Stability and Equilibrium Selection in a Link Formation Game

by

Rodrigo Harrison and Roberto Muñoz *

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Abstract

In this paper we use a non cooperative equilibrium selection approach as a notion of stability in link formation games. Specifically, we follow the global games approach first introduced by Carlsson and van Damme (1993), to study the robustness of the set of Nash equilibria for a class of link formation games in strategic form with supermodular payoff functions. Interestingly, the equilibrium selected is in conflict with those predicted by the traditional cooperative refinements. Moreover, we get a conflict between stability and efficiency even when no such conflict exists with the cooperative refinements. We discuss some practical issues that these different theoretical approaches raise in reality. The paper also provides an extension of the global game theory that can be applied beyond network literature.

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1 Introduction

The way that different agents interact has an important role in the outcome of many problems in economics and other social sciences. Recently, these interactions have been modeled using *network structures* or *graphs*, where the agents are represented by nodes and the arcs between nodes represent some specific kind of relation between the corresponding agents. This approach has proved to be successful in the study of many specific problems,[1] however, we do not have a unique and accepted theory to explain how the networks form, which properties they have in terms of social welfare and how robust are some results in specific environments when some of the assumptions are slightly modified. It is well known in the literature that, in general, a link formation game in strategic form can lead to the formation of multiple networks supported by multiple Nash equilibria. Even more, under some particular circumstances, any network can be supported by a Nash equilibrium of the game.[2] The use of traditional refinements is limited and depends on the details of the game, consequently, some stability notions have been used in order to refine the set of equilibria.

The stability notions used so far to refine the set of Nash equilibria in a link formation game have been based on cooperative game theory. The most prominent of them, from the strongest to the weakest, have been Strong Nash Equilibrium (SNE), Coalition Proof Nash Equilibrium (CPNE) and Pairwise Stability (PS). However, the applicability of these refinements lies critically on the feasibility of cooperation among agents. This assumptions may be too strong for a link formation game when, by definition, the network has not been formed.[3]

[1] For an excelent review of the main issues in network theory see Dutta and Jackson (2001) and Jackson (2001).

[2] See, for example, Slikker and van den Nouweland (2000).

[3] The feasibility of cooperation seems more appealing once the network has been formed and the agents interact among them.

In this paper we use a non cooperative[4] equilibrium selection approach as a notion of stability in link formation games. Specifically, we follow the *Global Games* approach pioneered by Carlsson and van Damme (1993),[5] to study the robustness of the set of Nash equilibria for a class of link formation games with supermodular payoff functions. In order to ilustrate this approach, let us suppose that G_x is a standard game of complete information where the payoffs depend on a parameter $x \in \mathbb{R}$, and also suppose that for some subset of the parameter x, G_x has a strict Nash equilibrium. Rather than observing the parameter x, suppose instead that each player observes a private signal $x_i = x + \sigma\varepsilon_i$ where $\sigma > 0$ is an scale factor and ε_i is a random variable with density ϕ. Denote this "perturbed game" by $G_x(\sigma)$, and let $NE(G_x)$ and $BNE(G_x(\sigma))$ denote the sets of Nash and Bayesian Nash equilibria of the unperturbed and perturbed games, respectively. Equilibrium selection is obtained when $lim_{\sigma\to 0}BNE(G_x(\sigma))$ is smaller than $NE(G_x)$. Carlsson and van Damme (1993) show, in fact, that for two-player, two-action games, this limit comprises a single equilibrium profile, and is obtained through iterated deletion of strictly dominated strategies. Recently these results have been extended by Frankel, Morris and Pauzner (2002) for games with many players and many actions, but it is limited to the case of games with strategic complementarities.

This paper extends the global game results to games with vector valued space of actions, such that each component of a player´s vector strategy represents a binary decision. Even though the application to a link formation game is very natural, our extension can be applied to other problems beyond the network literature. Therefore, in particular, this binary decision can be seen as player's *intention* of establishing a

[4]The non cooperative formation of networks has been studied in the literature by Bala and Goyal (2000), however, their approach is very different from ours, because the non cooperative notion in that paper is related with the possibility to establish links unilaterally, whithout the agreement of the partner. On the contrary, our model is in the tradition that the existence of a link requires both parties to agree.

[5]For an excellent desciption and survey of the ensuing literature see Morris and Shin (2002)

link with other player. A link will be formed if and only if both players want to form the link. Any Nash equilibria of this game will support a different network.

We study a general class of games where the link formation process follows the strategic form of Dutta, van den Noweland, and Tijs (1998),[6] such that if the payoff is parametrized by x, our main assumptions are: 1. *Increasing Differences:* player i's incentive to choose a *higher* action is non decreasing in the others players' action profile. 2. *Link Symmetry:* player i's incentive to choose an action depends on the number of links requested by each player and on the structure of the resulting network, but not on the identities of the players. *3.* Existence of *upper* and *lower dominance regions*: for sufficiently low (high) values of the parameter, the action vector such that shows link intention with nobody (everybody) is strictly dominant.

Under these assumptions, and some technical requirements, we prove that there exists a unique equilibrium profile surviving iterated elimination of strictly dominated strategies. The profile selected is independent of the noise size. The equilibrium strategy defines a unique k^* such that $\forall x_i < k^*$ each player chooses the action vector showing link intention with nobody, and $\forall x_i > k^*$ each player chooses the action vector showing link intention with everybody.

The selected Bayesian Nash profile is in conflict with those arising from the application of traditional cooperative refinements of the network literature: SNE, CPNE and PS. This difference shows that the cooperative notions of stability are not robust to incomplete information. Moreover, we show that the stability notions based on cooperative refinements do not conflict with *efficiency* in our class of payoff functions, however, the equilibrium selected under the global game approach does conflict. These differences raise some practical questions about which criteria should be satisfied by

[6] The strategic form approach of the link formation game was first proposed by Myerson (1991). The idea is that each player select a list of the other players he wants to form a link with. Then the lists are put together and if the link ij is required by both parts then it is formed.

networks that form in reality.

From an applied point of view, the paper highlights the importance of two standard assumptions in the link formation literature. First, the assumption of complete information can be the origin of the multiplicity of networks supported by Nash Equilibria in link formation games. This multiplicity disappears when we perturb the game introducing incomplete information. Second, the cooperative refinements have been used symmetrically to refine the multiplicity of equilibria in a link formation game and to argue that an existing network is stable to some cooperative deviations. However, the possibility of cooperation among coalitions of agents seems to be a more demanding assumption when the network is in formation than when the agents are maintaining or modifying an existing network. These observations raise some doubts about which is the pertinent equilibrium selection criteria in reality for a link formation game.

The paper is organized as follows. In section 2 we provide a simple example where we can show intuitively the main findings of the paper. Section 3 presents some basic background and notation in network theory which will be useful throughout the paper. In section 4 we describe our link formation game and we introduce the general class of supermodular payoff functions under study. Sections 5 introduces the most commonly used cooperative refinements and their application to our game. In section 6 we develop the alternative approach to equilibrium selection using the global game theory. In section 7 we specialized our payoff function in order to get some intuition about the results. The main conclusions are contained in sections 8. Finally, proofs of propositions are relegated to the appendix.

2 An Illustrative Example

The idea of this section is to provide a simple example and an intuitive explanation of the main results developed in the paper. We are not going to be formal and technical details are postponed to next sections.

Consider a link formation game of complete information G with three players, where the set of strategies for each player i is given by $A_i = \{0,1\}^2$. A strategy for player i is a two component column vector of zeros and ones which identifies the set of players he wants to form links with. A link between two players will be formed if and only if both players want to form the link. For example, if the strategies of the players are $a_i = (a_{ij} = 1, a_{ik} = 1)^t$, $a_j = (a_{ji} = 0, a_{jk} = 1)^t$, $a_k = (a_{ki} = 0, a_{kj} = 1)^t$, then only the link jk is created.[7] The payoff function for player i is defined by:

$$\pi_i = a_{ij}a_{ji}\left(x + a_{jk}a_{kj}\beta x\right) + (\alpha x - c)a_{ij} + a_{ik}a_{ki}\left(x + a_{kj}a_{jk}\beta x\right) + (\alpha x - c)a_{ik} \quad (1)$$

The variable x defines a level of profits which is assumed to be non negative, and c is a fixed parameter that represents a level of investment incurred by agent i for each link he wants to form. This investment is quasi specific to the partners, in the sense that if agent i incurs an investment to agent j, then even if j does not perform the reciprocal investment, and consequently the link ij is not formed, agent i receives a return $(\alpha x - c)$. The source of benefits αx is *independent* of other players' actions, in the sense that it can be obtained no matter the strategies the other players are following. On the other hand, if j also performs the quasi specific investment to i then the return to agent i increases to $(x + \alpha x - c)$. In other words, there is an extra *direct* benefit x from connection with each potential partner. Finally, agent i profits

[7] Note that the superscript t stands for transpose.

from the relation between j and k when they are connected and provided that i is connected with at least one of them. Note that if i is connected with both of them, this *indirect* benefit is duplicated. For example, in the complete network the total payoff for player i is given by $\pi_i = 2[x(1+\alpha+\beta) - c]$. In this sense, βx represent an indirect benefit or spillover that agent i is able to extract from the connection between his partners and their partners. It seems natural to assume that $1 > \alpha > 0$, $1 > \beta > 0$, because we are scaling the benefits in relation to those obtained from reciprocity (x).

One case where this kind of payoff function can be justified is in investment in R&D to reduce variable costs. In such a case, it has been empirically documented (see Goyal and Moraga-Gonzalez (2001)) that the firms tend to form alliances in pairs, represented by the links, but any reduction in cost obtained by i's partners can be imitated by i, no matter if such reduction was obtained due to R&D of i's partner or by a partner of i's partners. We can assume that these firms are not competitors in any final market, so no negative externalities from R&D will arise.

2.1 The Nash Equilibria

Given the symmetry of the problem we are going to consider the best response correspondence for player 1. This correspondence, and the Nash equilibria arising, are different depending on the values of x. Figure 1 provides a summary of the different network structures supported by Nash equilibria, $NE(G)$, for different values of x. Consider the following cases:

Case (a): Suppose that:

$$\frac{c}{1+\alpha} < x < \frac{c}{\alpha}$$

then the best response correspondence is given by:

$$BR_1(a_{-1}) = \begin{cases} a_{12} = 1, a_{13} = 1 & \text{if} \quad a_{21} = a_{31} = 1 \\ a_{12} = 1, a_{13} = 0 & \text{if} \quad a_{21} = 1, \ a_{31} = 0 \\ a_{12} = 0, a_{13} = 1 & \text{if} \quad a_{21} = 0, \ a_{31} = 1 \\ a_{12} = 0, a_{13} = 0 & \text{if} \quad a_{21} = 0, \ a_{31} = 0 \end{cases}$$

Note that, in this region, the strategies of agents 2 and 3 in relation to their connection does not affect the best response correspondence of agent 1. The intuition is that direct connections are enough to guarantee profitability. This characteristic leads to a multiplicity of Nash equilibria and, even more, it is possible to prove that all the feasible networks among the three agents can be supported by a Nash equilibrium.

Case (b): Suppose that:

$$\frac{c}{1+\alpha+\beta} < x < \frac{c}{1+\alpha}$$

then the best response correspondence is given by:

$$BR_1(a_{-1}) = \begin{cases} a_{12} = 1, a_{13} = 1 & \text{if} \quad a_{21} = a_{31} = a_{23} = a_{32} = 1 \\ a_{12} = 1, a_{13} = 0 & \text{if} \quad a_{21} = a_{23} = a_{32} = 1, \ a_{31} = 0 \\ a_{12} = 0, a_{13} = 1 & \text{if} \quad a_{21} = 0, \ a_{31} = a_{23} = a_{32} = 1 \\ a_{12} = 0, a_{13} = 0 & \text{otherwise} \end{cases}$$

Note that, in this case, the best response correspondence of player 1 is affected by the existence of the link between players 2 and 3. It is possible to prove that in this case only the empty and the complete network can be supported as a Nash equilibrium of the game.

Case (c): Finally, when $x < \underline{x} = c/(1+\alpha+\beta)$ a dominant strategy for any player

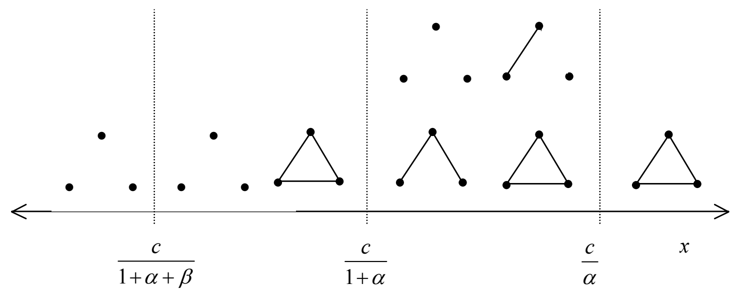

Figure 1: Network Structures supported by Nash Equilibria

i is to play $a_i = (0,0)^t \equiv \mathbf{0}$. Analogously, when $x > \overline{x} = c/\alpha$ a dominant strategy is to form links with all the other players $a_i = (1,1)^t \equiv \mathbf{1}$, leading to the complete network.

2.2 Equilibrium Selection using Cooperative Refinements

A strategy profile is called a Strong Nash Equilibrium (SNE) if it is a Nash equilibrium and there is no coalition of players that can strictly increase the payoffs of all its members using a joint deviation (Aumann (1959)). On the other hand, a strategy profile is called a Coalition Proof Nash Equilibrium (CPNE) if, as in an SNE, no coalition can deviate to a profile that strictly improves the payoffs of all the players in the coalition. However, in the CPNE the set of admissible deviations is smaller, because the deviation has to be stable with respect to further deviations by subcoalitions. Finally, a network is Pairwise Stable (PS) if no pair of agents has incentives to form or sever one link. A more formal treatment of this concepts is postponed to section 5.

The application of these cooperative refinements to our three players game G is very direct and a summary of results for the network supported by $SNE(G)$, $CPNE(G)$ and $PS(G)$ is given in figure 2.

First, it is possible to prove that, in this particular example, $SNE(G)$ coincides with $CPNE(G)$. Second, the analysis has to be performed in separated areas. It is easy to see that the strategy profile $a = (\mathbf{0}, \mathbf{0}, \mathbf{0}) \equiv [\mathbf{0}]$ is a $SNE(G)$ when $x < c/(1 + \alpha + \beta)$, because for this range of values each agent plays $\mathbf{0}$ as a dominant strategy and, consequently, no coalition of agents can improve upon.[8] On the other hand, $a = (\mathbf{1}, \mathbf{1}, \mathbf{1}) \equiv [\mathbf{1}]$ is a $SNE(G)$ when $c/(1 + \alpha + \beta) < x$. The intuition is that the grand coalition playing $a_i = a_j = a_k = \mathbf{1}$ (which is a Nash equilibrium) can improve upon any other strategy profile (Nash equilibrium or not) given the complementarities involved in the payoff functions and the fact that a positive payoff is guaranteed.

We have to be more careful in the analysis of pairwise stability. If $c/(1+\alpha+\beta) < x < c/(1 + \alpha)$ then the empty and the complete networks are pairwise stable and consequently, pairwise stability does not refine the set of Nash equilibria. This result is a consequence that for these low values of x the indirect connections are needed to make any connection profitable so, if nobody is making links, an agreement of two players to form a link is not enough to obtain a profitable relationship. On the other hand, if everybody is making links then no pair of agents benefits from severing a link. When $c/(1+\alpha) < x < c/\alpha$ any pair of agents which are not connected can profitably make a link and, consequently, only the complete network is pairwise stable. Finally, if $x < c/(1 + \alpha + \beta)$ then action $a_i = \mathbf{0}$ is a dominant strategy for player i and the unique pairwise stable network is the empty one. Analogously, if $x > c/\alpha$ then action $a_i = \mathbf{1}$ is a dominant strategy and the unique pairwise stable network is the complete one.

[8] In what follows $[\mathbf{0}]$ and $[\mathbf{1}]$ represents a matrix full of zeros or ones respectively. The dimensionality is given by the profile they are representing. For example, in this three players case, $a = [\mathbf{0}]$ is a 2x3 matrix of zeros representing a complete strategy profile and $a_{-i} = [\mathbf{0}]$ is a 2x2 matrix of zeros representing a strategy profile, which excludes player i's strategy.

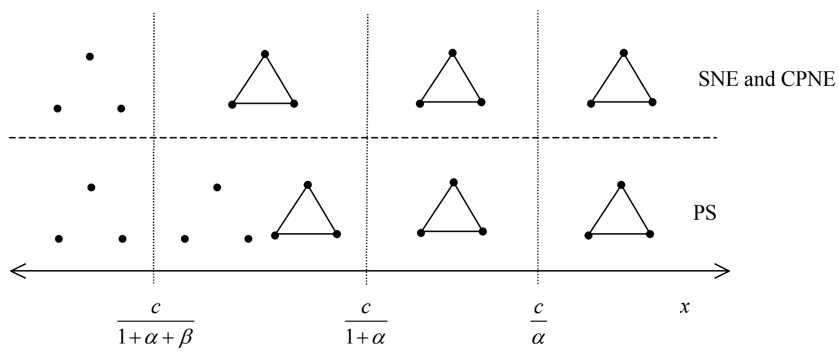

Figure 2: Network structures supported by Strong Nash Equilibrium (SNE), Coalition Proof Nash Equilibrium (CPNE) and Pairwise Stability (PS)

2.3 Equilibrium Selection using the Global Game Approach

Suppose now we allow some arbitrary amount of incomplete information in the payoff structure such that instead of observing the actual value of the level of profits x, each player just observes a private signal x_i, which contains diffuse information about x. The signal has the following structure: $x_i = x + \sigma\varepsilon_i$, where $\sigma > 0$ is a scale factor, x is drawn from $\left[\underline{X}, \overline{X}\right]$ with uniform density and ε_i is an independent realization of the density ϕ with support in $[-\frac{1}{2}, \frac{1}{2}]$. We assume ε_i is *i.i.d.* across the individuals.[9]

In this context of incomplete information, a Bayesian pure strategy for player i is a function $s_i : [\underline{X} - \frac{\sigma}{2}, \overline{X} + \frac{\sigma}{2}] \to A_i$, and $s = (s_1, s_2, s_3)$ is a pure strategy profile, where $s_i \in S_i$. Calling this game of incomplete information $G(\sigma)$, let us define as $BNE(G(\sigma))$ the set of Bayesian Nash equilibria of $G(\sigma)$.

Proposition 1: $\forall\ \sigma > 0$ *there exists a unique strategy profile* s^*, *that survives iterated elimination of strictly dominated strategies, where:*

$$s_i^*(x_i) = \begin{cases} \mathbf{1} & if \quad x_i > k^* \\ \mathbf{0} & if \quad x_i < k^* \end{cases} \qquad \forall i \ \ and \ k^* = \frac{4c}{2 + 4\alpha + \beta}$$

[9]Note that ϕ need not be symmetric around the mean nor even have zero mean.

Since the noise structure is $x_i = x + \sigma\varepsilon_i$, as $\sigma \to 0 \quad x_i \to x$, thus the unique equilibrium selected implies that $\forall\ x < k^*$ all agents play the action $\mathbf{0}$, so the empty network is formed, and $\forall\ x > k^*$ all the agents playing action $\mathbf{1}$ and, consequently, the complete network is formed. Conditional on the signal, figure 3 shows the networks supported by this equilibrium as the noise goes to zero.

This proposition is a particular case of proposition 4 (many players case), so we are not going to give a formal proof here. Instead, we are going to discuss the intuition behind the proposition. Consider players 2 and 3 using any strategy. It is common knowledge of the game that these strategies must consider playing the actions $\mathbf{0}$ and $\mathbf{1}$ in the previously identified dominance regions. It is possible to prove that agent 1's best response to such strategies is a strategy that considers playing $\mathbf{0}$ when $x_1 < \underline{x}^1$ and playing $\mathbf{1}$ when $x_1 > \overline{x}^1$ where $\underline{x} < \underline{x}^1$ and $\overline{x} > \overline{x}^1$. In other words, in equilibrium, the regions where $\mathbf{0}$ and $\mathbf{1}$ are played has been extended. Given the symmetry of the problem, all the agents perform the same analysis and consequently the regions where $\mathbf{0}$ and $\mathbf{1}$ are played are extended symmetrically for all the players. Iterating with this argument, it is possible to generate increasing and decreasing sequences $\{\underline{x}^n\}_{n=1}^{\infty}$ and $\{\overline{x}^n\}_{n=1}^{\infty}$, respectively, such that they have the same limit value, i.e., $\underline{x}^{\infty} = \overline{x}^{\infty} \equiv k^*$.

Finally, it is important to notice that the equilibrium profile selected in $G(\sigma)$ does not depend on the size of the noise. In this sense, we say that s^* is the unique equilibrium of the link formation game G, which is stable to incomplete information in the parameter x.

2.4 Efficient Allocation of the Game

The efficient allocation of the game $E(G)$, is defined for each x as the strategy profile that maximizes the sum of the payoffs for the players. It is easy to check that:

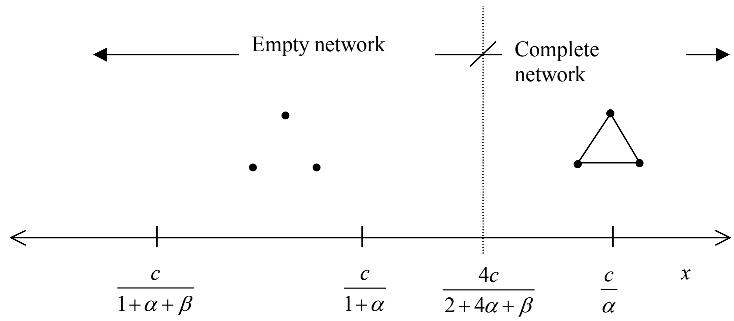

Figure 3: Equilibrium Selected using the Global Game Approach

$$E(G) = \begin{cases} \{[\mathbf{0}]\} & \text{if } x < c/(1+\alpha+\beta) \\ \{[\mathbf{1}]\} & \text{if } x > c/(1+\alpha+\beta) \end{cases}$$

and as a result, the networks supported by efficient allocation $E(G)$ coincide with those supported by the sets $SNE(G)$ and $CPNE(G)$ described in figure 2.

2.5 Discussion

The example developed illustrate the main results of the paper. First, the traditional cooperative refinements do not conflict with the efficient allocation for each level of x. However, the equilibrium selected using the global game approach clearly conflicts with efficiency when $\frac{c}{1+\alpha+\beta} < x < \frac{4c}{2+4\alpha+\beta}$. Second, in the interval $\frac{c}{1+\alpha} < x < \frac{4c}{2+4\alpha+\beta}$ all the cooperative refinements predict the formation of the complete network, however, our selected equilibrium predicts the empty network. This means that, for these values of x, it is impossible to satisfy the two stability conditions simultaneously. Giving that each stability notion leads to the selection of a different equilibrium, we have two implications. First, the cooperative refinements are not robust to incomplete information. Second, the feasibility of the equilibrium selected by each approach depends critically on the feasibility of the deviations considered in each stability condition. For example, if the agents are not able to cooperate in a link formation game, then it does not seem reasonable to select the equilibrium using a cooperative refinement and the global game approach could be more adequate.

In what follows we are going to show that these findings hold in a much more general setting defined by a family of payoff functions that satisfies a set of assumptions.

3 Some Preliminaries in Network Theory

The goal of this section is to provide some basic concepts and notation in network theory so that we will be able to discuss the insights of the paper in relation to the existing literature and using the standard language in the field.

3.1 Basic Background in Network Theory

Several authors have studied the theoretical foundations of network formation and its properties. Particular emphasis has been given to study the link formation process and the conflict between stability and efficiency in networks. The link formation literature precedes the stability/efficiency literature, however, the insights from the latter area have interacted and motivated more research in the former.

In the link formation literature, an important starting point is Myerson (1977) who departs from the traditional cooperative game theory imposing networks constraints to the role of the different agents in a coalition. Aumann and Myerson (1988) studied a particular link formation game in extensive form where the agents iteratively decide to offer or sever links to the others. This sequence of decisions proceeds in different rounds until no one wants to modify the selection of the previous round. They characterize the resulting networks as the subgame perfect Nash equilibria of the game. Most recently, Dutta, van den Noweland and Tijs (1998) studied a link formation game in strategic form and showed that the resulting networks differ from those of Aumann and Myerson (1988). The strategic form approach of the link formation game was first proposed by Myerson (1991). The idea is that each player select a list of the other players he wants to form a link with. Then the lists are put together and if the link ij is required by both parts then it is formed. Given the nature of the game, subgame perfection does not apply. Dutta, van den Noweland and Tijs (1998) used cooperative refinements to select among the multiplicity of Nash Equilibria of the game. Slikker and van den Noweland (2000) introduced a cost to establish links and showed that, in the game in extensive form of Aumann and Myerson, a decrease in this cost does not necessarily increase the number of links in equilibrium.

The study of the conflict between stability and efficiency of networks began with the paper of Jackson and Wolinsky (1996). This is a very important paper because they are the first to assign value directly to the network rather than to the coalition. This fact allows us to have different values for the same coalition depending on how the agents are connected.[10] In particular, this approach encompasses the case studied by Slikker and van den Noweland (2000). The focus of their paper is not on link formation but the conflict between stability and efficiency, where the stability notion they introduce is *pairwise stability (PS)*[11] and the efficiency notion is *strong efficiency.* Under PS they consider only the incentives of each pair of agents to form or sever one link and under strong efficiency the value of the network is maximized.[12] Their main result is that an anonymous and component balanced allocation rule does not exist[13] such that at least one strongly efficient graph is pairwise stable for every value function over the network. Dutta and Mutuswami (1997) adopt a mechanism design

[10] Technically, if the network is denoted by g, then the value function is denoted by $v(g)$.

[11] As Jackson and Wolinsky (1996) say, this concept is a weak notion of stability and it has been considered as a necessary condition.

[12] Jackson (2001) shows that the concept of efficiency could vary across problems depending on the degree of transferability of the value generated by the network.

[13] The allocation rule describes how the value of each network is distributed to the players. For a formal definition of the anonymity and component balance properties see Jackson and Wolinsky (1996).

approach to deal with the conflict between efficiency and stability. They show that, in some particular environments, it is possible to reconcile stability and efficiency with an adequate design of the allocation rule.

It is important to note how the discussion about the conflict between efficiency and stability has affected the modeling of the link formation game, which is the focus of this paper. The main relation is through the concept of stability. When we specify a game that model how the network forms, we can say that the Nash equilibria of the game satisfy, by definition, a stability condition; no one wants to unilaterally deviate from the equilibrium. However, this notion usually generates a multiplicity of equilibria supporting a multiplicity of networks structures, which lead us to consider some refinements. The pertinent kind of refinement depends on how we specify the game. For example, Aumann and Myerson (1988) used the subgame perfect equilibrium concept for their link formation game in extensive form. On the other hand, Dutta, van den Noweland and Tijs (1998) consider two kind of refinements for their strategic form game, the Strong Nash Equilibrium (SNE) and the Coalition Proof Nash Equilibrium (CPNE). Both of them are indeed stability concepts where the idea is to select the equilibria that survives against the possibility of deviations by coalitions. The first concept, however, is too demanding and it could be the case that the set of SNE is empty. Consequently, they use the CPNE as the concept of stability in networks and, therefore, as the relevant refinement in the link formation game. The pairwise stability notion of Jackson and Wolinsky (1996) has been argued to have an advantage because it is independent of the way that the link formation game is modeled. Although this is a weak notion of stability, it is enough to generate conflict between efficiency and stability and even more, there are situations where no pairwise stable network exists.[14]

[14] However, if the allocation rule is given by the Myerson value, there always exist a pairwise stable network.

In this paper we focus on the link formation game in strategic form of Dutta, van den Noweland and Tijs (1998) but we introduce a different, non cooperative, equilibrium selection approach. Our approach is in fact based on a different stability notion and consequently, we can analyze the properties of the selected equilibria and compare them with those from the cooperative results. We also discuss the traditional stability/efficiency conflict when our stability notion is being used.

3.2 Basic Notation in Network Theory

Following Jackson and Wolinsky (1996), we establish some basic notation concerning *graphs*.

Let $\mathcal{N} = \{1, ..., N+1\}$ be a finite set of players. The complete graph, denoted by g^{N+1}, is the set of all subsets of $\mathcal{N}$ of size 2. The set of all possible graphs on $\mathcal{N}$ is $\{g/g \subseteq g^{N+1}\}$. Let the *link* ij denotes the subset of $\mathcal{N}$ containing only i and j. We understand that $ij \in g$ if and only if the nodes i and j are directly connected. Moreover, we denote $g + ij = g \cup \{ij\}$ and $g - ij = g \backslash \{ij\}$.

Let $N(g) = \{i/\exists j$ s.t $ij \in g\}$ be the set of non isolated nodes and $n(g)$ be the cardinality of $N(g)$. A *path in* g *connecting* i_1 and i_n is a set of distinct nodes $\{i_1, ..., i_n\} \subseteq N(g)$ such that $\{i_1 i_2, i_2 i_3 ..., i_{n-1} i_n\} \subseteq g$.

The graph $g' \subset g$ is a *component* of g, if for all $i \in N(g')$ and $j \in N(g')$, with $i \neq j$, there exist a path in g' connecting i and j, and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies $ij \in g'$.

The *value of a graph* is represented by the function $v : \{g/g \subseteq g^{N+1}\} \rightarrow \mathbb{R}$. The set of all such functions is denoted by V. In some applications the value function is naturally defined as an aggregation of individual payoffs functions, $v(g) = \sum_i \pi_i(g)$, where $\pi_i : \{g/g \subseteq g^{N+1}\} \rightarrow \mathbb{R}$ is player i's payoff.

A graph $g \in \{g/g \subseteq g^{N+1}\}$ is *strongly efficient* if $v(g) \geq v(g')$ $\forall g' \in \{g/g \subseteq g^{N+1}\}$

An *allocation rule* $Y : \{g/g \subseteq g^{N+1}\} \times V \to \mathbb{R}^{N+1}$ is a rule that describes how the value of a graph is distributed to the individual players. $Y_i(g, v)$ is the payoff to player i from graph g under the value function v. In some specific contexts, as the one studied in this paper, the allocation rule fails to redistribute wealth, so $Y_i(g, v) = \pi_i(g)$.

With this concepts in mind we will be able to interpret our assumptions and results using the standard language of network theory.

4 The Link Formation Game

Consider the following general setup for an $N + 1$ person game G. There exists $N + 1$ players indexed by i, each player has a set of strategies $A_i = \{0, 1\}^N$. A strategy for player i is a column vector of zeros and ones which identify the set of players he wants to form links with. A link between two players will be formed if and only if both players want to form the link. For example, if players' strategies are such that $a_i = (..., a_{ij} = 1, a_{ik} = 1, ...)^t$, $a_j = (..., a_{ji} = 0, a_{jk} = 1, ...)^t$, $a_k = (..., a_{ki} = 0, a_{kj} = 1, ...)^t$, then the link jk is created. More generally, a strategy profile $a \in A = \bigtimes_{i=1}^{N+1} A_i$ defines the graph g formed according to:

$$g(a) = \{ij \subseteq g^{N+1} \;\; / \;\; a_{ij} = 1, \, a_{ji} = 1\} \tag{2}$$

With this notation, we are allowing the case that two or more different strategy profiles can define the same network. For example, it could be the case that $g(a) = g(a')$ with $a \neq a'$, and these profiles generate different values for the individual's payoff functions. For this reason we consider the payoff function $\pi_i : A \to \mathbb{R}$ and we will refer to the total value generated by a network g supported by a strategy profile a as: $v(a) = \sum_i \pi_i(a)$.

For simplicity we will assume symmetric players with a payoff function given by

$\pi(a_i, a_{-i}, x)$ where $a_i \in A_i$, $a_{-i} \in A_{-i} = \times_{j \neq i} A_j$,[15] and $x \in [\underline{X}, \overline{X}] \subset \mathbb{R}$ is an exogenous variable. We define $\Delta\pi(a_i, a_i'; a_{-i}, x) = \pi(a_i, a_{-i}, x) - \pi(a_i', a_{-i}, x)$ as agent i's payoff difference when he changes from action a_i' to action a_i.

Define $A_i^n \subset A_i$ such that, if $a_i \in A_i^n$ then a_i is N-dimensional vector which contain n components 1 and $N-n$ components 0. In fact, the family of sets $\{A_i^n\}_{n=0}^N$ defines a partition of A_i because $A_i = \bigcup_{n=0}^{N} A_i^n$ and $A_i^n \cap A_i^{n'} = \emptyset$ for all n and $n' \in \{0...N\}$ with $n \neq n'$. Additionally, it is easy to see that A_i^N and A_i^0 are singleton, therefore if $a_i \in A_i^N$ then $a_i = (1,1,1,...,1)^t \equiv \mathbf{1}$, and it is defined as the *highest* action vector. If $a_i \in A_i^0$ then $a_i = (0,0,0,...,0)^t \equiv \mathbf{0}$ is the *lowest* action vector. If $a_i \in \{\mathbf{1}, \mathbf{0}\} \subset A_i$, then a_i is an *homogenous* action vector, and $A_i^h \equiv A_i^0 \cup A_i^N = \{\mathbf{1}, \mathbf{0}\}$ is defined as players i' set of homogenous actions.

Similarly consider $A_{-i}^h = \times_{j \neq i} A_j^h$ and define $A_{-i}^{h,n}$ such that if M_n is an element of $A_{-i}^{h,n}$, then M_n is $N \times N$ matrix containing n columns $\mathbf{1}$ and $N-n$ columns $\mathbf{0}$. As we consider above, the family of sets $\left\{A_{-i}^{h,n}\right\}_{n=0}^{N}$ is a partition of A_{-i}^h because $A_{-i}^h = \bigcup_{n=0}^{N} A_{-i}^{h,n}$ and $A_{-i}^{h,n} \cap A_{-i}^{h,n'} = \emptyset$ for all n and $n' \in \{0...N\}$ with $n \neq n'$. In particular $M_N = [\mathbf{1}]$ is a $N \times N$ matrix of ones, and $M_0 = [\mathbf{0}]$ is a $N \times N$ matrix of zeros.

Let us consider the following assumptions in the payoff structure:

(A1). Increasing Differences (ID). Conditional on the value of the exogenous parameter x, the greater the other players' action profile the greater is player i's incentive to choose a higher action:

$\forall a_i \in A_i$ and $\forall a_{-i} \in A_{-i}$

[15] A_i is a partially ordered set:
$a_i \geq a_i'$ if $\forall j \neq i$ $a_{ij} \geq a_{ij}'$
$a_i > a_i'$ if $\forall j \neq i$ $a_{ij} \geq a_{ij}'$ and $a_{ij} > a_{ij}'$ for some j
In the same way A_{-i} is a partially ordered set:
$a_{-i} \geq a_{-i}'$ if $\forall j \neq i$ $a_j \geq a_j'$
$a_{-i} > a_{-i}'$ if $\forall j \neq i$ $a_j \geq a_j'$ and $a_j > a_j'$ for some j

If $a_i \geq a_i'$ and $a_{-i} \geq a_{-i}'$, $\Delta\pi(a_i, a_i'; a_{-i}, x) \geq \Delta\pi(a_i, a_i'; a_{-i}', x)$ $\forall x$

and in particular:

a. $\forall a_i \neq \mathbf{0}$, $\exists\, k \neq i$ with $a_{ik} = 1$. So if $a_{ki} = 0$ then:

$\Delta\pi(a_i, \mathbf{0}; a_{-i}, x) < \Delta\pi(a_i, \mathbf{0}; a_{-i}', x)$ $\forall x$

where $a_{-i}' \geq a_{-i} + e_{ki}$, and e_{ki} is a $N \times N$ matrix of 0, except the ki element which is 1.

b. $\forall a_i \neq \mathbf{1}$, $\exists\, k \neq i$ with $a_{ik} = 0$. So if $a_{ki} = 1$ then:

$\Delta\pi(\mathbf{1}, a_i; a_{-i}, x) > \Delta\pi(\mathbf{1}, a_i; a_{-i}', x)$ $\forall x$

where $a_{-i}' \leq a_{-i} - e_{ki}$

(A2). Continuity (C).

$\pi(a_i, a_{-i}, x)$ is a continuous function in x

(A3). Monotonicity (M). The greater the value of the exogenous parameter x, the greater is player i's incentive to choose a higher action:

$\exists\, c > 0$ $s.t.$ $\forall\, a_i > a_i' \forall\, a_{-i}$ and x, $x' \in [\underline{X}, \overline{X}]$ $x > x'$

$\Delta\pi(a_i, a_i'; a_{-i}, x) - \Delta\pi(a_i, a_i'; a_{-i}, x') > c\, \|a_i - a_i'\|\, (x - x')$

(A4). Links Symmetry (LS). Player i's incentive to deviate from an homogenous action depends on the number of links requested by each player and on the structure of the resulting network, rather than on the identities of the players:

a. $\forall\, a_i \in A_i^{n'}$ and $\forall a_i' \in A_i^{n'}$

$\sum_{a_{-i} \in A_{-i}^{h,n}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) = \sum_{a_{-i} \in A_{-i}^{h,n}} \Delta\pi(a_i', \mathbf{0}; a_{-i}, x)$

$\sum_{a_{-i} \in A_{-i}^{h,n}} \Delta\pi(\mathbf{1}, a_i; a_{-i}, x) = \sum_{a_{-i} \in A_{-i}^{h,n}} \Delta\pi(\mathbf{1}, a_i'; a_{-i}, x)$ $\forall n, n' = 0, ..., N$

It is important to notice that the role of network structure is incorporated through the summation over $a_{-i} \in A_{-i}^{h,n}$.

b. Moreover, the value of the incentive to deviate from the homogenous action $\mathbf{0}$ to any other action a_i varies proportionally with the elemental deviation that establishes just one link intention:

$\exists\ \lambda : A_i \to [0,\infty)$ satisfying $\lambda(\mathbf{0}) = 0$ and $\lambda(a_i) = 1\ \ \forall a_i \in A_i^1$, s.t. if $a_i \in A_i^n$ and $a_i' \in A_i^{n'}$, then:

$\lambda(a_i) > \lambda(a_i') \Leftrightarrow n > n'$,

$\lambda(a_i) = \lambda(a_i') \Leftrightarrow n = n'$

And $\forall a_i \in A_i$

$\sum_{a_{-i}\in\ A_{-i}^{h,n}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) = \lambda(a_i) \sum_{a_{-i}\in\ A_{-i}^{h,n}} \Delta\pi(a_i', \mathbf{0}; a_{-i}, x)\ \forall n = 0, ..., N\ \ \forall a_i' \in A_i^1$

(A5). Upper and Lower Indifference Signals (IS). If all other players are choosing the highest (lowest) action, there exists a unique value of x such that player i is indifferent between the lowest (highest) action and any other action.

$\forall a_i \in A_i$

$\exists!\ \underline{x} > \underline{X}\ s.t.\ \ \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$, and

$\exists!\ \overline{x}$ s.t. $\overline{X} > \overline{x} > \underline{x}\ s.t.\ \Delta\pi(\mathbf{1}, a_i; a_{-i} = [\mathbf{0}], \overline{x}) = 0$

Assumptions A1 (ID) parts a) and b) are required because we will need increasing differences being satisfied strictly under some circumstances. In A1 (ID) a, player i has link intention with player k but it is not reciprocal ($a_{ik} = 1$ but $a_{ki} = 0$). Then if k changes his strategy to request now a link with i and all the agents others than i are playing higher strategies ($a'_{-i} \geq a_{-i} + e_{ki}$), then player i's incentive to choose the original a_i is strictly higher. Intuitively,[16] in terms of our network notation, we are roughly saying that $\Delta\pi(g(a)) < \Delta\pi(g(a) + ik) \leq \Delta\pi(g')$, where $g(a) + ik \subseteq g'$. The intuition for A1 (ID) b is analogous.

Another important remark is that assumptions A1 (ID), A3 (M) and A5 (IS)

[16] This interpretation is not precise, because the same network can be supported by different strategy profiles generating different payoffs. However, the intuition is the same.

provide sufficient conditions for the existence of dominance regions, along which each action is strictly dominant, providing this setup with the necessary global game structure. i.e.

$\forall\, x < \underline{x}$, $\Delta\pi(a_i, \mathbf{0}; a_{-i}, x) < 0 \;\; \forall a \in A$, and

$\forall\, x > \overline{x}$, $\Delta\pi(\mathbf{1}, a_i; a_{-i}, x) > 0 \;\; \forall a \in A$

Finally, the following lemmas will be useful in the characterization of the equilibrium.

Lemma 1: For all $a_i \in A_i$ and for all $x \in [\underline{X}, \overline{X}]$ we have

$$\sum_{a_{-i} \in A^h_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) = \lambda(a_i) \sum_{a_{-i} \in A^h_{-i}} \Delta\pi(a'_i, \mathbf{0}; a_{-i}, x) \;\forall a'_i \in A^1_i$$

proof: It follows directly from A4 (LS) b. and because $\sum_{a_{-i} \in A^h_{-i}} = \sum_{n=0}^{N} \sum_{a_{-i} \in A^{h,n}_{-i}}$.

Lemma 2: *The values of* $\overline{x}$ *and* $\underline{x}$ *are independent of* a_i.

proof: It is a direct application of assumption A4 (LS) b.

5 Equilibrium Selection using Cooperative Refinements

In this section we are interested in applying some of the most commonly used stability concepts to our problem in order to refine the multiplicity of Nash equilibria that can arise in our game. Three concepts have been proposed: *Pairwise Stability (PS)*, *Coalition Proof Nash Equilibrium* (CPNE) and *Strong Nash Equilibrium* (SNE). These refinements are based in cooperative game theory, mainly because in the application to networks when we consider deviations from the Nash equilibrium, we must include the possibility of adding links, which requires the agreement of both parties. In what follows we provide a formal definition of each concept and we apply them to our problem.

5.1 Definitions

The graph g is *pairwise stable* with respect to the value function v and the allocation rule Y if:

(i) For all $ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ and $Y_j(g, v) \geq Y_j(g - ij, v)$

and

(ii) For all $ij \notin g$, if $Y_i(g, v) < Y_i(g + ij, v)$ then $Y_j(g, v) > Y_j(g + ij, v)$

The concept of pairwise stability is due to Jackson and Wolinsky (1996) and it is directly defined over the networks, independently of the link formation process. It says that the network will be pairwise stable when each pair of agents do not have incentives to add or sever a link. It is clear from the definition that adding a link requires both parties to agree, but any agent can sever a link unilaterally.

Jackson and Wolinsky (1996) claimed that pairwise stability has the advantage that it is independent of the formation process of the network, so no matter how the network is formed, pairwise stability will be meaningful. On the other hand, it has the disadvantage that it can be understood as a necessary condition for stability, but it is not sufficient because the concept does not consider either deviations of a bigger coalition of players or deviations where one player would want to add or sever more than one link. This inconvenience has motivated the introduction of stronger notions.

A strategy profile is called a Strong Nash Equilibrium (SNE) if it is a Nash equilibrium and there is no coalition of players that can strictly increase the payoffs of all its members using a joint deviation (Aumann (1959)). We are going to talk about *Strong Stability* to refer the case where the network g is formed by a SNE of the game.

Formally, let $\Gamma = (\mathcal{N}, \{A_i\}_{i \in \mathcal{N}}, \{u_i\}_{i \in \mathcal{N}})$ be a game in strategic form. A strategy profile $a^* \in A = \mathop{\times}\limits_{i=1}^{N+1} A_i$ is a *Strong Nash Equilibrium* (SNE) of Γ if there is no $T \subseteq \mathcal{N}$ and $a \in A$ such that:

(i) $a_i = a_i^* \quad \forall i \notin T$

(ii) $u_i(a) > u_i(a^*) \quad \forall i \in T$

The claimed advantage of this concept is that it can be understood as the strongest stability notion. Consequently when a network g is strongly stable it is virtually impossible to destabilize. The disadvantage, unfortunately is that an SNE does not always exist (see Slikker and van den Noweland (2000)). As a result, a weaker notion of stability is required.

In order to define the *coalition proof Nash equilibrium* (CPNE) we need some extra notation. Consider the game in strategic form $\Gamma = (\mathcal{N}, \{A_i\}_{i\in\mathcal{N}}, \{u_i\}_{i\in\mathcal{N}})$ as above. For every subset of players $T \subset \mathcal{N}$ and a fixed strategy profile $\hat{a}^{\mathcal{N}\setminus T} \in A^{\mathcal{N}\setminus T}$ for the players who do not belong to T, let $\Gamma(\hat{a}^{\mathcal{N}\setminus T})$ be the game induced on the players of T by the strategies $\hat{a}^{\mathcal{N}\setminus T}$, that is:

$$\Gamma(\hat{a}^{\mathcal{N}\setminus T}) = (T, \{A_i\}_{i\in T}, \{u_i^*\}_{i\in T})$$

where for all $i \in T$, $u_i^* : A^T \to \mathrm{I\!R}$ is given by $u_i^*(a^T) \equiv u_i(a^T, \hat{a}^{\mathcal{N}\setminus T})$ for all $a^T \in A^T$.

Now coalition proof Nash equilibria is defined inductively. In a one player game with player set $\mathcal{N} = \{i\}$, $\hat{a}^i \in A_i = A$ is a CPNE of the game $\Gamma = (\{i\}, A_i, u_i)$ if $\hat{a}^i$ maximizes u_i over A_i. Consider now a game Γ with $n > 1$ players. By induction, the CPNE has been defined for games with less than n players. Using this induction hypothesis, we say that a strategy profile $\hat{a} \in A^{\mathcal{N}}$ is *self enforcing* if for all $T \subset \mathcal{N}$, $\hat{a}^T$ is a CPNE of $\Gamma(\hat{a}^{\mathcal{N}\setminus T})$. Then, the strategy vector $\hat{a}$ is a CPNE of Γ if $\hat{a}$ is self enforcing and there is no other self enforcing strategy profile $a \in A^{\mathcal{N}}$ with $u_i(a) > u_i(\hat{a})$ for all $i \in \mathcal{N}$.

As in SNE, the CPNE demands that no coalition can deviate to a profile that strictly improves the payoffs of all the players in the coalition. However, in the CPNE the set of admissible deviations is smaller, because the deviation has to be stable with respect to further deviations by subcoalitions.

The advantage of this notion is that it is easier to satisfy than SNE. Even more, Slikker and van den Noweland (2000) have proved that in a three players game a CPNE always exists. The disadvantage is that a CPNE could be very difficult to find depending on the particular game.

It has been proved that,[17] for a general link formation game Γ under complete information, [18]:

$$\begin{aligned} SNE(\Gamma) &\subseteq PS(\Gamma) \subseteq NE(\Gamma) \\ SNE(\Gamma) &\subseteq CPNE(\Gamma) \subseteq NE(\Gamma) \end{aligned} \tag{3}$$

Finally, the set of *Efficient Allocations* of a game Γ, $E(\Gamma)$, is defined as:

$$E(\Gamma) = \{a^* \in A, \textit{ such that } a^* \in \arg\max_a \sum_{i \in \mathcal{N}} u_i(a)\} \tag{4}$$

It is clear that any $a^* \in E(\Gamma)$ defines a *strongly efficient* graph $g(a^*) \in G$ throughout (2). An important implication of the theoretical conflict between efficiency and stability is that, in general, $E(\Gamma) \nsubseteq PS(\Gamma)$ and $E(\Gamma) \nsubseteq CPNE(\Gamma)$.[19] We are going to show that this is not the case in our game.

[17] See Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997)

[18] Even when Pairwise Stability has been defined over graphs, we can talk about the set $PS(\Gamma)$ as the subset of Nash Equilibria leading to pairwise stable graphs through (2).

[19] See Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997), respectively.

5.2 The Cooperative Refinements

Consider the link formation game in strategic form defined in section 4 and satisfying the assumptions A1 to A5. In addition, we introduce the following assumption:

(A6). Status Quo Payoff (SQP)

$\pi(\mathbf{0}, a_{-i}, x) = 0 \quad \forall a_{-i} \in A_{-i}\ ,\ \forall x \in [\underline{X}, \overline{X}]$

This assumption is very natural in the sense that if agent i does not require any link, then he has neither benefits nor costs. The role of assumption A6 (SQP) is to permit us to write $\Delta\pi(a_i, \mathbf{0}; a_{-i}, x) = \pi(a_i, a_{-i}, x)$ so that all the assumptions over $\Delta\pi$ can be directly interpreted in terms of π.

Under this general set of assumptions it is not easy to give a detailed description of the set of Nash equilibria of the game. However, it is possible to show that some particular profiles are indeed Nash Equilibria, and even more, we can show that these equilibria are stable under the traditional cooperative refinements. In addition, we will show that the set $E(G)$ is always stable under the different cooperative notions.

Proposition 2: *Consider the link formation game G. Under assumptions A1 to A6 we have:*

a. *The set $E(G)$, satisfies:*

$$E(G) = \begin{cases} \{[\mathbf{0}]\} & \text{if } x < \underline{x} \\ \{[\mathbf{1}]\} & \text{if } x > \underline{x} \end{cases}$$

b. *If $a \in E(G)$ then a is stable under all the cooperative refinements.*

Proposition 2 shows that in the class of supermodular games defined in section 4 under assumptions A1 to A6 there is no conflict between efficiency and the cooperative notions of stability.

6 Equilibrium Selection using the Global Game Approach

Suppose now that the game is one of incomplete information in the payoff structure. Instead of observing the actual value of x, each player just observes a private signal x_i, which contains diffuse information about x. We assume that this is a game of private values, where each player gets utility directly from the signal rather than the actual value of the variable.[20]

The signal has the following structure: $x_i = x + \sigma\varepsilon_i$,where $\sigma > 0$ is a scale factor, x is drawn from the interval $[\underline{X}, \overline{X}]$ with uniform density, and ε_i is a random variable distributed according to a continuous density ϕ with support in the interval $[-\frac{1}{2}, \frac{1}{2}]$. We assume ε_i is *i.i.d.* across the individuals.

This general noise structure has been used in the global game literature, allowing us to model in a simple way the conditional distribution of the opponents signal, i.e. given a player's own signal, the conditional distribution of an opponent's signal x_j admits a continuous density f_σ and a cdf F_σ with support in the interval $[x_i - \sigma, x_i + \sigma]$. Moreover this literature establishes a significant result: when the prior is uniform, players' posterior beliefs about the difference between their own observation and other players' observations are the same, i.e. $F_\sigma(x_i \mid x_j) = 1 - F_\sigma(x_j \mid x_i)$.[21]

In this context of incomplete information, a Bayesian pure strategy for a player i is a function $s_i : [\underline{X} - \frac{\sigma}{2}, \overline{X} + \frac{\sigma}{2}] \rightarrow A_i$, and $s = (s_1, s_2, ..., s_N)$ is a pure strategy profile, where $s_i \in S_i$. Equivalently we define $s_{-i} = (s_1, s_2, ..s_{i-1}, s_{i+1}, ...s_N) \in S_{-i}$.

In particular, a switching strategy between the lowest and the highest action is a Bayesian pure strategy satisfying : $\exists\ k_i\ s.t.$

[20] Even though we have not proven that our main result is robust to this assumption, it is simple to model the private value case as a limit of the common values case (when players derive utilitity from the actual value of the variable) as the noise goes to zero ($\sigma \rightarrow 0$). This approach has been used in the global game literature. (Carlsson and van Damme (1993), Morris and Shin (2000) and Frankel, Morris and Pauzner (2002).)

[21] This property holds approximately when x is not distributed with uniform density but σ is small, i.e. $F(x_i \mid x_j) \approx 1 - F(x_j \mid x_i)$. See details in Lemma 4.1 Carlsson and van Damme (1993)

$$s_i(x_i) = \begin{cases} \mathbf{1} & if \quad x_i > k_i \\ \mathbf{0} & if \quad x_i < k_i \end{cases}$$

Abusing notation, we write $s_i(\cdot; k_i)$ to denote the switching strategy with threshold k_i.

Finally, if player i is observing a signal x_i and facing a strategy s_{-i} his expected payoff can be written as

$$\Pi_i(a_i, s_{-i}, x_i \mid x_i) = \int_{x_{-i}} \pi(a_i, s_{-i}(x_{-i}), x_i) dF_{\sigma(-i)}(x_{-i} \mid x_i)$$

Calling this game of incomplete information $G(\sigma)$, let us define $BNE(G(\sigma))$ as the set of Bayesian Nash equilibria of $G(\sigma)$. In addition, we assume:

(A7). Single Crossing (SC). There exists a unique value k^*, of the exogenous variable such that if player i receive a signal $x_i = k^*$ and he believes that all other players are using a switching strategy between **0** and **1** with threshold k^*, the expected value of his payoffs when he chooses **0** or **1** are the same:

$$\text{There exists a unique } k^* \text{ solving } \sum_{n=0}^{N} \sum_{a_{-i} \in A_{-i}^{h,n}} \{\Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, k^*)\} = 0$$

One of the main results of the paper proves that $G(\sigma)$ has a unique profile s^*, played in equilibrium $\forall \sigma > 0$, and in this profile every player will play a switching strategy $s_i(\cdot; k^*)$ with k^* according A7 (SC).

Proposition 3: *Consider the link formation game $G(\sigma)$. Under assumptions A1 to A5 and A7:*

$\forall\ \sigma > 0$ *there exists a unique strategy profile s^* surviving iterated elimination of strictly dominated strategies, where:*

$$s_i^*(x_i;k^*) = \begin{cases} \mathbf{1} & if \quad x_i > k^* \\ \mathbf{0} & if \quad x_i < k^* \end{cases} \qquad \forall i \;\; and \; \underline{x} < \;\; k^* < \overline{x}$$

The equilibrium strategy defines a unique k^* satisfying that $\forall x_i < k^*$ each player chooses the action vector that shows link intention with nobody, and $\forall x_i > k^*$ each player choose the action vector that shows link intention with everybody. It is important to notice that the equilibrium profile selected does not depend on the size of the noise σ, and it does not depend on the noise structure ϕ either. We have assumed that the parameter x is distributed according to a flat prior, but it is possible to prove that any prior can be treated as a flat prior when σ goes to zero. In this sense, we say that s^* is the unique equilibrium of the link formation game G, which is robust to incomplete information in the parameter x.

Even though the proposition proves that when $\sigma > 0$ each player is using a switching strategy s_i^*, the network formed depends on the size of the noise. In general, if some $x_i > \; k^* + \sigma$ then every player receives a signal greater than k^* and therefore the complete network is formed. Equivalently if some $x_i < \; k^* - \sigma$ the empty network is formed, but if some $x_i \in [k^* - \sigma, k^* + \sigma]$ any network can be formed depending on the realization of every player's signal. Following this analysis is easy to see that as σ goes to zero just two possibilities remain, the complete and the empty networks.

7 Application

As an example of the previous result, in this section we develop an application using a particular payoff structure. We consider a $N + 1$ player link formation game, such that each player has the same following payoff function:

$$\pi_i(a_i, a_{-i}, x) = \sum_{j \neq i} \left\{ a_{ij} a_{ji} \left(x + \sum_{k \neq i \neq j} a_{jk} a_{kj} \beta x \right) + (\alpha x - c) a_{ij} \right\} \tag{5}$$

which is a generalization of the payoff function described in equation (1). The interpretation of the different components of this function (independent, direct and indirect benefits) is the same as in section 2, where we interpreted it as the investment in R&D to reduce variable costs.

It is also clear that the game played is different depending on the values of x. In particular, when $x(1 + \alpha + \beta) < c$ a dominant strategy for any agent i is to play $a_{ij} = 0 \quad \forall i, j \in \{1, ..., N + 1\}$, $i \neq j$, forming the empty network. On the other hand, when $\alpha x > c$, then a dominant strategy is to play $a_{ij} = 1 \quad \forall i, j \in \{1, ..., N + 1\}$, $i \neq j$, forming the complete network.

Assumption A6 (SQP) holds trivially, while assumptions A1 to A5 can be directly checked as follows: the general statement for assumption A1 (ID) holds because the payoff function is supermodular in a and then, in particular, increasing differences is satisfied. (A1a) and (A1b) are satisfied due to the presence of a direct benefit x when the link exists. Assumptions A2 (C) and A3 (M) follow because the payoff function is "well behaved".Assumption (A4a) follows directly by the symmetry of the problem, while assumption (A4b) holds with $\lambda(a_i) = \sum_{j \neq i} a_{ij}$. Finally, assumption A5 (IS) holds with $\underline{x} = \frac{c}{(1+\alpha+\beta)}$ and $\overline{x} = \frac{c}{\alpha}$.

The problem of applying the different equilibrium selection approaches is reduced to verify assumption A7 (SC), which is done in the following proposition.

Proposition 4: *Consider the link formation game G when the payoff function has been specialized according to (5). The Single Crossing assumption A7 is satisfied with*:

$$k^* = \frac{Nc\sum_{n=0}^{N}\begin{pmatrix} N \\ n \end{pmatrix}}{\sum_{n=0}^{N}\begin{pmatrix} N \\ n \end{pmatrix}N\alpha + \left[(1+\beta)\sum_{n=0}^{N}\left\{\begin{pmatrix} N \\ n \end{pmatrix}n\right\} - N\beta\right]} \tag{6}$$

Proposition 4 permits us to apply the equilibrium selection by the global game approach to the payoff function defined by (5). The equilibrium selected generalizes the result discussed in section 2.

The identity $\sum_{n=0}^{N}\begin{pmatrix} N \\ n \end{pmatrix} = 2^N$ would permit us to simplify equation (6), however, the original formulation is more convenient to verify that $\frac{c}{1+\alpha} < k^* < \frac{c}{\alpha}$. Noting that $\underline{x} = \frac{c}{(1+\alpha+\beta)}$ then it is easy to see that the conflict between the equilibrium selection by cooperative concepts and the global game approach applies to this particular payoff function.

8 Conclusion

The goal of this paper is to use a non cooperative equilibrium selection approach as a notion of stability in link formation games. Specifically, we study the link formation game in strategic form of Dutta, Van den Noweland and Tijs (1998) where we constrain the payoffs to a class of supermodular functions defined by assumptions A1 to A5. Assumption A6 (SQP) is introduced to apply the traditional cooperative refinements and assumption A7 (SC) is introduced to apply the global game approach.

Our methodology is based on the global game theory, where the equilibrium selection is obtained through perturbations by allowing some arbitrarily small uncertainty in the payoff structure. Interestingly, the equilibrium selected with our stability concept is not only different, but is also in conflict with those predicted by the traditional cooperative refinements. As a consequence, a first insight of this paper is to show that the equilibria selected under the cooperative notions of stability are not robust to incomplete information.

In Proposition 2 we show that the set of Efficient Allocations is contained in the set of stable equilibria when the stability notions are cooperative. In other words, in our link formation game when the payoff functions belong to our class of supermodular functions, we do not have a conflict between stability and efficiency when cooperative refinements are used. On the contrary, from Proposition 3, we have that the conflict appears when our equilibrium selection technique is used.

From an applied point of view, the paper highlights the importance of two standard assumptions in the link formation literature. First, the assumption of complete information can be the origin of the multiplicity of networks supported by Nash Equilibria in link formation games. This multiplicity disappears in our environment under incomplete information because, from Proposition 3, there is a unique strategy profile that survives the iterative elimination of strictly dominated strategies and then any additional refinement is meaningless. Second, the possibility of cooperation among coalitions of agents seems to be a strong assumption in a link formation game. This observation, and the conflict between the equilibria selected under a cooperative and a global game approach, raise some doubts about which criteria is satisfied by the forming networks in reality.

In the three player example discussed in section 2, in the interval $\frac{c}{1+\alpha} < x < \frac{4c}{2+4\alpha+\beta}$, all the cooperative refinements predict the formation of the complete network, however, our approach predicts the formation of the empty network. In particular, pairwise stability implies that a couple of agents can be strictly better off if they cooperate, however the strategies required to support this behavior do not survive the iterated elimination of strictly dominated strategies under any level of incomplete information in the parameter x.

Finally, the conflict between efficiency and stability in networks under our global game approach could have some practical implications in the dynamic formation and destruction of markets. For example, if x is a variable affecting the benefits of the firms in a market under formation, then the firms would enter the market (or would form the network) at inefficiently high value of x. In a related paper we are studying the entry-exit decisions when the payoff function of the firm belongs to our supermodular class and the possibilities of cooperation are constrained to the "insiders" of the market. In such an environment we expect to have different trigger values of x affecting entry and exit decisions in equilibrium.

9 Appendix

Proof of Proposition 2

(b) First we have to prove that $a = [\mathbf{0}]$ and $a = [\mathbf{1}]$ are indeed Nash Equilibria of the game when $x < \underline{x}$ and $x > \underline{x}$ respectively.

Consider first $x > \underline{x}$ and $a_i \neq \mathbf{1}$. By A3 (M) we have:

$\Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) - \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) > c \left\| a_i - \mathbf{0} \right\| (x - \underline{x}) > 0$

and by A5 (IS) $\Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$, so $\Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) > 0$.

On the other hand, by A4 (LS) we have:

$\Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i} = [\mathbf{1}], x) = \frac{\lambda(\mathbf{1})}{\lambda(a_i)} \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) > \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x)$

Finally, using A6 (SQP) this means:

$$\pi(\mathbf{1}; a_{-i} = [\mathbf{1}], x) > \pi(a_i; a_{-i} = [\mathbf{1}], x) \quad \forall a_i \neq \mathbf{1}, x > \underline{x}. \tag{7}$$

In other words, when the others are playing $a_{-i} = [\mathbf{1}]$, then play $a_i = \mathbf{1}$ is a strict best response. As a result, $a = [\mathbf{1}]$ is a strict NE of the game when $x > \underline{x}$.

Now we are going to prove that $a = [\mathbf{1}]$ is a SNE of the game when $x > \underline{x}$ and then, by relations in (3), it is an stable equilibria under all the cooperative refinements.

We are going to prove that:

$\pi(\mathbf{1}; [\mathbf{1}], x) \geq \pi(a_i; a_{-i}, x) \quad \forall\ x > \underline{x}, \forall a \in A,\ a \neq [\mathbf{1}]$

which is a condition that implies that the strategy profile $a = [\mathbf{1}]$ is a Strong Nash Equilibrium (SNE).

Consider any $a \in A,\ a \neq [\mathbf{1}]$ and any $x > \underline{x}$. By A1 (ID) and A6 (SQP) we have:

$\pi(a_i; a_{-i}, x) = \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) \leq \Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], x) = \pi(a_i; [\mathbf{1}], x)$

and using equation (7) we have:

$\pi(a_i; [\mathbf{1}], x) \leq \pi(\mathbf{1}; [\mathbf{1}], x)$

which completes the proof that $a = [\mathbf{1}]$ is a SNE of the game when $x > \underline{x}$.

Now we analyze the case $x < \underline{x}$. In this case, the strategy $a_i = \mathbf{0}$ is a strictly dominant strategy for player i because $\forall a_i \neq \mathbf{0}$ and $\forall a_{-i} \in A_{-i}$, by A1 (ID), A3 (M) and A5 (IS) we have:

$\Delta\pi(a_i, \mathbf{0}; a_{-i}, x) \leq \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], x) < \Delta\pi(a_i, \mathbf{0}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$

and using A6 (SQP):

$\pi(a_i; a_{-i}, x) < 0 = \pi(\mathbf{0}; a_{-i}, x)$

In particular, considering $a_{-i} = [\mathbf{0}]$, we obtain that $a = [\mathbf{0}]$ is a strict NE of the game when $x < \underline{x}$.

Moreover, given any strategy profile $a \neq [\mathbf{0}]$ (not necessarily a Nash equilibrium) and any $x < \underline{x}$ we have:

$\pi(\mathbf{0}; a_{-i} = [\mathbf{0}], x) \geq \pi(a_i; a_{-i}, x)$ and then $a = [\mathbf{0}]$ is a SNE of the game.

Finally, when $x = \underline{x}$, the strategy profiles $a = [\mathbf{0}]$ and $a = [\mathbf{1}]$ lead to a payoff zero (by assumptions A6 (SQP) and A5 (IS) respectively), and using A1 (ID) and A5 (IS), for any strategy profile $a \in A$:

$\pi(a_i; a_{-i}, \underline{x}) \leq \pi(\mathbf{1}; a_{-i} = [\mathbf{1}], \underline{x}) = 0$

As a consequence there is no profitable deviation for player i from $a = [\mathbf{0}]$ or $a = [\mathbf{1}]$, so these profiles are Nash Equilibria. Using the same assumptions, there is no other profile where all the players in a coalition can obtain a positive payoff and, consequently, these profiles are also Strong Nash Equilibria. Moreover, if there exists any other efficient strategy profile under $x = \underline{x}$, the payoff for any player i would be zero and then, it would also be a SNE of the game.

(a) By definition, the set of Efficient Allocations of the game G is given by the strategy profiles that solves:

$$\max_{a \in A} \sum_{i=1}^{N+1} \pi_i(a_i; a_{-i}, x)$$

From the proof of part (b), we know that, when $x > \underline{x}$,.the strategy profile $a = [\mathbf{1}]$

is a Nash equilibrium satisfying:

$\pi_i(\mathbf{1};[\mathbf{1}],x) \geq \pi_i(a_i;a_{-i},x) \quad \forall\ x > \underline{x}\ ,\ \ a \neq [\mathbf{1}],\ \ i = 1...N+1$

But if the strategy profile $a \neq [\mathbf{1}]$ then there exists j so that $a_j \neq \mathbf{1}$ and for this agent, using A6 (SQP), A1 (ID) and A4 (LS) we have:

$\pi_j(a_j;a_{-j},x) \leq \pi_j(a_j;[\mathbf{1}],x) < \pi_j(\mathbf{1};[\mathbf{1}],x) \quad \forall\ x > \underline{x}$

and then the unique Efficient Allocation when $x > \underline{x}$ is given by the strategy profile $a = [\mathbf{1}]$.

An analogous argument leads us to prove that the unique Efficient Allocation when $x < \underline{x}$ is given by the strategy profile $a = [\mathbf{0}]$.■

Proof of Proposition 3

Denoting S_i^n the player i's set of strategies that survives n rounds of deletion of interim strictly dominated strategies, the process of iterated elimination is defined recursively as follows: set $S_i^0 \equiv S_i$ and for all $n > 0$

$$S_i^n \equiv \left\{ \begin{array}{c} s_i \in S_i^{n-1} :\ \nexists s_i^{'} \in S_i^{n-1}\ s.t.\ \Pi(s_i^{'}(x_i), s_{-i}, x_i \mid x_i) \geq \Pi(s_i(x_i), s_{-i}, x_i \mid x_i)\ \forall x_i \\ and\ with\ strict\ inequality\ for\ some\, x_i,\ \forall s_{-i} \in S_{-i}^{n-1} \end{array} \right\}$$

Consider a link formation game $G(\sigma)$. Under assumptions A1 to A5 and A7, we will argue by induction that set S_i^n satisfies:

$S_i^n = \{s_i : s_i(x_i) = \mathbf{0}\ if\ x_i < \underline{x}^n\ and\ s_i(x_i) = \mathbf{1}\ if\ x_i > \overline{x}^n\}\ ,$

where $\underline{x}_i$ and $\overline{x}_i$ are defined recursively as

$\overline{x}^n = \max\{x : \Delta\Pi(\mathbf{1},\mathbf{0};(s_j(x_j;\overline{x}^{n-1}))_{j\neq i}, x) = 0\}$

$\underline{x}^n = \min\{x : \Delta\Pi(\mathbf{1},\mathbf{0};(s_j(x_j;\underline{x}^{n-1}))_{j\neq i}, x) = 0\}$

The first round of elimination is described in the following lemma.

Lemma 3: For all $i\ \exists\ \underline{x}^1 > \underline{x}$ *and* $\overline{x}^1 < \overline{x}$ *s.t.*

$s_i \in S_i^1\ iff\ s_i(x_i) = \{\mathbf{0}\ if\ x_i < \underline{x}^1\ and\ \mathbf{1}\ if\ x_i > \overline{x}^1\}$

where

$$\overline{x}^1 = \max\{x : \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \overline{x}))_{j\neq i}, x) = 0\}$$

$$\underline{x}^1 = \min\{x : \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \underline{x}))_{j\neq i}, x) = 0\}$$

proof. Starting from the left: Player i (henceforth Pi) receive a signal $x_i = \underline{x}$, from A1 (ID), if s_i is a best response to a profile where every player is choosing a switching strategy $s_j(\cdot; \underline{x})$ $\forall j \neq i$, it will be a best response to any $s_{-i} \in S^0_{-i}$. Then player i' expected payoff difference between choosing action a_i rather than action $\mathbf{0}$ can be written as

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j; \underline{x}))_{j\neq i}, \underline{x}) = \int_{x_{-i}} \Delta\pi(a_i, \mathbf{0}; s_j(x_j; \underline{x}))_{j\neq i}, \underline{x}) dF_{\sigma(-i)}(x_{-i} \mid x_i)$$

or equivalently by

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j; \underline{x}))_{j\neq i}, \underline{x}) = \sum_{a_{-i}\in A^h_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, \underline{x}) \Pr(a_{-i} \mid (s_j(x_j; \underline{x}))_{j\neq i}, \underline{x})$$

where in general $\Pr(a_{-i} \mid (s_{-i}, x)$ represent player i' beliefs about the action profile a_{-i} conditional on other players' strategy s_{-i}.

Now, since, $\forall \sigma > 0$, $\forall a_{-i} \in A^h_{-i}$, $\Pr(a_{-i} \mid (s_j(x_j; \underline{x}))_{j\neq i}, \underline{x}) = \frac{1}{2^N} > 0$, then

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j; \underline{x}))_{j\neq i}, \underline{x}) = \frac{1}{2^N} \sum_{a_{-i}\in A^h_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, \underline{x})$$

By assuimptions A1 (ID) and A5 (IS) $\forall a_i \in A_i$, $\forall a_{-i} \in A^h_{-i}$ $\Delta\pi(a_i, \mathbf{0}; a_{-i}, \underline{x}) \leq 0$. By assumption A1 (ID) part a, at least one element is strictly negative, then

$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j; \underline{x}))_{j\neq i}, \underline{x}) < 0$. Therefore P$i$, upon receiving signal $x_i = \underline{x}$, will play action $a_i = \mathbf{0}$.

Now, if Pi receive a signal $x_i = \underline{x} + \sigma$

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j; \underline{x}))_{j\neq i}, \underline{x} + \sigma) = \Delta\pi(a_i, \mathbf{0}; s_{-i} = [\mathbf{1}], x_i = \underline{x} + \sigma)$$

By assumption A3 (M) $\Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], \underline{x} + \sigma) > \Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], x_i = \underline{x})$, and by assumptions A5 (IS) $\Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], x_i = \underline{x}) = 0$. Then $\Delta\pi(a_i, \mathbf{0}; [\mathbf{1}], x_i = \underline{x} + \sigma) > 0$.

Given continuity of the expected utility function and using the *intermediate value theorem:*

$\forall a_i$ and $\forall \sigma > 0$, $\exists\, \underline{x}^1$ s.t $\underline{x} < \underline{x}^1 < \underline{x} + \sigma$, where $\underline{x}^1 = \min\{x \mid \text{equation (8) holds}\}$

$$\Delta\Pi(a_i, \mathbf{0}; (s_j(x_j; \underline{x}))_{j\neq i}, x) = \sum_{a_{-i} \in A^h_{-i}} \Delta\pi(a_i, \mathbf{0}; a_{-i}, x) \Pr(a_{-i} \mid (s_j(x_j; \underline{x}))_{j\neq i}, x) = 0 \tag{8}$$

and from Lemma 2, we know that $\underline{x}^1$ is independent of a_i. Then in particular if $a_i = \mathbf{1}$

$$\underline{x}^1 = \min\{x \mid \Delta\Pi(\mathbf{1}, \mathbf{0}; (s_j(x_j; \underline{x}))_{j\neq i}, x) = 0\}$$

Starting from the right and using an equivalent argument we conclude that:

$\forall a_i$ and $\forall \sigma > 0$, $\exists\, \overline{x}^1$ s.t $\overline{x} > \overline{x}^1 > \overline{x} - \sigma$, where $\overline{x}^1 = \max\{x \mid \text{equation (9) holds}\}$

$$\Delta\Pi(\mathbf{1}, a_i; (s_j(x_j; \overline{x}))_{j\neq i}, x) = \sum_{a_{-i} \in A^h_{-i}} \Delta\pi(\mathbf{1}, a_i; a_{-i}, x) \Pr(a_{-i} \mid (s_j(x_j; \overline{x}))_{j\neq i}, x) = 0 \tag{9}$$

From Lemma 2, we know that $\overline{x}^1$ is independent of a_i. Then in particular if

$a_i = \mathbf{0}$

$$\overline{x}^1 = \max\{x \mid \Delta\Pi(\mathbf{1},\mathbf{0};(s_j(x_j;\overline{x}))_{j\neq i},x) = 0\} \blacksquare_{Lemma\ 3}$$

Repeating the process described in lemma 3, it is easy to prove by induction that $\exists\ \underline{x}^n > \underline{x}^{n-1}$ *and* $\overline{x}^n < \overline{x}^{n-1}$ *s.t.*

$S_i^n = \{s_i: \ s_i(x_i) = \mathbf{0}\ if\ x_i < \underline{x}^n\ and\ s_i(x_i) = \mathbf{1}\ if\ x_i > \overline{x}^n$

where

$$\overline{x}^n = \max\left\{x : \Delta\Pi(\mathbf{1},\mathbf{0};(s_j(x_j;\overline{x}^{n-1}))_{j\neq i},x) = 0\right\}$$

$$\underline{x}^n = \min\left\{x : \Delta\Pi(\mathbf{1},\mathbf{0};(s_j(x_j;\underline{x}^{n-1}))_{j\neq i},x) = 0\right\}$$

This process generates an increasing sequence $\{\underline{x}^n\}$ and a decreasing sequence $\{\overline{x}^n\}$. Let us suppose there exists limit points $\underline{x}^\infty$ and $\overline{x}^\infty$, then from equation (8) $\forall a_i$

$$\sum_{a_{-i}\in A_{-i}^h} \Delta\pi(a_i,\mathbf{0};a_{-i},\underline{x}^\infty)\Pr(a_{-i} \mid (s_j(x_j;\underline{x}^\infty))_{j\neq i},\underline{x}^\infty) = 0$$

Since $\Pr(a_{-i} \mid (s_j(x_j;\underline{x}^\infty))_{j\neq i},\underline{x}^\infty) = \frac{1}{2^N}$, then

$$\Delta\Pi(a_i,\mathbf{0};(s_j(x_j;\underline{x}^\infty))_{j\neq i},\underline{x}^\infty) = \frac{1}{2^N}\sum_{a_{-i}\in A_{-i}^h} \Delta\pi(a_i,\mathbf{0};a_{-i},\underline{x}^\infty) = 0$$

By assumption A5 (IS), in particular, this is true for $a_i = \mathbf{1}$, then

$$\sum_{a_{-i}\in A_{-i}^h} \Delta\pi(\mathbf{1},\mathbf{0};a_{-i},\underline{x}^\infty) = 0 \qquad (10)$$

Equivalently from equation 9, for the limit point $\overline{x}^\infty$ we get

$$\Delta\Pi(\mathbf{1}, a_i; (s_j(x_j; \overline{x}^{\infty}))_{j\neq i}, \overline{x}^{\infty}) = \frac{1}{2^N} \sum_{a_{-i}\in\, A^h_{-i}} \Delta\pi(\mathbf{1}, a_i; a_{-i}, \overline{x}^{\infty}) = 0$$

By assumption A5 (IS) this is in particular true for $a_i = \mathbf{0}$, then

$$\sum_{a_{-i}\in\, A^h_{-i}} \Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, \overline{x}^{\infty}) = 0 \tag{11}$$

Finally, it is easy to see that equations (10) and (11) are the same, and from assumption A7 (SC) $\underline{x}^{\infty} = \overline{x}^{\infty} = k^*$. Then $S^{\infty} = \bigcap_{n=0}^{\infty} S^n = \left\{(s_i(x_i; k^*))_{i=1}^{N+1}\right\}$ ■

Proof of Proposition 4

We must show that there exists a unique k^* solving,[22]

$$\sum_{a_{-i}\in\, A^h_{-i}} \{\Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, k^*)\} = 0 \tag{12}$$

where $\Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, k^*) = \sum_{j\neq i} a_{ji} \left(k^* + \sum_{k\neq i\neq j} a_{jk}a_{kj}\beta k^*\right) + N(\alpha k^* - c)$.

Then, equation (12) can be written as

$$\sum_{n=0}^{N} \sum_{a_{-i}\in\, A^{h,n}_{-i}} \left\{\sum_{j\neq i} \left(k^* + \sum_{k\neq j\neq i} a_{jk}a_{kj}\beta k^*\right) + N(\alpha k^* - c)\right\} = 0$$

Now, for all $n \neq 1$ it is easy to check that

[22] Remember $\sum_{n=0}^{N} \sum_{a_{-i}\in A^{h,n}_{-i}} \{\Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, k^*)\} = \sum_{a_{-i}\in\, A^h_{-i}} \{\Delta\pi(\mathbf{1}, \mathbf{0}; a_{-i}, k^*)\} = 0$

$$\sum_{a_{-i}\in\ A_{-i}^{h,n}} \left\{ \sum_{j\neq i} \left(k^* + \sum_{k\neq j\neq i} a_{jk}a_{kj}\beta k^* \right) + N(\alpha k^* - c) \right\}$$
$$= \left(\begin{array}{c} N \\ n \end{array} \right) (N(\alpha k^* - c) + n(k^* + \beta k^*))$$

and for $n = 1$

$$\sum_{a_{-i}\in\ A_{-i}^{h,n}} \left\{ \sum_{j\neq i} \left(k^* + \sum_{k\neq j\neq i} a_{jk}a_{kj}\beta k^* \right) + N(\alpha k^* - c) \right\} = \left(\begin{array}{c} N \\ 1 \end{array} \right) (N(\alpha k^* - c) + k^*)$$

then $\sum_{n=0}^{N} \sum_{a_{-i}\in\ A_{-i}^{h,n}} \left\{ \sum_{j\neq i} \left(k^* + \sum_{k\neq j\neq i} a_{jk}a_{kj}\beta k^* \right) + N(\alpha k^* - c) \right\} = 0$ is given by:

$$\sum_{n=0}^{N} \left\{ \left(\begin{array}{c} N \\ n \end{array} \right) (N(\alpha k^* - c) + n(k^* + \beta k^*)) \right\} - N\beta k^* = 0$$

Then $N\alpha k^* \sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) - N\alpha c \sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) + k^*(1+\beta) \sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) n - N\beta k^* = 0$

solving for k^* we get

$$k^* = \frac{N\alpha c \sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array} \right)}{N\alpha \sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) + (1+\beta) \sum_{n=0}^{N} \left(\begin{array}{c} N \\ n \end{array} \right) n - N\beta} \blacksquare$$

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