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RESEARCH MEMORANDUM

SOLVABILITY CONDITIONS, CONSISTENCY AND WEAK CONSISTENCY FOR LINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS AND TIME-INVARIANT SINGULAR SYSTEMS: THE GENERAL CASE

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SOLVABILITY CONDITIONS, CONSISTENCY AND WEAK CONSISTENCY FOR LINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS AND TIME-INVARIANT SINGULAR SYSTEMS: THE GENERAL CASE

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ABSTRACT

We present several solvability concepts for linear differential-algebraic equations (DAEs) with constant coefficients on the positive time-axis as well as for the associated singular systems, and investigate under which conditions these concepts are met. Next, we derive necessary and sufficient conditions for global **consistency** of initial conditions for the DAE as well as for the system, and generalize these conditions with respect to our concept of *weak* consistency. Our distributional approach enables us to generalize results in an earlier paper, where singular systems are assumed to have a regular pencil in the sense of Gantmacher. In particular, we will establish that global weak consistency in the system sense is equivalent to impulse controllability.

KEYWORDS

Linear differential-algebraic equation, singular system, impulsive-smooth distributions, solvability in the distribution and in the function sense, consistency, weak consistency.

1. Introduction.

In the present paper we consider Differential-Algebraic Equations (DAEs) on $\mathbb{R}^+ := [0, \infty)$ of the form

$$E\dot{x}(t) = Ax(t) + f(t) \qquad (1.1a)$$

and the associated linear systems

$$E\dot{x}(t) = Ax(t) + Bu(t) \qquad (1.1b)$$

with $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, arbitrary, and $x(t) \in \mathbb{R}^n$, $f(t) \in \mathbb{R}^l$, $u(t) \in \mathbb{R}^m$ for all $t \geq 0$.

If the forcing function f is given and E is invertible, then *every* point $x_0 \in \mathbb{R}^n$ is **consistent** [1] because

$$x(t) = \exp(E^{-1}At)x_0 + \int_0^t \exp(E^{-1}A(t-\tau))E^{-1}f(\tau)d\tau \quad (1.2)$$

is the solution of (1.1a) with $x(0^+) = x_0$ (assuming that f is at least locally integrable). In case of a singular matrix E, however, the set of consistent initial conditions may be unequal to the entire state space $\mathbb{R}^n$.

Example 1.1.

If $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is continuously differentiable, then the solution of the DAE $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + f$ is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -f_1 - \dot{f}_2 \\ -f_2 \end{bmatrix}$ [6], [17] and hence $\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$ can be called consistent only if $\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} = \begin{bmatrix} -f_1(0^+) - \dot{f}_2(0^+) \\ -f_2(0^+) \end{bmatrix}$.

Example 1.2.

Consider the singular DAE

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix},$$

with f sufficiently smooth. Then, apparently, this DAE has a solution only if $f_4 = 0$ [6], [17]. Assume this to be the case. Then x_4 may be **any** function. Next, we get $x_3 = -f_3$ and hence $-\dot{f}_3 = f_2$ [6], [17]. Again, assume this to be the case. If x_2 is **any** locally integrable function (e.g. take x_2 continuous), then $x_1 = x_{01} + \int_0^t [x_2(\tau) + f_2(\tau)]d\tau$, x_{01} arbitrary. Observe that x_{03} is consistent only if $x_{03} = -f_3(0^+)$.

Loosely speaking, a point x_0 is consistent if the DAE (1.1a) turns out to have a functional solution that starts in x_0 - in this paper we will provide an unambiguous definition for consistency in terms of *generalized* functions [15]. The two Examples show, that the set of consistent initial conditions for a singular DAE does not follow from *a priori* but from *a posteriori* observations. Again, consider Example 1.1 with f = 0. Only the origin is consistent. In other words, here a point x_0 may be called **inconsistent** if $x_0 \neq 0$; the DAE (with f = 0) has no functional solutions x that start in x_0 since x = 0 is the only one.

In [16] a simple electrical network with unit capacitor only is modeled by means of the system in Example 1.1 with f = 0, x_2 denoting the potential and x_1 the current; the open switch is closed at t = 0. If $x_{02} := x_{02}(0^-) \neq 0$ (and $x_{01} := x_{01}(0^-) = 0$), then it is claimed in [16] that $x_2 = 0$, but $x_1 = - x_{02}\delta(t)$ on $\mathbb{R}^+$ (with $\delta(t)$ denoting the Dirac delta function), and thus it is suggested that one may have an *impulsive* solution x of the DAE in Example 1.1 with f = 0 if an inconsistent initial condition x_0 is identified with the state value $x(0^-)$ of x immediately *before* starting the dynamical process. In this sense, $x_0 = x(0^-)$ may be called consistent if the DAE has a functional solution x with $x(0^+) = x_0 = x(0^-)$.

This interpretation of "initial condition" x_0 as the state value of x at $t = 0^-$ is used in e.g. [2], [5, § 22], [14], [16], [18]. Apparently (see the above), inconsistent initial conditions might give rise to impulses as solutions of the DAE (1.1a) even if the forcing function is zero. Therefore, certain authors on singular systems (e.g. [2]) allowed generalized functions (*distributions* [15]) as possible forcing functions and solutions of (1.1a), whereas others (e.g. [16]) based themselves on the Laplace transformation approach of Doetsch [5].

In [8] both viewpoints are joined by applying a special distributional framework to DAEs (1.1a) and systems (1.1b) on $\mathbb{R}^+$. The allowed class of distributions C_{imp}, proposed by Hautus in [13] for regular systems in connection with linear-quadratic control, turns out to be large enough to be representative for the solution's behaviour of (1.1) on one hand, but on the other C_{imp} is a commutative algebra over $\mathbb{R}$ with convolution of distributions as multiplication [12]. Since, moreover, C_{imp} has a lot of other nice properties (for details, see [12] - [13], also Section 2), the distributional setup in [8] allows a fully *algebraic* treatment of DAEs (1.1a) and systems (1.1b) on $\mathbb{R}^+$.

In addition, this framework turns out to cover Kronecker's interpretation of singular DAEs (see our Examples, [6], [17]). This was shown in [8, Theorem 2.13] if $\det(sE - A) \neq 0$ (the regular pencil $sE - A$ in the sense of Gantmacher [6]) and will be illustrated for general singular DAEs in Sections 2 and 3.

Other results for the case $\det(sE - A) \neq 0$ in [8], derived by means of the C_{imp}-approach, are on conditions for "global" consistency and "global" *weak* consistency in the "DAE" and the "system" sense. Loosely speaking (for details, see Section 4), given the forcing function f, then a point x_0 is *weakly* consistent (with f) if the *distributional* version of (1.1a) ([8], Section 2)

$$\delta^{(1)} * Ex = Ax + f + Ex_0\delta \tag{1.3}$$

has a *functional* solution x that *need not start in* x_0, i.e., $x(0^+)$ may be unequal to x_0 (here, $*$ denotes convolution and $\delta^{(1)}$ denotes the distributional derivative of δ). In the sequel we shall see that it is very well possible for the DAE (1.3) with forcing function f to have a functional solution x that does not start in x_0.

In the present paper, we want to generalize all results in [8] for DAEs and systems (1.1) with *arbitrary* coefficients E, A and B. Indeed, most of the statements in [8] will turn out to be special cases of related ones made here.

After the preliminaries in Section 2, we discuss **separate** solvability concepts for DAEs and systems (in the distribution as well as in the function sense) in Section 3. We will show that DAE-solvability of (1.3) in the distribution sense is **equivalent** to DAE-solvability of (1.1a) in the sense of our Examples 1.1 and 1.2, whereas solvability of (1.1b) in the function sense is clearly **stronger** than system solvability in the distribution sense. In Section 4, then, after having introduced **separate** concepts of consistency and weak consistency for DAEs and systems, we derive necessary and sufficient conditions for "global" consistency as well as "global" weak consistency for all concepts defined. In particular, we will establish that global weak consistency in the system sense is equivalent to Cobb's impulse controllability [4].

2. Preliminaries.

Let $\mathscr{D}_-$ be the space of test functions with upper-bounded support and let $\mathscr{D}_+'$ denote the dual space of real-valued continuous linear functionals on $\mathscr{D}_-$. Then the space $\mathscr{D}_+$ of test functions with lower-bounded support can be considered as a subspace of $\mathscr{D}_+'$ and every $u \in \mathscr{D}_+'$ has lower-bounded support [12]. With the "pointwise" addition and scalar multiplication, and with **convolution** $*$ of distributions as **multiplication**, $\mathscr{D}_+'$ is a commutative algebra over $\mathbb{R}$ with unit element δ, the Dirac delta distribution [12]. If $u^{(1)}$ denotes the distributional derivative of $u \in \mathscr{D}_+'$, then $u^{(1)} = (u * \delta)^{(1)} = u * \delta^{(1)}$. Any linear combination of δ and its distributional derivatives $\delta^{(l)}$, $l \geq 1$, is called *impulsive*. If $u \in \mathscr{D}_+'$ can be identified with an ordinary function (u, say) with support on $\mathbb{R}^+$ and this function u is smooth on $[0, \infty)$, then $u \in \mathscr{D}_+'$ is called *smooth*.

Linear combinations of impulsive and smooth distributions are called *impulsive-smooth* and the set of these distributions is denoted by $\mathcal{C}_{imp}$ [13, Def. 3.1]. This set $\mathcal{C}_{imp}$ is a subalgebra and hence it is closed under differentiation (= convolution with $\delta^{(1)}$) and closed under integration (= convolution with the inverse of $\delta^{(1)}$, the Heaviside distribution H) [12], [13, Section 3]. Since $u \in \mathcal{C}_{imp}$ is invertible within $\mathcal{C}_{imp}$ if and only if $u \notin \mathscr{D}_+$ [12, Theorem 3.11], it follows that every impulse is invertible. By defining [12, Def. 3.1] $p := \delta^{(1)}$, $p^k := p^{k-1} * p$ $(k \geq 2)$, $p^0 := \delta$, $p^{-1} := H$, $p^{-l} := p^{-(l-1)} * p^{-1}$ $(l \geq 2)$, we establish that $p^{k+l} = p^k * p^l$ $(k, l \in \mathbb{Z})$ and thus $(p^k)^{-1} = p^{-k}$, $(p^0)^{-1} = p^0 = \delta$; we will write $p^0 = 1$ and $\alpha\delta = \alpha$ $(\alpha \in \mathbb{R})$. Also, convolution will be denoted by juxtaposition. If $u = u_1 + u_2$, the (unique) decomposition of $u \in \mathcal{C}_{imp}$ in its impulsive part u_1 and its smooth part u_2, then $u(0^+) := \lim_{t \downarrow 0} u_2(t) = u_2(0^+)$. If $u \in \mathcal{C}_{imp}$ is smooth and $\dot{u}$ stands for the distribution that can be identified with the ordinary derivative of u on $\mathbb{R}^+$, then $pu = \dot{u} + u(0^+)$ (with $u(0^+) = u(0^+)\delta$). For more details on $\mathcal{C}_{imp}$, see [12], [13, Section 3], also [8] and [10]. For more details on distributions, see the work of Laurent Schwartz [15].

Let $\mathcal{C}_{p-imp}$, $\mathcal{C}_{sm}$ denote the subalgebras of pure impulses and smooth distributions, respectively, and let $\mathcal{C}_f$ denote the subalgebra of *fractional impulses*

$$\mathcal{C}_f := \{u \in \mathcal{C}_{imp} | u = u_1 u_2^{-1},\ u_{1,2} \in \mathcal{C}_{p-imp},\ u_2 \neq 0\},$$

then $\mathcal{C}_f$ is isomorphic to the commutative field of rational functions R(s) [10, Proposition 2.3]. Let k_1, k_2 be any two nonnegative integers and let $M^{k_1 \times k_2}(s)$, $M_f^{k_1 \times k_2}(p)$ denote the sets of $k_1 \times k_2$ matrices with elements in R(s), $\mathcal{C}_f$, respectively. Then we have the following **basic** result [10, Corollary 2.4].

Lemma 2.1.

Let $T(s) \in M^{k_1 \times k_2}(s)$, $\eta(s) \in M^{1 \times k_1}(s)$, $w(s) \in M^{k_2 \times 1}(s)$, and let $T(p)$, $\eta(p)$, $w(p)$ be the corresponding distributional matrices in $M_f^{k_1 \times k_2}(p)$, $M_f^{1 \times k_1}(p)$, $M_f^{k_2 \times 1}(p)$, respectively. Then

$$\eta(s)T(s) = 0 \iff \eta(p)T(p) = 0; \quad T(s)w(s) = 0 \iff T(p)w(p) = 0.$$

In particular, **T(s)** is left (right) invertible as a matrix with elements in R(s) if and **only if T(p)** is left (right) invertible as a matrix with elements in $\mathcal{C}_f$.

Now we present our *distributional* versions of (1.1a) and (1.1b) on $\mathbb{R}^+$ (compare (1.3)):

$$pEx = Ax + f + Ex_0, \tag{2.1a}$$

$$pEx = Ax + Bu + Ex_0. \tag{2.1b}$$

Here, $x_0 \in \mathbb{R}^n$ (Ex_0 stands for $Ex_0\delta$), $f \in \mathcal{C}_{imp}^l$ (the l-vector version of $\mathcal{C}_{imp}$) and $u \in \mathcal{C}_{imp}^m$. Together with (2.1), we define the *solution sets*

$$S(x_0, f) := \{x \in \mathcal{C}_{imp}^n | [pE - A]x = f + Ex_0\}, \tag{2.2a}$$

$$S_C(x_0, u) := \{x \in \mathcal{C}_{imp}^n | [pE - A]x = Bu + Ex_0\}, \tag{2.2b}$$

and we have attached an index C to the solution set of *state trajectories* for the system (2.1b) to indicate its C(ontrol) aspect; $u \in \mathcal{C}_{imp}^m$ is often called *input* or *control*.

Discussion.

First of all, we observe that the **form** of (2.1) is in line with earlier references on the use in singular systems of distributions (e.g. [2] - [3]) *and* on Laplace transforms (e.g. [5], [16]). Although (2.1) might seem nothing more than Laplace transformation of (1.1) in the sense of Doetsch [5], followed by substitution of s by p, we *stress* that (2.1a) may, in fact, be considered as an *initial value* problem for a linear DAE on $\mathbb{R}^+$ with constant coefficients *in the distribution sense* [8]. Here, x_0 plays the role of initial value - in standard cases. For instance, if E is invertible, then (2.1a) may be rewritten as

$$px = E^{-1}Ax + E^{-1}f + x_0 \tag{2.3}$$

and since $(sI - E^{-1}A)$ is invertible as a rational matrix, we find that for every pair $(x_0, f) \in \mathbb{R}^n \times C^1_{imp}$, (2.3) has exactly one solution, namely

$$x = (pI - E^{-1}A)^{-1}[E^{-1}f + x_0], \tag{2.4}$$

by Lemma 2.1. Now $(pI - E^{-1}A)^{-1}$ can be identified with the smooth function $\exp(E^{-1}At)$ on $\mathbb{R}^+$ [13, p. 375]. Thus, if $f \in C^1_{sm}$, then it follows directly that x in (2.4) corresponds to the function (1.2) on $\mathbb{R}^+$, and $x(0^+) = x_0$.

Next, let us consider our Examples 1.1 and 1.2 in the distributional version (2.1a).

Example 1.1 continued.

The DAE $p\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}f_1\\f_2\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_{01}\\x_{02}\end{bmatrix}$ has as solutions $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}- f_1 - pf_2 - x_{02}\\- f_2\end{bmatrix}$. If f_1 and f_2 are smooth, then $pf_2 = \dot{f}_2 + f_2(0^+)$. Hence, if $x_{01} = - f_1(0^+) - \dot{f}_2(0^+)$, $x_{02} = - f_2(0^+)$ (i.e., x_0 is consistent), then $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}- f_1 - \dot{f}_2\\- f_2\end{bmatrix}$ and $x_1(0^+) = x_{01}$, $x_2(0^+) = x_{02}$, in accordance with Kronecker, see Example 1.1. More generally, if $x_{02} = - f_2(0^+)$, x_{01} **arbitrary**, then, again, $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}- f_1 - \dot{f}_2\\- f_2\end{bmatrix}$, but not necessarily $x(0^+) = x_0$ - in fact, only $E(x(0^+)) = Ex_0$. Moreover, if $f_1 = f_2 = 0$, then $x_2 = 0$, $x_1 = - x_{02}$ $(= - x_{02}\delta)$, as was stated earlier [16].

Example 1.2 continued.

If f in the DAE

$$p\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \begin{bmatrix}f_1\\f_2\\f_3\\f_4\end{bmatrix} + \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_{01}\\x_{02}\\x_{03}\\x_{04}\end{bmatrix}$$

is smooth, then we get

$$x_1 = p^{-1}[x_2 + f_1 + x_{01}],$$

$$-\dot{f}_3 - f_3(0^+) = f_2 + x_{03},\ x_3 = -f_3,$$

$$0 = f_4.$$

Hence, if $f_4 = 0$, $f_2 = -\dot{f}_3$ and $x_{03} = -f_3(0^+)$ (consistent), x_2, $x_4 \in C_{sm}$ are taken arbitrarily (with initial values x_{02} and x_{04}, respectively), then x_3 corresponds to $-f_3$ and x_1 to $(x_{01} + \int_0^t [x_2(\tau) + f_1(\tau)]d\tau)$ on $\mathbb{R}^+$, in accordance with Example 1.2.

Our **Examples** clearly suggest that $S(x_0, f)$ contains at least one smooth solution x that actually starts in x_0 if x_0 is chosen consistently. In the next straightforward result we will prove that this is generally true.

Proposition 2.2.

Assume that, for a given smooth forcing function f, $x_0 \in \mathbb{R}^n$ is such that (1.1a) has a smooth solution x with $x(0^+) = x_0$. Then (the distribution) $x \in S(x_0, f)$.

Proof. We have $E\dot{x} = Ax + f$ and $x(0^+) = x_0$. Then $Ex(0^+) = Ex_0$ and thus $pEx = E\dot{x} + Ex_0 = Ax + f + Ex_0$, i.e., $x \in S(x_0, f)$.

Thus, our framework does not only cover e.g. [2], [5], [13] - [14], [16], [18], but also [6], [17]. Observe, moreover, that the special choice of **smooth** functions in C_{imp} obviates the problem of choosing the right solution set for (1.1a); without any *a priori* choice for the solution set in Example 1.2, x_4 might have been any function and x_2 might have been even discontinuous. The same difficulty occurs w.r.t. the forcing function f; if in Example 1.1 f_2 is continuously differentiable and f_1 continuous, then x is continuous, whereas in Example 1.2

x is continuous if f_1 is merely locally integrable. Note, in addition, that the question of (in)consistency is decided in the origin (our impulses have support in 0), and that smooth inputs do not limit the control possibilities in (2.1b) e.g. [3], [7], [9], [11], [13], [18]. On the other hand, a distributional setup for DAEs and systems (2.2), incorporating a larger class than C_{imp}, is certainly possible (see e.g. [4] and [8, Remark 2.5]), but it is our belief that then much of the method's elegance will be lost unnecessarily.

We will close this Section with our Main Lemma, together with Lemma 2.1 the building-stones in [10] and in this paper.

Main Lemma 2.3.

Let $x_0 \in \mathbb{R}^n$, $f = f_1 + f_2$, $f_1 \in C^l_{p-imp}$, $f_2 \in C^l_{sm}$, $x = x_1 + x_2 \in S(x_0, f)$, $x_1 \in C^n_{p-imp}$, $x_2 \in C^n_{sm}$. Then

$$pEx_1 + E(x_2(0^+)) = Ax_1 + f_1 + Ex_0, \tag{2.5a}$$

$$pEx_2 = Ax_2 + f_2 + E(x_2(0^+)). \tag{2.5b}$$

Proof. We have $pEx_1 + E(x_2(0^+)) + E[px_2 - x_2(0^+)] = Ax_1 + f_1 + Ex_0 + Ax_2 + f_2$ and $px_2 - x_2(0^+) = \dot{x}_2$, smooth.

Corollary 2.4.

Assume that $x \in S(x_0, f) \cap C^n_{sm}$, $f \in C^l_{sm}$. Then $Ex_0 = E(x(0^+))$.

Proof. Since $x_1 = 0$, $u_1 = 0$, the claim follows from (2.5a).

Remark 2.5.

The converse of Corollary 2.4 is not true; a counterexample is given in [10, Remark 2.7]. Corollary 2.4 expresses, that not so much the property $x(0^+) = x_0$ as its generalization $E(x(0^+)) = Ex_0$ is strongly related to the question of smoothness for solutions x of the DAE (2.1a) (see also Example 1.1 continued).

3. Solvability.

We consider the DAE

$$pEx = Ax + f + Ex_0 \qquad (3.1a)$$

and the associated system

$$pEx = Ax + Bu + Ex_0, \qquad (3.1b)$$

with $x_0 \in \mathbb{R}^n$, $f \in C^1_{imp}$, $u \in C^m_{imp}$, and the corresponding solution sets $S(x_0, f)$, $S_C(x_0, u)$ ((2.2)). In [8, Definitions 2.4, 4.1, 4.5] the following definitions of solvability for the DAE and the system are proposed.

Definition 3.1.

Let $f \in C^1_{imp}$ be given. Then the DAE (3.1a) is *solvable for f* if

$$\exists_{x_0} \in \mathbb{R}^n: S(x_0, f) \neq \emptyset.$$

If $f \in C^1_{sm}$, then (3.1a) is solvable for f *in the function sense* if $\exists_{x_0} \in \mathbb{R}^n: S(x_0, f) \cap C^n_{sm} \neq \emptyset$.

The system (3.1b) is *C-solvable* if

$$\forall_{x_0} \in \mathbb{R}^n \exists_u \in C^m_{imp} : S_C(x_0, u) \neq \emptyset.$$

The system (3.1b) is C-solvable *in the function sense* if

$$\forall_{x_0} \in \mathbb{R}^n \exists_u \in C^m_{sm} : S_C(x_0, u) \cap C^n_{sm} \neq \emptyset.$$

It is clear that DAE-solvability and C-solvability are two fully different concepts. Whereas, for a **given** f, the DAE is solvable if for **at least one** x_0, the solution set $S(x_0, f)$ is nonempty, C-solvability requires that for **every** x_0 there **exists an input** u such that $S_C(x_0, u) \neq \emptyset$. The latter definition finds its roots in the knowledge, that in many control problems x_0, interpreted as $x(0^-)$, may be arbitrary (unknown), as a result of which one may want to design some control law that does not depend **explicitly** on the initial condition, but rather works for all possible state values "in the same way" (feedback laws in control problems, for instance [3], [13], [18]).

The definition of DAE-solvability should be interpreted as a generalization in terms of distributions of earlier definitions for DAE-solvability in the function sense [6], [17]: In Example 1.1 **only one** initial condition x_0 is consistent; in other words, only for this x_0 the set $S(x_0, f)$ contains a smooth element that starts in x_0. If x_0 is called *consistent* in (3.1a) if $S(x_0, f)$ (f smooth) contains a smooth x with $x(0^+) = x_0$, then consistency in the ordinary sense can be identified with consistency in (3.1a) (see Proposition 2.2). Now, let us take a better look at our concept of DAE-solvability.

Lemma 3.2.

Let $f \in C^1_{sm}$ be given and $x_0 \in \mathbb{R}^n$ be such that $S(x_0, f)$ contains at least one smooth element x. Then there exists a consistent initial condition $\bar{x}_0$. In fact, $x \in S(\bar{x}_0, f)$ and $Ex_0 = E\bar{x}_0$.

Proof. Let $x \in S(x_0, f) \cap C^n_{sm}$. Then (Corollary 2.4) $E(x(0^+)) = Ex_0$ and hence $\bar{x}_0 = x(0^+)$ satisfies the requirements by the Main Lemma 2.3!

In particular, it follows from Lemma 3.2 that there exists a consistent initial condition for (3.1a) with given smooth f if (3.1a) is solvable for f in the function sense. In Theorem 3.3 we show that the existence of a consistent initial condition is, essentially, **equivalent** to DAE-solvability.

Theorem 3.3.

If $f = f_1 + f_2$, $f_1 \in C^1_{p\text{-imp}}$, $f_2 \in C^1_{sm}$ and $x \in S(x_0, f)$ for some $x_0 \in \mathbb{R}^n$, then $x(0^+)$ is consistent for f_2. In particular, if $f \in C^1_{sm}$, then

(3.1a) is solvable for f $\Leftrightarrow \exists_{x_0 \in \mathbb{R}^n}$: x_0 consistent for f.

Proof. If $x = x_1 + x_2$, $x_1 \in C^n_{p\text{-imp}}$, $x_2 \in C^n_{sm}$, then, by (2.5b), $x_2 \in S(x(0^+), f_2)$ and, obviously, $x_2(0^+) = x(0^+)$.

Theorem 3.3 states that the DAE (1.1a), with f smooth, is solvable in the sense of Kronecker [6], [17], i.e., there exists a consistent point x_0, if and only if our DAE (3.1a) is solvable for f in the distribution sense. Thus, our approach covers the usual conceptions of solvability in the function sense on one hand, but on the other it allows much more inputs as well as solutions for the DAE.

Example 1.2 continued.

Assume that $f_2 = f_{21} + f_{22}$ and $f_3 = f_{31} + f_{32}$, $f_{21}, f_{31} \in C_{p\text{-}imp}$, $f_{21} = \sum_{i=0}^{k} \alpha_i p^i$ ($k \geq 0$, all α_i real), and $f_{22}, f_{32} \in C_{sm}$. Then the DAE is solvable if $f_4 = 0$, $-\dot{f}_{32} = f_{22}$, $-pf_{31} = f_{21} - \alpha_0$; x_{03} must equal $-f_{32}(0^+) - \alpha_0$. If f is smooth, then the DAE (3.1a) is solvable if $f_4 = 0$, $-\dot{f}_3 = f_2$ and $x_{03} = -f_3(0^+)$. This agrees with earlier findings in Sections 1 and 2.

Example 1.2 illustrates that for an arbitrary DAE, with $f \in C^l_{imp}$ given, it seems very hard, if not impossible, to derive a condition that is not only sufficient, but also necessary for solvability, i.e., for the existence of a point x_0 such that $S(x_0, f) \neq \emptyset$. However, we can get very "close".

Lemma 3.4.

Assume that (3.1a) is solvable for $f \in C^l_{imp}$. Then there exists a $\bar{l} \in [0, l]$, a $\bar{f} \in C^{\bar{l}}_{imp}$ and $\bar{E}, \bar{A} \in \mathbb{R}^{\bar{l}\times n}$, $[\bar{E}, \bar{A}]$ of full row rank, such that, if

$$p\bar{E}x = \bar{A}x + \bar{f} + \bar{E}x_0, \qquad (3.2)$$

and $$\bar{S}(x_0, \bar{f}) := \{x \in C^n_{imp} \mid [p\bar{E} - \bar{A}]x = \bar{f} + \bar{E}x_0\} \qquad (3.3)$$

($x_0 \in \mathbb{R}^n$), then

$$x \in S(x_0, f) \Leftrightarrow x \in \bar{S}(x_0, \bar{f}).$$

Proof. Without loss of generality, we may assume that $[E\ A] = \begin{bmatrix} I_{\bar{l}} \\ Y \end{bmatrix} [\bar{E}\ \bar{A}]$ with $\bar{E}, \bar{A} \in \mathbb{R}^{\bar{l}\times n}$, $Y \in \mathbb{R}^{(l-\bar{l})\times\bar{l}}$, $[\bar{E}\ \bar{A}]$ of full row rank, and let $f = \begin{bmatrix} \bar{f} \\ g \end{bmatrix}$ be partitioned accordingly. Then, let $x_0 \in \mathbb{R}^n$ and $x \in C^n_{imp}$ be such that

$$p\begin{bmatrix} I \\ Y \end{bmatrix}\bar{E}x = \begin{bmatrix} I \\ Y \end{bmatrix}\bar{A}x + \begin{bmatrix} \bar{f} \\ g \end{bmatrix} + \begin{bmatrix} I \\ Y \end{bmatrix}\bar{E}x_0$$

(such x_0 and x exist!), then $-Y\bar{f} + g = 0$, i.e., $g = Y\bar{f}$. Hence

$$p\bar{E}x = \bar{A}x + \bar{f} + \bar{E}x_0.$$

The converse is now clear.

Example 1.2 continued.

If the DAE is solvable for f, then $f_4 = 0$. Here, we have

$$\bar{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ \bar{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

It follows from Lemma 3.4 that, without loss of generality, we may assume [E A] to be of full row rank if the DAE (3.1) is solvable for given $f \in C^l_{imp}$. Since, by Lemma 2.1,

[E A] full row rank $\Leftrightarrow$

[A - sE, E] right invertible as a rational matrix,

it is easily seen that, if [E A] is of full row rank, then, for *every* $f \in C^l_{imp}$, $\begin{bmatrix} x \\ x_0 \end{bmatrix} := \begin{bmatrix} R_1(p) \\ R_2(p) \end{bmatrix}(-f)$ is such that $pEx = Ax + f + Ex_0$ with $\begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix}$ a right inverse of [A - sE, E] (Lemma 2.1) - however, $x_0 = R_2(p)(-f)$ need not be constant (= constant times δ). This observation shows, that the condition

[E A] full row rank

is indeed very "close" to DAE-solvability - unfortunately, not close enough. However, conditions for "global" consistency and "global" weak consistency in the DAE-sense will be derived in Section 4.

As for C-solvability, we have the next result.

Theorem 3.5.

The system (3.1b) is C-solvable if and only if

$$\forall_{\eta(s)\,\in\, M^{1\times l}(s)}:\ \eta(s)[A - sE,\ B] = 0 \Leftrightarrow \eta(s)[E\ A\ B] = 0.$$

Proof. Without loss of generality, we may assume that $[E\ A\ B] = \begin{bmatrix} I_{\bar{l}} \\ Y \end{bmatrix}[\bar{E}\ \bar{A}\ \bar{B}]$ with $[\bar{E}\ \bar{A}\ \bar{B}]$ of full row rank. $\Leftarrow$ The condition is equivalent to right-invertibility of $[\bar{A} - s\bar{E},\ \bar{B}]$. If $\begin{bmatrix} \bar{R}_1(s) \\ \bar{R}_2(s) \end{bmatrix}$ is a right inverse, then, for every $x_0 \in \mathbb{R}^n$, $\begin{bmatrix} x \\ u \end{bmatrix} := \begin{bmatrix} \bar{R}_1(p) \\ \bar{R}_2(p) \end{bmatrix}(-\bar{E}x_0)$ is such that $[A - pE,\ B]\begin{bmatrix} x \\ u \end{bmatrix} = -Ex_0$ (Lemma 2.1). $\Rightarrow$ Assume that $\eta(s)[A - sE,\ B] = 0$. Then $\eta(p)[A - pE,\ B] = 0$ (Lemma 2.1) and hence, by definition of C-solvability, $\eta(p)Ex_0 = 0$ for all x_0, i.e., $\eta(p)[E\ A\ B] = 0$ and thus $\eta(s)[E\ A\ B] = 0$. This completes the proof.

Corollary 3.6.

If $[E\ A\ B]$ is of full row rank, then (3.1b) is C-solvable if and only if $[A - sE,\ B]$ is right invertible as a rational matrix.

In Theorem 3.3 we saw that DAE-solvability in the distribution sense is equivalent to DAE-solvability in the function sense. For C-solvability, things are less easy.

Example 3.7.

The system $p\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$ is C-solvable, but not C-solvable in the function sense: For every $x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$ we have $x_1 = 0$, $u = -x_{01}$, impulsive.

Section 4 contains a condition that is necessary and sufficient for C-solvability in the function sense. Example 3.7 does not satisfy this condition, whereas $[A - sE,\ B]$ is right invertible (Corollary 3.6).

4. Consistency and weak consistency.

In Section 3 a point x_0 is called DAE-consistent for (3.1a) with given smooth f if $S(x_0, f)$ contains a smooth x with $x(0^+) = x_0$. In Definition 4.1 we distinguish between consistency and its generalization, *weak* consistency [8, Definition 3.1].

Definition 4.1.

Consider (3.1a) with $f \in C^1_{sm}$.

A point $x_0 \in \mathbb{R}^n$ is called *DAE-consistent with f* if

$$\exists_x \in S(x_0, f) \cap C^n_{sm} : x(0^+) = x_0.$$

The set of these points is denoted by $I_{DAE}(f)$.

A point $x_0 \in \mathbb{R}^n$ is called *weakly DAE-consistent with f* if

$$S(x_0, f) \cap C^n_{sm} \neq \emptyset.$$

The set of these points is denoted by $I^W_{DAE}(f)$.

Consider (3.1b).

A point $x_0 \in \mathbb{R}^n$ is called *C-consistent* if

$$\exists_u \in C^m_{sm} \exists_x \in S_C(x_0, u) \cap C^m_{sm} : x(0^+) = x_0.$$

The set of these points is denoted by I_C.

A point $x_0 \in \mathbb{R}^n$ is called *weakly C-consistent* if

$$\exists_u \in C^m_{sm} : S_C(x_0, u) \cap C^m_{sm} \neq \emptyset$$

The set of these points is denoted by I^W_C.

Proposition 4.2.

The DAE (3.1a) is solvable for $f \in C^1_{sm} \Leftrightarrow I^W_{DAE}(f) \neq \emptyset$. The system (3.1b) is solvable in the function sense $\Leftrightarrow I^W_C = \mathbb{R}^n$.

Proof. $I^w_{DAE}(f) \neq \emptyset$ if and only if (3.1a) is solvable for f in the function sense (Definition 3.10); if (3.1a) is solvable for $f \in C^1_{sm}$, then $I_{DAE}(f) \neq \emptyset$ by Theorem 3.3 and $I^w_{DAE}(f) \supset I_{DAE}(f)$. The second claim is trivial, by definition.

Once more, we establish that DAE- and C-solvability are different concepts. This distinction is also apparent in the next Theorems on "global" consistency and "global" weak consistency.

Theorem 4.3.

Assume that in (3.1a), rank [E A] = l and $f \in C^1_{sm}$. Then

$$I_{DAE}(f) = \mathbb{R}^n \Leftrightarrow \text{im}(E) = \mathbb{R}^l, \qquad (4.1a)$$

$$I^w_{DAE}(f) = \mathbb{R}^n \Leftrightarrow \text{im}(E) + A(\ker(E)) = \mathbb{R}^l. \qquad (4.1b)$$

Proof. First statement. $\Leftarrow$ Assume without loss of generality that $E = [I_l\ 0]$, $A = [A_1\ A_2]$. If $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}$ are partitioned accordingly, then (3.1a) is of the form $px_1 = Ax_1 + Ax_2 + f + x_{01}$. If we choose $x_2 = p^{-1}x_{02}$ (smooth, $x_2(0^+) = x_{02}$), then $x_1 = (pI_l - A_1)^{-1}(Ax_2 + f + x_{01})$, smooth, and $x_1(0^+) = x_{01}$. $\Rightarrow$ Assume that $\eta E = 0$. It follows that $\eta Ax_0 + \eta f = 0$ for all x_0 and hence $\eta f = 0$, $\eta A = 0$. Thus, $\eta = 0$ since [E A] is of full row rank. Second statement. Assume that $\text{im}(E) \neq \mathbb{R}^l$. Then, without loss of generality, we may assume that (3.1a) is of the form

$$p\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_{01} \\ \bar{x}_{02} \end{bmatrix}. \qquad (4.2)$$

$\Leftarrow$ It follows that A_{22} is of full row rank; let A_{22}^+ be any right inverse. Let $\bar{x}_{01}$, $\bar{x}_{02}$ be arbitrary. The solution of

$$p\bar{x}_1 = [A_{11} - A_{12}A_{22}^+A_{21}]\bar{x}_1 + [f_1 - A_{12}A_{22}^+f_2] + \bar{x}_{01}$$

is smooth with $\bar{x}_1(0^+) = \bar{x}_{01}$, and $\bar{x}_2 = -A_{22}^+[A_{21}\bar{x}_1 + f_1]$ is smooth as well. We have shown that every point x_0 is weakly DAE-consistent with f. $\Rightarrow$ We must prove that A_{22} is of full row rank. Thus, let $\eta A_{22} = 0$. It follows that $\eta A_{21}\bar{x}_{01} + \eta f_2 = 0$ for all $\bar{x}_{01}$, because of Corollary 2.4. Hence $\eta f_2 = 0$, $\eta A_{21} = 0$. Since $[A_{21}\ A_{22}]$ is assumed to be of full row rank, we get $\eta = 0$.

Remark 4.4.

Observe that the conditions in (4.1) imply that [E A] is right invertible and that without loss of generality we may assume [E A] to be right invertible if the DAE is solvable (Section 3). If $\det(sE - A) \neq 0$, then [E A] is automatically of full row rank and Theorem 4.3 reduces to [8, Theorem 3.7]. In Examples 1.1 and 1.2 we have $I^{w}_{DAE}(f) \neq \mathbb{R}^n$.

Theorem 4.5.

Assume that in (3.1b), [E A B] is of full row rank. Then

$$I_C = \mathbb{R}^n \Leftrightarrow \operatorname{im}(E) + \operatorname{im}(B) = \mathbb{R}^l, \tag{4.3a}$$

$$I^w_C = \mathbb{R}^n \Leftrightarrow \operatorname{im}(E) + \operatorname{im}(B) + A(\ker(E)) = \mathbb{R}^l. \tag{4.3b}$$

Proof. First statement. If $\operatorname{im}(E) = \mathbb{R}^l$, we are done. Thus, let $\operatorname{im}(E) \neq \mathbb{R}^l$. Then we may assume that the system (3.1b) is in the form (4.2) with $f_i = B_i u$ $(i = 1, 2)$. $\Leftarrow$ The condition is equivalent to right-invertibility of B_2; let $B_2{}^+ = B_2{}'(B_2B_2{}')^{-1}$. If $\bar{x}_{01}$ and $\bar{x}_{02}$ are arbitrary, then the control $u = B_2{}^+(-A_{21}\bar{x}_1 - A_{22}\bar{x}_2)$, with $\bar{x}_2 = p^{-1}\bar{x}_{02}$ and $\bar{x}_1$ the solution of

$$pv = (A_{11} - B_1B_2{}^+A_{21})v + (A_{12} - B_1B_2{}^+A_{22})\bar{x}_2 + \bar{x}_{01},$$

is in C^m_{sm}, $\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} \in S_C(\begin{bmatrix}\bar{x}_{01}\\ \bar{x}_{02}\end{bmatrix}, u) \cap C^n_{sm}$ and $\bar{x}_1(0^+) = \bar{x}_{01}$, $\bar{x}_2(0^+) = \bar{x}_{02}$. $\Rightarrow$ We must show that B_2 is of full row rank. Thus, let $\eta B_2 = 0$. It follows that $\eta[A_{21}\ \ A_{22}]\begin{bmatrix}\bar{x}_{01}\\ \bar{x}_{02}\end{bmatrix} = 0$ since every x_0 is C-consistent. Hence $\eta[A_{21}\ A_{22}] = 0$, which yields $\eta = 0$, because [E A B] is of full row rank. Second statement. Again, assume that $\operatorname{im}(E) \neq \mathbb{R}^l$, and let (3.1b) be in the form (4.2) with $f_i = B_i u$ $(i = 1, 2)$. $\Leftarrow$ We have that $[A_{22}\ B_2]$ is of full row rank; set $R = A_{22}A_{22} + B_2B_2{}' > 0$. Let $\bar{x}_{01}$, $\bar{x}_{02}$ be arbitrary. The input $u = B_2{}'R^{-1}(-A_{21}\bar{x}_1)$ with $\bar{x}_1$ the solution of

$$pv = (A_{11} - [A_{12}\ B_1]\begin{bmatrix}A_{22}{}'\\ B_2{}'\end{bmatrix}R^{-1}A_{21})v + \bar{x}_{01},$$

is smooth and if $\bar{x}_2 = A_{22}'R^{-1}(-A_{21}\bar{x}_1)$, then $\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} \in C^n_{sm} \cap S_C(\begin{bmatrix}\bar{x}_{01}\\ x_{02}\end{bmatrix}, u)$ and $\bar{x}_1(0^+) = \bar{x}_{01}$. Hence we establish that every x_0 is weakly C-consistent. $\Rightarrow$ We must prove that $[A_{22}\ B_2]$ is right invertible. If $\eta[A_{22}\ B_2] = 0$, then $\eta A_{21}\bar{x}_{01} = 0$ for all $\bar{x}_{01}$ and hence $\eta[A_{21}\ A_{22}\ B_2] = 0$, i.e., $\eta = 0$. This completes the proof.

Remark 4.6.

The conditions in (4.3) imply right-invertibility of [A - sE, B] and hence also right-invertibility of [E A B]; note on the other hand that, without loss of generality, [E A B] may be assumed of full row rank in (3.1b). If $\det(sE - A) \neq 0$, then [A - sE, B] is right invertible, [E A B] is automatically of full rank and Theorem 4.5 reduces to [8, Theorem 3.8]. Example 3.7 does not satisfy (4.3b).

Example 4.7.

Consider the system

$$p\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_{01}\\ x_{02}\end{bmatrix}.$$

Clearly, $x_1 = p^{-1}x_{01}$, smooth, $x_1(0^+) = x_{01}$, and $u = -x_1$. Since for every x_{02} we can choose any smooth function x_2 with $x_2(0^+) = x_{02}$, we establish that every x_0 is C-consistent. Indeed, rank [E, B] = 1.

Example 4.8.

The system $p\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u + \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_{01}\\ x_{02}\end{bmatrix}$ is such that every x_0 is weakly C-consistent, but not C-consistent; if u = 0, then $x_1 = p^{-1}x_{01}$ and $x_2 = -x_1$. If $x_{01} + x_{02} \neq 0$, then there does not exist a smooth control u such that the unique state trajectory $x \in S_C(x_0, u)$ is smooth and $x(0^+) = x_0$.

Remark 4.9.

We have seen that the condition in (4.3b) is equivalent to the existence of a smooth control u and a smooth state trajectory x $\in S_C(x_0, u)$ for *every* initial condition x_0. In this sense the system (3.1b) may be called *impulse controllable* if (4.3b) is satisfied, since for every x_0 there exists a function u such that the solution set $S_C(x_0, u)$ has at least one element x that has no impulsive part. Although Cobb uses a different definition for impulse controllability in [4], he interprets it in the same way in [3] as we do here, and moreover, proves equivalence of his impulse controllability and (4.3b) by means of state space decomposition in [4, Theorem 4] for the case $\det(sE - A) \neq 0$. Our Theorem (4.5) shows the equivalence of (4.3b) and impulse controllability for arbitrary systems (3.1b) with [E A B] of full row rank. Also, observe that (4.3b) is expressed in the system coefficients only, without any extra parameter as in [18, Theorem 2].

Conclusions.

Our distributional framework for linear DAEs with constant coefficients and for singular systems on $\mathbb{R}^+$ covers well-known earlier DAE- and singular system interpretations. It enabled us to define satisfactory concepts for DAE- and system-solvability, in the distribution as well as in the function sense. We saw that DAE-solvability in the distribution sense is, essentially, equivalent to the usual concept of DAE-solvability, and derived a condition for system solvability. Then, consistency for DAEs and systems was redefined in terms of distributions and we introduced its generalization, *weak* consistency. Whereas a point is consistent if the corresponding solution set of the DAE contains a function that starts in that point, we call a point *weakly* consistent if this solution set merely contains a function. Finally, we presented conditions for global consistency and global weak consistency in the DAE and the system sense and established that global weak consistency in the system sense is equivalent to impulse controllability, i.e., to the possibility to find for every initial condition an input function that yields at least one functional state trajectory of the system. Because of linearity and of our special class of distributions, we could keep our treatment fully algebraic, and hence easily understandable.

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