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RELATIONAL CONSTRAINTS IN COALITION FORMATION

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by

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Abstract

Relations on the set of economic agents are natural social constraints with respect to individual and social behaviour of those agents. Although the society consists of a complex system of networks, such a relational structure is until now not used in general equilibrium theory. One of the main reasons is that such (finite) networks are determinants of social behaviour of the economic agents, which is not yet described very well in environments with infinitely many agents.

In this paper concepts are introduced that generate coalitions based on the positions of agents with respect to a given family of (relevant) networks. Our starting point therefore is a set of agents endowed with relations as the social characteristics of those agents. This will be called a relational structure. A coalitional structure can then be derived, using the instruments based on the mathematical notion of semi-ring as developed previously by Gilles (1987). The connection of a relational structure with standard economic general equilibrium theory is thus established.

Key Words : Relational structure, coalition formation, networks, cooperative behaviour, general equilibrium, positive modelling.

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1. INTRODUCTION

One of the main problems of contemporary economic theory is to describe the notion of competition between agents in an economy in a formal way. Until recently perfectly competitive economies were solely described by models consisting of an atomless measure space of agents. Although this measure theoretic context is considered to be purely technical, it describes an environment in which an individual agent has measure zero and therefore has no observable influence on the economic and social processes. Such a definition excludes all at most countable subsets of agents for having an observable influence on the outcome of such economic or social processes. This definition therefore contradicts the observation, that in daily life situations many finite subsets of agents, such as managers or union leaders, have a very noticeable influence on economic processes.

Recently the problem as described above has been tackled from several points of view. The most direct approach has been developed by Kaneko and Wooders (1987a and b) and Hammond, Kaneko and Wooders (1987). They introduce the concept of the f-core to describe a situation in which only finite coalitions in a continuum of agents are allowed to block a proposed allocation. Thus, although these finite coalitions remain insignificant in the continuum, they become significant in the economic process as described by the f-core. However this concept does not recognize the social position of an agent in the continuum, which is one of the main determinants of the agents' power to influence the economic processes: in the f-core all agents are treated equally. This contradicts our intuition that a union leader, e.g., has more influence than a plain consumer.

In this paper we try to solve this "inequality between agents" problem as mentioned above in an indirect way. Our main reference point will be the theory as developed by Gilles (1987 and 1988). There it is observed that the mathematical concept of σ-algebra, which underlies until recently all definitions of coalition forming behaviour, cannot be given a senseful economic interpretation and moreover gives rise to certain inconsistencies between cooperative and non-cooperative approaches to the phenomenon of perfect competition. Gilles (1987) introduces the concept of semi-ring, which is weaker than the notion of σ-algebra. Based on this mathematical instrument one defines a coalitional structure as the economic primitive concept of the model. A coalitional structure consists of a set of agents endowed with a semi-ring of so-called primitive coalitions. These primitive coalitions are the fundamental particles in the process of coalition formation. It is assumed that a coalitional structure describes the social environment of each agent in the economy: membership of a family, an enterprise, a club and a country is reflected in memberships of certain primitive coalitions. Apart from these social characteristics, each agent is also assumed to have individual characteristics such as preferences, capacities and wealth. In Gilles (1988) it is shown that under certain conditions perfect individual competition coincides with complete social pluriformity of such an economy.

In this model the main blocking coalitions are the so-called realizable coalitions. These kind of coalitions are formed by cooperation between a finite number of pairwise disjoint primitive coalitions. It is therefore implicitly assumed that certain members of the participating primitive coalitions are organizing such a realizable coalition. (Mostly these members are the managers or leaders of these participating primitive coalitions.)

In this paper we give a more basic approach to this kind of coalition formation by describing the social environment of an agent more fundamentally. Therefore we also model explicitly the organizing capacities of certain agents. Our theory eventually generates a coalitional structure as developed by Gilles.

Our starting point is a relational structure in which agents have specific positions in possibly repeating patterns of relations. These relations are assumed to be observable through measurable characteristics of the agents such as location, profession, income, etc.. This assumption is called the **axiom of positive modelling**. It says that there exists an embedding of the relational structure in some topological space based on observable characteristics, such that properties of this relational structure can at least locally be inferred from observations.

In contrast to this axiom of positive modelling we can define an axiom of normative modelling. In that case there exist patterns of interaction between agents ("forces") such that properties of a relational structure can be inferred or rationalized from assumptions on pattern formation. We shall explore the possibilities of this second axiom in another paper.

This paper is organised as follows.

The next section treats the embedding of a relational structure, given by a pair ($\mathcal{A},\mathcal{R}$), into a topological space T = (V,τ) by some surjective (or one-to-one) mapping. We assume that this mapping locally gives a description of relations between agents by means of observable features or characteristics. The rest of the paper will be based upon the map (A,R) of the relational structure, which will be called a relational model.

In section 3 we introduce the concept of communication between agents in the context of a relational model. Next we introduce the important notion of a **connected** relational model, taking into account that A may be subdivided into a countable number of disjoint connected subsets of the topological subspace $(A,\tau|A) \subset T$. It turns out that connectedness is equivalent to the property that every two agents in the relational model are able to communicate with each other. The important economic concept of a network can then be defined. It is a set of communicating agents whose relations cover the set of all agents A. It is established that the existence of such a network is equivalent to connectedness of the relational model. Furthermore we will formulate conditions on (A,R) under which there exist countable or even finite networks. A network will be called relevant if it also has an economic interpretation.

The impact of relevant networks on coalition formation is discussed and shown in subsection 4.1. Such a family of relevant networks $\mathcal{U}$ generates a class of certain coalitions S($\mathcal{U}$), called a service structure generated by that family of networks. In those structures the positions of the agents are essential, even if there are only finitely many agents in the generating networks. Thus the coalitions in the service structure S($\mathcal{U}$) obey a social constraint based upon the relational model or structure. Next we apply the machinery as developed by Gilles (1987) to arrive at a coalitional structure. The semi-ring closure of S($\mathcal{U}$) and a measure μ on this semi-ring just form a coalitional structure of agents (A,γ($\mathcal{U}$),μ). Conditions for its existence will be formulated. The coalitional structure is fully based on positions of agents with respect to certain (relevant) networks.

Although the members of these generating networks are individually negligible, they play a very important role in the process of coalition formation. In certain respect the primitive coalitions, which are the outcome of this process, are organized by these agents. This type of coalition formation coincides with the intuitive foundations of the model of Gilles (1987 and 1988) and cannot be embedded in the traditional approach as mainly described by Aumann (1964) and Hildenbrand (1974).

Future research has to focus on models in which the members of generating networks, who play such an important intermediary role in coalition formation and blocking, are earning profits from their membership of such a generating network. Here we may have a link with the work of Hammond (1987).

In subsection 4.2 we focus on the special case of a relational model consisting of a single compact and connected subset of a certain topological space. In that subsection we will apply the main theorems as developed in the previous sections. Finally in subsection 4.3 an example is given for a simple health economy with a continuum of agents, but with finitely many doctors. In this example we will describe a relational structure in which these doctors play a fundamental role in the process of coalition formation. Furthermore a first attempt is formulated in describing a new cooperative equilibrium concept in the setting of an economy based on a relational structure.

2. RELATIONAL STRUCTURES: THE MODEL

In this section we develop the primitive concepts of our model. The basic economic notion of the model is the so-called relational structure. It describes a set of agents, who have a specific pattern in their economic relationships. These relationships are crucial in the formation of coalitions in this kind of economic environments. An economy which has a relational structure as its primitive concept, is called a <u>relational economy</u>.

In this paper we will not explicitly state the definition of a relational economy and its characteristic patterns, but we will mainly be dealing with cooperative behaviour in relational structures. First we define the economic notion of relational structure.

<u>2.1 Definition</u>

A <u>relational structure</u> is a pair ($\mathcal{A},\mathcal{R}$) where

$\mathcal{A}$ is a set of agents ;

$\mathcal{R} \subset \mathcal{A} \times \mathcal{A}$ is a symmetric and reflexive relation on $\mathcal{A}$.

□

A relational structure is in fact a mathematical graph. However this graph can be infinite, even uncountable. Especially in uncountable relational structures we have to be able to modify the model. Therefore we embed the relational structure in a topological space. This embedding also has some nice economic interpretations with repect to the positive aspect of economic theory.

<u>2.2 Definition</u>

Let ($\mathcal{A},\mathcal{R}$) be a relational structure.

($\mathcal{A},\mathcal{R}$) can be <u>embedded</u> in a topological space if there exists a topological space $T := (V,\tau)$, where V is a set and τ is a topology on V, and a surjective mapping $g : \mathcal{A} \longrightarrow V$ such that for every $a \in A$, there exists an open neighbourhood $U_a \in \tau$ such that for every agent $b \in U_a \cap A$: $(a,b) \in R$,

where

$A := g(\mathcal{A}) \subset V$ and
$R := \{ (a,b) \in A \times A \mid (g^{-1}(a), g^{-1}(b)) \in \mathcal{R} \}$.

□

As one can see, an embedded relational structure has as specific property, that the topological structure is <u>locally</u> able to describe the relations between agents in the relational structure. So we actually assume that in uncountable relational structures, which can be embedded, there are some features or characteristics from which partially/locally the relationships between agents can be inferred.

We can however construct embedded relational structures which do not fulfill the intuitive economic foundations of this property, i.e. the embedding is economically trivial. Very simple examples are embeddings in discrete topological spaces, i.e. topological spaces in which the sets $\{a\}$, $a \in V$, are all open. However there also are less trivial topologies which do not give additional relevant information on relations between agents if a relational structure is embedded in such a space. For example take $\mathcal{A} := [0,1] = I$ and let $\mathcal{R} \subset I \times I$ be a relation on $\mathcal{A}$. Evidently $(\mathcal{A},\mathcal{R})$ is a relational structure. Now for the topological space take $V := I^I$ and τ is the product topology on this set emerging from the Euclidean topology on I. So $T := (V,\tau)$ is a compact topological space. Now we take $g : \mathcal{A} \longrightarrow V$ given by $g(a) = f_a$, $a \in \mathcal{A}$, where $f_a : I \longrightarrow I$ is defined by $f_a(a) = 1$ and $f_a(b) = 0$, $b \neq a$. It is quite obvious that the relational structure $(\mathcal{A},\mathcal{R})$ can be embedded in T by the mapping g. However this embedding does not satisfy the intuitive economic foundations of the property that it locally explains relations between agents, because there exists a neighbourhood of $a \in A$, say U, for which $(U\setminus\{a\}) \cap A = \emptyset$.

From these examples we conclude that we need additional conditions on an embedding. Before we modify the embedding property, we mention that all structural properties of the original relational structure remain in the embedding. We state this property in the next lemma.

2.3 Lemma

Let $(\mathcal{A},\mathcal{R})$ be a relational structure which can be embedded in some topological space $T = (V,\tau)$, and let (A,R) be the embedding of this relational structure in T. Then:

$R \subset A \times A$ is a reflexive and symmetric relation on A.

□

The previous remarks on the embedding property and its economic purpose, which is to describe the relations between agents in a relational structure locally by some features or characteristics, lead to the following modification of the embedding property.

2.4 Definition

Let $(\mathcal{A},\mathcal{R})$ be a relational structure.

The relational structure $(\mathcal{A},\mathcal{R})$ can be embedded properly if there exists a topological space $T = (V,\tau)$ and a surjective mapping $g : \mathcal{A} \longrightarrow V$ such that $(\mathcal{A},\mathcal{R})$ can be embedded in T by this mapping g and additionally for its embedding $(A,R) \subset (V,V \times V)$ there exists an at most countable sequence $(C_n)_{n \in \mathbb{N}}$ of pairwise disjoint topologically connected subsets of T, such that $A = \bigcup_{n=1}^{\infty} C_n$.

□

From mathematics we now know that $(A,\tau|A)$ consists of at most countable components, i.e. maximal topologically connected sets. The (unique) sequence of components of $(A,\tau|A)$, denoted by $(A_n)_{n \in \mathbb{N}}$, is called the subdivision of A.

If a relational structure can be embedded properly, then either $\mathcal{A}$ is at most countable or the pair $(\mathcal{A},\mathcal{R})$ can be described as the union of a countable number of connected subsets in a topological space. This topological space describes locally - as defined in definition 2.2 - characteristics or features, which determine the relationship between agents. In this respect the embedding-property is a notion of positive modelling: at least locally, we can describe relations between agents in a relational structure by some set of characteristics or features.

In a relational structure, which can be embedded properly, we therefore deal with relations which can be described (locally) by some features or characteristics. If the subdivision of the embedding consists of at least two components, then we deal with at least one discrete characteristic, embedded in the topological structure of the embedding of the relational structure. Examples of discrete and continuous characteristics are easily constructed. For example income, wealth and geographic residence are continuous characteristics, while nationality, profession and sex are discrete characteristics. In cases with at least one discrete characteristic, we deal with embeddings consisting of at least two disjoint components, and thus we deal with a subdivided embedding in the sense that the subdivision consists of several components.

In the sequel we will only deal with relational structures which can be embedded properly in a topological space. We call this assumption the axiom of positive modelling. Therefore we only have to deal with so-called relational models.

2.5 Definition **(Axiom of positive modelling.)**
Let $T = (V,\tau)$ be a topological space.
The pair $(A,R) \subset (V,V\text{x}V)$ is called a relational model if there exists a relational structure $(\mathcal{A},\mathcal{R})$ for which there exists a one-to-one mapping $g : \mathcal{A} \longleftrightarrow A$ which properly embeds the pair $(\mathcal{A},\mathcal{R})$ into the pair (A,R).

□

In positive modelling we have given an economic relational structure, which can be embedded in a structure having some nice properties, such as a topological space. This relational model is observed in its characteristics and, using the axiom of positive modelling, properties of the relational structure are inferred from these observations. In normative modelling we deal with given characteristics and its associated relational structure. In such a context we try to explain the relational structure from axioms on pattern formation. This procedure also leads to an axiom of modelling, namely the **axiom of normative modelling**. This kind of modelling will be discussed in a forthcoming paper.

Positive modelling has some nice applications in empirical economics, because it can describe coalition formation based on relations between agents in the economy, and investigates these relations and coalition formation empirically. Normative modelling tries to explain and understand why these relationships exist.

All definitions which are formulated in terms of relational models, without using explicitly its topological structure, can be transformed into properties of the relational structure, i.e. the graph itself. Therefore in the sequel we only deal with relational models, and abstract from its economic foundations, the relational structure.

Within the setting of such a relational model we define and describe a very natural form of communication between agents, which is completely based on the existing relations between agents. In fact we assume a "shaking hands" kind of communication. This kind of communication forms the foundation of the coalition formation as described in the next sections.

<u>2.6 Definition</u>

Let (A,R) be a relational model.

Then the <u>relation-mapping of (A,R)</u> is the multifunction $F : A \longrightarrow 2^A$ given by

$$F(a) := \{ b \in A \mid (a,b) \in R \}, \; a \in A.$$

□

We can now derive some trivial properties of the relation-mapping of a relational model (A,R).

<u>2.7 Proposition</u>

Let (A,R) be a relational model. Then we have the following properties:

(i) For every agent $a \in A$, $a \in \text{int}(F(a))$, where $\text{int}(S)$ is the relative interior of a subset $S \subset A$, i.e. with respect to the subspace $(A,\tau|A) \subset T = (V,\tau)$.

(ii) For every pair of agents $a,b \in A$:

$$a \in F(b) \text{ if and only if } b \in F(a).$$

□

Now we come to the definition of communication in a relational model. First we define a technical tool to describe our notion of communication in a proper fashion.

2.8 Definition

Let (A,R) be a relational model.

(a) Let $E,F \subset A$ be two sets of agents and let $n \in \mathbb{N}$.
A finite sequence of subsets of A, denoted by $C_1,\dots,C_n \subset A$ is called an irreducible chain between E and F if it satisfies the following properties:
(i) $E = C_1$ and $F = C_n$;
(ii) $C_j \cap C_{j+1} \neq \emptyset$ for $j = 1,\dots,n-1$;
(iii) $C_h \cap C_j = \emptyset$ for $|h - j| > 1$.

(b) Let $a,b \in A$ be two agents in the relational model.
The agents a and b are said to be able to communicate within the setting of the relational model if there exists an integer $n \in \mathbb{N}$ and a finite sequence of agents $a_1,\dots,a_n \in A$ such that the sequence of sets of relations $F(a_1),\dots,F(a_n) \in 2^A$ is an irreducible chain between F(a) and F(b).

□

It is quite obvious that the capability to communicate is a mathematical equivalence relation on the set A. Our purpose is to state the conditions under which all agents in the relational model are able to communicate to each other. In such a case this mathematical equivalence relation generates exactly one equivalence class, which of course is equal to the set of all agents. This line of research will be pursuited in the next section of this paper.

Note that finiteness of the communication line is essential in the definition above. It is assumed that one agent can reach the other with some information in a finite number of steps. The chain of agents as defined in definition 2.8 (b) can also be called an n-intermediate chain between the agents a and b. The number n denotes the length of the communication chain between the two agents a and b.

The reason why we can justify this definition of communication in a relational model is given in the next proposition, which shows that the intuitive concept of "shaking hands" communication is equivalent to the definition of communication as given above.

2.9 Proposition

Let (A,R) be a relational model and let $a,b \in A$ be two agents in the relational model (A,R). Then the following assertions are equivalent:

(i) a and b are able to communicate in (A,R) ;

(ii) There exists a finite sequence of agents $a_1,\ldots,a_n \in A$ such that

$a_1 = a$ and $a_n = b$,

$a_{j+1} \in F(a_j)$ for $j = 1,\ldots,n-1$.

□

Proofs of these properties are omitted, because these can easily be derived from the definitions of F and communication.

3. NETWORKS IN RELATIONAL MODELS

We now have a full description of communication as defined in a relational structure and in its embedding relational model, and therefore also for a large class of relational structures. This communication is based on the intuitive notion of "shaking hands" communication. As shown in the previous section finiteness of the communication chains or lines is essential in this description of communication.

We now come to the question under which conditions agents are able to communicate in a relational model, especially under which conditions all agents in such a model are able to communicate with each other. The conditions are defined in the context of a relational model and turn out to be quite global, as will be shown in the next theorems.

3.1 Definition

Let (A,R) be a relational model, and let $(A_n)_{n \in \mathbb{N}}$ be the subdivision of the set of agents A.

(a) The pair $(\bar{A},\bar{R})$, where $\bar{A} \subset V$ and $\bar{R} \subset \bar{A} \times \bar{A}$, is called a condensation of the relational model (A,R) if there exists a mapping $\text{cond} : A \longrightarrow \bar{A}$, which is surjective and satisfies the following properties:

(i) For every integer $n \in \mathbb{N}$ and any pair of agents $a,b \in A_n$:
$$\text{cond}(a) = \text{cond}(b) \;;$$

(ii) For any two integers $n,m \in \mathbb{N}$, $n \neq m$, and all agents $a \in A_n$ and $b \in A_m$ it holds that
$$\text{cond}(a) \neq \text{cond}(b) \;;$$

(iii) $(a,b) \in \bar{R}$ if and only if there exist integers $n,m \in \mathbb{N}$, $n \neq m$, and two agents $c \in A_n$ and $d \in A_m$ such that:
$$\text{cond}(c) = a \quad \text{and} \quad \text{cond}(d) = b,$$
$$(c,d) \in R \;.$$

(b) The relational model (A,R) is said to be <u>connected</u> if there exists a condensation $(\bar{A},\bar{R})$ of (A,R) such that for any two points $a,b \in \bar{A}$, there exists an integer $n \in \mathbb{N}$ and a finite sequence of points $a_1,\dots,a_n$ in $\bar{A}$ such that

$$a = a_1 \quad \text{and} \quad b = a_n$$
$$(a_j,a_{j+1}) \in \bar{R} \quad \text{for every } j = 1,\dots,n-1.$$

□

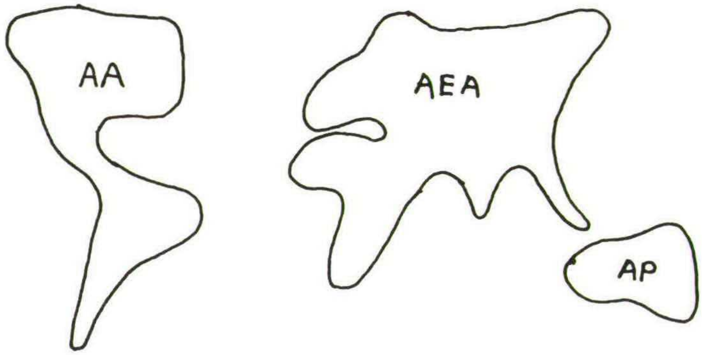
Connectedness is quite a natural condition on relational models. It just prescribes that there exist communication lines between all groups of agents in which the set of all agents in the model can be divided. Moreover mathematically, connectedness of the relational model is equivalent to the fact that the condensation of the relational model is a graph, which is finitely connected.

It is obvious that for every relational model there exists a "unique" condensation, in which uniqueness is guaranteed up to a transformation in the topological space T.

To underline the statement that connectedness is quite a natural condition on a relational model, we give a real life example.

<u>3.2 Example</u>

We describe the concepts as developed in the previous section and above with a historical example. Let $\mathcal{A}_1$ be the set of people living on the earth in the year 1400 and let $\mathcal{A}_2$ be the set of people living in the year 1900. Furthermore let $\mathcal{R}_i$ represent the set of relations between agents in $\mathcal{A}_i$ (where i = 1,2). Assume that both relational structures ($\mathcal{A}_1,\mathcal{R}_1$) and ($\mathcal{A}_2,\mathcal{R}_2$) geographicly can be embedded in the two dimensional Euclidean space $\mathbb{R}^2$:



Let AA be the American continent, AEA the African-European-Asian continents and AP the Australian-Pacific islands. Since historically the oceans precluded communication in 1400, but not in 1900, we thus may conclude that this relational model of ($\mathcal{A}_1, \mathcal{R}_1$) is not connected, while this relational model of ($\mathcal{A}_2, \mathcal{R}_2$) is.

□

We now come to the main theorems of this paper. In these theorems we discuss the nature of communication in a relational model, and therefore in a relational structure. The first theorem deals with the question as formulated previously: under which conditions are all agents in the relational model able to communicate with each other. The answer to this question turns out to be very elegant.

<u>3.3 Theorem</u>

A relational model (A,R) is connected if and only if any two agents in A are able to communicate.

Proof

Let $(A_n)_{n \in \mathbb{N}}$ again denote the subdivision of the relational model (A,R). Moreover let the pair $(\bar{A},\bar{R})$ denote a condensation of (A,R) and let cond be the mapping as defined in definition 3.1.

Now we can describe $\bar{A}$ as follows: $\bar{A} = \{ \bar{a}_n \in V \mid \bar{a}_n = \mathrm{cond}(A_n),\ n \in \mathbb{N} \}$.

Only if

Let (A,R) be connected. For the proof that all agents in (A,R) are able to communicate we need the following claim:

<u>CLAIM</u>

For any integer $m \in \mathbb{N}$: Every pair of agents $a,b \in A_m$ are able to communicate, i.e. there exists a finite sequence of agents $(a_1,\dots,a_n)$ in A_m such that $a = a_1$, $b = a_n$, $(a_j,a_{j+1}) \in R$ for $j = 1,\dots,n-1$, and finally $(a_j,a_h) \notin R$ if $|j - h| > 1$.

We proceed by proving the claim. Therefore we apply lemma (10.3.8) of Csaszar (1978) for the sets $A = F(a)$, $B = F(b)$ and $C = A_m$ by taking as open covering of A_m the class $\mathcal{B} := \{ \mathrm{int}(F(a)) \mid a \in A_m \}$. Now by the

(topological) connectedness of A_m this lemma asserts the existence of an irreducible chain of elements of $\mathcal{B}$ between F(a) and F(b).
Let $\{ F(a_j) \mid j = 1,\dots,n \}$ be the set or sequence, which can be derived from this irreducible chain. Now the sequence $(a_1,\dots,a_n)$ is just the requested finite sequence as asserted in the claim. Hence the claim has been proved.

To complete the proof of the only if-part of the theorem, we take two agents $a,b \in A$. We have to consider two cases.

I - There exists an integer $n \in \mathbb{N}$ such that $a \in A_n$ as well as $b \in A_n$.
In this case the claim asserts that a and b are able to communicate.

II - There exist two distinct integers $k,\ell \in \mathbb{N}$ such that $a \in A_k$ and $b \in A_\ell$.
By definition of connectedness of (A,R), there exists a finite sequence in $\bar{A}$, say $(b_1,\dots,b_n)$ such that $b_1 = \bar{a}_k$, $b_n = \bar{a}_\ell$, $(b_j,b_{j+1}) \in \bar{R}$ for $j = 1,\dots,n-1$, and $(b_h,b_j) \notin \bar{R}$ if $|j - h| > 1$.
Now we are able to construct the following finite sequence:

i) $c_{11} = a$

ii) $c_{j2} \in \mathrm{cond}^{-1}(b_j)$ for $j = 1,\dots,n-1$
$c_{j1} \in \mathrm{cond}^{-1}(b_j)$ for $j = 2,\dots,n$
such that $(c_{j2},c_{j+1,1}) \in R$ for $j = 1,\dots,n-1$ (The existence of such pairs is garanteed by the definition of a condensation and connectedness.)

iii) $c_{n2} = b$.

For every $j \in \{1,\dots,n\}$ the claim asserts the existence of a finite sequence of intermediate agents between the pair $c_{j1},c_{j2} \in \mathrm{cond}^{-1}(b_j) = A_j$ and thus, since $(c_{j2},c_{j+1,1}) \in R$, the union of all these finte sequences is again a finite sequence, which is irreducible. Hence, a and b are able to communicate as defined in 2.8.

If

Assume that any pair of agents $a,b \in A$ are able to communicate.
Then we only have to check the definitions of a condensation and connectedness to arrive at the conclusion that the relational structure has to be connected. We leave this to the reader.

□

The proof of the previous theorem leads to some additional insights into the nature of relational models and the communication within the context of such models. One such additional is formulated in the next corollary, which states the nature of communication in a relational model. We also link this corollary with example 3.2, which discusses a situation of non-communication.

3.3' Corollary

Let (A,R) be a relational model.

The set of agents A can be divided into an at most countable number of pairwise disjoint subsets $(B_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, any two agents $a, b \in B_n$ are able to communicate, and moreover there is no communication between any two agents $a \in B_n$ and $b \in B_k$ when $n \neq k$.

Proof

This follows immediately from the fact that communication is a mathematical equivalence relation and the claim in the proof of theorem 3.3.

□

We will now define the notion which is the main tool in the derivation of a coalitional structure from a relational model. In fact we define a form of coalition formation, purely based on this tool. Within the setting of a relational structure, the tool is called a "network".

3.4 Definition

Let (A,R) be a relational model.

A subset $N \subset A$ of agents in the relational model is called a network if it satisfies the following conditions:

(i) $\{ F(a) \mid a \in N \}$ is a covering of A ;

(ii) Every pair of agents in the subset N is able to communicate **within** the network, i.e. for every two agents $a,b \in N$ there exists a finite sequence of agents in N, say $(c_1,\dots,c_n)$ such that $c_1 = a$, $c_n = b$, $(c_j,c_{j+1}) \in R$ for $j = 1,\dots,n-1$, and finally $(c_j,c_h) \notin R$

if $|j - h| > 1$ (In other words: $(N, R \cap N \times N)$ is a **connected** relational model.) ;

(iii) N is **minimal** in the sense that for every $a \in N$, the subset $N\setminus\{a\}$ does not satisfy as well (i) as (ii).

□

Next we can construct from the previous definition the following collections of networks in the relational model (A,R):

$$\mathcal{N} := \{ N \subset A \mid N \text{ is a network in } (A,R) \}$$
$$\mathcal{N}_c := \{ N \in \mathcal{N} \mid N \text{ is at most countable} \}$$
$$\mathcal{N}_f := \{ N \in \mathcal{N} \mid N \text{ is finite} \}.$$

It is obvious that for any relational model (A,R) its collections of networks is ordered as follows: $\mathcal{N}_f \subset \mathcal{N}_c \subset \mathcal{N}$.

It is clear that networks are something telling of the organisation of the communication within a relational model, and therefore in a relational structure. If there exists a network, then all communication can be done through this network. So if agent a wants to communicate with agent b, he can do this by passing on the message through members of the network only.

In fact we actually want to deal with a special subclass of networks, namely the relevant networks in a relational structure. Relevant networks do not only play a communicative role, but also have an economic purpose, especially with respect to some service. Examples of such relevant networks are bakers, hospitals, and plumbers. These kind of agents form networks which are servicing all other agents in the relational structure, and also can guarantee communication within the relational structure. It is quite clear that these networks are only relevant if the relational structure can be properly embedded in a topological space, which is describing the relevant characteristics. A network of plumbers does not seem to be very relevant if all agents in the relational structure are embedded in a topological space which is describing agents only with respect to their medical characteristics. For some further discussion of these kind of networks we refer to the example of health economies in the next section.

Since we are dealing with positive modelling, we cannot state anything about the existence of such relevant networks. We can only provide a first step in the analysis of such relational structures by providing a full description of the conditions under which the existence of certain kinds of networks is guaranteed. The next theorems all deal with this problem. The first existence result is especially nice since the existence of networks turns out to be equivalent to the communication problem as dealt with in theorem 3.3.

3.5 Theorem **(General existence of networks.)**
Let (A,R) be a relational model.

$$\mathcal{N} \neq \emptyset \text{ if and only if } (A,R) \text{ is connected.}$$

Proof

If

Define $\mathcal{S} := \{ N \subset A \mid N$ satisfies both (i) and (ii) of definition 3.4 $\}$. First we note that $A \in \mathcal{S}$, since by assumption (A,R) is connected and connectedness is equivalent to communication between all agents, as is shown in theorem 3.3. This last property shows that 3.4 (ii) is satisfied.

Next we take a collection $\mathcal{B} \subset \mathcal{S}$ such that for every pair $N_1, N_2 \in \mathcal{B}$ it holds that either $N_1 \subset N_2$ or $N_2 \subset N_1$. Now we will prove that the intersection of this subcollection also satisfies 3.4 (i) and (ii), i.e. $\cap \mathcal{B} \in \mathcal{S}$. Hence we will check whether this intersection satisfies both properties.

(i) The covering property is naturally satisfied, since suppose there exists an agent $a \in A$ which is not covered by $\{ F(i) \mid i \in \cap \mathcal{B} \}$. Then there exists a set $N \in \mathcal{B}$ such that a is not covered by the collection $\{ F(i) \mid i \in N \}$. This is a violation of the definition of $\mathcal{S}$.

(ii) Suppose that there exist two agents $a,b \in \cap \mathcal{B}$ who are not able to communicate within the set $\cap \mathcal{B}$. Then, again by the totally ordering of the collection $\mathcal{B}$, there exists a set $N \in \mathcal{B}$ for which a and b are neither able to communicate through N. Again this leads to a contradiction, which asserts that the intersection also satisfies property 3.4 (ii).

So by the application of the lemma of Zorn (see for example Quigley (1970) or Csaszar (1978)), there exists a minimal element in $\mathcal{S}$ with respect to inclusion. This is the desired network of (A,R). Hence we have proved that the collection $\mathcal{N} \neq \emptyset$.

Only if

Suppose that $N \in \mathcal{N}$ and let the pair $(\bar{A},\bar{R})$ be a condensation of the relational model (A,R). We will prove that $(\bar{A},\bar{R})$ is a finitely connected graph, thus proving connectedness.

We define $\bar{N} := \text{cond}(N) \subset \bar{A}$. Obviously $\bar{N}$ is at most countable and finitely connected within $(\bar{A},\bar{R})$, i.e. for every two agents $a,b \in \bar{N}$ there exists a finite sequence in $\bar{N}$, say $(c_1,\dots,c_n)$, such that $c_1 = a$, $c_n = b$, and $(c_j,c_{j+1}) \in \bar{R}$ for $j = 1,\dots,n-1$.

Take $a,b \in \bar{A}$, then by definition of a network there exists $c,d \in \bar{N}$ such that $(a,c) \in \bar{R}$ and $(b,d) \in \bar{R}$. Moreover by the statement above, there exists a finite sequence between c and d within $\bar{N}$, say $(s_2,\dots,s_{n-1})$. Now take $s_1 := a$ and $s_n := b$, then by property of the sequence between c and d and the relationship between a and c, and b and d, for every $j = 1,\dots,n-1$ $(s_j,s_{j+1}) \in \bar{R}$. So we have proved the finite connectedness of the condensation $(\bar{A},\bar{R})$ and hence we have proved the theorem.

□

We now come to the question whether there exist countable or even finite networks in relational models. It is obvious that we have to impose some severe conditions and restrictions to achieve such existence results.

<u>3.6 Theorem</u> **(Existence of countable networks.)**

Let (A,R) be a relational model such that A is a Lindelöf subspace of the topological space $T = (V,\tau)$. Then:

$$\mathcal{N}_c \neq \emptyset \text{ if and only if } (A,R) \text{ is connected.}$$

(For a definition of Lindelöf spaces we refer to the appendix.)

<u>Proof</u>

Only if

Since $\mathcal{N}_c \subset \mathcal{N}$ and we assume that $\mathcal{N}_c \neq \emptyset$ it easily follows from theorem 3.5 that (A,R) is connected.

If

Assume that (A,R) is connected. We define

$$\mathcal{S}_c = \{ N \in \mathcal{S} \mid N \text{ is countable} \}$$

where $\mathcal{S}$ is as defined in the proof of theorem 3.5. First we show that the collection $\mathcal{S}_c \neq \emptyset$.

Let $(A_n)_{n \in \mathbb{N}}$ be the subdivision of A.

Moreover let $(\bar{A},\bar{R})$ be a condensation of the relational model (A,R).

By definition of a condensation, there exists a point-to-set mapping denoted by $P : \bar{R} \longrightarrow 2^A$ with $P((\bar{a},\bar{b})) := \{a,b\}$ where $\bar{a} = \text{cond}(a)$, $\bar{b} = \text{cond}(b)$ and $(a,b) \in R$.

Since $\bar{R}$ is at most countable we know that $P(\bar{R}) \subset 2^A$ is also at most countable.

We define the collection $\mathcal{B} := \{ \text{int}(F(a)) \mid a \in A \}$. Then $\mathcal{B}$ is an open covering of A, since by definition of a relational model $a \in \text{int}(F(a))$. Since A is a Lindelöf subspace of the topological space $T=(V,\tau)$, there exists a countable subcovering of $\mathcal{B}$, say $\mathcal{B}_0$. We denote by N_0 the (at most countable) set of agents such that $\mathcal{B}_0 = \{ \text{int}(F(a)) \mid a \in N_0 \}$.

Next we define $N_2 := N_0 \cup N_1$, where N_1 is such that for every two agents $a,b \in N_0$: if $\text{int}(F(a)) \cap \text{int}(F(b)) \neq \emptyset$ and $a \neq b$, then there exists a **unique** agent $c \in N_1$ such that $c \in \text{int}(F(a)) \cap \text{int}(F(b))$.

It is obvious that such a set N_1 exists and can be derived from the set N_0. It also is quite clear that N_1 is at most countable, and thus N_2 is also at most countable.

Now define the set $N := N_2 \cup P(\bar{R})$. Hence $N \subset A$, and moreover N satisfies the covering property 3.4 (i) by the covering property of $\mathcal{B}_0$ and hence of $N_0 \subset N_2$. Furthermore it also satisfies condition 3.4 (ii) by definition of $P(\bar{R})$, the covering property of N_0, i.e. for every agent $a \in P(\bar{R})$ there exists an agent $b \in N_0$ such that $a \in F(b)$, and the definition of N_2, which guarantees communication within N_2.

This shows that $N \in \mathcal{S}$.

Moreover since N is the union of two at most countable sets we have shown that $N \in \mathcal{S}_c$.

Next we apply a similar course of reasoning as is followed in the proof of theorem 3.5. Thus it can easily be shown that the collection $\mathcal{S}_c$ satisfies the conditions of the lemma of Zorn, and hence we have established the existence of a minimal element in $\mathcal{S}_c$. And therefore we have shown that there exists at least one countable network in (A,R).

□

Finally we state our last existence theorem, which investigates the conditions under which there exists a finite network in the relational model (A,R). These conditions turn out to be quite strong.

3.7 Theorem **(Existence of finite networks.)**
Let (A,R) be a relational model for which A is a compact set in the topological space T. Then:

$$\mathcal{N}_f \neq \emptyset \text{ if and only if } (A,R) \text{ is connected.}$$

(Relational models which are compact and connected are called continuum relational models.)

Proof
The **only if**-part is quite trivial since $\mathcal{N}_f \subset \mathcal{N}$. (Apply theorem 3.5.)
Therefore we only have to show that the **if**-part is true.
Let $\mathcal{S}_f := \{ N \in \mathcal{S} \mid N \text{ is finite} \}$.
As in the proof in theorem 3.6 we show that this collection is non-empty, and then apply Zorn's lemma on this collection to prove the existence of a minimal element, which is a finite network.

Since A is a compact subset of a topological space, we know that there exists a finite subcovering of the open covering $\mathcal{B} = \{ \text{int}(F(a)) \mid a \in A \}$ of A. We denote the set of the participating agents within this subcovering as the set $N_0 \subset A$. Moreover, since A is compact, we know that the relational model (A,R) has a finite subdivision. (Again by simply applying the existence of a finite subcovering of any open covering of A.) Therefore we know

that (A,R) has a finite condensation $(\bar{A},\bar{R})$. By the definitions and arguments as given in the proof of theorem 3.6, there exists a similar set $P(\bar{R}) \subset A$. In this case we know that this set is finite, since $\bar{R}$ is finite. So we may construct, as is done in the proof of theorem 3.6, a finite set of agents $N := N_0 \cup N_1 \cup P(\bar{R})$, where N_0 and $P(\bar{R})$ are as defined above, while N_1 is chosen such that for any two agents $a,b \in N_0$, $a \neq b$, for which $int(F(a)) \cap int(F(b)) \neq \emptyset$, there is a **unique** agent $c \in N_1$ such that $c \in int(F(a)) \cap int(F(b))$.
Again it is simply shown that the set N satisfies as well condition 3.4 (i) as 3.4 (ii). Hence by the finiteness of N and these properties we know that $N \in \mathcal{S}_f$.

By applying Zorn's lemma on the collection $\mathcal{S}_f$ we establish the existence of a finite network in (A,R).

□

4. APPLICATIONS AND EXAMPLES

In this section we give some applications and examples of the model as developed in the previous sections. The main application, which we present in this section, is the formulation of a way starting from a relational structure and ending in a coalitional structure, as developed by Gilles (1987 and 1988). This application thus gives an explicit model of coalition formation, based on relations between agents only. This application will be discussed in subsection 4.1.

In subsections 4.2 and 4.3 we give some examples of the theory as developed in the previous sections, and the application of the model on the theory of coalition formation as developed in subsection 4.1. The first example considers the special situation in which the subdivision of the relational model (A,R) consists of one compact and connected subset of the topological space $T = (A, A|\tau)$, i.e. (A,R) is a continuum model consisting of a continuum of agents and a relation on that continuum. The second example considers a health economy and shows the relevancy of certain networks in describing such kind of specialized economic environments. Both examples shed some light on the new concepts as developed in the previous sections.

4.1 Coalition formation

The starting point of this subsection on coalition formation is a relational structure $(\mathcal{A}, \mathcal{R})$, which can be embedded properly in a relational model (A,R) in some topological space $T = (V,\tau)$. (Hence it satisfies the axiom of positive modelling.) Further we denote by $\mathcal{N}$, $\mathcal{N}_c$ and $\mathcal{N}_f$ the collections of respectively all possible networks, all countable networks and all finite networks of the relational model (A,R).

First we develop some technical tools to describe the environment in which we state our theory on coalition formation starting from the relational structure as described above.

4.1.1 Definition

Let S be a set and let $\emptyset \neq \Phi \subset 2^S$ be a collection of subsets of S.

(a) Φ is a <u>σ-algebra on S</u> if it satisfies the following properties:

i - $S \in \Phi$;

ii - For every sequence $(E_n)_{n \in \mathbb{N}} \subset \Phi$: $\bigcup_{n=1}^{\infty} E_n \in \Phi$;

iii - For every two elements $E, F \in \Phi$: $E \backslash F \in \Phi$.

(b) Φ is a <u>ring on S</u> if for every pair $E, F \in \Phi$ it holds that

$$E \backslash F \ , \ E \cup F \in \Phi \ .$$

(c) Φ is a <u>half-ring on S</u> if for every pair $E, F \in \Phi$:

$$E \backslash F \ , \ E \cap F \in \Phi \ .$$

(d) Φ is a <u>semi-ring on S</u> if it satisfies the following properties:

i - $\emptyset \in \Phi$;

ii - For every pair $E, F \in \Phi$:

$E \backslash F \ , \ E \cap F \in \Omega(\Phi)$,

where $\Omega(\Phi) := \{ \bigcup_{n=1}^{N} E_n \mid N \in \mathbb{N}, \ E_n \in \Phi \ (n=1,\dots,N)$ pairwise disjoint $\}$.

(e) The collection $\sigma(\Phi)$ [$\omega(\Phi)$, $\eta(\Phi)$, $\gamma(\Phi)$] $\subset 2^S$ is the σ-algebra [ring, half-ring, semi-ring] which is <u>generated by Φ</u> if it is the smallest σ-algebra [ring, half-ring, semi-ring] which contains Φ as a subcollection.

□

For a full exposition of some of these (mathematical) concepts we refer to Janssen and van der Steen (1984). We mention however the following evident properties of these kind of collections of subsets. Let Φ be a collection of subsets of some set S, then:

(a) Any σ-algebra is a ring.

(b) Any ring is a half-ring.

(c) Any half-ring is a semi-ring.

(d) It holds that

$$\Phi \subset \gamma(\Phi) \subset \eta(\Phi) \subset \omega(\Phi) \subset \sigma(\Phi) \subset 2^S \ .$$

(e) If Φ is a semi-ring, then

$$\Omega(\Phi) = \omega(\Phi) \ .$$

Examples show that semi-rings can be strictly smaller than half-rings, although semi-rings are harder to grasp than half-rings, especially in economic environments. We mention the example in which one takes the following set $S := [0,2] \times [0,3] \subset \mathbb{R}^2$, and the collection $\Phi := \{ [0,2] \times [0,2], [0,2] \times [1,3], [0,1] \times [0,3] \}$ of subsets of S. We leave it to the reader to find out that $\gamma(\Phi) \subsetneq \eta(\Phi)$. (This is true since the set $[0,2] \times [1,2]$ is divisible in $[0,1] \times [1,2]$ and $(1,2] \times [1,2]$, but does not need to belong to the semi-ring $\gamma(\Phi)$. It however does belong to the half-ring $\eta(\Phi)$.)

We now define the main concept as developed in Gilles (1987 and 1988), namely the concept of coalitional structure. For a full description of the properties of a coalitional structure in an economic environment we refer to the papers as mentioned above.

4.1.2 Definition

Let A be a set of agents.
The triple (A,Γ,μ) is called a coalitional structure (of agents) if

(a) $\Gamma \subset 2^A$ is a semi-ring on A ;

(b) $\mu : \Gamma \longrightarrow [0,1]$ is a normalised measure on (A,Γ), i.e.

- i - $\mu(\emptyset) = 0$;
- ii - $\sup \{ \Sigma \mu(E_n) \mid E_n \in \Gamma \ (n \in \mathbb{N}) \text{ pairwise disjoint} \} = 1$;
- iii - For every sequence $(E_n)_{n \in \mathbb{N}} \subset \Gamma$ of pairwise disjoint elements in Γ it holds that

$$\text{if } \bigcup_{n=1}^{\infty} E_n \in \Gamma, \text{ then } \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) .$$

□

The purpose of this subsection is to develop a method which derives a coalitional structure from a relational model. Then we have also constructed a model of coalition formation in case of a relational structure. We now come to the main tools in the development of this theory of coalition formation in the special case of a relational structure or model.

4.1.3 Definition

Let (A,R) be a relational model and let $\mathcal{N}$ be the collection of its networks. Futhermore let $\emptyset \neq \mathcal{M} \subset \mathcal{N}$ be a nonempty family of networks of (A,R).

(a) The class $S(\mathcal{M}) := \{ F(a) \mid \exists N \in \mathcal{M} : a \in N \} \subset 2^A$ is called the service structure of, or generated by, $\mathcal{M}$.

(b) The semi-ring $\gamma(\mathcal{M}) := \gamma(S(\mathcal{M}))$ is the coalitional structure generated by $\mathcal{M}$ if there exists a normalised measure $\mu : \gamma(\mathcal{M}) \longrightarrow [0,1]$ such that the triple $(A, \gamma(\mathcal{M}), \mu)$ is a coalitional structure as defined in 4.1.2.

□

From the definition we learn that usually we do not choose any family of networks of the relational model, but we choose a particular family of networks. Especially the case in which we choose some family of relevant networks of (A,R) can be very interesting. These kinds of families of networks are determined by economic features such as economic activity or positions of the agents in some structure which is based on certain characteristics, for example medical ones. (This example will be extended in subsection 4.3.) This also explains the name of the class $S(\mathcal{M})$: it describes the specific structure of how the agents are serviced by the (relevant) networks in the family $\mathcal{M}$. (All members of such a family have the same economic purpose or service.)

As provided by the service structure we base coalition formation, as described in definition 4.1.3 (b), on positions of agents within the service structure of the family $\mathcal{M}$ of certain relevant networks. We take some service structure as given and then the primitive coalitions, i.e. the members of the semi-ring $\gamma(\mathcal{M})$, are taken as those groups of agents in the relational model, who have the same position with respect to the service structure and thus with respect to the chosen networks.

Especially if we take the half-ring generated by the service structure $S(\mathcal{M})$ instead of the semi-ring, we explicitly use all available information of the relational model with respect to the position of the agents with respect to that particular family of (relevant) networks with that specific economic service purpose.

Our final problem in this model of coalition formation based on a relational structure or model is to solve the question under which conditions there exists a well defined measure on the generated semi-ring of some non-empty family of networks in the relational model. Therefore we need a stronger version of the axiom of positive modelling.

4.1.4 Strong axiom of positive modelling

Let ($\mathcal{A},\mathcal{R}$) be a relational structure.

($\mathcal{A},\mathcal{R}$) can be modelled properly if it can be embedded properly in some relational model (A,R) in a topological space T = (V,τ) such that the following properties are satisfied:

(i) (A,R) is a connected relational model ;

(ii) Let $(A_n)_{n \in \mathbb{N}}$ be the subdivision of A. Then the following properties are satisfied.

* There exists a sequence $(\delta_n)_{n \in \mathbb{N}}$ of real numbers such that $\delta_n \geq 0$ $(n \in \mathbb{N})$, and $\Sigma\, \delta_n = 1$, which describes the fraction δ_n of the agents in A who are member of component A_n.
* Moreover, for every integer $n \in \mathbb{N}$ the component $A_n \subset A$ is a compact subset in the topological space T.

□

Obviously if a relational structure can be modelled properly, then we not only assume that there is enough information available to describe the relations locally by some set of features or characteristics, but additionally we asssume that there is information about the size of the groups of agents who are "close" to each other with respect to those features or characteristics. ("Closeness" is meant here as closeness in the decription by the chosen characteristics as reflected in the topological space in which the relational structure is embedded.) This additional information is reflected in two forms, namely topologically and measure theoretically.

Topologically we reflect this additional information in the assumption that the components in the subdivision of the model are topologically compact. So we actually assume that in some way the class of agents in the relational structure, who are "close" to each other, is not unbounded.

(Especially in Hausdorff- or metric spaces this is the case, while in those spaces compactness implies closedness. In metric spaces it even implies boundedness.)

In **measure theoretic** terms we assume that the size of a component of the subdivision of A can be expressed explicitly by some nonnegative number or percentile. It expresses the size of the classes of "close" agents with respect to the chosen set of features or characteristics.

We now come to the main conclusion of our modelling. Before we are able to formulate this conclusion we have to define some additional tool.

4.1.5 Definition

The relational model (A,R) in the topological space $T = (V,\tau)$ is called <u>measurable</u> if for every agent $a \in A$:

$$F(a) \in \sigma(\tau|A),$$

where $\tau|A := \{ E \cap A \mid E \in \tau \}$ is the topology restricted to the subspace A of T.

□

Measurability can be interpreted as a "natural" condition on a relational model. With this property we are able to formulate our main existence theorem of coalitional structures based on relational structures.

4.1.6 Existence theorem

Let ($\mathcal{A},\mathcal{R}$) be a relational structure.

If ($\mathcal{A},\mathcal{R}$) can be modelled properly in some measurable relational model (A,R), such that $(A,\tau|A)$ is a locally connected and metrizable subspace of some topological space $T = (V,\tau)$, then for any non-empty family of networks $\emptyset \neq \mathcal{M} \subset \mathcal{N}$, there exists a non-trivial measure $\mu : \gamma(\mathcal{M}) \longrightarrow [0,1]$ such that $(A,\gamma(\mathcal{M}),\mu)$ is a coalitional structure.

<u>Proof</u>

Let (A,R) be the relational model as described in the theorem and let the sequence $(A_n)_{n \in \mathbb{N}}$ be its subdivision.

For any integer $n \in \mathbb{N}$, the set A_n is a connected, compact, locally connected and metrizable subspace of the topological space $(A,\tau|A)$. (Locally connectedness of A_n follows from theorem (10.2.3) of Csazsar (1978) and definition 2.4 which garantees that A_n is as well closed as open in the space $(A,\tau|A)$.) By the Hahn-Mazurkiewicz theorem (see the appendix) there exists a continuous function $f_n : I \longrightarrow A_n$ - where $I = [0,1]$ - which is surjective.

Now we define the following mapping:
$\bar{\mu} : \sigma(\tau|A) \longrightarrow [0,1]$ is a measure fully defined by

$$\bar{\mu}(E) := \sum_{n=1}^{\infty} \delta_n . \lambda(f_n^{-1}(E \cap A_n)),$$

where $E \in \sigma(\tau|A)$ and λ is the Lebesgue measure on $(I, \sigma(\mathcal{E}))$. ($\mathcal{E}$ is the Euclidean topology on the unit-interval I and $\sigma(\mathcal{E})$ the collection of Borel-sets on the unit-interval.)
It is evident that this definition is in order and $\bar{\mu}$ indeed is a measure on the σ-algebra $\sigma(\tau|A)$, since for every integer $n \in \mathbb{N}$ the mapping f_n is continuous and thus for any set $F \in \sigma(\tau|A_n)$, $f_n^{-1}(F)$ is a Borel-measurable set.

Now take $\emptyset \neq \mathcal{M} \subset \mathcal{N}$. (Such non-empty families exist, because all conditions of theorem 3.5 are statisfied and hence $\mathcal{N} \neq \emptyset$.) Then $\gamma(\mathcal{M})$ is a non-trivial semi-ring. Next take for the measure $\mu : \gamma(\mathcal{M}) \longrightarrow [0,1]$ the restriction of the measure $\bar{\mu}$ to the subclass $\gamma(\mathcal{M}) \subset \sigma(\tau|A)$. This definition is proper, because the relational model (A,R) satisfies the measurability property as defined in 4.1.5, i.e. for every agent $a \in A$, $F(a) \in \sigma(\tau|A)$ and hence $S(\mathcal{M}) \subset \sigma(\tau|A)$. Therefore the semi-ring generated by this class $S(\mathcal{M})$, the service structure of $\mathcal{M}$, consists only of measurable sets.

□

From the proof and the statement of the theorem it is obvious that we also may take the collections $\eta(\mathcal{M})$, $\omega(\mathcal{M})$ or even $\sigma(\mathcal{M})$ instead of the semi-ring $\gamma(\mathcal{M})$ in the formulation of the theorem. It is also quite obvious that only the formulations with the semi-ring $\gamma(\mathcal{M})$ and the half-ring $\eta(\mathcal{M})$ are economically useful. This is the purpose of one of the next subsections, which discusses a health economy based on a relational structure and the concept of coalition formation as developed in the previous sections, this subsection, and especially the previous theorem.

4.2 An application: simple structures

In this subsection we will consider as an example, a specific class of relational models. On this class we will apply the main existence theorems as developed in this paper.

One of the major observations which can be found by considering contemporary economic theory, is that in many models in general equilibrium theory one assumes that the set of agents is a continuum. (Hence the set of agents is assumed to be a compact and connected topological space, endowed with an atomless measure on the σ-algebra of Borel-sets generated by the topology.) In most cases one takes the unit-interval as the set of agents in the economy. In this subsection we consider precisely such a situation, which we shall call "simple" or "Euclidean".

4.2.1 Definition

Let $(\mathcal{A},\mathcal{R})$ be a relational structure which is embedded in a relational model (A,R) in some topological space T = (V,τ). Now we define:

(a) (A,R) is simple if $(A,\tau|A) \subset T$ is a topological continuum and (A,R) is a measurable relational model.

(b) (A,R) is Euclidean if A is a continuum in a finite dimensional Euclidean space and furthermore (A,R) is a measurable relational model.

□

It is easily established that every Euclidean relational model is simple. Therefore all properties derived on simple relational models can be applied on Euclidean relational models.

4.2.2 Corollary

Let the relational model $(\mathcal{A},\mathcal{R})$ be embedded in a simple relational model (A,R). Then the following properties are satisfied:

(a) $(\mathcal{A},\mathcal{R})$ and (A,R) satisfy the strong axiom of positive modelling.

(b) There exists a finite network in the relational model (A,R), i.e. $\mathcal{N}_f \neq \emptyset$.

(c) Let $\emptyset \neq \mathcal{M} \subset \mathcal{N}$ be a non-empty family of networks in (A,R). If $(A,\tau|A) \subset T = (V,\tau)$ is a locally connected and metrizable subspace of the topological space T, then there exists a normalised measure $\mu\colon \gamma(\mathcal{M}) \longrightarrow [0,1]$ such that $(A,\gamma(\mathcal{M}),\mu)$ is a coalitional structure.

Proof

By observing that any simple relational model is connected and checking the necessary conditions, we conclude that any simple relational model satisfies theorem 3.7 and definition 4.1.4. Therefore (a) and (b) are true. satisfied. Next observe that a simple relational structure therefore also satisfies the conditions of theorem 4.1.6 if $(A,\tau|A) \subset T$ is a locally connected and metrizable topological space. Hence, by application of that theorem, we may conclude that assertion (c) is also satisfied.

□

For Euclidean relational models we can simplify the formulation of assser- tion 4.2.2(c) considerably.

4.2.3 Corollary

Let (A,R) be an Euclidean relational model, and let $\emptyset \neq \mathcal{M} \subset \mathcal{N}$ be a non-empty family of networks. Then there exists a measure $\mu\colon \gamma(\mathcal{M}) \longrightarrow [0,1]$ such that $(A,\gamma(\mathcal{M}),\mu)$ is a coalitional structure.

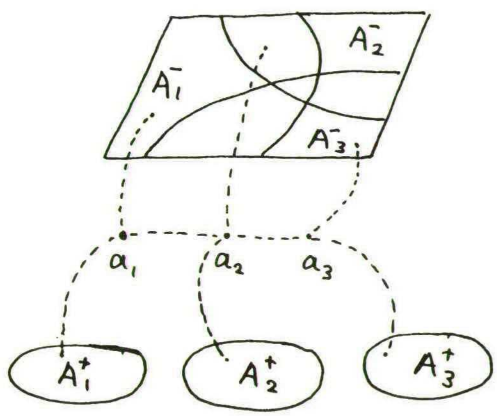
□

4.3 An application: a health economy

In this example we consider a health economy, reflected in some relational health structure $(\mathcal{A},\mathcal{R})$, which can be modelled properly in a relational model (A,R), where A is a subset of some finite-dimensional Euclidean space. (Note that this model does not have to be Euclidean in the sense of the previous subsection.)

There are ℓ different illnesses or diagnoses, each having a set of patients, denoted by A_i^-, $i = 1,\dots,\ell$, which are called diagnostic related groups (DRG), and a set of specialised helpers, denoted by A_i^+, $i = 1,\dots,\ell$. The helpers may be nurses, administrators, people working in hospitals and/or other institutions. Sets of helpers are called diagnostic related institutions (DRI). We complete our model by introducing a set of medical doctors, who determine a diagnosis. We assume that there are only a finite number of medical doctors, each related to some illness or diagnosis, and we denote them by a_i, $i = 1,\dots,\ell$.

So we may conclude that $A = A^- \cup (\bigcup_{i=1}^{\ell} A_i^+) \cup \{ a_1,\dots,a_\ell\}$, where $A^- = \bigcup_{i=1}^{\ell} A_i^-$ is a connected and compact set consisting of patients. Moreover it is clear that the set A^-, the DRI's and the doctors form a subdivision of A. (See figure below.)



With this in mind we can describe the relations between the agents in this model. (For simplicity we define $A^+ := \bigcup_{i=1}^{\ell} A_i^+$ as the set of agents working in the medical institutions or DRI's.)

Each DRI A_i^+, $i \in \{1,\dots,\ell\}$, is related to some medical doctor a_j, so each institution has an alliance with some doctor. Furthermore all doctors are related to each other and cover the set of patients A^-.

Normalised weights $\{ \delta_j \mid j = 1,\dots,2\ell+1 \}$ are given to the components in the subdivision of A as described above. It is also clear that this model is connected and therefore the strong axiom of positive modelling is satisfied. Now let $\mathcal{U}$ be the class of health-relevant networks. Then it is evident that such a network $N \in \mathcal{U}$ always contains the finite set of medical doctors. No coalition consisting of patients and institutions only is feasible as a primitive coalition. The relational structure then requires that a doctor is also a member. (However with the instruments as developed in Gilles (1987), such coalitions may exist as realizable coalitions and thus may play an important role in the blocking process as described by the semi-core.) This means that an agent who possibly is negligible, has a crucial position in coalition formation on the lowest level. The next step in our modelling should be to formulate equilibrium concepts in which the agents can earn a profit from such crucial positions. (See also Hammond (1987).)

Although defining an equilibrium concept is not the purpose of this paper, we think it is illustrative for the power of our approach to make a first attempt. Consider therefore an economy in which the agents can find themselves in three positions: patient, nurse or doctor, resp. x, y and z. There are ℓ different illnesses, cured by ℓ corresponding institutions. The relations are described by the model above, where nurses are assumed to be the only members of the medical institutions (DRI). Each patient is endowed with an ordered vector of illnesses $v \in (\{0,1\})^{\ell}$ and a wealth scalar $w \in [0,1]$. Each nurse in institution i can trade an illness treated in that institution v_i for money income w, but only if there is a doctor related to that institution, who also earns money w.

Let the utility function of those groups be defined as follows:

$$U_x(v,w) > U_x(v',w') \text{ iff } v < v' \text{ and } [w > w' \text{ if } v \geq v'],\ x \in A^- \ ;$$

$$U_y(v,w) > U_y(v',w') \text{ iff } v > v' \text{ and } w > w',\ y \in A^+ \ ;$$

$$U_z(v_i,w) > U_z(v_i',w') \text{ iff } w > w',\ z \in \{a_1,\dots,a_\ell\} \ .$$

An <u>allocation</u> is a function f(x), g(y), h(z) assigning (v,w) to each agent in the economy. Let $\bar{f}$, $\bar{g}$, $\bar{h}$ be the initial endowment.

Next we define in the setting as described above an absolutely minimal requirement for equilibrium allocations, namely allocations which cannot be blocked by any primitive coalition. Let $C \in \gamma(\mathcal{M})$ be a primitive coalition in the semi-ring generated by the medical relevant networks in the economy as described above. For simplicity assume that **all doctors** a_i in the economy are **not negligible**, i.e. $\delta_{z_i} > 0$. Now the primitive coalition C is said to be able to <u>block</u> a given allocation (f,g,h) if there exists a re-allocation (f',g',h') such that the following properties are satisfied:

(i) $\int_C (f',g',h')\, d\mu \leq \int_C (\bar{f},\bar{g},\bar{h})\, d\mu$;

(ii) $U_a(v'(a),w'(a)) > U_a(v(a),w(a))$ for almost every agent $a \in C$.

Note that a primitive coalition is only able to block some allocation if it has as well patients as nurses as its members. But the relational structure then requires that at least one doctor also has to be a member of that coalition. Hence again we see the intermediary and crucial role of doctors in this model.

Note that in the specific case that **all doctors** are **negligible**, i.e. $\delta_z = 0$, the blocking conditions reduce to:

(i) $\int_C (f',h')\, d\mu \leq \int_C (\bar{f},\bar{h})\, d\mu$;

(ii) $U_a(v'(a),w'(a)) > U_a(v(a),w(a))$ for almost every agent $a \in C \cap A^- \cap A^+$.

However doctors remain as crucial in coalition formation as before, so they have blocking power on their own and this leads to the following additional property which has to be satisfied also:

(iii) $U_z(v'(a_i),w'(a_i)) > U_z(v(a_i),w(a_i))$ for every doctor $a_i \in \{a_1,\dots,a_\ell\} \cap C$.

This additional property sketches the importance of the doctors in the process of coalition formation, especially primitive coalitions, in this economy. It also sheds some light on the problems which arise in defining a proper cooperative equilibrium concept in an economy with a relational structure. Although agents may be negligible, they can play a crucial role in coalition formation, and therefore earn large profits.

Finally we return to the positive modelling. The strong axiom of positive modelling requires that each fraction δ_n of agents in the economy, who are member of component A_n, has a nonnegative value. We can observe these fractions as the corresponding characteristics or features are observable. From the relations between characteristics we can infer (statistically) relations between agents in the economy. The economy as described above is a nice example of a situation in which such a statistical estimation can be performed.

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APPENDIX : TOPOLOGICAL SPACES

All definitions and theorems as formulated in this appendix can be found in Csazsar (1978). For elaborations and further properties we therefore refer to this book on general topology.

Let (E,τ) be a topological space, i.e. E is a set and τ is a collection of open sets, which contains the empty set $\emptyset$ and is closed for taking arbitrary unions and finite intersections.

A.1 Definition

(E,τ) is separable if there exists a subset $D \subset E$, consisting of a countable number of elements, such that its closure is E itself, i.e. it holds that $\overline{D} = E$. (Hence D is a countable **dense** subset of E.)

A.2 Definition

(E,τ) is a Lindelöf space if from each open covering of E one can select a countable subcovering.

A.3 Theorem

(a) Every separable metric space is a Lindelöf space.

(b) **(Original) Lindelöf theorem**
Any Euclidean space is a Lindelöf space.

□

A.4 Definition

(a) (E,τ) is connected if there do not exist two open disjoint sets $A,B \in \tau$, $A \cap B = \emptyset$, such that $E = A \cup B$.

(b) (E,τ) is a locally connected space if every point $x \in E$ has a neighbourhood base consisting of connected sets only.

(c) (E,τ) is a continuum if it is a compact and connected topological space.

A.5 Theorem

(a) A topological space is locally connected if and only if it has a base consisting of connected sets only.

(b) Let (E_1,τ_1) and (E_2,τ_2) be two Hausdorff- or T_2-spaces, and let $f : E_1 \longrightarrow E_2$ be a continuous surjection. If (E_1,τ_1) is a locally connected continuum, then (E_2,τ_2) has the same property.

(c) **Hahn-Mazurkiewicz**
Every locally connected metrizable continuum is a continuous image of the Euclidean space $(I, \mathcal{E})$ with $I = [0,1]$ the unit-interval, and $\mathcal{E}$ the Euclidean topology on I.

□

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