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OPTIMAL TAXATION AND NORMALISATIONS

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Abstract

There still seems to be some confusion about the consequences of normalisations in the optimal taxation litterature. We claim that: 1) Normalisations do not matter for the real solution of optimal taxation problem. 2) Normalisations do matter for good characterisations of the solutions to optimal taxation problems. Whereas the first point is uncontroversial, the second one is less well understood.

By postponing the normalisation of consumer prices, we detail how normalisations of consumer prices affect the characterisation of optimal commodity taxes, derive the preferred characterisation, and show how it depends on the normalisation.

Key words: Optimal taxation, normalisation, marginal cost of public funds.
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1 Introduction

There still seem to be some confusion about the the role of normalisations in the optimal taxation literature, even with perfect competition, complete markets and full information. We set out to clarify this matter, and claim that, with all income generated by endowments:

1. Normalisations do not matter for the real solution of optimal taxation problem.[1]

2. Normalisations do matter for (good) characterisations of the solutions to optimal taxation problems.

The first point is uncontroversial. The second one is also fairly obvious, but, as far as I can see, not acknowledged in the literature. Indeed the standard opinion seems to be that since the real solution is independent of normalisations, so should characterisations, although I have not found anyone saying this explicitly.

The distinction is clarified by an analogy with the view (analogous to the normalisation) of a (3-dimensional) object (analogous to the solution). For such objects, it is clear that:

1. The view does not change the object.
2. The view is important for seeing the essential properties of the object.

In our case the homogenity of demand in consumer prices introduces one degree of freedom in choosing the "view" on the solution.

We support this distinction by showing in detail how normalisations affect the characterisation, and arrive at a canonical (preferred) normalisation.

The above is our main point. But it might also be wortwile to repeat that one also needs to distinguish between different kinds of taxation of endowments (usually only leisure):

1. Taxation of own consumption of initial endowments (e.g. leisure).
2. Taxation of sales of initial endowments (e.g. labour).

In standard optimal taxations models, untaxed own consumption of initial endowments is usually what makes first-best unattainable, and thus a basic trait of the models. Untaxed sales of initial endowments is only a normalisation of consumer prices, though, as we will argue, the canonical one for interpretations. More specifically, we claim that to get the best interpretation, consumer prices should be normalised so that the *tax income from initial endowments is zero* at the optimum. This is of course a straightforward generalisation of the standard case, where leisure is the only endowment and labour is untaxed. Using this normalisation, we arrive at a generalised inverse elastisity rule, generalising slightly the characterisation in Deaton (1979). The rule says that one should tax complements to the initial endowments harder. The intuitive explanation is that doing this is an indirect way of introducing a tax on the own consumption of endowments.

[1] This presupposes demand and supply functions which are homogenous of degree 0 in consumer and producer prices. If not, Munk (1978) and Dixit and Munk (1977) show that normalisations also matter for the real solution.

For other purposes other normalisations might be better. E.g., when it comes to solving the problem, the proposed normalization is useless, as it generally presupposes the optimum.

The assumption of perfect competition is important, for, as noted by Gabszewicz and Vial (1972), with imperfect competition the price normalization also matters for the real solution of the taxation problem. Additionally, one are usually not able to solve such general equilibrium models without making a normalization.

In the following, after discussing the literature, we first set up the commodity tax model, then repeats the simple points concerning the second distinction using the homogeneity of demand. Finally, we detail how normalisations of consumer prices affect the characterisation of optimal commodity taxes, and arrive at our canonical normalisation for interpretations.

1.1 Some examples from the literature

Is not the main distinction above well-known? In one sense it is, as most people use the canonical normalisations and the resulting intuitive interpretations. People explicitly discussing normalisations in an optimal taxation context, however, tend to miss at least the first distinction, as I will try to show.

Except for the interpretation issue, this article build on the work of the people we criticise here, as the model presentation is only a slight variation of that of Deaton (1979) and Auerbach (1985). The variation is that we use the economically more intuitive compensated willingness to pay function instead of their distance function, and define what we call endowment elasticities, which helps the interpretation.

Auerbach Auerbach in his excellent handbook article (1985, p. 89) clearly notes the difference between the two types of taxation of leisure. He seems to disagree to our main point about the canonical normalisation for characterisations, however. Thus in the case with an endowment of leisure only, while we argue that the canonical normalisation is to make sales of leisure untaxed, as this gives the most intuitive interpretation, he warns (on p. 90) that making leisure untaxed leads to a "loss of distinction between untaxable and untaxed goods." As mentioned, this only happens if one does not discriminate between *own consumption* of leisure (untaxable by model setup) and *sale* of leisure (untaxed due to the normalisation). Indeed, Auerbach seems (implicitely) to require that characterisations should be independent of the normalisation, at least he provides such a characterisation, which then becomes weaker than necessary, as we will see.

Deaton Even Deaton (1981, p. 1256) seems confused about the role of normalisations. Discussing the characterisation of optimal taxes (equation (15) below) he states:

> Note the special place occupied by good 0, leisure. ... the asymmetry is due to the numeraire role of labor (or leisure). Since leisure is untaxed, government revenue is implicitly measured in labor units so that by taxing complements with the revenue good, taxation is rendered easier. In general, the government will presumably wish to

> purchase goods other than labor and this would lead to a different tax rule. For example, a king who must pay a tribute of oxen to a neighboring conqueror would do well to levy relatively high taxes on goods complementary with oxen.

Stern Stern (1986, p. 298) claims that "there have been a temptation "to elevate the innocent normalization ... into something of of real substance". He then points out the error in Deatons above claim. He is, however, not quite precise about what is going on as he states that

> The crucial reason for the central role of complementarity with leisure in the results concerning the optimum proportion of tax in price is that there is an endowment of leisure which cannot be taxed in this second-best problem.

The root of the problem is as we have said that own consumption of leisure is untaxed. That labor (sales of leisure) is untaxed is an innocent normalisation. Furthermore, the fact that normalisations are irrelevant for the solution of the optimal taxation problem does not imply that they are irrelevant for the characterisation of the problem.

It looks like Stern does not discriminate between the task of *understanding* the principles of taxation and the task of *solving* a taxation problem. The first task one needs an intuitive characterisation, where normalisations matter, whereas for the second task normalisations does not matter (except maybe for the speed of finding the solution) – but characterisations usually does not matter either. This is different from the case with explicit analytic solutions, where one wants to arrange the solution in an easily interpretable way.

Stern also lists some other questionable reservations against Deaton's approach. One is that the symmetry between goods and factors are lost on Deatons approach. This is true, but that symmetry is replaced by the more fundamental one between positive and negative net trades. Another is, if I understand him correctly, is that the duality results underlying Deaton's approach presuppose positive lump-sum income. This is wrong, as the duality results only presupposes positive (full) income, and this is for all practical purposes fulfilled without lump-sum income. Finally, he claims that it is hard to measure or even define the endowment of leisure. But the measurement of the endowment of goods is more a problem when it comes *solving* an optimal taxation problem than for *understanding* the principles of taxation. For the last task, I think Deatons's approach is the preferred one.

Fullerton Fullerton (1997), discussing the double dividend hypothesis, is quite close to our position. He shows clearly how the *double dividend hypothesis* (that the pollution tax should exceed marginal environmental damages) depends on the normalization used. He seems, however, less inclined to recommend the normalisation where labour is untaxed for interpretations, despite it giving the natural double dividend hypothesis.

Myles In contrast to most of the above authors, Myles (1995, p. 122), is also confused about the different kinds of taxations of leisure, but here we consentrate on our main point:

> It has been shown that in an economy with constant returns to scale, consumer and producer prices can be normalised separately and that the standard procedure is to make one good the numeraire and set its consumer and producer prices equal. This normalisation also has the effect of setting the tax on that good to zero. The latter fact is clearly seen to be of no consequence whatsoever since the zero tax is just a result of the normalisation rule. In particular, the zero tax carries no implications about the nature of the good nor about the ability to tax that good. This follows since the good with zero tax can be chosen arbitrarily from the set of available goods.

The "no consequences whatsoever" might be interpreted as saying that interpretations should be independent of normalisations, thus missing our first distinction. He continues (on p. 123):

> Particular examples of this are found in Corlett and Hague (1953) 'By taxing those goods complementary with leisure, one is to some extent taxing leisure itself' (p. 26) .... This, of course, is a false interpretation.

Here he seems to infer that interpretations based on a specific normalisation is illegitimate, and thus our first distinction is lost.

2 The optimal commodity taxation model

We use the standard optimal commodity taxation model with one price-taking individual,[2] except that we allow for initial endowments, $\overline{\mathbf{x}} \geq \mathbf{0}$, of all goods.[3] Assume fixed firm (producer) prices, $\overline{\mathbf{p}}$.[4] With quantity taxes, $\mathbf{t}$, on market transactions, $\mathbf{x}-\overline{\mathbf{x}}$, and no lump sum tax, prices to individuals (consumer prices) are $\mathbf{p} = \overline{\mathbf{p}} + \mathbf{t}$, and the income of the individual (at these prices), $m = \overline{m} + \mathbf{p}'\overline{\mathbf{x}}$, where $\overline{m} = 0$ is exogenously given income. The problem for efficiency is as usual that own consumption of initial endowments is untaxed.

Remark 1 *We talk about firm prices instead of producer prices and prices to the individual instead of consumer prices, to emphasise that these are all prices to firms and individuals, and not only prices of the goods they produce or consume.*

Remark 2 *With our tax system, $t_i > 0$, means that good i is taxed if the individual is a net demander of good i and that it is subsidised if the individual is a net supplier of it.*

Assume that the public sector maximises the utility of the individual given an exogenously given tax requirement, $\overline{T}$. Normalise firm prices by setting income

[2] This is of course unrealistic. It is usually justified by saying that one are interested in efficiency results in an economy with many individuals, but this is not efficiency in the Pareto sense.

[3] We also assume a complete set of markets, price-taking actors and no profit. Without these assumptions, even the real solution to the problem might depend on the normalisation.

[4] This corresponds to a (constant scale) Leontief technology. The assumption of fixed firm prices can be replaced with a linear technology without any consequences for the results. The only difference is that then we also have to make the firm price normalisation at the prices in optimum, as we do for the individual price normalisation below.

at firm prices, $\overline{\mathbf{p}}'\overline{\mathbf{x}} = 1$. Then $\overline{T}$ is the tax requirement as a share of the income at firm prices. Also let $v(\mathbf{p}, m)$ be the indirect utility function and $\mathbf{x}(\mathbf{p}, m)$ the demand of the individual. The governments problem is then:

$$V(\overline{\mathbf{p}}, \overline{m}, \overline{T}) = \max_{\mathbf{t}} v(\mathbf{p}, m) \quad \text{s.t.} \quad \mathbf{t}'(\mathbf{x}(\mathbf{p}, m) - \overline{\mathbf{x}}) - \overline{T} \geq 0 \quad (\mu). \tag{1}$$

The marginal social utility of (exogenously) income to the individual**,** $V_{\overline{m}} = v_m + \mu \mathbf{t}'\mathbf{x}_m$, and the marginal social utility of increasing the tax income requirement, $V_{\overline{T}} = -\mu$. Thus μ is the marginal social utility of reducing the public tax requirement, and $\theta = (\mu - V_{\overline{m}})/\mu$ is the value of transferring one pound (exogenously) from the individual to the public, in units of the public budget. All these concepts are evaluated at consumer prices.

Before characterising the solution, we discuss different normalisations of consumer prices.

2.1 Normalisation of consumer prices

With all income generated by endowments, $\overline{\mathbf{x}}$, demand, $\mathbf{x}(\mathbf{p}, \mathbf{p}'\overline{\mathbf{x}})$, is homogeneous of degree 0 in consumer prices, $\mathbf{p}$. Thus we can scale (normalise) the consumer prices, $\mathbf{p}$, without affecting demand. This homogeneity is often used to set the consumer price on one good equal to its firm price, so that the tax on this good is 0. We illustrate the consequences of different choices with an example.

Example 1 *With a (constant scale) Leontief technology, choose units so that all goods have firm price* 1*. There are two goods,* $i = 1, 2$ *(without endowments) in addition to leisure (good* 0*). Assume that with leisure untaxed (the canonical normalisation, as we shall see), the tax on good* 2 *is larger than that on good* 1*, i.e.* $t_2 > t_1$*. The consumer prices are then* $\mathbf{p}^0 = (1, 1 + t_1, 1 + t_2)'$*.*
What happens if we instead choose good 2 *as untaxed, by dividing consumer prices by* $1 + t_2$*? We get new consumer prices,* $\mathbf{p}^2 = (\frac{1}{1+t_2}, \frac{1+t_1}{1+t_2}, 1)'$*. But then* $p_0^2, p_1^2 < 1$*, thus we have a subsidy on good* 1 *and tax on good* 0*. There is a tax on leisure since the price of leisure is smaller to the consumer than the firms.*

The result in the case with good 2 as untaxed is hard to interpret: To obtain the required public revenue, one should subsidise good 1! In contrast, the case with leisure untaxed is more comprehensible, with two taxed goods. I think this example already points to the importance of the normalisation for a good characterisation. We get briefly back to the example at the end of the paper.

Tax systems In our context, define a *tax system* as the abstract entity which are invariant under different consumer price normalisations. Choosing a normalisation then only give an instance of the tax system.

I can then restate my main claim as follows: At least for *interpretation* purposes, the canonical instance of a tax system, is the one making the individual's income equal to one at the optimum.

As noted by Fullerton (1997), for *implementing* such a tax system, other considerations, like which goods it is feasible to tax, might make another normalisation more appropriate.

3 Characterising optimal commodity taxes

As mentioned, we postpone the consumer price normalisation, to get a clearer view of it's effect. The first order condition of our taxation problem, (1), with respect to the tax rates, $\mathbf{t}$, after transposing, and using the symmetry of the Slutsky matrix, $\mathbf{x}_{\mathbf{p}}^{H}$, is

$$v'_{\mathbf{p}} + \mu\left(\mathbf{x} - \overline{\mathbf{x}} + \mathbf{x}_{\mathbf{p}}\mathbf{t}\right) = \mathbf{0}.$$

With initial endowments, Roy's theorem and the Slutsky equation are $v_{\mathbf{p}} = -v_m(\mathbf{x}-\overline{\mathbf{x}})'$ and $\mathbf{x}_{\mathbf{p}} = \mathbf{x}_{\mathbf{p}}^{H} - \mathbf{x}_m(\mathbf{x}-\overline{\mathbf{x}})'$, where $\mathbf{x}^{H}$ is the compensated demand. Inserting from this into the first-order condition gives

$$-v_m(\mathbf{x}-\overline{\mathbf{x}}) + \mu\left(\mathbf{x} - \overline{\mathbf{x}} + \left(\mathbf{x}_{\mathbf{p}}^{H} - \mathbf{x}_m(\mathbf{x}-\overline{\mathbf{x}})'\right)\mathbf{t}\right) = \mathbf{0}.$$

Collecting the terms with $\mathbf{x}-\overline{\mathbf{x}}$, dividing by μ and inserting for $V_{\overline{m}}$ and θ, we get the *Ramsey rule*:

$$\theta(\mathbf{x}-\overline{\mathbf{x}}) + \mathbf{x}_{\mathbf{p}}^{H}\mathbf{t} = \mathbf{0}. \tag{2}$$

A direct interpretation of the Ramsey rule is that an equal relative increase in all tax rates (at optimum) should give an equal relative reduction in the compensated quantity of all non-endowed goods.[5] This interpretation, however, does not convey much information about the implied taxes rates as these are quite implicit in the characterisation. But some stronger assumptions give simpler interpretations.

The (compensated) inverse elasticity rule Assume that leisure (good 0) is the only endowment, and no compensated cross price elasticities between goods, except towards leisure. Then Ramsey rule, (2), gives the *inverse elasticity rule* (for $k \neq 0$):[6]

$$\frac{t_k}{p_k} = -\frac{\theta}{El_{p_k}x_k^H} = \frac{\theta}{El_{p_0}x_k^H}. \tag{3}$$

The first form of this rule is mostly used, but the second gives the best interpretation. It says that one should *tax relative complements to leisure harder*.

The Corlett-Hague rule Assume that leisure (good 0) is the only endowment and that there are only two other goods. In this case, the Ramsey rule, (2), gives the *Corlett-Hague rule*:[7]

$$\frac{\frac{t_1}{p_1}}{\frac{t_2}{p_2}} = \frac{El_{p_0}x_2^H + El_{p_2}x_1^H + El_{p_1}x_2^H}{El_{p_0}x_1^H + El_{p_1}x_2^H + El_{p_2}x_1^H}. \tag{4}$$

On the right hand side, only the first terms are different in the nominator and denominator. Thus the interpretation is again that we should tax relative complements to leisure harder.[8]

[5] The characterisation gets somewhat more complex for endowed goods.

[6] Here $El_b a = (b/a)\,(da/db)$ is the elasticity of a with respect to b.

[7] Dalton and Sadka (1979) gives a more direct generalisation of the Corlett-Hague rule than the one by Deaton which we advocate, but in their case there are still only two taxed goods.

[8] In contrast to the assumptions of the inverse elasticity rule, the assumptions of the Corlett-Hague rule allows complements to leisure.

The importance of the common interpretation of the second form of the inverse elasticity rule and the Corlett-Hague rule is argued by Sandmo (1987). The argument below, essentially due to Deaton (1979), gives a general characterisation of optimal commodity taxes with the same intuitive interpretation. Thus the argument outdates Sandmo's (1976, p. 46) remark: "Elasticity formulae become very complicated in the general case and provide little intuitive insight into the structure of taxation."

Auerbach (1985, p. 92) notes that if one interpret x_i^H as excess demand, the Corlett-Hague rule does not change even if leisure is not good 0 (the numeraire). Our point is that the *interpretation* of the Corlett-Hague rule only makes sense if sales of leisure is untaxed, and that this therefore is the canonical choice of normalisation. Taxing more heavily relative complements with leisure makes sense, as the untaxed own consumption of leisure is the reason for not obtaining the first-best in this model. To give an intuitive explanation of why one should tax relative complements with an arbitrary consumer good, taken as numeraire, on the other hand, seems close to impossible. As we will see, it is only with leisure untaxed, that we can generalize the Corlett-Hague rule, in the sense that we get a general characterisation with essentially the same interpretation.

3.1 Inverting the Ramsey rule

We use the Antonelli matrix, which is the derivative of the (compensated) marginal willingness to pay function, to invert the Ramsey rule. First, however, we recall these concepts.

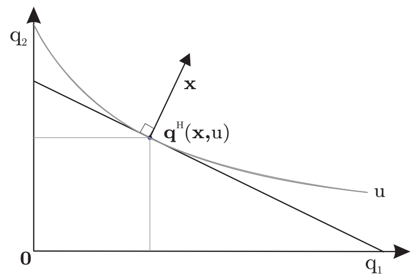
3.1.1 The (compensated) marginal willingness to pay

First introduce prices, $\mathbf{q} = \mathbf{p}/\mathbf{p}'\overline{\mathbf{x}}$, scaled so that income is one. With these prices, $\mathbf{q}$, the indirect utility function can be written $U^*(\mathbf{q}) = v(\mathbf{p}/\mathbf{p}'\overline{\mathbf{x}}, 1)$.

The (compensated) *willingness to pay function* (or *indirect expenditure function*), e^*, is the least one is willing to pay for quantities $\mathbf{x}$, to keep utility at or below u, i.e.:[9]

$$e^*(\mathbf{x}, u) = \min_{\mathbf{q}} \mathbf{q}'\mathbf{x} \quad \text{subject to} \quad U^*(\mathbf{q}) - u \leq 0.$$

The solution to this problem, $\mathbf{q}^H(\mathbf{x}, u)$, is the (compensated) *marginal willingness to pay* for $\mathbf{x}$ (given utility level u) or the (compensated) *inverse demand for* $\mathbf{x}$. By the envelope property, $e^*_{\mathbf{x}}(\mathbf{x}, u) = \mathbf{q}^H(\mathbf{x}, u)$. We draw a picture of the problem for a given level curve, $U^*(\mathbf{q}) = u$. With prices on the axes, utility is increasing towards origin in the figure:



[9] Deaton and Auerbach actually use the equivalent distance function instead, but the willingness to pay function is more intuitive.

The (compensated) willingness to pay function is concave, homogenous of degree 1, and nondecreasing in $\mathbf{x}$, and additionally continuous for $\mathbf{x} \gg \mathbf{0}$, just as the expenditure function (in $\mathbf{p}$).

Compensated demand, $\mathbf{x}^H(\mathbf{p}, u)$, and compensated marginal willingness to pay (inverse demand), $\mathbf{q}^H(\mathbf{x}, u)$, are generalised inverse functions of prices in the following sense:[10]

$$\mathbf{q}^H(\mathbf{x}^H(\mathbf{p}, u), u) = \frac{\mathbf{p}}{e(\mathbf{p}, u)}. \tag{5}$$

This relation says that given prices $\mathbf{p}$, with expenditure 1 to reach utility level u, we get back to $\mathbf{p}$ by first taking the compensated demand at these prices, and then the compensated marginal willingness to pay, given this demand. If expenditure to reach utility level u is different from 1, we only get back to original prices, scaled to expenditure 1.

Taking the derivative of the above expression wrt. prices $\mathbf{p}$, we get that the *Slutsky matrix*, $\mathbf{x}^H_{\mathbf{p}}$, and the *Antonelli matrix*, $\mathbf{q}^H_{\mathbf{x}}$, are *generalised inverses*, i.e.

$$(\mathbf{p}'\overline{\mathbf{x}})\, \mathbf{q}^H_{\mathbf{x}} \mathbf{x}^H_{\mathbf{p}} = \mathbf{I} - \mathbf{q}\mathbf{x}', \tag{6}$$

where $\mathbf{I}$ is the identity matrix.

3.1.2 The inverse Ramsey rule

Let $\overline{\theta} = \theta \mathbf{p}'\overline{\mathbf{x}}$.[11] Multiply the Ramsey rule, (2), with $\mathbf{p}'\overline{\mathbf{x}}$ and use the definition of $\overline{\theta}$ to get

$$\overline{\theta}(\mathbf{x} - \overline{\mathbf{x}}) = (\mathbf{p}'\overline{\mathbf{x}})\, \mathbf{x}^H_{\mathbf{p}} \mathbf{t}.$$

Left multiply by the Antonelli matrix, $\mathbf{q}^H_{\mathbf{x}}$, using the generalized inverse property, (6), to get

$$\overline{\theta}\mathbf{q}^H_{\mathbf{x}}(\mathbf{x} - \overline{\mathbf{x}}) = (\mathbf{p}'\overline{\mathbf{x}})\, \mathbf{q}^H_{\mathbf{x}} \mathbf{x}^H_{\mathbf{p}} \mathbf{t} = (\mathbf{I} - \mathbf{q}\mathbf{x}')\mathbf{t}.$$

As $\mathbf{q}^H(\mathbf{x}, u)$ is homogenous of degree 0 in quantities $\mathbf{x}$, $\mathbf{q}^H_{\mathbf{x}}\mathbf{x} = \mathbf{0}$. Inserting from the public budget constraint, $(\mathbf{x} - \overline{\mathbf{x}})'\mathbf{t} = \overline{T}$. This gives the (unnormalised) *inverse Ramsey rule*:[12]

$$\mathbf{t} = (\overline{T} + \overline{\mathbf{x}}'\mathbf{t})\mathbf{q} + \overline{\theta}\mathbf{q}^H_{\mathbf{x}}\overline{\mathbf{x}}. \tag{7}$$

Before proceeding, we introduce some definitions.

3.1.3 Endowment elasticities

The (compensated) *endowment elasticity of good* k,

$$\iota_k \overset{def}{=} -\frac{\tau}{q_k} \frac{\partial q^H_k(\tau\mathbf{x} + (1-\tau)\,\overline{\mathbf{x}}, u)}{\partial \tau} \Big|_{\tau=1} .$$

This is the elasticity of the (compensated) marginal willingness to pay for good k with respect to an adjustment of consumption (from the optimum $\mathbf{x}$) in the direction of the endowments, $\overline{\mathbf{x}}$.[13]

[10] A simple proof is given in an appendix. Another proof of (6) below is given in Salvas-Bronsard, Leblanc and Bronsard (1977).

[11] We get back to the interpretation of $\overline{\theta}$.

[12] This is essentially Auerbach's (1987) formula (6.6). Thus, so far there is nothing new.

[13] In the standard case with an endowment only of leisure, the endowment elasticity of a good is simply the elastisity of the (compensated) marginal willingness to pay for that good wrt. the quantity of leisure.

Expanding this definition, using homogeneity and symmetry, we get:

$$q_k \iota_k = \sum_j \frac{\partial q_k^H}{\partial x_j}(\overline{x}_j - x_j) = \sum_j \frac{\partial q_k^H}{\partial x_j}\overline{x}_j = \sum_j \overline{x}_j \frac{\partial q_j^H}{\partial x_k}. \tag{8}$$

Let $\beta_k \overset{def}{=} q_k \overline{x}_k$ be the *income share* of good k, evaluated at the marginal willingness to pay at optimum, and $\overline{\iota}^\beta = \sum_j \beta_j \iota_j$ the *average* (compensated) *endowment elasticity*, with the income share weights.

More on $\overline{\theta}$ Right multiplying (7) by $\overline{\mathbf{x}}$, using that $\mathbf{q}'\overline{\mathbf{x}} = 1$, we get an expression for the marginal value of transferring one pound (exogenously) from the individual to the public (in units of the public budget):[14]

$$\overline{\theta} = \frac{\overline{T}}{-\overline{\mathbf{x}}'\mathbf{q}_{\mathbf{x}}^H\overline{\mathbf{x}}} \geq 0. \tag{9}$$

The quadratic form in the denominator of (9), however, equals the average marginal willingness to pay elasticity, as by the definition of β_k, (8), and the definition of $\overline{\iota}^\beta$:

$$\begin{aligned}\overline{\mathbf{x}}'\mathbf{q}_{\mathbf{x}}^H\overline{\mathbf{x}} &= \sum_{k,j} \overline{x}_k \frac{\partial q_j^H}{\partial x_k}\overline{x}_j = \sum_{k,j} q_k \overline{x}_k \frac{\overline{x}_j}{q_k}\frac{\partial q_j^H}{\partial x_k} \\ &= \sum_k \beta_k \sum_j \frac{\overline{x}_j}{q_k}\frac{\partial q_j^H}{\partial x_k} = \sum_k \beta_k \iota_k = \overline{\iota}^\beta.\end{aligned}$$

Thus from (9),

$$\overline{\theta} = \frac{\overline{T}}{-\overline{\iota}^\beta}. \tag{10}$$

We saw above that θ is the value of transferring one pound exogenously, *at consumer prices*, from the individual to the government. Then $\overline{\theta} = \theta\mathbf{p}'\overline{\mathbf{x}}$ is the value of transferring one pound exogenously, *at firm prices*, from the individual to the government. This is so, since with our firm price normalisation, the income at firm prices, $\overline{m} = \overline{\mathbf{p}}'\overline{\mathbf{x}} = 1$, while income at prices to the individual, $m = \mathbf{p}'\overline{\mathbf{x}}$. Thus $m = \mathbf{p}'\overline{\mathbf{x}}\,\overline{m}$, so $dm/d\overline{m} = \mathbf{p}'\overline{\mathbf{x}}$.

The marginal efficiency loss of taxation By its interpretation, θ looks like a natural measure of the marginal efficiency loss of taxation. As pointed out by Håkonsen (1998), however, this is problematic, as one would like such a measure to be independent of the consumer price normalisation, and θ is not.[15] But from (9), $\overline{\theta}$ is independent of the consumer price normalisations and is therefore a natural measure of the marginal efficiency loss of taxation. Again, with our (canonical) normalisation below, $\theta = \overline{\theta}$, so with *this* normalisation, θ also is also a measure of the marginal efficiency loss of taxation.

[14] The inequality follows as $\mathbf{e}^*(\mathbf{x}, u)$ is concave i $\mathbf{x}$, thus $\mathbf{q}_{\mathbf{x}}^H$ is negative semidefinit.

[15] Håkonsen then proceeds to define the (total) efficiency loss by means of a pair of dual optimal value functions for the optimization problem.

3.2 The normalisation and inverse characterisation

We are now ready for our (canonical) normalisation, being a straightforward generalisation of the standard practice with leisure as the only endowment, and untaxed labour.

Normalise consumer prices by setting income at the *optimal* consumer prices equal to one, $\mathbf{p}'\overline{\mathbf{x}} = 1$. Then, at optimum, the initial endowments are a nontaxed (composite) good, i.e. $\mathbf{t}'\overline{\mathbf{x}} = 0$. Then also $\mathbf{p} = \mathbf{q}$, and (7) simplifies to the *inverse Ramsey rule*:

$$\mathbf{t}' = \overline{T}\mathbf{p}' + \overline{\theta}\overline{\mathbf{x}}'\mathbf{q}^H_{\mathbf{x}}. \tag{11}$$

On component form, (11) is:

$$t_k = \overline{T}p_k + \overline{\theta}\sum_j \overline{x}_j \frac{\partial q^H_j}{\partial x_k}.$$

Dividing by p_k, inserting for ι_k from (8) and $\overline{\theta}$ from (10), gives the *generalised inverse elasticity rule*:

$$\frac{t_k}{p_k} = \overline{T}\left(1 + \frac{\iota_k}{-\overline{\iota}^{\beta}}\right). \tag{12}$$

Thus the *public tax requirement* (as a share of income at firm prices) is a *basic tax rate.* Deviations from this base tax rate is for each good proportional to its endowment elasticity.

Call a good k an *endowment substitute* if $\iota_k < 0$, and an *endowment complement* if $\iota_k > 0$. The basis tax rate should then be raised for endowment complements and lowered for endowment substitutes.[16] The intuition is clear: A tax on own consumption of endowments removes the efficiency loss. Thus a tax on endowment complements is an indirect way of taxing own consumption of endowments, thereby reducing the efficiency loss. Taxing endowment substitutes, on the other hand, only increases the own use of untaxed endowments. This is of course the same intuition as one get both from the second form of the inverse elasticity rule, (3), and the Corlett and Hague rule, (4), above. It does, however, avoid the restrictive assumptions of both these characterisations.

3.2.1 Auerbachs alternative

Without the canonical normalisation, from (7), we get the characterisation:

$$\frac{t_k}{p_k} = \frac{1}{\mathbf{p}'\overline{\mathbf{x}}}\left(\mathbf{t}'\overline{\mathbf{x}} + \overline{T}\left(1 + \frac{\iota_k}{-\overline{\iota}^{\beta}}\right)\right). \tag{13}$$

In this characterisation especially the term for the tax value of endowments, $\mathbf{t}'\overline{\mathbf{x}}$, is hard to interpret, as the other new term, $(\mathbf{p}'\overline{\mathbf{x}})^{-1}$, is only a proportionality factor. In example 1, it is this term which explains why we get negative tax rate under the last normalisation, as it is negative.

Formula (13) is essentially formula (6.7) in Auerbach (1985) (except that we have introduced the endowment elasticities), obtained by setting an arbitrary tax rate equal to 0. Auerbach, however, gets around the problems caused by the tax value of endowments, by looking at the differences between the relative tax

[16] With only endowments of one good, this is essentially Hicks' (1956) notions of q-compements and q-substitutes with respect to this good.

rates in his formula (6.8), before interpreting the results. Thus he essentially considers

$$\frac{t_k}{p_k} - \frac{t_j}{p_j} = \frac{\overline{T}}{\mathbf{p}'\overline{\mathbf{x}}\left(-\overline{\iota}^{\beta}\right)}\left(\iota_k - \iota_j\right) \backsim \iota_k - \iota_j. \tag{14}$$

This, however, is a weaker characterisation, than the generalized inverse elastisity rule, (12). Auerbach has obviously seen the possibility of using our canonical characterisation. Thus is looks like the reason he use (14) instead is that he thinks that characterisations (and interpretations) should be independent of choice of normalisation, as (14) is, except for the proportionality factor.

3.2.2 Only one endowment

With endowments of only one good, 0 (say leisure), from (8) using symmetry, the (compensated) endowment elasticity of good is the elasticity of (compensated) marginal willingness to pay wrt. leisure times the endowment share of consumption:

$$\iota_k = \frac{\overline{x}_0}{q_k}\frac{\partial q_k^H}{\partial x_0} = \frac{\overline{x}_0}{x_0} El_{x_0} q_k^H.$$

In this case $\beta_0 = 1$. Thus from the definition:

$$\overline{\iota}^{\beta} = \iota_0 = \frac{\overline{x}_0}{x_0} El_{x_0} q_0^H.$$

Hence the generalized inverse elasticity rule (12) simplifies to:[17]

$$\frac{t_k}{p_k} = \overline{T}\left(1 + \frac{El_{x_0} q_k^H}{-El_{x_0} q_0^H}\right). \tag{15}$$

The departure of the tax rate on good k from the base tax rate is for each good proportional to the (compensated) elasticity of the marginal willingness to pay wrt. the price of leisure.

A comparison with the inverse elasticity rule The generalized inverse elasticity rule, (15), is similar to the second form of the inverse elasticity rule, (3), except that it involves the elasticity of the (compensated) marginal willingness to pay (i.e. inverse demand) wrt. leisure, whereas the standard inverse elasticity rule involves the inverse value of the elasticity of (compensated) demand with respect to the price of leisure.

Appendix: The generalized inverse property

Here, we verify the generalized inverse property used in the main text, using the duality between the compensated direct and inverse demand. We state the basic result for demand correspondences, with respect to income normalized prices and quantities.

Given a utility level u, define the *compensated demand correspondence*, c^u, by $\mathbf{x} \in c^u(\mathbf{q})$ if $\mathbf{qx} \leq 1$, $U(\mathbf{x}) \geq u$ and for all $\mathbf{x}'$ such that $U(\mathbf{x}') \geq u$, $\mathbf{qx}' \geq 1$, and the *compensated inverse demand correspondence*, c^{*u}, by $\mathbf{q} \in c^{*u}(\mathbf{x})$ if $\mathbf{qx} \leq 1$, $U^*(\mathbf{q}) \leq u$ and and for all $\mathbf{q}'$ such that $U^*(\mathbf{q}) \leq u$, $\mathbf{q}'\mathbf{x} \geq 1$. The

[17] This is Deaton's (1979) formula (51).

following straightforward proposition states that the two concepts are dual in a simple way.

Proposition 1 *Assume monotone preferences. Then* $\mathbf{x} \in c^u(\mathbf{q})$ *if and only if* $\mathbf{q} \in c^{*u}(\mathbf{x})$.

Proof. $\Rightarrow$: Assume $\mathbf{x} \in c^u(\mathbf{q})$. To show that $\mathbf{q} \in c^{*u}(\mathbf{x})$. Trivially $\mathbf{qx} \leq 1$, so we need to show first that $U^*(\mathbf{q}) \leq u$ and secondly that if $U(\mathbf{q}') \leq u$, then $\mathbf{q}'\mathbf{x} \leq 0$.

1. Assume $U^*(\mathbf{q}) > u$. Then since $U^*(\mathbf{q}) = \sup_{\mathbf{x}} \{U(\mathbf{x}) | \mathbf{qx} \leq 1\}$, there is $\mathbf{x}'$ such that $U(\mathbf{x}') > u$ and $\mathbf{qx}' \leq 1$. Hence by monotonicity there is $\mathbf{x}''$ such that $U(\mathbf{x}'') > u$ and $\mathbf{qx}'' < 1$, contradicting $\mathbf{x} \in c^u(\mathbf{q})$.

2. Let $U(\mathbf{q}') \leq u$, and assume that $\mathbf{q}'\mathbf{x} > 1$. Then by monotonicity, there is $\mathbf{q}''$ such that $U(\mathbf{q}'') < u$ and $\mathbf{q}''\mathbf{x} > 1$. Since $U(\mathbf{x}) = \inf_{\mathbf{q}} \{U^*(\mathbf{q}) | \mathbf{qx} \geq 1\}$, then $U(\mathbf{x}) \leq U(\mathbf{q}'') < U(\mathbf{q}) = u$, contradicting $\mathbf{x} \in c^u(\mathbf{q})$.

$\Leftarrow$: This is essentially the same argument. ■

If both the above correspondences are single-valued, we essentially have the compensated demand and inverse demand *functions*, $\mathbf{x}^H(\mathbf{q}, u)$ and $\mathbf{q}^H(\mathbf{x}, u)$, ie. $\mathbf{x} \in c^u(\mathbf{q})$ can be written $\mathbf{x} = \mathbf{x}^H(\mathbf{q}, u)$ and $\mathbf{q} \in c^{*u}(\mathbf{x})$ can be written $\mathbf{q} = \mathbf{q}^H(\mathbf{x}, u)$. In this case, a consequence of the proposition is the generalised inverse property (5), as

$$\mathbf{q}^H\left(\mathbf{x}^H(\mathbf{q}, u), u\right) = \mathbf{q} = \frac{\mathbf{p}}{\mathbf{p}'\mathbf{x}(\mathbf{p}, u)} = \frac{\mathbf{p}}{e(\mathbf{p}, u)}.$$

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