG)\mathcal{L}\subset\mathcal{L},(C+DG)\mathcal{L}=0,(S+TG)\mathcal{L}=0$. If $G\in\mathcal{G}_z(\Sigma):=\{H:\mathbf{R}^n\to\mathbf{R}^m|(A_H)\mathcal{V}_z\subset\mathcal{V}_z,(C_H)\mathcal{V}_z=0,(S_G)\mathcal{V}_z=0\}$, then $\mathcal{V}_z=<\ker\begin{bmatrix} C+DG \\ S+TG\end{bmatrix}|A+BG>$. Furthermore, $\mathcal{G}_z(\Sigma)\cap\mathcal{G}(\Sigma)\neq\emptyset$.

Proof. All claims, except for the last one, are in [1, Section III]. Next, let $G\in\mathcal{G}_z(\Sigma)\neq\emptyset$. Then the map $G|\mathcal{V}_z$ can be extended on $\mathcal{V}$ in such a way that the resulting extension, $\bar{G}:\mathcal{V}\to\mathbf{R}^m$, is such that $(A+B\bar{G})\mathcal{V}\subset\mathcal{V},(C+D\bar{G})\mathcal{V}=0$ (e.g. [4]). If $\tilde{G}$ is an arbitrary extension of $\bar{G}$ on $\mathbf{R}^n$, then $\tilde{G}\in\mathcal{G}_z(\Sigma)\cap\mathcal{G}(\Sigma)$.

Now let $G\in\mathcal{G}_z(\Sigma)\cap\mathcal{G}(\Sigma)$, then, with $v(i)=u(i)-Gx(i)$, (1.1)-(1.2) transform into

$$x(i+1)=A_Gx(i)+Bv(i),y(i)=C_Gx(i)+Dv(i), \tag{1}$$

$$z(i)=S_Gx(i)+Tv(i). \tag{2}$$

Suppose that $\mathcal{X}_2$ is such that $\mathcal{V}_z\oplus\mathcal{X}_2=\mathcal{V}$, and that $\mathcal{X}_3$ is such that $\mathcal{V}\oplus\mathcal{X}_3=\mathbf{R}^n$. Moreover, let $[\ker(\begin{bmatrix} D \\ T\end{bmatrix})\cap B^{-1}(\mathcal{V}_z)]\oplus\mathcal{U}_2=[\ker(D)\cap B^{-1}(\mathcal{V})]$, and let $[\ker(D)\cap B^{-1}(\mathcal{V})]\oplus\mathcal{U}_3=\mathbf{R}^m$. Then, with respect to suitably chosen bases, (2.1) - (2.2) transform into

$$\begin{bmatrix} x_1(i+1) \\ x_2(i+1) \\ x_3(i+1)\end{bmatrix}=\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{A}_{13} \\ 0 & \bar{A}_{22} & \bar{A}_{23} \\ 0 & 0 & \bar{A}_{33}\end{bmatrix}\begin{bmatrix} x_1(i) \\ x_2(i) \\ x_3(i)\end{bmatrix}+\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 0 & B_{22} & B_{23} \\ 0 & 0 & B_{33}\end{bmatrix}\begin{bmatrix} v_1(i) \\ v_2(i) \\ v_3(i)\end{bmatrix}, \tag{3a}$$

$$y(i)=[0\ 0\ \bar{C}_3]\begin{bmatrix} x_1(i) \\ x_2(i) \\ x_3(i)\end{bmatrix}+[0\ 0\ D_3]\begin{bmatrix} v_1(i) \\ v_2(i) \\ v_3(i)\end{bmatrix}, \tag{3b}$$

$$z(i)=[0\ \bar{S}_2\ \bar{S}_3]\begin{bmatrix} x_1(i) \\ x_2(i) \\ x_3(i)\end{bmatrix},+[0\ T_2\ T_3]\begin{bmatrix} v_1(i) \\ v_2(i) \\ v_3(i)\end{bmatrix}, \tag{4}$$

with $\ker(B_{33})\cap\ker(D_3)=0$ and $\ker(\begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33}\end{bmatrix})\cap\ker(\begin{bmatrix} 0 & D_3 \\ T_2 & T_3\end{bmatrix})=0$, *by construction.* Note that $y(i)$ is generated by the subsystem for $x_3(i)$, whereas $y(i)$ and $z(i)$ *jointly* are generated by the subsystem for $x_2(i)$ *and* $x_3(i)$. These subsystems are *strongly observable* [1, p. 344] *by construction*, i.e., the associated *system matrices* are of full column rank for every $s\in\mathbf{C}$ [1, Section III], [5]. Thus, these subsystems are *strongly detectable* [1, p. 354], i.e., if $y(i)\to 0$, then $x_3(i)\to 0$, and if $\begin{bmatrix} y(i) \\ z(i)\end{bmatrix}\to\begin{bmatrix} 0 \\ 0\end{bmatrix}$, then $\begin{bmatrix} x_2(i) \\ x_3(i)\end{bmatrix}\to\begin{bmatrix} 0 \\ 0\end{bmatrix}(i\to\infty)$, *irrespective of inputs and initial states* [1, Section III], [5]. This key observation leads us to the first main result.

Theorem 2.3

$$\forall x_0\in\mathbf{R}^n: J_z(x_0)<\infty\Leftrightarrow\mathcal{X}_s(A)+<A|im(B)>+\mathcal{V}_z=\mathbf{R}^n. \tag{5}$$

Assume this to be the case. Then there exists a unique matrix $K_z\in\Gamma_0$ such that, for all $x_0, J_z(x_0)=x_0'K_zx_0$ and $K_z\mathcal{V}_z=0$. In addition, if $K\in\Gamma, K\mathcal{V}_z=0$, then $K\leq K_z$.

Proof. Let $x_0\in\mathbf{R}^n, J(x_0,u)<\infty$ and $z(i)\to 0(i\to\infty)$ for some u. Then, also, $y(i)\to 0(i\to\infty)$. Now, consider (2.3)-(2.4). From the foregoing, then, $\begin{bmatrix} x_2(i) \\ x_3(i)\end{bmatrix}\to\begin{bmatrix} 0 \\ 0\end{bmatrix}(i\to\infty)$, i.e., the Euclidean distance between $x(i)$ and $\mathcal{V}_z$ converges to zero as i tends to infinity. As, for every $x_0\in\mathcal{V}_z, J_z(x_0)=0$, we establish that $J_z(x_0)<\infty$ for *every* $x_0\in\mathbf{R}^n$ *only if*

$$(\begin{bmatrix} \bar{A}_{22} & \bar{A}_{23} \\ 0 & \bar{A}_{33}\end{bmatrix},\begin{bmatrix} B_{22} & B_{23} \\ 0 & B_{33}\end{bmatrix})\text{ is stabilizable} \tag{6}$$

[6, Ch. VI]. Assume this to be the case. Then there *exists* a feedback $\begin{bmatrix} v_2(i) \\ v_3(i) \end{bmatrix} = \begin{bmatrix} H_{22} & H_{23} \\ H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} x_2(i) \\ x_3(i) \end{bmatrix}$ such that the resulting closed-loop matrix for $x_2(i)$ and $x_3(i)$ has all its eigenvalues *within the unit circle* [6, Ch. VI]. Therefore there *exists* a matrix $M_z \geq 0$, with $\mathcal{V}_z \subset \ker(M_z)$, such that, for all $x_0, J_z(x_0) \leq x_0' M_z x_0$, and hence there also *exists* a unique $K_z \geq 0$ such that, for all $x_0, J_z(x_0) = x_0' K_z x_0$ [2, Lemma 5], and $K_z \mathcal{V}_z = 0$. From Proposition 1.1, then, $K_z \in \Gamma_0$. On the other hand, if V_z denotes a basis matrix for $\mathcal{V}_z$, then, obviously, by the Hautus test, (2.6) $\Leftrightarrow (A_G, [B\ V_z])$ is stabilizable $\Leftrightarrow \mathcal{X}_s(A) + < A|im([B\ V_z]) >= \mathbf{R}^n$ [6, Ch. VI] $\Leftrightarrow \mathcal{X}_s(A) + < A|im(B) > + \mathcal{V}_z = \mathbf{R}^n$, since, $< A|im([B\ V_z]) >=< A|im(B) > + \mathcal{V}_z + A(\mathcal{V}_z) + ... + A^{n-1}(\mathcal{V}_z) =< A|im(B) > + \mathcal{V}_z$, as $(A + BG)\mathcal{V}_z \subset \mathcal{V}_z$. Finally, assume that (2.5) holds and let $K \in \Gamma, K\mathcal{V}_z = 0$. If $J(x_0, u) < \infty$ and $z(i) \to 0 (i \to \infty)$, then $x_2(i) \to 0$ and $x_3(i) \to 0 (i \to \infty)$ in (2.3)-(2.4), and hence $x'(i) K x(i) \to 0 (i \to \infty)$. Thus, from (1.8), $x_0' K_z x_0 = J_z(x_0) \geq x_0' K x_0$ and $K_z \geq K$. This completes the proof.

Definition 2.4.

Σ is *output stabilizable* if $\mathcal{X}_s(A) + < A|im(B) > + \mathcal{V} = \mathbf{R}^n$.

Corollary 2.5.

Let J_+, J_- denote the optimal costs for the LQCP with and without stability, respectively. For every $x_0 \in \mathbf{R}^n, J_+(x_0) < \infty$ if and only if Σ is stabilizable. If this is the case, for all $x_0, J_+(x_0) = x_0' K_+ x_0$ with $K_+ \in \Gamma_0$ and, if $K \in \Gamma$, then $K \leq K_+$. For every $x_0 \in \mathbf{R}^n, J_-(x_0) < \infty$ if and only if Σ is output stabilizable. If this is the case, then, for all $x_0, J_-(x_0) = x_0' K_- x_0$ with $K_- \in \Gamma_0$, $\ker(K_-) = \mathcal{V}$, and, if $K \in \Gamma$ and $K\mathcal{V} = 0$, then $K \leq K_-$. Moreover, $u(i) = F(K_-)x(i)\ (i \geq 0)$ is optimal.

Proof. If $\ker(S) = 0$ and $T = 0$, then $\mathcal{V}_z = 0$; if $s = 0$, then $\mathcal{V}_z = \mathcal{V}$. Now, combine Theorem 2.3 with Proposition 1.5.

Corollary 2.6.

$\{K \in \Gamma_0 | K \geq 0\} \neq \emptyset \Leftrightarrow \mathcal{X}_s(A) + < A|im(B) > + \mathcal{V} = \mathbf{R}^n$.

Proof. $\Rightarrow$ let $\phi(K) = 0, K \geq 0$. Then, by (1.8), (1.13), $J(x_0, u) \leq x_0' K x_0$ if, for all $i \geq 0$ we take $u(i) = F(K)x(i)$. Now, apply Corollary 2.5. $\Leftarrow$ Corollary 2.5.

It should be stressed that Theorem 2.3 determines K_z *unambiguously* in terms of solutions of Γ; if $\tilde{K} \in \Gamma, \tilde{K}\mathcal{V}_z = 0$ and $K \leq \tilde{K}$ if $K \in \Gamma, K\mathcal{V}_z = 0$, then $K_z \leq \tilde{K} \leq K_z$. In words, one can say that K_z is the *largest element* of $\{K \in \Gamma | K\mathcal{V}_z = 0\}$. Yet, $K_z \in \Gamma_0$ and K_z corresponds to $\begin{bmatrix} 0 & 0 \\ 0 & \bar{K}_+ \end{bmatrix}$, with $\bar{K}_+$ the *largest* solution of the LMI that corresponds to the subsystem for $x_2(i)$ *and* $x_2(i)$ in (2.3) (Proposition 1.7 (a)). If $s = 0$, then $\bar{K}_+ = \begin{bmatrix} 0 & 0 \\ 0 & K_{33} \end{bmatrix}$, and K_{33} denotes the *unique* positive semidefinite solution of the ARE that is associated with the subsystem for $x_3(i)$ [1, Section IV].

3. Existence of optimal controls for left invertible systems.

In the final Section we assume that $\rho = m$ [1, p. 350-351]. Moreover, we assume (2.5) to be valid and hence (Corollary 1.4) $D'D + B'K_zB > 0$. For notational ease, set $A_z := A_{F(K_z)}, C_z := C_{F(K_z)}, S_z := S_{F(K_z)}, D_1 := \{z \in \mathbb{C} || z| \leq 1\}, D_1^0 := \{z \in \mathbb{C} || z| < 1\}$.

Proposition 3.1.

$\forall x_0 \in \mathbf{R}^n \exists u : J_z(x_0) = J(x_0, u)$ and $z(i) \to 0 (i \to \infty) \Leftrightarrow \sigma(A_z | \mathbf{R}^n / \mathcal{V}_z) \subset D_1^0$.

Proof. $\Rightarrow$ From Corollary 1.6, for every $x_0 \in \mathbf{R}^n, u(i) = F(K_z)x(i)\ (i \geq 0)$ and hence (1.1)-(1.2) transform into $x(i+1) = A_z x(i), y(i) = C_z x(i), z(i) = S_z x(i)$. From Proposition 1.7 (c), then, $F(K_z) \in \mathcal{G}_z(\Sigma)$ (take $G \in \mathcal{G}_z(\Sigma)$, and recall that $K_z\mathcal{V}_z = 0$). Thus, if $\mathcal{V}_z \oplus \mathcal{X}_2 = \mathbf{R}^n$, then the transformed system (1.1)-(1.2) can be partitioned into $\begin{bmatrix} x_1(i+1) \\ x_2(i+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix}, y(i) = C_2 x_1(i), z(i) = S_2 x_1(i)$, and $\sigma(A_{22}) = \sigma(A_z | \mathbf{R}^n / \mathcal{V}_z)$. As

$x_2(i) \to 0(i \to \infty)$ for *every* x_{02} (see proof of Theorem 2.3), it follows that $\sigma(A_{22}) \subset D_1^0$. $\Leftarrow$ If $\sigma(A_{22}) \subset D_1^0$, then choosing $u(i) = F(K_z)x(i)$ $(i \geq 0)$ yields that $x_2(i) \to 0$ for every x_{02} and thus, for every $x_0 \in \mathbf{R}^n, z(i) \to 0(i \to \infty)$. Now, apply Proposition 1.5.

Proposition 3.2.

$\sigma(A_z|\mathbf{R}^n/\mathcal{V}_z) \subset D_1$.

Proof. Consider (2.3). Since $\rho = m$, ker $(D) \cap B^{-1}(\mathcal{V}) = 0$ [1, p. 352], and thus the first two columns of B in (2.3) are not appearing, and $D = D_3$. Let, further, $\bar{A} = \begin{bmatrix} \bar{A}_{22} & \bar{A}_{23} \\ 0 & \bar{A}_{33} \end{bmatrix}, \bar{B} = \begin{bmatrix} B_{23} \\ B_{33} \end{bmatrix}, \bar{C} = [0, C_3]$. As $(\bar{A}, \bar{B})$ is stabilizable by (2.5), the largest solution $\bar{K}_+$ of the LMI associated with $\bar{\Sigma} = (\bar{A}, \bar{B}, \bar{C}, D)$ exists (Corollary 2.5), and $K_z = \begin{bmatrix} 0 & 0 \\ 0 & \bar{K}_+ \end{bmatrix}$ (see end of Section 2). Moreover, $D'D + B'K_zB = D'D + \bar{B}'\bar{K}_+\bar{B} > 0$, and it follows that $\sigma(A_z|\mathbf{R}^n/\mathcal{V}_z) = \sigma(\bar{A} - \bar{B}(D'D + \bar{B}'\bar{K}_+\bar{B})^{-1}(D'\bar{C} + \bar{B}'\bar{K}_+\bar{A}))$. Hence we are done if the *largest* solution K_+ of Γ satisfies $\sigma(A_{F(K_+)}) \subset D_1$ if (A, B) is *stabilizable and* $\rho = m$ (existence of K_+ is clear by Corollary 2.5). Now, consider, besides (1.1), the system $\Sigma_n (n \geq 1)$ with, instead of $y(i), y_n(i) = [y'(i)\ (1/n)x'(i)]'$. The system matrix for Σ_n is left invertible for every $s \in \mathbb{C}$, as ker $(B) \cap$ ker $(D) = 0$, and hence the *unique* positive semidefinite solution of $n^{-2}I + \phi(K) = 0, K_n$, is such that $\sigma(A_{F(K_n)}) \subset D_1^0$ [1, Section IV]. As $K_{n_1} \geq K_{n_2} \geq K_+ (n_1 \geq n_2)$, by (1.4) and Corollary 2.5, we establish that $\bar{K} := lim_{n\to\infty} K_n \geq K_+$, but also $\bar{K} \leq K_+$, since $P(K_n) + \begin{bmatrix} n^{-2}I & 0 \\ 0 & 0 \end{bmatrix} \geq 0$, and thus, by continuity, $\bar{K} \in \Gamma$. Hence $K_+ = \bar{K}$ and therefore $\sigma(A_{F(K_+)}) \subset D_1$, again by continuity (note that, for all $n \geq 1, D'D + B'K_nB \geq D'D + B'K_+B > 0$).

Proposition 3.3.

If $G_{1,2} \in \mathcal{G}(\Sigma)$, then $\sigma(A_{G_1}|\mathcal{V}) = \sigma(A_{G_2}|\mathcal{V})$. Let $G \in \mathcal{G}_z(\Sigma) \cap \mathcal{G}(\Sigma)$. Then $\sigma(A_z|\mathbf{R}^n/\mathcal{V}_z) \cap (D_1 \backslash D_1^0) = \emptyset \Leftrightarrow \sigma(A_G|\mathcal{V}/\mathcal{V}_z) \cap (D_1 \backslash D_1^0) = \emptyset$.

Proof. First claim: [1, p. 353]. Second claim: $\Rightarrow$ Let $(A_G - \lambda I)v \in \mathcal{V}_{z'} v \in \mathcal{V}, v \notin \mathcal{V}_{z'} |\lambda| = 1$. Then, with $K = K_z$ in Proposition 1.7 (a), we find that $B'K_zv = 0$, as $D'D + B'K_zB > 0$. Hence, by Proposition 1.7 (c), $A_zv = A_Gv$, which contradicts our assumption. $\Leftarrow$ Let $(A_z - \lambda I)v \in \mathcal{V}_{z'} v \notin \mathcal{V}_{z'} |\lambda| = 1$. Then, by Proposition 1.7 (b) with $K = K_z, v \in <$ ker $(C_z)|A_z > \subset \mathcal{V}(u(i) = F(K_z)x(i)$ yields $y(i) = 0$ for all $i \geq 0$ if $x_0 = v)$. Now K_- exists, by Corollary 2.5. Thus, by Proposition 1.7 (c), $A_{F(K_-)}v = A_Gv$. On the other hand, again by Proposition 1.7 (c), $A_{F(K_-)}v = A_zv - B(D'D + B'K_-B)^{-1}(D'C_z + B'K_-A_z)v = A_zv$, and we have a contradiction with our assumption.

Theorem 3.4.

Assume that (2.5) is valid and that Σ is left invertible. Then for every $x_0 \in \mathbf{R}^n$ an optimal control for (LQCP $)_z$ exists if and only if $\sigma(A_G|\mathcal{V}/\mathcal{V}_z) \cap (D_1 \backslash D_1^0) = \emptyset$, with $G \in \mathcal{G}_z(\Sigma) \cap \mathcal{G}(\Sigma)$. If this is the case, then this optimal control is *unique*, and it can be given by the *unique* feedback law $u(i) = F(K_z)x(i)$, for all $i \geq 0$.

Proof. Combine Propositions 3.1-3.3.

Remarks.

1. Observe that Theorem 2.3 links *any* (LQCP $)_z$ to *one* $K_z \in \Gamma_0$. Thus, Theorem 2.3 *maps* the set $\{(S, T) \in (\mathbf{R}^{sxn}, \mathbf{R}^{sxm})|s \geq 0\}$ to $\{K \in \Gamma_0 | K \geq 0\}$ if (A, B) is stabilizable. One can show that this map is, in fact, *onto*; for every $0 \leq K \in \Gamma_0, J_z(x_0) = x_0'Kx_0$ on $\mathbf{R}^n$, with $z(i) = Kx(i)$ $(i \geq 0)$.
2. Corollary 2.6 generalizes [7, Theorem 3.1].
3. Proposition 3.2 is *untrue* if $\rho \neq m$. For example, let $A = 2, B = 1, C = D = 0, S = 1, T = 0$. The ARE is $0 = 4K - K - 4K = -K; K_+ = 0$ and $A_{F(K_+)} = 2 \notin D_1$; note that $T(z) = 0$.

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