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Dynamic firm behaviour and  
financial leverage clienteles

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## Abstract

In this paper we will study the impact of both corporate and personal taxation on the optimal evolution pattern of a firm. The firm is described by an optimal control model. It may attract two kinds of money capital: equity by retaining earnings and debt. We will discuss the successive stages in the evolution of the firm, derive some optimal decision rules and show the similarities with the theory of 'financial leverage clienteles'.

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## Dynamic firm behaviour and financial leverage clienteles

G..J.C.Th. van Schijndel

1. Introduction

The effect of corporate taxes on the market value of a levered firm continues to be a central issue in recent contributions in finance theory. After the publication of the Modigliani & Miller (1963) tax correction paper many writers have sought to reconcile the M&M maximum leverage prediction with observed capital structure.

Although the idea of financial leverage clienteles has appeared in literature before Miller (1977) used it to argue that personal taxes could offset corporate taxes such that in equilibrium the value of any individual firm would be independent of its leverage. This approach caused a stream of contributions like Kim, Lewellen & McConnel (1979), DeAngelo & Masulis (1980) and Kim (1982), studying and extending the Miller-hypothesis that companies following a low leverage strategy would find a market along investors in high tax brackets and the stock of highly levered firms would be held by investors with low personal tax rates.

The purpose of this paper is to add another dimension to this discussion: dynamics. As the survey of Feichtinger (1982a) and the collections of Bensoussan, Kleindorfer & Tapiero (1978) and Feichtinger (1982b) very well show many recent papers extend the theory of the firm using optimal control techniques to solve real dynamic models in an analytical way. Those models provide insight of the relevance of time and the evolution of variables in course of time.

According to the first part of this introduction our research is related to the financial models like those of Lesourne (1973), Leland (1972), Sethi (1978), Van Loon (1983) and the survey of Lesourne & Leban (1982), which are dealing with the dynamic problems of finance, dividend and investment. Although many authors are aware of the disturbing influence

personal taxation possibly provides, they do not care. Research to this subject has been done by Ylä-Liedenpohja (1978), but assuming an infinite time horizon and taken debt financing not especially into account a number of interesting topics are left out of consideration.

So, the aim of this contribution is to extend this part of the dynamic theory by introducing both corporate and personal taxation in a way we can transfer the discussion of financial leverage clientele into a dynamic environment.

In section 2, therefore, we will describe a deterministic dynamic model of a firm which behaves as if it maximizes its value conceived by tax homogeneous shareholders. We will distinguish different personal and corporate tax rates. Using Optimal Control theory an analytical solution of the problem will be derived in section 3. The sections 4 and 5 present an economic interpretation and a further analysis of the results. As we distinguish personal tax rates on capital gain and dividend income the firm turns out to be able to invest in projects with a net return less than the discount rate of the shareholders. Besides this, the firm will at last finance all investments with only equity even when debt financing is cheap compared to equity. Finally in section 6 we will present the results of a sensitivity analysis concerning parameters that are interesting for economic analysis such as the personal tax rates and the discount rate. This analysis will also illustrate the similarity with the Miller-hypothesis.

## 2. The model

In the deterministic model of the firm, we will present in this section, the firm concerned has only one production factor: capital goods  $K$ . The firm may attract two kinds of money capital, equity  $X$  and debt  $Y$ , so from the balance sheet we derive that

$$(1) \quad K(T) = X(T) + Y(T)$$

in which

$T$  = time.



We further assume that the firm is operating under decreasing returns to scale caused by the imperfect output markets and/or increasing marginal costs of organizing the production due to the increasing scale of the firm. Therefore, revenue before interest and taxation is a concave function of the amount of capital goods. No transaction costs are incurred when borrowing or paying off debt money, corporate tax is proportional to profit and is paid at once. Issues of new shares are not allowed. Earnings after corporate taxation are used to issue dividends or to increase the value of equity through retained earnings:

$$(2) \quad \dot{X} := \frac{dX}{dT} = (1 - \tau_c)[O(K) - rY] - D$$

in which

$O(K)$  = operating income, i.e. revenue before interest and corporate taxation,  $O(K) > 0$ ,  $\frac{dO}{dK} > 0$ ,  $\frac{d^2O}{dK^2} < 0$

$D(T)$  = dividend payments

$\tau_c$  = corporate tax rate

$r$  = market interest rate.

Depreciation is assumed to be proportional to capital goods. The impact of investments on the production capacity is described by the general used formulation of net investments:

$$(3) \quad \dot{K} := \frac{dK}{dT} = I - aK$$

in which

$I(T)$  = gross investments

$a$  = depreciation rate.

Next we will limit the amount of debt. On page 58 of his book Ludwig (1978) represents an interesting summary of alternative ways to formulate the limits of borrowing. We will introduce an upperbound on debt in terms of a maximum debt of equity rate:

$$(4) \quad Y \leq hX$$

in which

$h$  = maximum debt to equity rate.

Together with the interest rate  $r$  this is a way to deal with uncertainty within the framework of a deterministic model. Because the level of  $r$  is an indication of the risk class to which the firm belongs expression (4) may be conceived as a condition on the financial structure of the firm that must be fulfilled in order to stay in the relevant risk-class (see Van Loon (1983)).

The shareholders of the firm are assumed to have personal tax rates on dividend  $\tau_d$  and capital gain  $\tau_g$  such that the ratio  $(1-\tau_g)/(1-\tau_d)$  is the same for all shareholders. This case occurs e.g. if investor and manager are one and the same person.

Finally we assume that the firm behaves as if it maximizes the shareholders' value of the firm. As we like to separate dividend income and capital gain explicitly this value consists in the tradition of Ludwig (1978) and Van Loon (1983) of the present value of the net dividend stream plus the net discounted gain on equity at the end of the planning period:

$$(5) \quad \max_{I, Y, D} (1-\tau_d) \int_{T=0}^z D(T) e^{-iT} dT + X(z) e^{-iz} - \tau_g (X(z) - X(0)) e^{-iz}$$

in which

$\tau_d$  = personal tax rate on dividend

$\tau_g$  = personal tax rate on capital gain

$z$  = planning horizon

$i$  = discount rate of the shareholder after personal taxation.

Formulating the objective in this way not only the difference between the value of the personal tax rates will be of crucial importance but also the time lag expressed by the discount rate. In accordance with most of the tax regimes the tax rate on dividend is assumed to exceed the tax rate on capital gain.

Following the above discussion we may formulate the optimization problem for the firm as follows:

$$(6) \quad \max_{I, Y, D} (1-\tau_d) \int_{T=0}^z D(T) e^{-iT} dT + (1-\tau_g) X(z) e^{-iz}$$

$$(7) \quad \text{s.t.} \quad \dot{X} = (1-\tau_c)(O(K)-rY) - D$$

$$(8) \quad \dot{K} = I - aK$$

$$(9) \quad 0 \leq K = X + Y$$

$$(10) \quad 0 \leq Y \leq hX$$

$$(11) \quad D \geq 0$$

### 3. Additional conditions for optimality

The model as formulated in paragraph 2 can be solved in an analytical way by using 'optimal control' techniques. The state of the system is described by the amount of equity and capital and is controlled by investments, dividends and the amount of debt. The aim of this control is to reach a maximum value of the objective function.

To solve the model we will use the procedure of Van Loon (1983), which is based on the maximum principle of Russak (1970). This method is convenient because the model also contains state-constraints.

To avoid non interesting stages we make following assumptions:

$$(12) \quad \text{if } K(T) = 0 \text{ then } (1-\tau_c) \frac{dO}{dK} > \max \{(1-\tau_c)r, i\}$$

The marginal revenue of the first product to be sold exceeds each of the financial costs implying that the firm will consider only those alternatives that are profitable from the start.

$$(13) \quad i \neq (1-\tau_c)r$$

In this way we avoid degenerated solutions. Moreover it is coincidental when the discount rate and the net cost of debt equal each other. Since a deterministic framework without taxation requires equality between the discount rate and the interest rate we rewrite in the tradition of Brealey & Myers (1981) expression (13) into:

$$(1-\tau_r)r \neq (1-\tau_c)r$$

in which

$$\tau_r = \text{personal tax rate on interest.}$$

So in fact only the case  $\tau_r = \tau_c$  is excluded by assumption (13).

(14) Finally we assume that the firm owns a certain initial amount of equity and debt such that

$$Y(0) = hX(0)$$

$$X(0) + Y(0) = K(0) > 0$$

To avoid jumps in state variables  $X$  and  $K$  we need a closed control region by putting artificial boundaries to dividend and investments. We assume however that these control variables never pass these sufficiently large boundary values and therefore we omit them.

The necessary and in our case also sufficient conditions acquiring an optimal solution are presented in the appendix. Based on these conditions we may discern a number of different paths or stages, each characterized by its own combination of active and inactive restrictions. Although the complementary slackness conditions allow eight different combinations this number will be reduced to five by assumptions (12) through (14). In the next section we will discuss these remaining stages.

#### 4. Feasible paths and optimal trajectories

As each path is characterized by its own combination of active and inactive restrictions, i.e. non-zero and zero artificial and Lagrange-parameters, its properties can be represented by different policies concerning capital structures, growth speed and/or dividend payments. Although we have derived these characteristic properties in an analytical way it suffices here to summarize the findings in the next table.

path	$\dot{\mu}_1$	$\dot{\mu}_2$	$\lambda$	Y	D	$\dot{X}$	K	feasible
1	0	-	+	hX	0	+	$\dot{K} > 0$	always
2	0	0	+	+	0	+	$K_{YX}^*$	always
3	-	0	+	0	0	+	$\dot{K} > 0$	always
4	-	0	0	0	max	0	$K_X^*$	$i < (1-\tau_c)r$
5	0	-	0	hX	max	0	$K_Y^*$	$i > (1-\tau_c)r$

Table 1: Characteristic properties of feasible paths.

(Definition of the variables  $\mu_1$ ,  $\mu_2$  and  $\lambda$  in the appendix.)

in which

$$K = K_{YX}^* \rightarrow (1-\tau_c) \frac{dO}{dK} = (1-\tau_c)r$$

$$K = K_X^* \rightarrow (1-\tau_c) \frac{dO}{dK} = i$$

$$k = K_Y^* \rightarrow (1-\tau_c) \frac{dO}{dK} = \frac{1}{1+h} i + \frac{h}{1+h} (1-\tau_c)r$$

A complete economic interpretation is presented on the next page, where the coupling of stages to master trajectories is described. As an example we derive from table 1 that path 1 has excellent possibilities for growth. Under this policy the firm will attract as much debt as possible, retain all earnings and invest this money in order to realize a maximum increase of the amount of equity.

By now we like to couple these different stages in order to obtain optimal evolution patterns. A convenient method to handle this problem is



that of Van Loon (1983) implying like dynamic programming a start at the end of the planning period: first we search for feasible final stages and secondly for previous stages (for examples see the appendix). A feasible final path satisfies the so-called 'transversality conditions'. Using the characteristic properties of the several stages we derive following conditions:

path	final path if
1	$i > (1-\tau_c)r$ and $1 < (1-\tau_g)/(1-\tau_d) < i/(1-\tau_c)r$
2	$i > (1-\tau_c)r$ and $(1-\tau_g)/(1-\tau_d) = i/(1-\tau_c)r$
3	$i > (1-\tau_c)r$ and $(1-\tau_g)/(1-\tau_d) > i/(1-\tau_c)r$
5	$i > (1-\tau_c)r$ and $(1-\tau_g)/(1-\tau_d) = 1$
3	$i < (1-\tau_c)r$ and $(1-\tau_g)/(1-\tau_d) > 1$
4	$i < (1-\tau_c)r$ and $(1-\tau_g)/(1-\tau_d) > 1$

Table 2: Conditions for final stages.

First remark that on a final stage dividend is issued only if the personal tax rates equal each other, which is in complete agreement with the results of models without personal taxation (Ludwig (1978) and Van Loon (1983)). Secondly we distinguish a financial and a fiscal condition. The first one represents the state of the art, the latter one specifies the investors' attitude towards the wanted kind of income. The bigger the difference between the personal tax rates the more he prefers capital gain instead of dividend. So, based on the financial condition we discern two different master trajectories both ending with path 3. The first one occurs if  $i < (1-\tau_c)r$  and is represented by figure 1. Although debt financing is more expensive than selffinancing the firm borrows the maximum amount of debt that is possible given the size of equity. In case of acquiring more equity only by making more profit, it is advantageous for the firm to borrow as long as the resulting marginal cost is less than marginal revenue. In this case, borrowing increases profit and thus, it raises the rate of growth of equity till  $T = t_{12}$ .

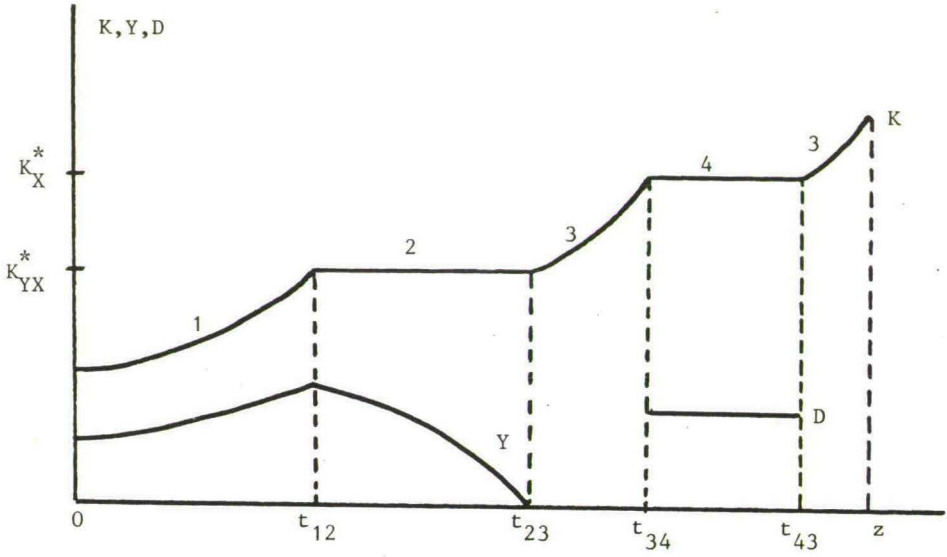


Figure 1. Master trajectories if  $i < (1 - \tau_c)r$

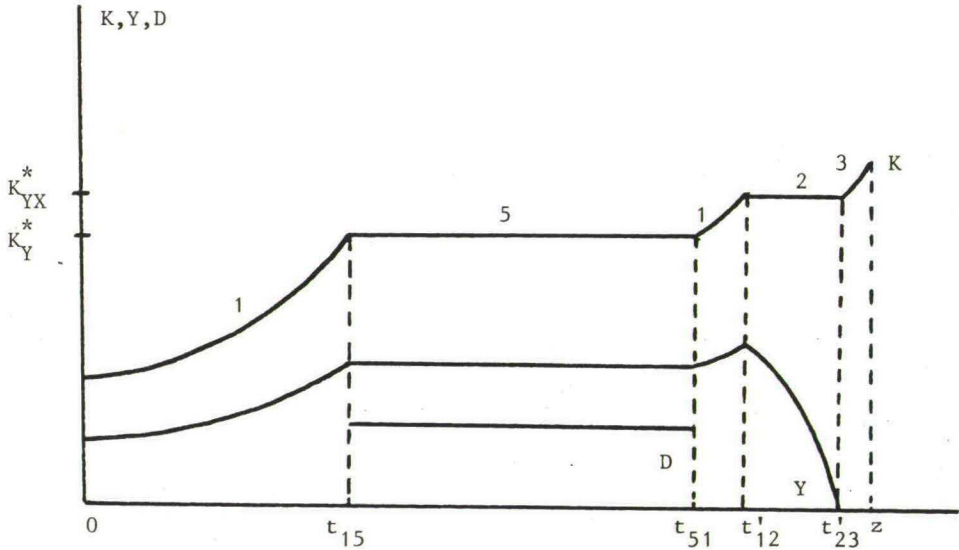


Figure 2. Master trajectories if  $i > (1 - \tau_c)r$

On this path the firm grows at maximum speed in order to realize a maximum growth of the income stream and thus a maximum increase of the amount of cheap equity. At  $T = t_{12}$  the size of the stock of capital goods is such that

$$(15) \quad K = K_{YX}^* \leftrightarrow (1-\tau_c) \frac{dO}{dK} = (1-\tau_c)r$$

At this level it is profitable to use retained earnings to pay back debt money, for

- issuing earnings is valued by the shareholders according the discount rate  $i$ ;
- continuing expansion investment yields a net revenue of  $(1-\tau_c) \frac{dO}{dK} < (1-\tau_c)r$ ;
- paying back debt money saves  $(1-\tau_c)r$  rent payments per unit.

So the firm stops its expansion for a while in order to use all its income to replace debt equity (path 2).

After this period of consolidation it still makes sense to expand the amount of capital goods, because it is financed now by equity only. Because no dividend is issued the firm starts increasing again as fast as possible (path 3). In this way the firm reaches the state of maximum dividend pay-out in a self-financing regime as quick as possible:

$$(16) \quad K = K_X^* \leftrightarrow (1-\tau_c) \frac{dO}{dK} = i$$

At this moment it is useless to continue expansion investments because additional net cost exceeds net revenue (path 4).

However, a moment  $T = t_{43}$  arrives at which the advantage of the lower tax rate on capital gain ( $\tau_g < \tau_d$ ) exceeds the drawback of discounting the salvage value of equity at the end of the planning period. From this moment no dividend is issued anymore and the firm starts expansion investments again (path 3). It is true, that marginal net revenue is less than the discount rate. This loss, however, is counterbalanced by the tax advantage got by the shareholders at the end of the planning period. As we see, the discount rate not longer equals the cost of equity per unit.

The second optimal master evolution pattern (see figure 2) occurs if debt money is cheap compared to equity. The start of the pattern is the same as the start of the previous one: due to the cheapness of debt money ( $i > (1-\tau_c)r$ ) the firm borrows the maximum amount of debt that is possible and invests all money in capital goods in order to realize a maximum growth of the income stream (path 3).

It is worth investing at the maximum level till  $K = K_Y^*$ , because at levels lower than  $K_Y^*$  the marginal income after taxation exceeds marginal financing costs in the case of maximum debt financing.

As soon as the amount of capital goods equals  $K_Y^*$ , this accelerated growth is cut off abruptly at  $T = t_{51}$ , because marginal net revenue equals the weighted sum of net costs of debt and equity:

$$(17) \quad (1-\tau_c) \frac{dO}{dK} = \frac{1}{1+h} i + \frac{h}{1+h} (1-\tau_c)r$$

Investments fall down to the replacement level and remaining earning are issued to the shareholders (path 5). Corresponding with the previous evolution pattern a moment arrives at which the firm stops dividend payments and start expansion investments by retaining earnings. Besides this the firm borrows as much as possible because borrowing increases profit and raises the rate of growth till  $K = K_{YX}^*$  (path 1). According to the case of relatively expensive debt, the firm now drops debt to save  $(1-\tau_c)r$  interest payments per unit (path 2) and after that it starts growing in a self financing regime till the end of the planning periode (path 3).

The evolution pattern after termination of path 5 depends among others on the difference between the personal tax rates, in other words: the tax advantage yielded by the shareholders receiving capital gain instead of dividend. A lower value of  $(1-\tau_g)/(1-\tau_d)$  will not only alter the final stage but also postpone the moment  $T = t_{51}$  and decline the number of stages to pass through.



### 5. A further analysis

By now we have described the optimal solution of our model in a way most of the publications on dynamics of the firm do. Besides this we like to do another way of analysis: a derivation of two global decision rules, which together constitute the policy of the firm (see Van Loon (1983)):

#### a) Financial decisions rule.

The financial structure is characterized by the relative amounts of equity and debt. The latter one is restricted by the first. So, the financial structure has two extreme cases: the case that the assets are financed by equity only and the case that the firm is financed by means of the maximal allowed amount of debt. Which of both is the optimal one depends on the marginal return to equity. The firm will try to realize such a financial structure as to maximize marginal return to equity, which implies:

choose for  $\left\{ \begin{array}{l} \text{self financing} \\ \text{maximum debt} \end{array} \right.$  if  $K \left\{ \begin{array}{l} > \\ < \end{array} \right. K_{YX}^*$

#### b) Dividend/investment decision rule.

The firm can spend its earnings in two ways: to pay out dividend or to retain it in order to finance investments and/or to pay back debt money. The last mentioned decision has been discussed implicitly in the previous decision rule: redemption of debt starts as soon as the firm attains the  $K_{YX}^*$ -level on which self financing becomes optimal instead of maximum debt financing.

The second possibility is certainly preferable as long as marginal return to equity exceeds the discount rate of the shareholders, for this rate represents the rate of return that the shareholders can obtain elsewhere. As soon as marginal return to equity equals the discount rate the firm will in general pay out dividend and invest only on the replacement level.

So far so good, but what happens if the discount rate exceeds marginal return to equity? In absence of personal taxation one can prove that



this situation only occurs if the initial amount of equity and capital goods are too high to make production at that level profitable. Van Loon (1983) argues that in such a case the optimal policy is to decrease the capital good stock and to pay out all earnings. We join this statement but on top of it the introduction of personal taxation brings on a second possible situation, which results in an additional dividend/investment decision rule to point out.

On the stationary stages 4 and 5 the firm has to decide whether it will continue to issue dividend or start to retain earnings. If we suppose that the firm holds this earnings only in cash and obtains no revenue from them, the shareholders value the first possibility by  $(1-\tau_d)e^{-iT}$  and the latter one by  $(1-\tau_g)e^{-iz}$ . This means that capital gain on one hand will be more profitable in view the tax advantage ( $\tau_g < \tau_d$ ) but on the other hand less due to the time lag  $z-T > 0$ . So, the decision to continue dividends or to retain earnings and held cash money depends on the sign of

$$(18) \quad (1-\tau_g)/(1-\tau_d) = e^{i(z-T)}$$

It is obvious that given the values of the tax rates, the discount rate and the planning horizon only one value, i.e.  $T = t_b^*$ , will satisfy the equality of expression (18).

However, this situation is not the optimal one. In spite of the decreasing marginal return on equity expansion investments still acquire positive revenue  $0 < (1-\tau_c) \frac{dO}{dK} < i$ , which can be used again to finance more investments. So, the shareholders will not only receive the retained dollar earning but also the increasement of equity during the time interval  $[T, z]$ . The value of this increasement depends on the leverage of the firm. As in the optimal solution shareholders value an increasement of equity and capital goods corresponding the co-state variables, retaining one dollar at time  $T = t$  and using it to finance new investments raises in the case  $i < (1-\tau_c)r$  the value of the firm with

$$(19) \quad \int_{T=t}^z (\psi_1(T) + \psi_2(T))(1-\tau_c) \frac{dO}{dK} dT$$

in which

$\psi_j(T)$  co-state variable to equity and capital goods respectively (see the appendix).

In a situation of maximum debt financing an increase of equity with one dollar allows a rise of debt with  $h$  dollar. So, a retention of one dollar results in an increase of the amount of capital goods with  $(1+h)$  dollar, which use results in a revenue of

$$(20) \quad (1+h)(1-\tau_c) \frac{dO}{dK} - hr = (1-\tau_c) \frac{dO}{dK} + h(1-\tau_c) \left( \frac{dO}{dK} - r \right)$$

So in a situation of maximum debt financing a retention of one dollar earning on the stationary stage raises the value of the firm with

$$(21) \quad \int (\psi_1 + (1+h)\psi_2) \left[ (1-\tau_c) \frac{dO}{dK} + h(1-\tau_c) \left( \frac{dO}{dK} - r \right) \right] dT$$

Note that no boundaries are indicated as in this situation ( $i > (1-\tau_c)r$ ) the firm will pass through both stages with maximum as without debt after termination of the stationary stage.

The discussion above results in the following additional dividend investment decision rule:

continue to issue dividend and make only investments on the replacement level	}	if $T \begin{cases} < \\ > \end{cases} t_b^*$
stop to pay out dividends and spend all earnings on expansion investments	{	

in which

$t_b^*$  is determined by

- in the case  $i < (1-\tau_c)r$

$$(1-\tau_d)e^{-it_b^*} = \int_{T=t_b^*}^z (\psi_1 + \psi_2)(1-\tau_c) \frac{dO}{dK} dT + (1-\tau_g)e^{-iz}$$

- in the case  $i > (1-\tau_c)r$  and e.g. path 1 final stage

$$(1-\tau_d)e^{-it_b^*} = \int_{T=t_b^*}^z (\psi_1 + (1+h)\psi_2) [(1-\tau_c) \frac{dO}{dK} + h(1-\tau_c) (\frac{dO}{dK} - r)] dT + (1-\tau_g)e^{-iz}$$

It is possible to show that in absence of personal taxation the point  $t_b^*$  equals the planning horizon  $z$ , which means that contrary to our model with different personal taxes the firm issues dividend on a final stage.

## 6. Sensitivity analysis

In this section we study the influence of environmental changes on six different features of the evolution process of the firm. This is a sensitivity analysis concerning parameters that are interesting for economic analysis: the personal tax rates  $\tau_d$  and  $\tau_g$ , the corporate tax rate  $\tau_c$ , the interest rate  $r$  and the discount rate  $i$ . The features of the evolution process are the level of capital goods on the stationary stage  $K^*$ , the speed of growth of equity  $X$ , the level of capital gain on equity, the points in time of start and termination of the stationary stage,  $t_a^*$  and  $t_b^*$  respectively, and the leverage  $Y/X$ .

### 6.1. Fiscal parameters

#### a. Corporate taxes

A reduction of the corporate tax rate causes two direct consequences:

- a rise of the net cost of debt due to the tax deductibility;
- possibilities to increase (expansion) investments in view lower tax payments.

The first one enables a switch of the sign of  $i > (1-\tau_c)r$  resulting in a policy change of the firm into a low leverage strategy.

Contrary to marginal revenue, however, net marginal costs of finance are influenced only partially. Thus, such a reduction results into increas-

ing profits and a larger amount of capital goods on stationary stages. A reduction of the corporate tax rate also increases earnings after tax payments from which (expansion) investments are to be paid. In this way growth gathers speed which is also demonstrated by

$$(22) \quad - \frac{\partial \dot{X}}{\partial \tau_c} = (O(K) - rY) \geq 0$$

On top we can derive that such a reduction will put forward the termination point of the stationary stage. No conclusion, however, is possible with relation to the start of the stationary stage and the amount of dividend. because a reduction of the corporate tax rate causes two opposite influences: a rise of the speed of growth and a larger amount of capital goods to be reached.

The consequences of a rise of the corporate tax rate are summarized in table 3.

#### b. Personal tax rates

The impact of changes in the value of the personal tax rates on the features of the firm are obvious. A rise of the tax rates on dividend causes an increasing interest in capital gain finding expression in putting forward the termination point of the stationary stage  $t_b^*$  as a result of a change in final stage showed by table 2. As the speed of growth does not changes the total amount of net dividend will decline due to the higher tax payments and the shorter period of issuing dividends by the firm.

A rise of the tax rate on capital gain presents almost the opposite picture: the stationary stage will grow in length, but the level of dividend payments after personal taxation does not alter.

On top of the results above table 3 shows the impact of the two financial parameters on the feature of the evolution process.

a rise of impact on		$\tau_c$	$\tau_d$	$\tau_g$	$r$	$i$
		level stationary stage	$K^*$	-	0	0
speed of growth	$\dot{X}$	-	0	0	-	0
capital gain		-	+	-	-	-
point of starting stationary stage	$t_a$	?	0	0	?/+	-
point of ending stationary stage	$t_b$	+	-	+	+/0	+
leverage	$Y/X$	+	0	0	-	+

Table 3. Results of sensitivity analysis.

in which

- + = rise of the feature value
- = fall of the feature value
- 0 = no influence on the relevant feature
- ? = no conclusion possible due to opposite influences
- ./ = conclusion before separation sign applies only if  $i < (1-\tau_c)r$ , after separation sign if  $i > (1-\tau_c)r$ .

## 6.2. Financial parameters

As earlier research has been done in sensitivity analysis with relation to financial parameters (Van Loon (1983)) we discuss the results only briefly and refer for a comprehensive discussion to the research mentioned.

A rise of the interest rate  $r$  causes increasing costs of finance implying a drop in the speed of growth in stages with both debt and equity. In the case of expensive debt compared to equity the termination point of the stationary stage will be postponed. A switch of the sign of  $i > < (1-\tau_c)r$  resulting in a change of the policy of the firm belongs also to the possibilities.



A change in the value of the discount rate has no influence on the speed of growth. The level of the capital goods stock in stationary stages will decline and the termination point of these paths will be postponed due to the larger revenue for the investor elsewhere.

### 6.3. Combined influence of fiscal and financial parameters

As we discussed in earlier sections the discount rate expresses the rate of return after taxation that the shareholders can obtain elsewhere. In a deterministic framework this implies that the discount rate depends on the market interest rate and the personal tax rate on ordinary income. As in most of the tax regimes the tax rate on dividend is positively correlated with the tax rate on ordinary income we can write

$$(23) \quad i = i(\tau_d, r) \quad \frac{\partial i}{\partial \tau_d} < 0 \quad \frac{\partial i}{\partial r} > 0$$

Considering this dependence we will once more study the impact of the tax rate on dividend on the features of the optimal evolution patterns of the firm by means of two extreme cases.

First we assume that the tax rate on dividend of investor A,  $\tau_{dA}$ , differs much from the tax rate on capital gain  $\tau_g$  and has such a value that

$$(24) \quad i_A = i_A(\tau_{dA}, r) < (1 - \tau_c)r$$

According to the Miller hypothesis these investors would prefer firms following a low leverage strategy, low dividend payments and high capital gains. With relation to our model figure 3 represents the optimal evolution pattern.

In the second case we assume that the tax rate on capital gain remains the same, but the tax rate on dividend of investor B,  $\tau_{dB}$ , differs only little and has such a value that

$$(25) \quad i_B > (1 - \tau_c)r \text{ and } (1 - \tau_g)/(1 - \tau_{dB}) < i_B/(1 - \tau_c)r$$

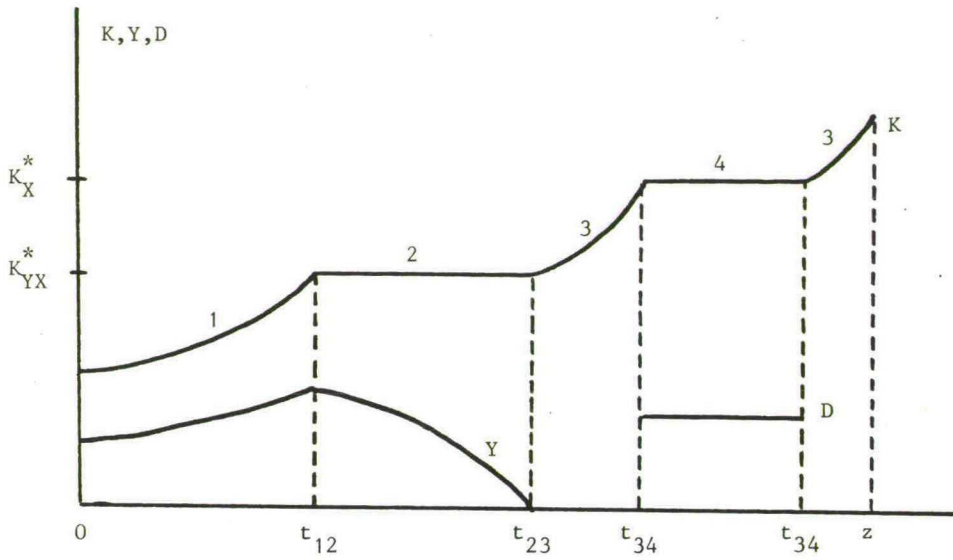


Figure 3. Master trajectories if  $i_A < (1-\tau_c)r$  and  $(1-\tau_g)/(1-\tau_{dA}) > 1$

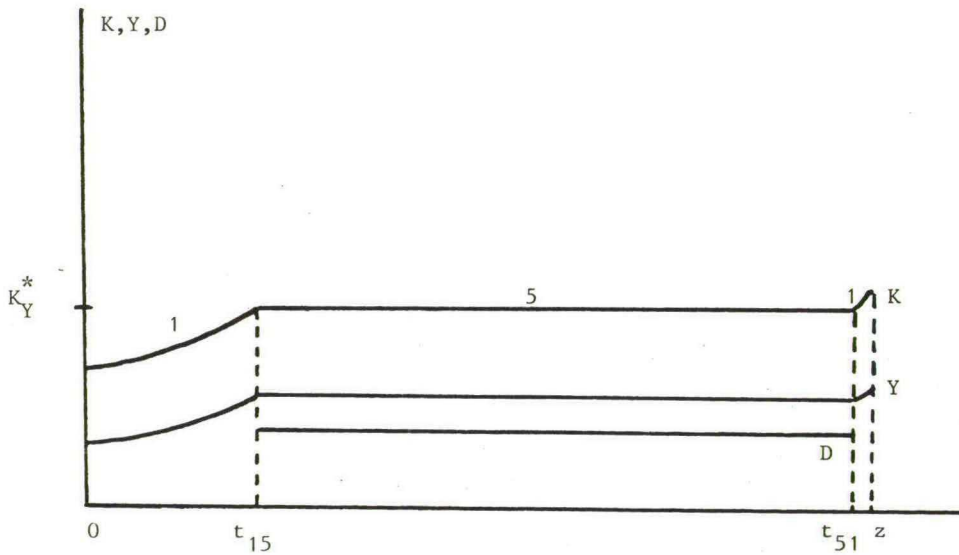


Figure 4. Master trajectories if  $i_B > (1-\tau_c)r$  and  $1 < (1-\tau_g)/(1-\tau_{dB}) < i_B/(1-\tau_c)r$

in which

$$i_B = i_B(\tau_{dB}, r) > i_A$$

Table 2 shows that under these circumstances the optimal evolution pattern, represented by figure 4, will end with path 1. In the opinion of investor B the firm has to choose for a high leverage strategy and has to issue as much dividend as possible. Similar to the Miller hypothesis we discover clear differences between these two optimal master trajectories. On top of this similarity the Miller hypothesis is enlarged by the introduction of dynamics: investors in low tax brackets prefer not only a high leverage strategy and dividend above capital gain, but they also like to receive earnings as quick as possible, while investors in high tax brackets are willing to postpone earnings in view their tax advantage and discount rate.

## 7. Summary

In this paper we considered especially the influence of tax systems on the optimal dynamic policy of the firm by introducing both corporate and personal taxation. In this way we extended the dynamic theory of the firm and we did a first investigation on the relevance of financial leverage clienteles in a dynamic framework.

After presentation of the model we derived an analytical solution using 'optimal control' techniques. The results differ from those of dynamic models without personal taxation in three ways:

- in final stages no dividend will be issued;
- at last the firm finances investments with only equity, even when debt is cheap compared to equity;
- the discount rate of the shareholders equals no longer the marginal cost of equity which enables the firm to invest in projects with a net return less than the discount rate.

Sensitivity analysis showed the impact of the personal tax rates and other parameters on six features of the optimal master trajectories. As we assumed that the discount rate depends on the interest rate and the personal tax rate on dividend we saw similarities with the results of static models. Especially the Miller hypothesis seems to be confirmed: shareholders in high tax brackets prefer a policy of low dividend payments and high capital gains by retaining earnings.

Contrary to Miller we did not yet consider market equilibrium situations, at which we will pay attention in future contributions.

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Appendix. A reduced form of the model and its conditions for optimality.

In order to simplify the solution procedure, we will first leave out a mathematically superfluous element in the formulation (6)-(11) by eliminating the control variable  $Y$ . We can rewrite (9) in:

$$(0.1) \quad Y = K - X$$

Substitution of the above expression in (6) through (11) results in the next reduced form of the model:

$$(0.2) \quad \max_{I, D} (1-\tau_d) \int_{T=0}^Z D e^{-iT} dT + (1-\tau_g) X(z) e^{-iz}$$

$$(0.3) \quad \dot{X} = (1-\tau_c)(O(K)-rK+rX) - D$$

$$(0.4) \quad \dot{K} = I - aK$$

$$(0.5) \quad K - X \geq 0$$

$$(0.6) \quad (1+h)X - K \geq 0$$

$$(0.7) \quad D \geq 0$$

$$(0.8) \quad X \geq 0$$

Combining expressions (0.5) and (0.6) however makes (0.8) superfluous, so we will leave it out.

Let the Hamiltonian be

$$(0.9) \quad H = (1-\tau_d) e^{-iT} D + (\psi_2 + \mu_1 - \mu_2)(I - aK) + \\ (\psi_1 - \mu_1 + (1+h)\mu_2)[(1-\tau_c)(O(K) - rK + rX) - D]$$

and let the Lagrangian be

$$(0.10) \quad L = H + \lambda D$$

in which

$\psi_j = \psi_j(T)$  adjoint variable or co-state variable which denotes the marginal contribution of the  $j$ -th level state variable to the performance level

$\lambda = \lambda(T)$  dynamic Lagrange multiplier representing the dynamic 'shadow price' of the dividend restriction

$\mu_\ell = \mu_\ell(T)$  artificial variable.

To find the optimal solution we now apply the theorems for necessary and sufficient conditions as presented by Van Loon (1983, pp. 139-141).

For an optimal control history  $(D^*, I^*)$  of the problem formulated by (0.2) through (0.7) and the resulting state trajectory  $(K^*, X^*)$  to be optimal, it is necessary that there are functions  $\psi_j(T)$ ,  $\lambda(T)$  and  $\mu_\ell(T)$  such that:

$$\begin{aligned} H_{\text{optimal}} &:= H(K^*, X^*, D^*, I^*, \psi_j, \lambda, \mu_\ell, T) \\ &= \text{maximum}_{D, I} \{ H(K^*, X^*, D, I, \psi_j, \lambda, \mu_\ell, T) \} \end{aligned}$$

for each  $T$ ,  $0 \leq T \leq z$

and, except at points of discontinuity of  $(D^*, I^*)$ :

$$(0.11) \quad \dot{\psi}_1 = - \frac{\partial L}{\partial X} = -(\psi_1 - \mu_1 + (1+h)\mu_2)(1-\tau_c)r$$

$$(0.12) \quad \dot{\psi}_2 = - \frac{\partial L}{\partial K} = (\psi_1 - \mu_1 + (1+h)\mu_2)(1-\tau_c)(r - \frac{dO}{dK}) + (\psi_2 + \mu_1 - \mu_2)a$$

$$(0.13) \quad \frac{\partial L}{\partial D} = (1-\tau_d)e^{-iT} - (\psi_1 - \mu_1 + (1+h)\mu_2) + \lambda = 0$$

$$(0.14) \quad \frac{\partial L}{\partial I} = \psi_2 + \mu_1 - \mu_2 = 0$$

$$\begin{array}{l}
 (0.15) \quad \psi_1(z) = (1-\tau_g)e^{-iz} \\
 (0.16) \quad \psi_2(z) = 0 \\
 (0.17) \quad \mu_1(z)[K(z) - X(z)] = 0 \\
 (0.18) \quad \mu_2(z)[(1+h)X(z) - K(z)] = 0
 \end{array}
 \left. \vphantom{\begin{array}{l} (0.15) \\ (0.16) \\ (0.17) \\ (0.18) \end{array}} \right\} \text{transversality conditions}$$

$$\begin{array}{l}
 \dot{\mu}_1(K-X) \\
 (0.19) \quad \dot{\mu}_2((1+h)X-K) = 0 \\
 \lambda D = 0
 \end{array}
 \left. \vphantom{\begin{array}{l} \dot{\mu}_1(K-X) \\ (0.19) \\ \lambda D = 0 \end{array}} \right\} \text{complementary slackness conditions}$$

(0.20)  $\psi_j(T)$  are continuous with piecewise continuous derivatives

(0.21)  $\lambda(T)$  is non-negative and continuous on intervals of continuity of  $\{D, I\}$

(0.22)  $\mu_1(T)$  is continuous when  $\dot{(K-X)}$  is discontinuous

(0.23)  $\mu_2(T)$  is continuous when  $\dot{(1+h)X - K}$  is discontinuous

(0.24)  $\mu_\ell(T)$  is continuous on intervals of continuity of  $\{D, I\}$ , not negative and not increasing

As  $H_{\text{optimal}}$  is a concave function of  $(K, X)$  these conditions are also sufficient.

From these conditions we can derive that  $\mu_\ell(T)$  and  $\lambda(T)$  are continuous functions.

Combining (0.22) and (0.24) then  $\mu_1(T)$  is continuous

(0.25) if  $\{D, I\}$  continuous and/or  $\dot{(K-X)}$  discontinuous.

We can rewrite (0.25) with (0.2) and (0.3) into

(0.26) if  $\{D, I\}$  continuous and/or

$$I - (1-\tau_c)(0-rX) + D - [(1-\tau_c)r + a]K \text{ discontinuous}$$

Due to the closed control region,  $X$  and  $K$  are continuous, so at least one of the control variables must be discontinuous in order to get a discontinuity of expression (0.26).

So the two parts of expression (0.26) are complementary to each other and (0.25) will always be fulfilled.

We may conclude that  $\mu_1$  is continuous in the above optimality conditions.

In the same way we can derive the continuity of  $\mu_2$  and with the help of (0.13) and (0.20) the continuity of  $\lambda$ .

#### Example of final stage

path 5:  $\dot{\mu}_1 = \lambda = 0$ ,  $\dot{\mu}_2 < 0$ ,  $Y = hX$ ,  $D > 0$ ,  $K > 0$ .

if path 5 will be final stage it has to fulfill the transversality conditions (0.15)-(0.18). Therefore

$$(0.15') \quad \psi_1(z) = (1-\tau_g)e^{-iz}$$

$$(0.16') \quad \psi_2(z) = 0$$

$$(0.17') \quad \mu_1(z) = 0 \quad \text{as} \quad K(z) - X(z) = Y(z) > 0$$

$$(0.18') \quad \mu_2 \geq 0 \quad \text{as} \quad (1+h)X(z) - K(z) = 0$$

Substitution of these necessary conditions in (0.13) and (0.14) gives:

$$(0.13') \quad (1-\tau_d)e^{-iz} - (1-\tau_g)e^{-iz} - (1+h)\mu_2(z) = 0$$



$$(0.14') \quad \mu_2(z) = 0$$

which means that path 5 can be final stage only if  $\tau_d = \tau_g$ .

Example of coupling procedure

To couple two paths we use the continuity properties of relevant variables.

Problem: on which conditions can path 5 precede the string path 1 → path 2 → path 3?

We have to consider especially the continuity property of  $\lambda$  as this variable is positive on path 1 and equal to zero on path 5. So,  $\lambda$  can only be continuous if

$$(0.27) \quad \dot{\lambda} < 0 \text{ when } \lambda = 0 \text{ on path 1.}$$

From (0.11)-(0.14) we can derive

$$(0.28) \quad \dot{\lambda}/(1+h) = -(1-\tau_d)e^{-iT} \left[ (1-\tau_c) \left( \frac{dO}{dK} - \frac{h}{1+h} r \right) - \frac{1}{1+h} i \right] \\ - \lambda(1-\tau_c) \left[ \frac{dO}{dK} - \frac{h}{1+h} r \right].$$

So (0.27) can be fulfilled if

$$(0.29) \quad (1-\tau_c) \frac{dO}{dK} > \frac{h}{1+h} r + \frac{1}{1+h} i$$

which means that on path 1 holds

$$(0.30) \quad K(T) < K_y^*$$

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