ERSITY
KE
UNIVERSITEIT
BRABANT

POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

[redacted]

DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

ON THE DIFFERENTIABILITY OF THE SET OF EFFICIENT (μ,σ^2) COMBINATIONS IN THE MARKOVITZ PORTFOLIO SELECTION METHOD

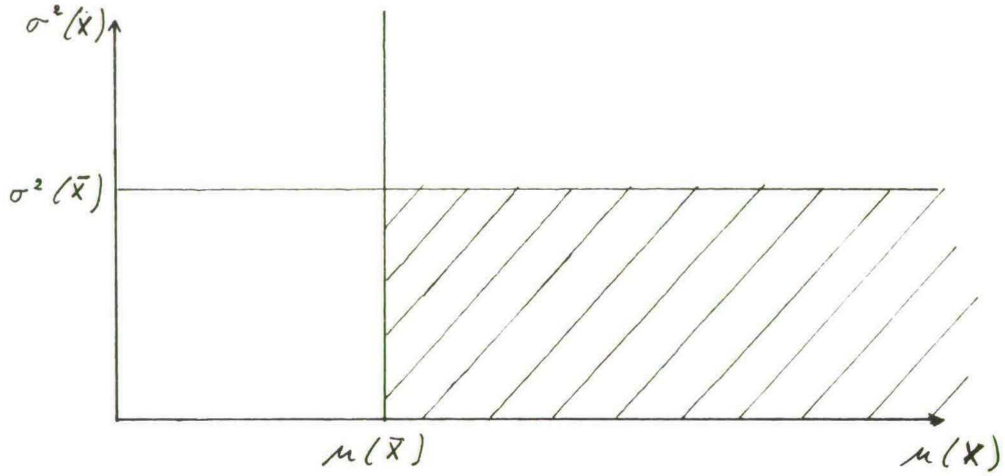
J. Kriens

FEW 420

Errata FEW 420

p. 1 This Memorandum earlier appeared as a chapter in "Twenty-five years of operations research in the Netherlands": Papers Dedicated to Gijs de Leve, C.W.I. Tract 70, edited by Jan Karel Lenstra, Henk Tijms, Ton Volgenant, Centre for Mathematics and Computer Science, Amsterdam (1989).

p. 2



p. 4 line 13 .... this nature

line 14 .... this nature

On the differentiability of the set of efficient (μ,σ^2) combinations in the Markowitz portfolio selection method

J. Kriens*)

Tilburg University, Department of Econometrics
PO Box 90153, 5000 LE Tilburg, The Netherlands

Contents

Abstract

In this paper differentiability properties of the set of efficient (μ,σ^2) combinations are discussed. After a review of statements made in the literature, two conditions for nondifferentiable points are derived and illustrated with some numerical examples.

---

*) I'm grateful to H. Geerts and A. Volgenant for their cricital remarks on an earlier version of this paper and to L.W.G. Strijbosch for his assistance in making the computations.

1. General

Markowitz studied the following investment problem, cf. H.M. Markowitz (1956), (1959). An investor wants to invest an amount b in the securities 1,...,n. If he invests an amount x_j in security j, then

$$(1.1)\qquad \sum_{j=1}^{n} x_j = b.$$

There may be more linear constraints; suppose

$$(1.2)\qquad \mathcal{A}X = B$$

and

$$(1.3)\qquad X \geqq \mathcal{O}$$

should be satisfied with $\mathcal{A}$ an (m×n)-matrix, B an m-vector and $X' = (x_1,\dots,x_n)$.
The yearly revenue on one dollar invested in security j is a random variable $\underline{r}_j$ with $\mathcal{E}\,\underline{r}_j = \mu_j$; the covariance matrix of the $\underline{r}_j$ equals $\mathcal{C}$. Denote the yearly revenue of a portfolio $X = (x_1,\dots,x_n)'$ by $\underline{r}(X)$, the expected value of $\underline{r}(X)$ by $\mu(X)$, its variance by $\sigma^2(X)$ and let $M' = (\mu_1,\dots\mu_n)$. Then

$$(1.4)\qquad \mu(X) = M'X$$

and

$$(1.5)\qquad \sigma^2(X) = X'\,\mathcal{C}\,X.$$

In order to find "good" solutions of the problem, a risk averse investor may put a restriction on $\mu(X)$ and then minimize $\sigma^2(X)$, or put a restriction on $\sigma^2(X)$ and next maximize $\mu(X)$. Markowitz studies the problem from a more general point of view and introduces the notion of <u>efficient portfolio</u>. A feasible portfolio $X = \bar{X}$ is efficient if:

a) no feasible portfolio has a revenue with larger or equal expected value and smaller variance, and

b) no feasible portfolio has a revenue with smaller or equal variance and larger expected value;

cf. H.M. Markowitz (1959), p. 310.

In the (μ,σ^2)-plane this means that if a portfolio $X = \bar{X}$ is efficient, there do not exist feasible portfolios with corresponding $(\mu(X),\sigma^2(X))$ points in the closed rectangle $\leqq \sigma^2(\bar{X})$ and $\geqq \mu(\bar{X})$, cf. fig. 1.1.

Fig. 1.1. No feasible portfolio with $(\mu(X),\sigma^2(X))$ in the shaded area.

According to Markowitz all efficient portfolios can be derived by solving

$$(1.6)\qquad \min_{X}\{X'\,\mathcal{C}\,X - \lambda M'X \mid \mathcal{A}\,X = B \wedge X \geqq \mathcal{O}\}$$

for all $\lambda \geqq 0$, cf. H.M. Markowitz (1959), p. 315-316. A precise and more general statement of the theorem underlying the algorithm is given by J. Kriens en J.Th. Van Lieshout (1988). In our case their theorem reduces to:

Theorem

A feasible portfolio $X = \bar{X}$ is efficient if and only if

a) there exists a $\bar{\lambda} > 0$, such that

$$(1.7)\qquad \min_{X}\{X'\,\mathcal{C}\,X - \bar{\lambda} M'X \mid \mathcal{A}\,X = B \wedge X \geqq \mathcal{O}\} = \bar{X}'\,\mathcal{C}\,\bar{X} - \bar{\lambda} M'\bar{X},$$

or b)

$$(1.8) \quad \max_X [M'X \mid X' \mathcal{C} X = \min_Y \{Y' \mathcal{C} Y \mid \mathcal{A} Y = B \wedge Y \geqq 0\}] = M'\bar{X},$$

or c)

$$(1.9) \quad \min_X [X' \mathcal{C} X \mid M'X = \max_Y \{M'Y \mid \mathcal{A} Y = B \wedge Y \geqq 0\}] = \bar{X}' \mathcal{C} \bar{X}.$$

Note that strictly speaking condition c) can be omitted because $M'X$ is a linear function of X.
Usually one starts with setting $\lambda = 0$ in (1.7), thus with determining the minimum value possible of the variance. Next λ is raised to get new efficient portfolios. For specific values of λ there is a change in the basis; suppose these values are $\bar{\lambda}_1, \ldots, \bar{\lambda}_k$ and that the corresponding efficient solutions are $\bar{X}_1, \ldots, \bar{X}_k$. We form the (sub) sequence $\bar{X}_{j_1}, \ldots, \bar{X}_{j_\ell}$ $(\ell \leqq k)$ from $\bar{X}_1, \ldots, \bar{X}_k$ for which the $(\bar{\mu}, \bar{\sigma}^2)$ combinations are different. This (sub) sequence is <u>the set of corner portfolios</u>.

The set of all $(\mu(\bar{X}), \sigma^2(\bar{X}))$ points in the (μ, σ^2)-plane corresponding to efficient portfolios $\bar{X}$ is <u>the set of efficient (μ, σ^2) combinations</u> of the problem. Between the (μ, σ^2) points of two adjacent corner portfolios it is part of a strictly convex parabola, cf. J. Kriens and J.Th. van Lieshout (1988), p. 185.
The question discussed in this paper concerns the differentiability properties of this set in the (μ, σ^2) points corresponding to corner portfolios. Section 2 reviews some statements made in the literature, section 3 summarizes the expressions given by J. Kriens and J.Th. van Lieshout (1988) for the values of the basic variables in a basic feasible solution and section 4 presents an example of nondifferentiability. Next necessary and sufficient conditions are derived for getting points of nondifferentiability, which conditions are verified for some numerical examples in section 6.

2. Driving through the literature on differentiability properties

Markowitz himself is not very clear in his statements on differentiability properties of the set of efficient (μ,σ^2) combinations. In his book he writes, cf. H.M. Markowitz (1959), p. 153:
"The set of points representing efficient portfolios turns a corner, forms a sharp kink, as our passenger transfers from one critical line to another. There is typically no such kink, however, in the curve describing the relation between E and V for efficient portfolios. ..... The relationship between V and E transfers from one parabola to the other without discontinuity or kink" (E is in our notation μ and V is σ^2).
And then two paragraphs further down:
"It is, however, possible for the curve relating efficient V to efficient E to have a kink. ..... Whenever a kink occurs, it must be of this nature rather than of this nature ."
Markowitz does not give a numerical example with a point in which the set of efficient (μ,σ^2) points is not differentiable.

After the book by Markowitz many articles and books appeared with statements on the differentiability properties of the set of efficient (μ,σ^2) combinations. It is not planned to revue them all but just to mention a few of the "highlights" in the literature. Keep in mind: the function in question is not necessarily differentiable everywhere, cf. the example in section 4.

An amusing mixture of mathematical and economic arguments is given by E.F. Fama and M.H. Miller (1972), p. 243. In a footnote they remark:
"We should note, for the mathematically more sophisticated, that the efficient set curve need not be differentiable everywhere, so that, strictly speaking, the representation of equilibrium in terms of a "tangency" could be incorrect. It can be shown, however, that the maximum number of points at which the efficient set curve is not differentiable cannot be greater than the numer N of available assets. With infinitely divisible assets, the number of efficient portfolios is infinite; that is, the efficient set curve is continuous. Thus these nondifferentiable points do not greatly

detract from our conclusions; in mathematical terms, they constitute a set of measure 0."
As stated at the end of section 1, between two corner portfolios the set is part of a convex parabola (as already shown by Markowitz); from the algorithm based on (1.6) it follows directly that the number of corner portfolios is finite, so Fama and Miller's conclusion is trivial and not very informative.

G.P. Szegö (1980) devotes chapter 12 to the investment problem with only the constraints (1.1) and (1.3). He introduces the notion "region of admissable portfolios $\mathcal{R}^n$ in the (μ,σ^2) plane", defined parametrically by the equations (1.4) and (1.5) subject to (1.1) and (1.3). The boundary $\bar{\mathcal{B}}^n$ of this region is defined by the minimal values of (1.5) subject to (1.1), (1,3) and (1.4) and therefore coincides with the set of efficient (μ,σ^2) points. His conclusion about the differentiability of this set runs (cf. p 135): "In all circumstances, however, if follows that "The boundary $\bar{\mathcal{B}}^n$ of the region of admissable portfolios with nonnegativity constraints on the allocation vector ..... is represented on the plane (ν,π) by a continuously differentiable curve composed of a sequence of arcs of parabolas each of which belongs to the boundary of the region of admissable portfolios of a subset of the set of n investments". (the plane (ν,π) is our (μ,σ^2) plane).

The "proof" is based on Szegö's analysis of the properties of $\bar{\mathcal{B}}^n$. He also developes an algorithm to identify $\bar{\mathcal{B}}^n$.
The argument is rather lengthy and will not be repeated here. Moreover his conclusion on p. 135 that "their common points ..... are true tangency points" is not generally correct as is shown by the example in section 4..

The last author to be quoted is J. Vörös. He states: "It can easily be seen that parabolas describing efficient return-variance connection at intervals $[c_{i-1},c_i]$ and $[c_i, c_{i+1}]$ respectively have the same values at c_i and do not intersect each other.
Otherwise the solution would not be optimal at interval $[c_{i-1},c_i]$. Thus we

can state the following theorem. The function $Z^{+}(C)$ is continuously differentiable and convex", cf. J. Vörös (1986), p. 298 (c is in our notation μ and $Z^{+}(c)$ is $\sigma^2(\mu)$).
To be sure he modifies this statement in a subsequent contribution, cf. J. Vörös (1987), p. 305. The theorem now runs: "The efficient frontier $Z^2_+(c)$ is continuously differentiable except in points where $a_i = a_j$ for all $i,j \in M$" (a_i is μ_i in our notation, $Z^2_+(c)$ is again $\sigma^2(\mu)$ and M is the set of x_j-variables being in the basis). Because the condition $a_i = a_j$ for all $i,j \in M$ only makes sense if M contains at least 2 elements, as Vörös assumes indeed, the restriction in the theorem relates to efficient (μ,σ^2) points with 2 or more x_j variables in the basis. The proof does not take into account cases in which M contains only one element, and then the set may be indifferentiable as the example in section 4 shows. So, this mere point already implies that the formulation as well as the proof of the theorem is not correct.

Vörös develops the same algorithm for identifying the efficient (μ,σ^2) points as Szegö did, but both do not prove that all efficient points are actually found in this way. Note that this algorithm is different from the algorithm based on (1.6). As a matter of fact the solution presented for the second problem in J. Vörös (1986) is incorrect; the efficient point with minimum variance is the point $(8.3\times10^3;\ 10.53\times10^6)$ and not the point shown in Vörös' figure 2.

3. Explicit expressions for efficient portfolios

Starting from the Kuhn-Tucker conditions for the optimal solution of (1.6), Kriens and Van Lieshout (1988) derive an expression for the values of the basic variables which, if $\mathcal{C}$ is positive definite, holds for every efficient portfolio. With constraints

$$(3.1) \qquad \mathcal{A}\, X \leqq B$$

rather than (1,2), the Kuhn-Tucker conditions run

$$(3.2) \qquad -2\,\mathcal{C}\,\bar{X} \qquad - \mathcal{A}'\bar{U} + \bar{V} = -\bar{\lambda}M$$

$$(3.3) \qquad \mathcal{A}\,\bar{X} \qquad + \bar{Y} \qquad = B$$

$$(3.4) \qquad \bar{V}'\bar{X} = 0,\ \bar{U}'\bar{Y} = 0,\ \bar{X},\bar{Y},\bar{U},\bar{V} \geqq \mathcal{O}\ ;$$

$\bar{Y}$ contains the values of the slack variables, $\bar{U}$ and $\bar{V}$ the values of the vectors of Lagrange multipliers.
Omitting bars to get variables X, Y, U and V, the equations (3.2) and (3.3) can be summarized as

(3.5)

X'	Y'	U'	V'	
$-2\,\mathcal{C}$	$\mathcal{O}$	$-\mathcal{A}'$	$\mathcal{I}$	$-\bar{\lambda}M$
$\mathcal{A}$	$\mathcal{I}$	$\mathcal{O}$	$\mathcal{O}$	B

.

If

$$(3.6) \qquad Z_b' = (X_b', Y_b', U_b', V_b')$$

denotes the set of basic variables for a given efficient portfolio, (3.5) can be partitioned into

X'_b	X'_{nb}	Y'_b	Y'_{nb}	U'_b	U'_{nb}	V'_b	V'_{nb}	
$-2\ \mathcal{C}_{b_1}$	$-2\ \mathcal{C}_{nb_1}$	$\mathcal{O}$	$\mathcal{O}$	$-\ \mathcal{A}'_{b_1}$	$\mathcal{A}'_{b_2}$	$\mathcal{O}$	$\mathcal{I}$	$-\bar{\lambda}M_{b_1}$
$-2\ \mathcal{C}_{b_2}$	$-2\ \mathcal{C}_{nb_2}$	$\mathcal{O}$	$\mathcal{O}$	$-\ \mathcal{A}'_{nb_1}$	$\mathcal{A}'_{nb_2}$	$\mathcal{I}$	$\mathcal{O}$	$-\bar{\lambda}M_{b_2}$
$\mathcal{A}_{b_1}$	$\mathcal{A}_{nb_1}$	$\mathcal{O}$	$\mathcal{I}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	B_{b_1}
$\mathcal{A}_{b_2}$	$\mathcal{A}_{nb_2}$	$\mathcal{I}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	$\mathcal{O}$	B_{b_2}

The matrix $-2\ \mathcal{C}$ is partitioned into the square matrices $-2\ \mathcal{C}_{b_1}$ and $-2\ \mathcal{C}_{nb_2}$ corresponding to basic and non-basic variables x_j and into $-2\ \mathcal{C}_{b_2}$ and $-2\ \mathcal{C}_{nb_1}$ with $\mathcal{C}_{b_2} = \mathcal{C}'_{nb_1}$. $\mathcal{A}_{b_1}$, $\mathcal{A}_{nb_1}$ and B_{b_1} respresent the active constraints, $\mathcal{A}_{b_2}$, $\mathcal{A}_{nb_2}$ and B_{b_2} the non-active constraints. Therefore there are identity matrices in the fourth place of the Y'_b column and in the third place of the Y'_{nb} column. The matrix of coefficients of basic variables is

$$(3.8)\quad \mathcal{B} = \begin{bmatrix} -2\ \mathcal{C}_1 & \mathcal{O} & -\ \mathcal{A}'_{b_1} & \mathcal{O} \\ -2\ \mathcal{C}_{b_2} & \mathcal{O} & -\ A'_{nb_1} & \mathcal{I} \\ & & & \\ \mathcal{A}_{b_1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{A}_{b_2} & \mathcal{I} & \mathcal{O} & \mathcal{O} \end{bmatrix} .$$

To facilitate computations Kriens and Van Lieshout reshuffle (3.8) into

$$(3.9)\qquad \mathcal{B}_v = \begin{bmatrix} -2\ \mathcal{C}_{b_1} & -\ \mathcal{A}'_{b_1} & \mathcal{O} & \mathcal{O} \\ \mathcal{A}_{b_1} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ & & & \\ -2\ \mathcal{C}_{b_2} & -\ \mathcal{A}'_{nb_1} & \mathcal{I} & \mathcal{O} \\ \mathcal{A}_{b_2} & \mathcal{O} & \mathcal{O} & \mathcal{I} \end{bmatrix} .$$

The values of the basic variables are

$$(3.10)\quad \bar{Z}_{bv} = \mathcal{B}_v^{-1} \begin{bmatrix} \mathcal{O} \\ B_{b_1} \\ \mathcal{O} \\ B_{b_2} \end{bmatrix} - \bar{\lambda}\, \mathcal{B}_v^{-1} \begin{bmatrix} M_{b_1} \\ \mathcal{O} \\ M_{b_2} \\ \mathcal{O} \end{bmatrix}$$

with $\bar{Z}'_{bv} = (\bar{X}'_b, \bar{U}'_b, \bar{V}'_b, \bar{Y}'_b)$. Explicit expressions for the values of the basic variables are found by computing $\mathcal{B}_v^{-1}$:

$$(3.11)\quad \mathcal{B}_v^{-1} = \left[\begin{array}{c|c} \begin{bmatrix} -2\,\mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1} & \mathcal{O} \\ \hline -\begin{bmatrix} -2\,\mathcal{C}_{b_2} & -\mathcal{A}'_{nb_1} \\ \mathcal{A}_{b_2} & \mathcal{O} \end{bmatrix} \begin{bmatrix} -2\,\mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1} & \begin{bmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{I} \end{bmatrix} \end{array}\right]$$

with

$$\begin{bmatrix} -2\,\mathcal{C}_{b_1} & -\mathcal{A}'_{b_1} \\ \mathcal{A}_{b_1} & \mathcal{O} \end{bmatrix}^{-1} =$$

$$(3.12)\quad \left[\begin{array}{c|c} -\frac{1}{2}\,\mathcal{C}_{b_1}^{-1} + \frac{1}{2}\,\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1} & \mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1} \\ \hline -(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1} & -2(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1} \end{array}\right].$$

Substituting (3.12) into (3.11) and the result into (3.10), they find

(3.13) $\bar{X}_b = A + D\bar{\lambda}$

with

(3.14) $A = \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1} (\mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1})^{-1} B_{b_1}$

and

(3.15) $D = \frac{1}{2}\left[\mathcal{C}_{b_1}^{-1} - \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1} (\mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1})^{-1} \mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \right] M_{b_1}.$

The corresponding values $\mu(\bar{X}_b)$ and $\sigma^2(\bar{X}_b)$ are

(3.16) $\mu(\bar{X}_b) = M'_{b_1} A + M'_{b_1} D\bar{\lambda}$

(3.17) $\sigma^2(\bar{X}_b) = A' \mathcal{C}_{b_1} A + 2A' \mathcal{C}_{b_1} D\bar{\lambda} + D' \mathcal{C}_{b_1} D\bar{\lambda}^2.$

For the proofs, see appendix A of their contribution.

4. Looking at an example of nondifferentiability

The following example has a point of nondifferentiability; it originates with Markowitz and was handed to me by Vörös. The data are

$$(4.1)\qquad M = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix},\ \mathscr{C} = \begin{bmatrix} 3 & 3 & -1 \\ 3 & 11 & 23 \\ -1 & 23 & 75 \end{bmatrix},\ \mathscr{A} = (1 \quad 1 \quad 1),\ B = (1).$$

For this problem conditions (3.2),...,(3.4) reduce to

$$(4.2)\qquad \begin{array}{rrrrrrrl} -6x_1 & -6x_2 & +2x_3 & -u_1 & +v_1 & & & = -\lambda \\ -6x_1 & -22x_2 & -46x_3 & -u_1 & & +v_2 & & = -3\lambda \\ 2x_1 & -46x_2 & -150x_3 & -u_1 & & & +v_3 & = -5\lambda \end{array}$$

$$(4.3)\qquad x_1 + x_2 + x_3 + y_1 = 1$$

$$(4.4)\qquad \sum_{j=1}^{3} v_j x_j = 0,\ u_1 y_1 = 0,\ X,Y,U,V \geqq \mathcal{O}\ ;$$

the bars denoting optimal values are omitted.

In order to perform the portfolio solution analysis a user written subroutine has been linked to the linear optimization package LINDO. In that subrountine special features of LINDO like the parametric analysis option have been used.

Table 4.1

Basic solutions of the example

$\bar{\lambda}$	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{\mu}$	$\bar{\sigma}^2$
0	0.950	0	0.050	1.20	2.80
3	0.875	0	0.125	1.50	3.25
4	0.500	0.500	0	2.00	5.00
8	0	1.000	0	3.00	11.00
12	0	1.000	0	3.00	11.00
52	0	0	1.000	5.00	75.00

With formulae (3.16) and (3.17) the relationships between $\mu(\bar{X}_b)$, $\bar{\lambda}$ and $\sigma^2(\bar{X}_b)$, $\bar{\lambda}$ can be derived. It is found that for the corner portfolio $\bar{X}' = (0 \;\; 1 \;\; 0)$ with $(\bar{\mu},\bar{\sigma}^2) = (3,11)$ the left hand side derivative of the efficient (μ,σ^2) set equals 8 in (3,11) whereas the right hand side derivative equals 12. So the set of efficient (μ,σ^2) points is not differentiable in the point (3,11). In the computations this property is revealed by the production of 2 successive bases with different values of $\bar{\lambda}$ but the same optimal $\bar{X}$-vector. The results are also in agreement with

$$(4.5) \qquad \left[\frac{d\sigma^2}{d\mu}\right]_{(\bar{\mu},\bar{\sigma}^2)} = \bar{\lambda}$$

if the set is differentiable, $\lim\limits_{\mu\uparrow 3} \dfrac{d\sigma^2}{d\mu} = 8$ and $\lim\limits_{\mu\downarrow 3} \dfrac{d\sigma^2}{d\mu} = 12$.

However, the algorithm does not show any computational problems, this as opposed to a conjecture by Vörös concerning his own algorithm: "This counterexample shows that the procedure suggested by Szegö and of the author may not be valid so generally as it is stated.....", cf. J. Vörös (1987), p. 305.

5. Evident necessary and sufficient conditions for nondifferentiability

Inspection of the example in section 4 makes clear that a point of nondifferentiability in the set of efficient (μ,σ^2) points comes into being if for a range of $\bar{\lambda}$ values the vector $\bar{X}_b$ remains the same. From (3.13) it follows that this is the case if and only if D equals $\mathcal{O}$. Define $\mu_{min} := \min_i \mu_i$ and $\mu_{max} := \max_i \mu_i$; then for an efficient (μ,σ^2) point with $\mu \in (\mu_{min},\mu_{max})$ a necessary and sufficient condition for nondifferentiability runs $D = \mathcal{O}$. The next 2 theorems exploit this property for the problem with only the restrictions (1.1) and (1.3).

Theorem 5.1

If in the investment problem subject to (1.1) and (1.3) $\mathcal{C}$ is positive definite and a corner portfolio with $\mu \in (\mu_{min},\mu_{max})$ contains only one x-variable > 0, then the set of efficient (μ,σ^2) points is nondifferentiable in that point.

Proof

Suppose $\bar{x}_i > 0$, then $\bar{x}_i = b$, $\mathcal{C}_{b_1} = (c_{ii})$, $\mathcal{A}_{b_1} = (1)$, $M_{b_1} = (\mu_i)$.
From (3.15) it follows

$$(5.1)\qquad D = \frac{1}{2}\,\mathcal{C}_{b_1}^{-1}[\,\mathcal{I} - \mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}]M_{b_1}.$$

Substitution of the values of $\mathcal{A}_{b_1}$ en $\mathcal{C}_{b_1}^{-1}$ shows

$$(5.2)\qquad \mathcal{I} - \mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1} = \mathcal{O}\,.$$

so $D = \mathcal{O}$ and $\bar{X}_b = A$, cf. (3.13). q.e.d.

Now suppose a corner portfolio contains as basic variables $X'_b = (x_1,\ldots,x_k)$ $(k > 1)$. Define

(5.3) $\mathcal{M} = (m_{ij}) := \mathcal{C}_{b_1}^{-1}$

(5.4) $f := \sum_{i=1}^{k} \sum_{j=1}^{k} m_{ij}$

(5.5) $d := \sum_{i=1}^{k} (\sum_{j=1}^{k} m_{ij}\mu_j).$

Theorem 5.2

If in the investment problem subject to (1.1) and (1.3) $\mathcal{C}$ is positive definite and a corner portfolio with $\mu \in (\mu_{min}, \mu_{max})$ contains k (> 1) variables > 0, then the set of efficient (μ,σ^2) points is nondifferentiable in that point if and only if all corresponding μ-values are equal to $\frac{d}{f}$.

Proof

Let $X_b = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$, $\mathcal{C}_{b_1} = \begin{bmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & & \vdots \\ c_{k1} & & c_{kk} \end{bmatrix}$, $\mathcal{A}'_{b_1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $M_{b_1} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix}$,

then

(5.6) $(\mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1})^{-1} = \frac{1}{f}$

and $D = \mathcal{O}$ can be reduced to

(5.7) $f\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix} - \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$,

or

(5.8) $\forall_{i \in \{1,\ldots,k\}}\ \mu_i = \frac{d}{f}$.

So D = $\mathcal{O}$ if and only if (5.8) holds. q.e.d.

Remark, As a consequence of these theorems, D may be a zero vector and therefore the statement by Kriens and Van Lieshout (1988) that $M'_{b_1}.D$ is always $\neq 0$ (p. 187) cannot be generally correct. In their "proof", see appendix B of the article, the matrix $\mathcal{B}^*_v$ not necessarily has an inverse as is illustrated by the example in section 4: for the efficient portfolio (0 1 0) their matrix $\mathcal{B}^*_v$ equals

$$(5.9) \qquad \mathcal{B}^*_v = \left[\begin{array}{cccc|c} -22 & -1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ -6 & -1 & 1 & 0 & 1 \\ -46 & -1 & 0 & 1 & 5 \\ \hline 3 & 0 & 0 & 0 & 0 \end{array}\right].$$

6. Verification of the conditions in some examples

In this section forementioned formulae and conditions are illustrated with the help of some examples.

Example 6.1: data see section 4.

In the case of corner portfolio $\bar{X}' = (0\ 1\ 0)$ there is only one x-variable > 0 and the set of efficient (μ,σ^2) points is indeed nondifferentiable in the corresponding point $(\bar{\mu},\bar{\sigma}^2) = (3,11)$. Substitution of the data in (3.15) leads to $D = \mathcal{O}$.
The behaviour of the dual variables is also clear. If (3.12) is substituted in (3.11) and the result into (3.10), we get

$$(6.1) \quad \bar{U}_b = -2(\mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1})^{-1} B_{b_1} + \bar{\lambda}(\mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1})^{-1} \mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} M_{b_1} .$$

For $\bar{\lambda} = 0$, x_1 and x_3 are basic variables and then

$$\bar{u}_1 = 2(\mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1})^{-1} B_{b_1} = -5.6;$$

if we look at the corner portfolio $\bar{X}' = (0\ 1\ 0)$, then only x_2 is basic variable and from (6.1) it follows

$$\bar{u}_1 = -44 + 3\bar{\lambda},$$

so if $\bar{\lambda}$ rises from 8 to 12, the value of $\bar{u}_1$ rises from -20 to -8.
In the same way the values of $\bar{V}_b$ can be derived from the third "row" in (3.10). Therefore we need the elements in the third "row" of (3.11). The first two elements in this "row" of $\mathcal{B}_v^{-1}$ are

$$(6.2) \quad \mathcal{C}_{b_2}\{- \mathcal{C}_{b_1}^{-1} + \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1} (\mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1})^{-1} \mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1}\}$$

$$- \mathcal{A}'_{nb_1} (\mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1} \mathcal{A}'_{b_1})^{-1} \mathcal{A}_{b_1} \mathcal{C}_{b_1}^{-1}$$

and

(6.3) $2\,\mathcal{C}_{b_2}\{\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\} + \mathcal{A}'_{nb_1}\{-2(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\};$

the third element equals $\mathcal{I}$ and the fourth $\mathcal{O}$. So

(6.4) $$\bar{V}_b = [2\,\mathcal{C}_{b_2}\{\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\} + \mathcal{A}'_{nb_1}\{-2(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\}] .B_{b_1} - \lambda[\{\mathcal{C}_{b_2}\{-\mathcal{C}_{b_1}^{-1} + \mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1})^{-1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\} - \mathcal{A}'_{nb_1}(\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1}) . \mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\}M_{b_1} + \mathcal{I}.M_{b_2}].$$

If conditions (1.2) only consist of $\sum_{j=1}^{n} x_j = 1$, then, using (5.4) and (5.6), (6.4) can be simplified to

(6.5) $$\bar{V}_b = \frac{2}{f}(\mathcal{C}_{b_2}\mathcal{C}_{b_1}^{-1}\mathcal{A}'_{b_1} - \mathcal{A}'_{nb_1}) - \bar{\lambda}[\{\mathcal{C}_{b_2}\mathcal{C}_{b_1}\{-\mathcal{I} + \frac{1}{f}\mathcal{A}'_{b_1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\} - \frac{1}{f}\mathcal{A}'_{nb_1}\mathcal{A}_{b_1}\mathcal{C}_{b_1}^{-1}\}.M_{b_1} + \mathcal{I}\,M_{b_2}].$$

In the case of the efficient portfolio (0 1 0) in the example (6.5) reduces to

(6.6) $$\bar{V}_b = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_3 \end{bmatrix} = \begin{bmatrix} -16 + 2\bar{\lambda} \\ 24 - 2\bar{\lambda} \end{bmatrix}.$$

If $\bar{\lambda}$ is raised, for $\bar{\lambda} = 8$, x_1 leaves the basis and v_1 comes in and for $\bar{\lambda} = 12$, $\bar{v}_3 = 0$, so for $\bar{\lambda} > 12$, v_3 leaves the basis and x_3 comes in; cf. also table 4.1.

Example 6.2

Assume

$$(6.7) \quad M = \begin{bmatrix} 1 \\ 5 \\ 5 \\ 10 \end{bmatrix}, \ \mathscr{C} = \begin{bmatrix} 1 & 1.4 & 1.5 & 16 \\ 1.4 & 4 & 0 & 32 \\ 1.5 & 0 & 8 & 6 \\ 16 & 32 & 6 & 400 \end{bmatrix}, \ \mathscr{A} = (1 \ \ 1 \ \ 1 \ \ 1), \ B = (1).$$

Starting from the conditions (3.2),...,(3.4) the LINDO optimization routine generates the basic solutions presented in table 6.1.

Table 6.1

Basic solutions of example 6.2

$\bar{\lambda}$	$\bar{x}_1$	$\bar{x}_2$	$\bar{x}_3$	$\bar{x}_4$	$\bar{\mu}$	$\bar{\sigma}^2$
0	1.00	0	0	0	1.00	1.00
0.200	1.00	0	0	0	1.00	1.00
0.227	0.98	0.02	0	0	1.10	1.02
0.617	0	0.67	0.33	0	5.00	2.67
8.267	0	0.67	0.33	0	5.00	2.67
36.471	0	0	0.76	0.24	6.18	28.98
157.600	0	0	0	1.00	10.00	400.00

The set of efficient (μ,σ^2) points is not differentiable in the point $(\bar{\mu},\bar{\sigma}^2)$ = (5.00,2.67) corresponding to the efficient portfolio $\bar{X}'$ = (0 0.67 0.33 0). According to theorem 5.2 this behaviour was to be expected. The set of corresponding $\bar{\lambda}$ values equals $[0.617 \leqq \bar{\lambda} \leqq 8.267]$. Substitution of

$$(6.8) \quad M_{b_1} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \ \mathscr{C}_{b_1} = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}, \ \mathscr{A}_{b_1} = (1 \ \ 1), \ M_{b_1} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

in (3.15) results in $D = \mathscr{O}$.

The values of the dual variables can be derived by substituting (6.8) into (6.1) and (6.5) respectively. In the latter case we find

$$(6.9) \quad \bar{V}_b = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_4 \end{bmatrix} = \begin{bmatrix} -2.467 + 4\bar{\lambda} \\ 41.333 - 5\bar{\lambda} \end{bmatrix} ;$$

so for $\bar{\lambda} = 0.617$ $\bar{v}_1$ is > 0 and enters the basis, whereas for $\bar{\lambda} = 8.267$ $\bar{v}_4$ becomes < 0 and leaves the basis.

The last example is designed by H. Geerts; in this case the theorems of section 5 do not apply because besides condition (1.1) there is one more constraint.

Example 6.3

Let

$$(6.10)\quad M = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathscr{A} = \begin{bmatrix} 1 & 1 \\ \frac{8}{7} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ \frac{15}{14} \end{bmatrix}.$$

Using the conditions (3.2),...,(3.4) the basic solutions presented in table 6.2 are found.

Table 6.2
Basic solutions of example 6.3

$\bar{\lambda}$	$\bar{x}_1$	$\bar{x}_2$	$\bar{\mu}$	$\bar{\sigma}^2$
0	0	0	0	0
1.333	0.333	0.667	1.333	0.889
3.000	0.500	0.500	1.500	1.250
3.333	0.500	0.500	1.500	1.250
8.750	0.938	0	1.875	3.516

The set of efficient (μ,σ^2) points is nondifferentiable in $(\bar{\mu},\bar{\sigma}^2) = (1.500, 1.250)$, the corresponding values of $\bar{\lambda}$ are $[3.000 \leq \bar{\lambda} \leq 3.333]$. The value of D equals $\mathcal{O}$ because as all reciprocals exist the expression between square brackets in (5.1) is the zero matrix. There are no basic variables v_j whereas the expression for $\bar{U}_b$ follows from (6.1) and runs

$$(6.11)\quad \bar{U}_b = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} 20 - 6\bar{\lambda} \\ -21 + 7\bar{\lambda} \end{bmatrix}.$$

7. Evidential matter

Fama, E.F. and M.H. Miller (1972), The Theory of Finance, Holt, Rinehart and Winston, New York.

Kriens, J. and J.Th. van Lieshout (1988), Notes on the Markowitz portfolio selection method, Statistica Neerlandica 42, 181-191.

Markowitz, H.M. (1956), The Optimization of a Quadratic Function subject to Linear Constraints, Naval Research Logistics Quarterly 3, 111-133.

Markowitz, H.M. (1959), Portfolio Selection, John Wiley and Sons, New York.

Szegö, G.P. (1980), Portfolio Theory with Application to Bank Asset Management, Academic Press, New York.

Vörös, J. (1986), Portfolio analysis - An analytic derivation of the efficient portfolio frontier, European Journal of Operational Research 23, 294-300.

Vörös, J. (1987), The explicit derivation of the efficient portfolio frontier in the case of degeneracy and general singularity, European Journal of Operational Research 32, 302-310.

IN 1989 REEDS VERSCHENEN

368 Ed Nijssen, Will Reijnders
"Macht als strategisch en tactisch marketinginstrument binnen de distributieketen"

369 Raymond Gradus
Optimal dynamic taxation with respect to firms

370 Theo Nijman
The optimal choice of controls and pre-experimental observations

371 Robert P. Gilles, Pieter H.M. Ruys
Relational constraints in coalition formation

372 F.A. van der Duyn Schouten, S.G. Vanneste
Analysis and computation of (n,N)-strategies for maintenance of a two-component system

373 Drs. R. Hamers, Drs. P. Verstappen
Het company ranking model: a means for evaluating the competition

374 Rommert J. Casimir
Infogame Final Report

375 Christian B. Mulder
Efficient and inefficient institutional arrangements between governments and trade unions; an explanation of high unemployment, corporatism and union bashing

376 Marno Verbeek
On the estimation of a fixed effects model with selective non-response

377 J. Engwerda
Admissible target paths in economic models

378 Jack P.C. Kleijnen and Nabil Adams
Pseudorandom number generation on supercomputers

379 J.P.C. Blanc
The power-series algorithm applied to the shortest-queue model

380 Prof. Dr. Robert Bannink
Management's information needs and the definition of costs, with special regard to the cost of interest

381 Bert Bettonvil
Sequential bifurcation: the design of a factor screening method

382 Bert Bettonvil
Sequential bifurcation for observations with random errors

383 Harold Houba and Hans Kremers
Correction of the material balance equation in dynamic input-output models

384 T.M. Doup, A.H. van den Elzen, A.J.J. Talman
Homotopy interpretation of price adjustment processes

385 Drs. R.T. Frambach, Prof. Dr. W.H.J. de Freytas
Technologische ontwikkeling en marketing. Een oriënterende beschouwing

386 A.L.P.M. Hendrikx, R.M.J. Heuts, L.G. Hoving
Comparison of automatic monitoring systems in automatic forecasting

387 Drs. J.G.L.M. Willems
Enkele opmerkingen over het inversificerend gedrag van multinationale ondernemingen

388 Jack P.C. Kleijnen and Ben Annink
Pseudorandom number generators revisited

389 Dr. G.W.J. Hendrikse
Speltheorie en strategisch management

390 Dr. A.W.A. Boot en Dr. M.F.C.M. Wijn
Liquiditeit, insolventie en vermogensstructuur

391 Antoon van den Elzen, Gerard van der Laan
Price adjustment in a two-country model

392 Martin F.C.M. Wijn, Emanuel J. Bijnen
Prediction of failure in industry
An analysis of income statements

393 Dr. S.C.W. Eijffinger and Drs. A.P.D. Gruijters
On the short term objectives of daily intervention by the Deutsche Bundesbank and the Federal Reserve System in the U.S. Dollar - Deutsche Mark exchange market

394 Dr. S.C.W. Eijffinger and Drs. A.P.D. Gruijters
On the effectiveness of daily interventions by the Deutsche Bundesbank and the Federal Reserve System in the U.S. Dollar - Deutsche Mark exchange market

395 A.E.M. Meijer and J.W.A. Vingerhoets
Structural adjustment and diversification in mineral exporting developing countries

396 R. Gradus
About Tobin's marginal and average q
A Note

397 Jacob C. Engwerda
On the existence of a positive definite solution of the matrix equation $X + A^T X^{-1} A = I$

398 Paul C. van Batenburg and J. Kriens
Bayesian discovery sampling: a simple model of Bayesian inference in auditing

399 Hans Kremers and Dolf Talman
Solving the nonlinear complementarity problem

400 Raymond Gradus
Optimal dynamic taxation, savings and investment

401 W.H. Haemers
Regular two-graphs and extensions of partial geometries

402 Jack P.C. Kleijnen, Ben Annink
Supercomputers, Monte Carlo simulation and regression analysis

403 Ruud T. Frambach, Ed J. Nijssen, William H.J. Freytas
Technologie, Strategisch management en marketing

404 Theo Nijman
A natural approach to optimal forecasting in case of preliminary observations

405 Harry Barkema
An empirical test of Holmström's principal-agent model that tax and signally hypotheses explicitly into account

406 Drs. W.J. van Braband
De begrotingsvoorbereiding bij het Rijk

407 Marco Wilke
Societal bargaining and stability

408 Willem van Groenendaal and Aart de Zeeuw
Control, coordination and conflict on international commodity markets

409 Prof. Dr. W. de Freytas, Drs. L. Arts
Tourism to Curacao: a new deal based on visitors' experiences

410 Drs. C.H. Veld
The use of the implied standard deviation as a predictor of future stock price variability: a review of empirical tests

411 Drs. J.C. Caanen en Dr. E.N. Kertzman
Inflatieneutrale belastingheffing van ondernemingen

412 Prof. Dr. B.B. van der Genugten
A weak law of large numbers for m-dependent random variables with unbounded m

413 R.M.J. Heuts, H.P. Seidel, W.J. Selen
A comparison of two lot sizing-sequencing heuristics for the process industry

IN 1990 REEDS VERSCHENEN

419 Bertrand Melenberg, Rob Alessie
A method to construct moments in the multi-good life cycle consumption model