On the Size and Structure of Group Cooperation

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Abstract

When time preferences are heterogeneous and bounded away from one, how "much" cooperation can be achieved by an ongoing group? How does group cooperation vary with the group's size and structure? This paper examines characteristics of cooperative behavior in the class of symmetric, repeated games of collective action. These are games characterized by "free rider problems" in the level of cooperation achieved. Repeated Prisoner's Dilemma games are a special case.

We characterize the level of *maximal average cooperation (MAC)*, the highest average level of cooperation, over all stationary subgame perfect equilibrium paths, that the group can achieve. The MAC is shown to be increasing in monotone shifts, and decreasing in mean preserving spreads of the distribution of discount factors. The latter suggests that more heterogeneous groups are less cooperative on average. Finally, in a class of Prisoner's Dilemma games, we show under weak conditions that the MAC exhibits increasing returns to scale in a range of heterogeneous discount factors. That is, larger groups are more cooperative, on average, than smaller ones. By contrast, when the group has a common discount factor, the MAC is invariant to group size.

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1 Introduction

Successful cooperation at some level is required in any group. In most contexts, teams, clubs, and partnerships function most effectively when their members get along with one another. When they do not, substantial social conflicts often result.

This paper examines characteristics of group cooperation in repeated, collective action games. Consider a group with n members. Each group member, $i = 1, \ldots, n$, has discount factor δ_i in the repeated game. In each period Member i chooses a normalized level of effort or contribution, $p_i \in [0, 1]$, which indicates how cooperative he is toward other members of the group. As is standard in collective action games, the individual is better off the more cooperative are all members in the aggregate. However, each has a (static) incentive to behave uncooperatively, i.e., to "free ride." Moreover, the marginal incentive to free ride weakly increases in the aggregate cooperation of the group. Consequently, $p_i = 0$ for all members is the unique Nash equilibrium of the stage game. One such example is the Prisoner's Dilemma game. Voluntary provision of a public good is another.

How "much" cooperation can this group achieve on average? How does the mix of patient and impatient group members affect the level of cooperation in equilibrium? Does the size of the group matter?

The answers to these questions are straightforward in polar cases. For instance, if for all members i and j, $\delta_i = \delta_j$ is close to zero (uniform impatience), then free rider incentives of the stage game take over, and so joint cooperation cannot be sustained. On the other hand, if $\delta_i = \delta_j$ is close to one (uniform patience), equilibria exist in which all group members are "fully" cooperative each period.[1] In both of these cases, group size plays no substantive role.

The intermediate cases are harder and not well understood. We therefore depart from the traditional focus on uniformly patient players in repeated games.[2] Consider, instead, an arbitrary profile of discount factors for the group. We examine paths arising from the set of stationary subgame perfect equilibria (SSPE), i.e., equilibria in which, after any history, a stationary profile of actions is played thereafter. Specifically, consider a stationary profile of actions, $p = (p_1, \ldots, p_n)$, that maximizes the average level of cooperation

$$\frac{1}{n}\sum_{i=1}^{n} p_i \tag{1}$$

over SSPE paths. The value of this program is a level of cooperation, $p^*(n, \delta) \in [0, 1]$, which we call the *maximal average cooperation (MAC).*

[1] Clearly, full cooperation is a particular application of the Folk Theorem. See, as a standard reference, Fudenberg and Maskin (1986).

[2] Other work which departs from the uniform patience assumption is discussed later in this Section.

The maximal average cooperation (MAC) describes the highest average level of cooperation attainable by the group each period when behavior is stationary. We are interested in it, rather than in group welfare, because it is cardinal and is calculated from behavior directly. Direct measurement of behavior is desirable in firms, teams, and other organizations. Such organizations are often hierarchical; individuals are organized in teams that self-monitor individual behavior while the aggregate outcome is evaluated by a supervisor. Players' payoffs are not directly observed. In such cases, a group's effectiveness might be judged by its average cooperative effort.

As a measure of group cooperation the MAC is, in some sense, restrictive. Clearly, it is meaningful only if individual actions have an agreed upon cardinal interpretation. Moreover, because it is derived from stationary equilibria, it entails some loss of generality when discounting is heterogeneous. A recent paper by Lehrer and Pauzner (1999) shows that the feasible payoff set changes when discounting is heterogeneous. In particular, equilibrium payoffs typically exist outside the convex hull of the stage game payoff set if individuals are sufficiently but heterogeneously patient.

Despite the imperfections, we argue that the MAC is a useful concept for understanding group cooperation when discounting is heterogeneous. The MAC and, in particular, the stationary equilibria from which the MAC is derived, can be justified on two main grounds.

(1) First and foremost, so little is known about equilibria when discounting is bounded away from either one or zero that it is sensible to start with an approach that admits a transparent comparison with standard models. In particular, because the stationary feasible payoff set does not change if discounting is heterogeneous, *stationary equilibria can isolate changes due to equilibrium behavior rather than due to changes in feasibility constraints*. Indeed, when discounting is heterogeneous, the problem with *non*stationary equilibria is precisely one of sorting out these two effects — feasibility and equilibrium. For this reason, comparative statics exercises are less informative when nonstationary equilibria are considered.

(2) Stationary equilibria facilitate objective measurement of the group by an external agent. This is especially relevant in a firm or an organization. By contrast, nonstationary equilibria present problems for measurement since it is unclear how cooperation early in the game should be weighted against cooperation later if there is no common discount factor.

Surprisingly, the literature on repeated games with intermediate and/or heterogeneous discounting is sparse. The literature studying a *common* discount factor taking intermediate values is more common. Sorin (1986), Cave (1987), Stahl (1991), and Mailath, Obara, and Sekiguchi (2002) all study the payoff set of repeated Prisoner's Dilemma with a common discount factor bounded away from one.

The literature on heterogeneous discounting typically studies behavior in the "heterogeneous limit," meaning that relative differences between the different agents' discount factors are maintained while discounting approaches one. Lehrer and Pauzner (LP) (1999) characterize equilibrium payoffs in two player games in the limit as period length goes to zero while maintaining a fixed log ratio $\frac{\log(\delta_1)}{\log(\delta_2)}$ between the two discount factors. Heterogeneous limits are also examined by Fudenberg, Kreps, and Maskin (1990), who prove a Folk Theorem for a subset of the players when the other players' discount factors are 0, and by Fudenberg and Levine (1989), Celantani, et al. (1995), and Aoyagi (1996) all of whom examine reputation-building by a sufficiently patient, long run player who faces a sequence of short run players in a repeated game. Harrington (1989) characterizes the bounds on heterogeneous discount factors required to achieve collusion in oligopolies. In a precursor to the present paper, Haag and Lagunoff (2005a) examine stationary trigger strategies in a local interaction network with heterogeneous discounting.

The present paper does not restrict discounting to be common or asymptotically close to one. Section 2 demonstrates how the MAC varies across different discounting profiles in a simple example of a two-player Prisoner's Dilemma game. Section 3 introduces a general class of symmetric collective action stage games.[3] Our main results are contained in Sections 4-6. Section 4 defines the MAC for the class of collective action games. We show that the MAC is characterized by a maximal fixed point of a particular function. Properties of this function reveal how and to what degree the equilibrium can accommodate intermediate levels of cooperation.

Section 5 examines the effect on cooperation when the composition of the group changes (holding size fixed). The main finding in this Section is that mean preserving spreads *reduce* the MAC. In other words, greater group heterogeneity lowers cooperation. Increased social conflict is generated by increasingly different time preferences. The intuition, roughly, is that because a relatively patient individual is already cooperating more than a relatively impatient one, he faces less cooperation from his rivals. Consequently, he gains less on the margin from a given decrease in cooperation than would an impatient individual. An increment in the patient player's discount factor therefore has a smaller net effect than a corresponding decrement in the impatient player's.

Unfortunately, the class of collective action games includes all sorts of stage games that are not neutral to size. In order to separate effects of the repeated game from "built-in" effects of the stage game, Section 6 restricts attention to a subclass of "size-neutral" stage games that conform to n-Player Prisoner's Dilemma. We show that under fairly broad conditions and over a nondegenerate range of discount factor profiles, the MAC in the repeated PD game exhibits *increasing returns to scale*, i.e., the average level of cooperation of the scaled up group exceeds that of the original group.

The surprising thing about the result is that the static PD game itself exhibits a "non-increasing returns to scale" property: a player's marginal payoff in the PD game, hence his marginal incentive to free ride, is increasing in size. Hence, the result on increasing returns

[3] We restrict attention to symmetric games so that the source of heterogeneity among players is isolated to the discount factors.

comes precisely from the repetition of the game. Just as significantly, the heterogeneous discounting is critical to the result. The assumptions do not hold, and the result fails, when discounting is homogeneous. Specifically, when players' time preferences are the same, the MAC is invariant to group size.

The intuition for the result can be seen in a 2-person group with a "patient" and an "impatient" player. If this group is replicated once, doubling in size, then the patient player faces one other patient player and two impatient players. Whereas before, he faced one impatient player. Hence the proportion of patient to impatient players has changed favorably for the patient player. Consequently, the marginal response to a change in scale is positive above a fixed threshold discount factor and is negative below it. However, the magnitudes of these marginal responses are different for patient and impatient players. Impatience has a dampening effect: an impatient player cares less about the future responses of others to his current action. Consequently, the positive response of the patient player to an increase in group size exceeds in absolute value the negative response of the impatient one.

The size result casts some doubt on the prevailing wisdom about free rider problems that comes primarily from analysis of static games. Beginning with Olson (1965), the standard wisdom is that free rider problems worsen with size.[4] This "Olson Conjecture," as it has come to be known, is discussed along with related literature in Section 7. This section also summarizes our findings and discusses limitations and possibilities for future work. Finally, Section 8 is an Appendix which contains all the proofs of the results.

2 A Prisoner's Dilemma Example

To see how the MAC is determined, we begin with what we view as the quintessential collective action problem: the Prisoner's Dilemma game given by the matrix below.

	C	D
C	c, c	$-\ell, d$
D	$d, -\ell$	$0, 0$

Figure 1: Prisoner's Dilemma

For Prisoner's Dilemma we require that $d > c > 0 > -\ell$, and $2c > d - \ell$. For now, to make the PD game consistent with n-player collective action games examined later, we also

[4] To be fair, Olson outlined several possible reasons, most of which are not taken up formally here or elsewhere in the literature, for the ineffectiveness of larger groups. See Section 7 for a discussion of the literature.

assume submodularity, i.e., $d - \ell > c$. This means that a player's net payoff from switching to D is larger if his rival chooses C.[5]

Suppose players can choose mixed strategies where p_i is the probability of choosing C by Player $i = 1, 2$. Then Player i's payoff in the mixed extension of the PD game is:

$$p_i(p_j c - (1 - p_j)\ell) + (1 - p_i)p_j d \tag{2}$$

In the repeated game it is sometimes assumed that the history of mixed strategies p_i is itself observed. Observability of mixed strategies is not a problem in the Folk Theorem literature since payoffs in the limiting case where $\delta \to 1$ can always be replicated by time averaging. The approximation deteriorates, however, when discount factors move away from one. Other than as an approximation, we can think of no obvious example where players' mixed actions are observable. Hence, in the general model in Section 3, p_i is a continuous *pure* action, and the payoff expression in (2) is that of the base, stage game. We only use the mixed strategy interpretation in this Section for its familiarity to readers and for its usefulness as an illustrative device.

Whether p_i is pure or mixed, Player i's payoff is decreasing in his own action, and increasing in his rival's. The unique Nash equilibrium is $(p_1, p_2) = (0, 0)$. However, when the PD game is infinitely repeated, it is a standard exercise to show that full cooperation, i.e., $p_i = 1$ for each i, can be sustained as a subgame perfect equilibrium if both individuals are sufficiently patient. We examine how well they do if one or both discount factors falls below the "patience threshold?"

Let δ_i denote the discount factor of individual $i = 1, 2$. Each individual uses δ_i to calculate his average discounted sum of payoffs. A pair (p_1, p_2) constitutes a stationary subgame perfect equilibrium (SSPE) if, for each $i = 1, 2$, and $j \neq i$,

$$p_i(p_j c - (1 - p_j)\ell) + (1 - p_i)p_j d \geq (1 - \delta_i)p_j d \tag{3}$$

The right hand side of (3) is the one shot gain from a deviation to the most uncooperative action, $p_i = 0$, with the consequence that both members permanently revert thereafter to the one shot Nash equilibrium, (D, D).

Inequality (3) yields this system of inequalities

$$\begin{aligned} \delta_1 p_2 &\geq p_1(p_2\mu + (1 - p_2)\gamma) \\ \delta_2 p_1 &\geq p_2(p_1\mu + (1 - p_1)\gamma) \end{aligned} \tag{4}$$

where $\mu \equiv (d - c)/d$ and $\gamma \equiv \ell/d$.[6] The first inequality is Player 1's incentive constraint against deviating to D. The second inequality is the identical incentive constraint for

[5] The case of $d - \ell < c$ is examined later in this Section.

[6] Note that by our submodularity assumption, $\mu > \gamma$ since $d > c + \ell$.

Player 2. Clearly, full cooperation, i.e., $(p_1, p_2) = (1, 1)$ solves this system whenever $\delta_i \geq \mu \equiv (d - c)/d$ for both $i = 1, 2$. In this case the Folk Theorem property applies.

However, when $\delta_i < \mu$ for either or both players, partial cooperation is still possible. In any SSPE, (p_1, p_2), the constraints bind for either, both, or neither player depending on the value of the discount factors. Clearly, if a constraint does not bind, then for that player i, it must be that $p_i = 1$ since otherwise, his cooperation can be increased without violating the incentive constraint of the other player. Solving the system (4) for each possible case, the maximal average cooperation (over all SSPE pairs (p_1, p_2)) is given by:

$$p^* = \begin{cases} 1 & if \quad \delta_i \geq \mu, \text{ for } i = 1, 2 \\ \frac{1}{2}\left[\frac{\delta_1}{\mu} + 1\right] & if \quad \delta_1 < \mu \text{ and } \delta_1\delta_2 \geq \mu\gamma + \delta_1(\mu - \gamma) \\ \frac{1}{2}\left[1 + \frac{\delta_2}{\mu}\right] & if \quad \delta_2 < \mu \text{ and } \delta_1\delta_2 \geq \mu\gamma + \delta_2(\mu - \gamma) \\ \frac{1}{2}\left[\frac{\delta_1\delta_2 - \gamma^2}{(\delta_2+\gamma)(\mu-\gamma)} + \frac{\delta_1\delta_2 - \gamma^2}{(\delta_1+\gamma)(\mu-\gamma)}\right] & if \quad \gamma^2 \leq \delta_1\delta_2 < \mu\gamma + \min\{\delta_1, \delta_2\}(\mu - \gamma) \\ 0 & if \quad \delta_1\delta_2 < \gamma^2 \end{cases} \tag{5}$$

In expression (5), the MAC takes on distinct values in each of five regions in discount factor space. These regions are exhibited in Figure 2a. For example, the conditions which generate a MAC of $\frac{1}{2}\left[\frac{\delta_1}{\mu} + 1\right]$ are those in which Player 2's constraint does not bind, and Player 1's does bind. In this case, Player 2 cooperates with certainty, while Player 1 "partially cooperates," choosing C with probability less than one. Solving for Player 1's binding constraint when $p_2 = 1$ gives $p_1 = \delta_1/\mu \equiv \delta_1/\frac{d-c}{d}$. Hence, a player who discounts to 20% of the full cooperation threshold μ is uncooperative 20% of the time.

A few remarks about (5) are warranted. First, note that p^* is (weakly) increasing and continuous in the pair (δ_1, δ_2). It is also decreasing in (arithmetic) mean preserving spreads of the original pair (δ_1, δ_2). This means that the larger is the difference in the two players' time preferences, other things equal, the less cooperative is the group on average. To see this for two special cases, suppose first both players are at the critical discount factor for full cooperation, i.e., $\delta_i = \mu$, for both $i = 1, 2$. A mean preserving spread takes at least one individual below his full cooperation region, thereby lowering the MAC . Alternatively, suppose that both players are just at the critical discount factor for positive cooperation, i.e., $\delta_i = \gamma$, for both $i = 1, 2$. Then a mean preserving spread reduces the MAC to 0 since the product of the two discount factors with arithmetic mean of γ is maximized at $\delta_1 = \delta_2 = \gamma$.

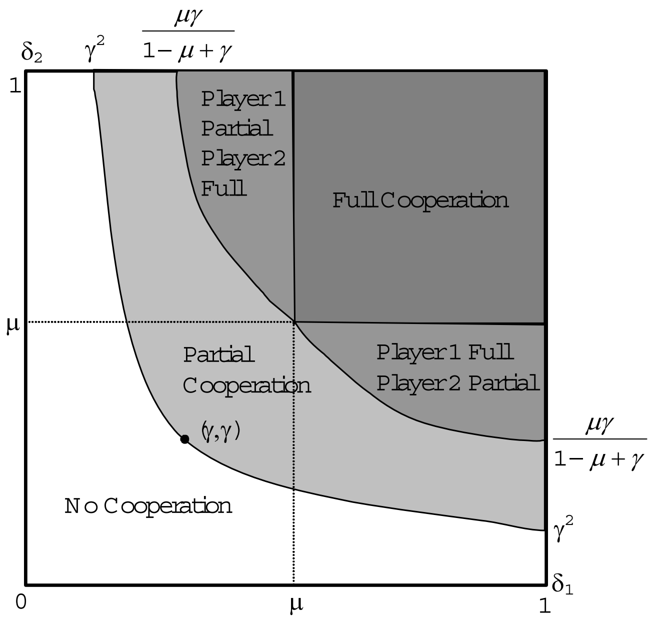

Fig. 2a: The Submodular Case

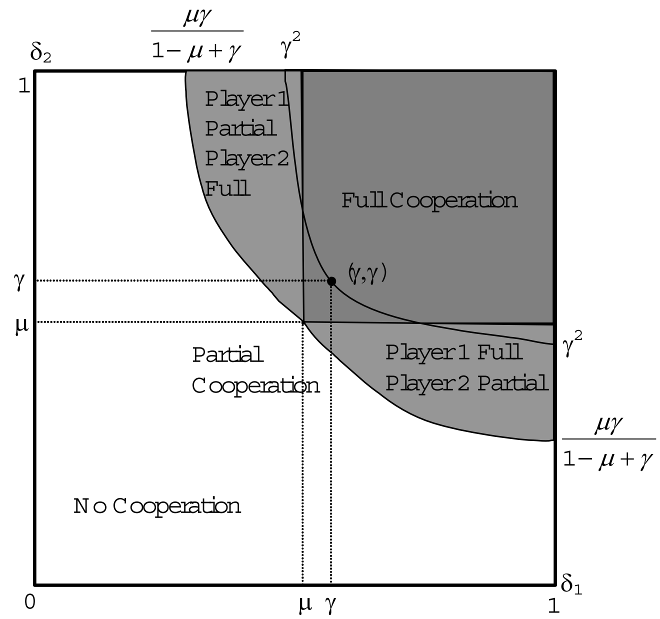

Fig. 2b: The Supermodular Case

Second, a noteworthy special case of Equation (5) is the common discount factor case where $\delta_1 = \delta_2 = \bar{\delta}$. In this case (5) reduces to Equation (6) below.[7]

$$p^* = \begin{cases} 1 & if \quad \bar{\delta} \geq \mu \\ \frac{\bar{\delta}-\gamma}{\mu-\gamma} & if \quad \gamma \leq \bar{\delta} < \mu \\ 0 & if \quad \bar{\delta} < \gamma \end{cases} \tag{6}$$

Third, 2-player games are somewhat special since there is always a re-ordering of one player's strategy set that renders any game submodular. Hence, the supermodular case of $\mu < \gamma$ yields similar, but not identical, results. In the supermodular game the incentives to cooperate increase precisely when the other player is more cooperative. In this case, the region in which both players cooperate disappears except for a nongeneric boundary case of $\delta_1\delta_2 = \mu\gamma + \min\{\delta_1, \delta_2\}(\mu - \gamma)$ (see Figure 2b). The regions that define partial cooperation for one player and full cooperation for another are the same. The region in which there is no cooperation is given by $\delta_1\delta_2 < \mu\gamma + \min\{\delta_1, \delta_2\}(\mu - \gamma)$. As before, cooperation always (weakly) increases in delta. Typically, n-player collective action games are submodular, and

[7] For some reason, many of the studies of repeated games with common discounting examine a knife-edged case of Prisoner's Dilemma where $\mu = \gamma$ (Stahl (1991) is an exception). This assumption, in our case, would lead to the flawed conclusion that partial cooperation is never sustainable. We show that it is, for both players, when $\mu > \gamma$. Partial cooperation for exactly one player is sustainable when $\mu < \gamma$.

so this case is of limited interest.

Fourth, this example cannot address questions of group size. To answer these, the general n-group model is developed in the subsequent section.

3 Collective Action Games

A collection of n individuals play an infinitely repeated game in discrete time $t = 0, 1, \ldots$. We refer to this collection as the "n-group" or, more simply, "the group." In each period, the stage game is as follows. Each member i of the n-group chooses a number, p_i, from a set $[0, 1]$ of feasible actions. The action p_i determines member i's chosen level of cooperation toward the group. Notationally, we will let $P = \sum_j p_j$ denote the aggregate level of cooperation, and $P_{-i} = \sum_{j \neq i} p_j$, the aggregate level of cooperation excluding group member i. An action profile is given by $p = (p_1, \ldots p_n)$. Individual i's payoff of profile p is given by $\bar{\pi}_i(p)$.

Given an infinite repetition of a game with payoffs $(\bar{\pi}_i)_{i=1}^n$, let $p(t)$ denote the action profile taken in period t. An individual's dynamic payoff then is:

$$\sum_{t=0}^{\infty} (1 - \delta_i)\delta_i^t \bar{\pi}_i(p(t)) \tag{7}$$

where δ_i denote the discount factor of member i. Let $\delta = (\delta_1, \ldots, \delta_n)$ denote the (arbitrary) profile of discount factors of the group.

Our analysis is restricted to the symmetric class of *collective action* stage games. Roughly, a collective action game is a game in which the payoff $\bar{\pi}_i$ for individual i is increasing in the aggregate level of cooperation of all members of the group, but assumed to decrease in his own level of cooperation in the aggregate. Moreover, one's incentive to cooperative is assumed to decrease in the aggregate level of cooperation.

We follow the canonical convention in assuming that a collective action game is anonymous. Specifically, we assume a game represented by a payoff function $\pi_i : \mathbb{R}_+^2 \to \mathbb{R}$ for i satisfying $\pi_i(p_i, P) = \bar{\pi}_i(p)$ where, recall, $P = \sum_j p_j$ is the aggregate level of cooperation. A *collective action game* is any profile $(\pi_i)_{i=1}^n$ that satisfies properties (A1)-(A5) below.

(A1) (Symmetry) (π_i) is a symmetric game, i.e., $\pi_i = \pi$ for all i.

(A2) (Monotonicity) For each i, π is C^2, strictly increasing in its second argument, strictly decreasing in its first argument, and for all p_i, all P, and all $x > y > 0$,

$$\frac{\pi(p_i, P - x)}{\pi(p_i + x, P)} > \frac{\pi(p_i, P - y)}{\pi(p_i + y, P)} > 1.$$

(A3) (Concavity) π is weakly, jointly concave in all variables.

(A4) (Submodularity) π is weakly submodular in p and P in the natural order:[8] if $p_i' \geq p_i$ and $P' \geq P$, then

$$\pi(p_i', P') - \pi(p_i, P') \leq \pi(p_i', P) - \pi(p_i, P).$$

(A5) (Normalization) $\pi(0, 0) = 0$ and there exists $1 < \bar{P} < n$, such that $\pi(1, \bar{P}) = 0$.

Other than the normalization in (A5), which is largely a technical condition, these assumptions describe standard properties of games with free rider problems. Namely, each individual always has an incentive to "under-contribute." This incentive generates the unique, pure strategy Nash equilibrium of $p_i = 0$ for all i. At the same time, some positive contribution by all individuals is Pareto preferred to the Nash equilibrium.[9]

Assumption (A2) assumes monotonicity in both levels and relative differences. Specifically, with a simple relabeling, the second inequality in (A2) may be expressed as $\pi(p_i, P) > \pi(p_i + y, P + y)$ for all $y > 0$. In other words, identical increases in the private and aggregate contributions decrease one's payoff. The first inequality expresses the same idea in log differences. It entails that payoffs decrease *proportionately* faster in the first argument (one's own contribution) than they increase in the second (the aggregate contribution).

Finally, Assumption (A4) together with (A2) capture the "Olsonian" intuition that free rider problems worsen the larger is the aggregate contribution. Notice that these games have the property that the Nash equilibrium payoff coincides with the minmax payoff for each player. This property can be relaxed, but at a significant technical cost (see Footnote 12).

It is easy to check that the Prisoner's Dilemma game in Section 2 is a special case. Another special case is the game of voluntary provision of a public good. A canonical representation of this is given by the payoff $\pi(p_i, P) = F(P) - G(p_i)$ where F is concave and G is convex.

4 Maximal Average Cooperation

We assume that the histories $(p(1), \ldots, p(t))$ are publicly observable and restrict attention to stationary subgame perfect equilibria (SSPE). These are subgame perfect equilibria in which, after any history, there is some profile of actions such that this profile is chosen in

[8] Equivalently, p and P are strategic substitutes in the natural order.

[9] In the unique equilibrium, $\pi(0, 0) = 0$, while by Assumption (A5), for each $P > \bar{P}$, then $\pi(p_i, P) > 0$ for all p_i.

each period thereafter.[10] Because histories are publicly observed and the minmax payoff is sustained by a Nash equilibrium, any SSPE path in this class of games can be implemented by an SSPE which uses simple "trigger strategies" in which any deviation is met with permanent reversion to the one shot, "uncooperative" equilibrium. Hence, there is no confusion when we refer to the stationary path profile, p, as a *stationary subgame perfect equilibrium (SSPE)*.[11]

Let $E(\delta)$ denote the set of SSPE profiles given δ. Our particular interest is in the profile p that solves

$$\max_{p \in E(\delta)} \frac{1}{n} \sum_{i=1}^{n} p_i \tag{8}$$

The optimal value of (8) is an action $p^* \in [0, 1]$ that describes the *maximal average co-operation (MAC)*. In Section 5, the MAC will be denoted by $p^*(n, \delta)$ to mark the explicit dependence on group's size, n, and characteristics, δ.

In what follows, we fix the identity i of a particular individual in the group. Fix a type profile δ and a stationary path, $p(t) = p$, $\forall t$. Using (16), we will say that a pair $(p, P) \in [0, 1]^n \times [0, n]$ describes a SSPE if and only if $\sum_{i=1}^{n} p_i = P$ and for each $i = 1, 2, ...n$,

$$\pi(p_i, P) \geq (1 - \delta_i)\pi(0, P - p_i)$$

which we rewrite as

$$Q(p_i, P; \delta_i) \equiv \delta_i \pi(0, P - p_i) + [\pi(p_i, P) - \pi(0, P - p_i)] \ \geq \ 0 \tag{9}$$

The reason (9) characterizes an SSPE is clear. Given that the minmax payoff of 0 coincides with the Nash equilibrium payoff, it suffices to verify the equilibrium against the worst possible punishment — permanent reversion to the one shot equilibrium.[12]

By (A2), one can readily verify that Q is continuous in all variables. The following Lemmatta establish some useful properties of the incentive constraint. The proofs of these and all subsequent results are in the Appendix.

Lemma 1 *Q is strictly decreasing in p_i.*

[10] One could easily relax this definition to a weaker SSPE where stationarity applies only *on* the equilibrium path. In that case off-path punishments need not be stationary. In our framework, however, the two are equivalent: stationary punishments may be assumed without loss of generality — see footnote 11.

[11] The argument for why there is no loss of generality is standard: since the minmax payoff is 0, any equilibrium path remains an equilibrium path if it is enforced by the threat of permanent reversion to the one-shot equilibrium in which everyone in the group chooses "D" thereafter.

[12] In the general case where the minmax payoff differed from the Nash equilibrium, we would have to verify additional out-of-equilibrium "perfection" constraints. Nevertheless, Inequality (9) would still characterize the on-path constraint.

Next, fix some P and define $\underline{\delta}(P)$ by

$$\underline{\delta}(P) = \inf\{r \in [0, \infty] : \; Q_i(p_i, P; r) \geq 0, \; \forall p_i \leq \min\{1, P\}\} \tag{10}$$

That is, $\underline{\delta}(P)$ is the infimum over the extended reals that satisfies the incentive constraint for all levels, p_i, of i's cooperation.

Lemma 2 *For each i:*

$$\textit{if} \;\; 0 \leq \delta_i \leq \underline{\delta}_i(P) \quad \textit{then} \quad \textit{there exists} \;\; p_i \in [0, \min\{1, P\}] \;\; \textit{such that} \quad Q(p_i, P; \delta_i) = 0$$

$$\textit{if} \;\; \delta_i > \underline{\delta}_i(P) \geq 0 \quad \textit{then} \quad Q(p_i, P; \delta_i) > 0, \; \forall p_i \in [0, 1]$$

Moreover, if $P > 1$, then $\underline{\delta}(P)$ satisfies

$$\pi(1, P) = (1 - \underline{\delta}(P))\pi(0, P - 1) \tag{11}$$

The purpose of these two results can now be made clear. We can now combine these two Lemmatta to produce the following: for any P, if $0 \leq \delta_i \leq \underline{\delta}_i(P)$ then there exists $p_i \in [0, 1]$ such that $Q(p_i, P; \delta_i) = 0$ and $\frac{\partial Q(p_i, P; \delta_i)}{\partial p_i} \neq 0$. The Implicit Function Theorem thus implies existence of a locally continuous function $R(\cdot; \delta_i)$ such that $p_i = R(P; \delta_i)$ and $R(P; \delta_i)$ satisfies $Q(R_i(P; \delta_i), P; \delta_i) = 0$ (i.e., incentives bind at $p_i = R(P; \delta_i)$).

Now define $\psi : [0, n] \times [0, 1] \to [0, 1]$ by

$$\psi(P; \delta_i) = \begin{cases} R(P, \delta_i) & if \quad 0 \leq \delta_i \leq \underline{\delta}_i(P) \\ 1 & if \quad \delta_i > \underline{\delta}_i(P) \geq 0 \end{cases}$$

For each group member i, $\psi(\cdot; \delta_i)$ is the *maximal response* to P. It describes the maximal level of i's cooperation consistent with i's incentive constraint given aggregate cooperation P. Finally, define $\Psi : [0, n] \times [0, 1]^n \to [0, n]$ by

$$\Psi(P; \delta) = \sum_{i=1}^{n} \psi(P; \delta_i) \tag{12}$$

Here, $\Psi(\cdot; \delta)$ is the aggregate maximal response to aggregate cooperation P.

Theorem 1 *For any repeated collective action game, the MAC is given by*

$$p^* \;=\; \max\left\{\frac{P}{n} \in [0, 1] : \; P = \Psi(P; \delta)\right\} \tag{13}$$

where the solution to (13) is nonempty and $\Psi(P) < P$, $\forall P > P^*$.

According to the result, the MAC is the maximal fixed point of Ψ divided by n. Letting $P^* = np^*$, the property $\Psi(P) < P$, $\forall P > P^*$ is a tranversality property. It means that the MAC crosses the 45° line and so it varies continuously with small perturbations the game's structure.

While the fixed point characterization is somewhat reminiscent of standard fixed point logic for static games, the meaning and derivation are obviously different. Rather than coming from a best response correspondence, Ψ is an aggregate equilibrium response along the equilibrium path of the repeated game.

5 Composition Effects

The Characterization Result (Theorem 1) will prove useful because it is easier to evaluate parametric changes in the fixed point mapping Ψ, than to evaluate these changes in the MAC directly. For example, if Ψ is increasing in δ, then it follows that the MAC is increasing in δ. The former is much easier to establish directly than the latter.

Theorem 2 *For any repeated collective action game, let $\delta' = (\delta'_i)$ and $\delta'' = (\delta''_i)$ denote two distinct profiles of discount factors, and let $p^*(n, \delta')$ and $p^*(n, \delta'')$ denote their respective MAC s.*

Suppose either

1. $\delta' > \delta''$ *or*
2. δ'' *is a mean preserving spread of* δ' *in the sense that*

$$\frac{1}{n}\sum_i \delta''_i = \frac{1}{n}\sum_i \delta'_i \tag{14}$$

and for each member i, $\delta''_i \geq \delta'_i$ iff $\delta'_i \geq \frac{1}{n}\sum_i \delta'_i$.

Then $p^*(n, \delta'') \leq p^*(n, \delta')$, with strict inequality if $0 < p^*(n, \delta') < 1$.

The Proof shows that for each i, ψ_i is increasing and concave in δ_i. This establishes the first part. The second part then follows from a standard, second order stochastic dominance argument. Note that one cannot generally infer strict inequality between the two since δ'' may already place all individuals in the region of full cooperation (i.e., $p_i = 1$, $\forall i$), and so further increases in δ have no effect.

The fact that average cooperation "increases" in each member's patience is, of course, to be expected. However, the second statement in the Theorem is less obvious. Increased heterogeneity increases the degree of social conflict. More patient individuals, it turns out, are less responsive to increases in patience than impatient ones. The intuition, roughly, is the following. The effect of increasing a player's discount factor is to increase the weight on future punishment following a deviation. But since relatively more patient individual is already cooperating more than an impatient individual, he faces less cooperation from his rivals. Consequently, the patient player's cooperation gains him less on the margin after an increment in his discount factor than an impatient player loses after a decrement in his.

The result is reminiscent of a result in a recent paper by Dal Bo (2001). In a Bertrand game with a common, but stochastic discount factor, he shows that higher variability in the time distribution of discounting lowers prices and profits of firms. Whereas our model displays heterogeneity in agents and homogeneity in time, his model displays the opposite. The mechanics of the two arguments are quite different in any event.

6 Size Effects

The issue of scale of cooperation is more complicated. A corporate merger, for example, requires the merger of two groups of workers, each of a different size and each with different characteristics. It is certainly possible for the merger to increase both the size *and* the social variability relative to each group. Hence, to isolate the effect of size alone, we consider the effect of increasing n while fixing the distribution on δ. The simplest construction replicates the group α times, for some positive integer α. Specifically, if the group size is scaled from n to αn, then there are α "copies" of each individual i with discount factor δ_i. Formally, let $\delta(n) = (\delta_1, \ldots, \delta_n)$ denote the profile of discount factors in order to express the explicit dependence on group size. Then

$$\delta(\alpha n) = (\overbrace{\delta(n), \delta(n), \ldots, \delta(n)}^{\alpha}\ )$$

Recall that the MAC *exhibits increasing (decreasing) returns at scale factor* α if

$$p^*(\alpha n, \delta(\alpha n)\ ) \ > \ (<)\ p^*(n, \delta(n)\ ) \tag{15}$$

Inequality (15) can be expressed equivalently as $P^*_{\alpha n} \ > \ (<)\ \alpha P^*_n$. Hence, increasing (decreasing) returns to cooperation exist if maximal *aggregate* cooperation is superadditive (subadditive).

One might reasonably ask if or when the MAC in repeated collective action games exhibits some form of returns to scale. Unfortunately, there can be no general answer because these games have no "built-in" assumptions regarding size. For example, the payoff functions, $\pi^1 = P - G(p_i)$ and $\pi^2 = \frac{P}{n} - G(p_i)$, both satisfy the axioms in Section 3. Yet, each has a different implications for group size. In the first case, payoffs increase in the aggregate contribution, while in the second, payoffs increase in the *average* contribution. Clearly, the public good in the second example is congestable.

Given these limitations, we examine, as a starting point, effects of size in a subclass of games that generalizes 2-player Prisoner's Dilemma games in Section 2. An n-player game in this subclass has payoffs given by

$$\pi(p_i, P) = (P - p_i)d - p_i(P - p_i)(d - c - \ell) - p_i(n-1)\ell \tag{16}$$

Here, c is the reward to mutual cooperation, d is the reward to (unilaterally) uncooperative behavior, and ℓ is the loss incurred from others' uncooperative behavior. As before, $d > c > 0 > -\ell$, and $2c > d - \ell > 0$. It is easy to check this payoff satisfies (A1)-(A5), and is equivalent to the mixed extension of PD game in which $p = (0, \ldots, 0)$ is the unique, stage Nash equilibrium.[13]

This "PD" game is a canonical candidate for examining the Olson Conjecture. The payoff function has the interesting characteristic that the (marginal) incentive to free ride increases with group size.[14] Yet, despite the *decreasing returns to scale* of this stage game, the MAC in the repeated version of this n-player PD is *invariant* to size when players are homogeneous, i.e, $\delta_i = \bar{\delta},\ \forall i$. To see this, observe that when player's discount factors are homogeneous, the incentive constraints are identical and the SSPE is symmetric. The incentive constraint then reduces to

$$Q(p_i, P; \delta_i) \equiv p_i\left((n-1)pc + (n-1)(1-p)(-\ell)\right) + (1-p_i)(n-1)pd - (1-\bar{\delta})(n-1)pd \geq 0 \tag{17}$$

where p is the symmetric probability that each of the other members cooperates, and $\bar{\delta}$ is the uniform discount factor. Obviously, this constraint is invariant to n. Hence, the MAC is also invariant to n, and is described by (6).

In order to examine the general case, it will prove more helpful if the scale factor α is treated as a continuous, rather than a discrete, variable. First, we write an individual's response as $\psi(P, n, \delta_i)$ to explicitly express the dependence on group size and on discount factor. Then, a group member's *marginal response to scale* α is given by

$$\frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha}. \tag{18}$$

[13] One can verify that (16) is equivalent to

$$\bar{\pi}_i(p) = \sum_{j \neq i} \left[p_i\left(p_j c + (1-p_j)(-\ell)\right) + (1-p_i)p_j d\right]$$

[14] This is easily verified. Observe: $\frac{\partial \bar{\pi}}{\partial p_i} = -P_{-i}(d-c-\ell) - (n-1)\ell = -P(d-c-\ell) - (n-1)\ell + p_i(d-c-\ell) > -\alpha P(d-c-\ell) - (\alpha n - 1)\ell + p_i(d-c-\ell)$. for any $\alpha > 1$.

where $P_n^* \equiv np^*(n, \delta^n)$ is the maximal aggregate cooperation in the n-group. If this partial derivative is positive (negative), then member i increases (decreases) his contribution if both group size and its aggregate contribution is scaled up by $\alpha < 1$.

Theorem 3 *Given any PD game with payoffs described by (16), let* p *be a SSPE such that* $\frac{1}{n}\sum_i p_i = p^*(n, \delta)$*. Then, for any scale factor* $\alpha > 1$*, and for any group member* $i = 1, \ldots, n$ *with* $0 < p_i < 1$*,*

$$\frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha} \begin{cases} > 0 & if \quad \delta_i \ > \ p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma \\ = 0 & if \quad \delta_i \ = \ p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma \\ < 0 & if \quad \delta_i \ < \ p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma \end{cases} \tag{19}$$

In words, the sign of a member i's marginal response to scale α depends on whether or not his discount factor lies above the threshold $p^*(n, \delta)\mu + [1 - p^*(n, \delta)\ ]\gamma$. The scale factor α does not appear in (19), meaning that sign of the individual's marginal response is invariant to scale. Of course, the *magnitude* of one's marginal response does depend on the magnitude of the change in scale.

An obvious corollary to the Theorem 3 is that an interior MAC exhibits increasing (decreasing) returns at every scale factor $\alpha > 1$ if

$$\delta_i \ > \ (<)\ p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma, \qquad \forall i = 1, \ldots, n. \tag{20}$$

Of course, (20) is only a sufficient condition, and a strong one at that. Yet, it turns out that a natural asymmetry exists between the increasing an decreasing returns cases. To show increasing returns, one need only compare the *average* discount factor with the threshold.

Theorem 4 *Let* $0 < p^*(n, \delta) < 1$ *denote the MAC of any PD game of size* n*. Order the discount factors in ascending order:*

$$\delta_1 < \delta_2 < \cdots < \delta_n$$

and let i^* *be the largest index* i *for which* $\psi(P_n^*, n, \delta_i) < 1$ *(i.e., the response of each member* $i = 1, \ldots, i^*$ *is less than one).*

Then, if $\frac{n}{(n-1)^2} < \gamma$*, and if*

$$\frac{1}{i^*}\sum_{i=1}^{i^*} \delta_i \quad > \quad p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma, \tag{21}$$

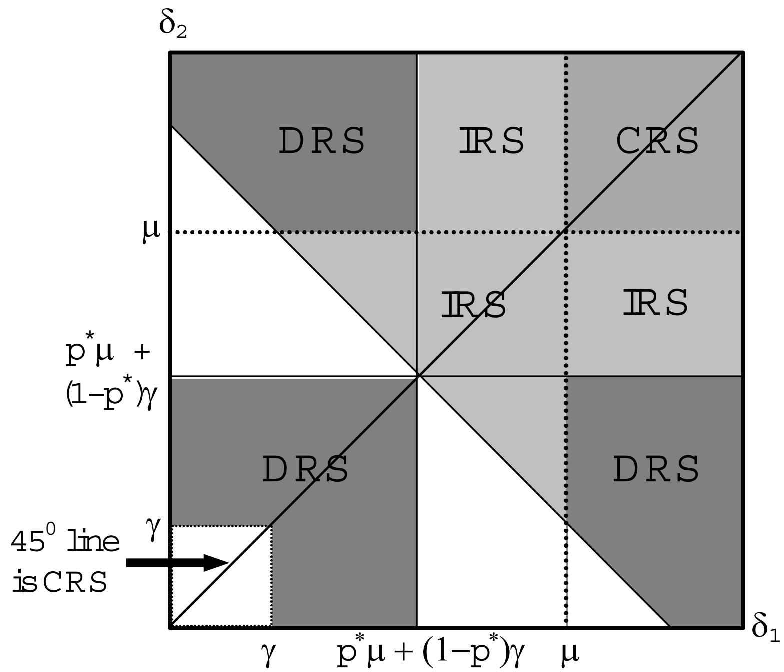

Figure 3: Returns to Scale in the 2-Member Group

then the MAC exhibits increasing returns to scale at every scale factor $\alpha > 1$*.*

According to the Theorem, the MAC exhibits increasing returns to scale if Inequality (21) holds and the initial group is large enough. The left hand side of (21) is the average discount factor among those who do not cooperate fully. The right hand side is the same threshold condition as in (19) which determines whether an individual's marginal response to scale is positive or negative. Hence, (21) requires that the conditional average discount factor of the group is above this threshold.

Significantly, the Inequality (21) fails when discount factors are identical. In that case, recall that the MAC is given by (6) which is invariant to size. One can check in that case that the left and right hand sides in (21) are equal, and so the marginal response to scale of every member is zero. Figure 4 exhibits returns to scale in the 2-member group. Constant returns occur in the "full cooperation" region and along the 45 degree line where discount factors are identical. Increasing returns occur where (21) holds. Decreasing returns occur where the threshold condition for both individuals fail or where it fails for one individual and the other is no longer an "interior" cooperator.

The proof combines the previous Theorem 3 with a convexity argument. Roughly, we prove that an individual's marginal response to scale is convex. A simple Jensen's Inequality argument establishes that if the conditional average discount factor exceeds the threshold

$p^*(n,\delta)\mu+[1-p^*(n,\delta)\ ]\gamma$ then the group's marginal response to a change in scale is positive. Hence, the MAC exhibits returns to scale.

To get a better idea for why result indicates increasing rather than decreasing returns, consider two individuals, with discount factors δ_H, δ_L, respectively, with $\delta_H > \delta_L$. The high or patient type has a discount factor above the threshold: $\delta_H > p^*(n,\delta)\mu + [1 - p^*(n,\delta)\ ]\gamma$. The impatient type lies below it. If the incentive constraint in the "PD" game in (16) binds with equality, then it is not hard to show

$$p_i \;=\; \frac{\delta_i}{\mu + \gamma\frac{(n-1)-P_{-i}}{P_{-i}}} \tag{22}$$

A simple inspection of (22) reveals that p_i increases in one's own discount factor, and decreases in the ratio of uncooperative to cooperative behavior of others, $(n-1-P_{-i})/P_{-i}$. In this two person case, this ratio is $(1-p_L)/p_L$ for the patient player and is faces $(1-p_H)/p_H$ for the impatient player.

If the group size, say, doubles, then the patient individual faces another patient type as well as two impatient types. The ratio of uncooperative to cooperative behavior becomes $(3-2p_L-p_H)/(2p_L+p_H)$ which is smaller than before. Hence, the patient player responds positively. The opposite is true of the impatient player who responds negatively in equilibrium. Returns to scale exist if the positive response of the patient player exceeds the negative response of the impatient one. This turns out to be the case. The reason is simple. The incentive constraints measure current gains to deviating against future losses from punishment in the equilibrium continuation. The cooperation ratio $(n-1-P_{-i})/P_{-i}$ is used to determine each member's discounted average payoff in equilibrium. An impatient player is less responsive to changes in this ratio because he is less responsive to the changes in the resulting equilibrium path. Hence, although his cooperation ratio is inferior when the group size doubles, his response (as reflected in (22)) is ultimately dampened by his impatience.

7 Summary and Discussion

This paper defines and characterizes maximal average cooperation (MAC) for a group of individuals in a class of repeated, collective action games. Because the equilibrium from which the MAC is calculated is stationary, any changes in the MAC are presumably due only to changes in the equilibrium set. Hence, performance of groups of differing size and composition may be compared. The results show that homogeneous groups are, ceteris parabis, more cooperative, and for certain stage games, larger groups are more cooperative as well.

The results do not, of course, imply that social planners should construct purely homogeneous groups. Nor should they attempt to form groups of arbitrarily large size. Returns to homogeneity may be balanced by benefits to diversity along other dimensions. Returns to size may be balanced by increasing costs of retaining individuals in larger groups. Moreover, the assumption of perfect monitoring in the model is probably harder to justify with larger groups. Depending on the game, imperfect monitoring may bound the group away from the Pareto frontier. Finally, it may not be possible to keep constant the distribution of types when adding to the group. Our results say little about how mergers of differently composed groups perform.

Interest in group performance in collective action problems date back at least to Olson (1965). The ensuing literature comparing the size of a group with the degree of cooperation in *static* collective action games is vast. See Sandler (1992) or Marwell and Oliver (1993) for surveys. Pecorino (1999) takes up the issue of group size in repeated games with uniform discounting. He calculates the threshold δ required for full cooperation. Though the threshold increases in n, Pecorino finds that for certain quasi-linear payoffs, as $n \to \infty$ the limiting threshold δ is less than one in the limit. Consequently, full cooperation is sustainable for arbitrarily large groups.

There are also studies that find, as in the present paper, reversals of the Olson Conjecture. These include Esteban and Ray (2001), McGuire (1974), and Chamberlin (1974). They show that an increase in group size may increase the total provision of the public good, even though individual contributions may fall. The present model shows something stronger. Namely, an increase in group size may lead to increase in the *average* contribution.

It is worth emphasizing that it is the *repeated* nature of the problem, coupled with heterogeneous discounting, which accounts for this increase. Our paper provides an interesting contrast with static Bayesian environments when the number of agents goes to infinity. When valuations are privately observed, well known results of Rob (1989) and Mailath and Postlewaite (1990) show that the likelihood of efficient provision of a public project decreases (and goes to zero) as the number of agents increases. In these models, each player becomes informationally small as the number of players increase, but not small enough relative declining likelihood of being pivotal in determining the outcome. In the present model, the repeated game keeps each player pivotal even as n increases. Any player who deviates in our model can shift the continuation payoff for everyone in a discontinuous way. Whether our results extend more broadly is an open question.

8 Appendix

8.1 Proofs of Lemma 1 and Lemma 2

Proof of Lemma 1 We prove that Q is decreasing in p_i. Observe that by Monotonicity (A2), $\pi(0, P - p_i)$ is strictly decreasing in p_i. Hence, it suffices to show that $[\pi(p_i, P) - \pi(0, P - p_i)]$ is strictly decreasing in p_i. Let $\tilde{p}_i > p_i$. Observe that

$$\begin{aligned} \pi(\tilde{p}_i, P) - \pi(p_i, P) \ &\leq \ \pi(\tilde{p}_i, P - p_i) - \pi(p_i, P - p_i) \ \leq \pi(\tilde{p}_i - p_i, P - p_i) - \pi(0, P - p_i) \\ < \ \pi(\tilde{p}_i - p_i - (\tilde{p}_i - p_i), P - p_i - (\tilde{p}_i - p_i)) - \pi(0, P - p_i) \ &= \pi(0, P - \tilde{p}_i) - \pi(0, P - p_i) \end{aligned} \tag{23}$$

Here, the first inequalities follows from Submodularity (A4). The second inequality follows from Concavity (A3). The third from (joint) Monotonicity (A2). Re-arranging terms gives

$$\pi(\tilde{p}_i, P) - \pi(0, P - \tilde{p}_i) < \pi(p_i, P) - \pi(0, P - p_i)$$

which gives the desired monotonicity result. ∎

Proof of Lemma 2 We first verify (11) if $P > 1$. Notice that (11) is just the incentive constraint for $p_i = 1$ evaluated at equality and without restricting r to be in $[0, 1)$. By Lemma 1, $Q(p_i, P; r) \geq 0$ for all p_i if and only if $Q(1, P; r) \geq 0$. Hence, $\underline{\delta}(P)$ is the infimum among r that satisfies $Q(1, P; r) \geq 0$. In particular, since for all such r,

$$Q(1, P; r) \equiv \pi(1, P) - (1 - r)\pi(0, P - 1) \geq 0$$

and since $\pi(1, P) < \pi(0, P - 1)$ by Monotonicity, Equation (11) holds for $r = \underline{\delta}(P)$.

Next, note that by construction of $\underline{\delta}(P)$ in (10), if $\delta_i > \underline{\delta}(P)$, then $Q(1, P; \delta_i) > 0$.

Finally, if $\delta_i \leq \underline{\delta}(P)$, then $Q(\min\{1, P\}, P; \delta_i) \leq 0$. If this were not the case, i.e., if $Q(1, P; \delta_i) > 0$ for some $\delta_i \leq \underline{\delta}(P)$, then since Q is increasing it follows that $Q(\min\{1, P\}, P; \underline{\delta}(P)) > 0$. In that case, $\underline{\delta}(P)$ could not be the infimum as defined in (10). We conclude that $Q(\min\{1, P\}, P; \delta_i) \quad \leq 0$ for $\delta_i \leq \underline{\delta}(P)$. Since $Q(0, P; \delta_i) > 0$ the Intermediate Value Theorem implies the existence of a $p \in [0, \min\{1, P\}]$ such that $Q(p, P; \delta_i) = 0$. ∎

8.2 Proof of Theorem 1

Fix δ. To simplify notation for the proof, we define $R_i(P) = R(P; \delta_i)$, $\psi_i(P) = \psi(P; \delta_i)$, and $\Psi(P) = \Psi(P; \delta)$. First, we show that ψ_i is continuous. Using the previous Lemmatta, observe, first, that $\underline{\delta}(P) = \infty$ for all $P \in [0, 1]$ and $\underline{\delta}(\cdot)$ is a decreasing, continuous function on $(1, n]$. Consequently, $\delta_i \leq \underline{\delta}(P)$ for all $P \in [0, P'_i]$ where $\underline{\delta}(P'_i) = \delta_i$. Note that $P'_i > 1$. By the Implicit Function Theorem, the implicit function R_i is therefore continuous on $P \in [0, P'_i]$. Observe also that $R_i(0) = 0$ and $R_i(P'_i) = 1$. Finally, since $\psi_i(P) = 1$ for all $P > P'_i$ (i.e., $\delta_i > \underline{\delta}(P)$), we have established that ψ is continuous.

Given the continuity of ψ for all i, it is clear that Ψ is continuous. Hence, By Brouwer's Theorem, Ψ has a fixed point. Now observe that the set of fixed points corresponds to the intersection of the graph of Ψ with the graph of the 45 degree line. Since both are closed sets in $[0, n]^2$, the graph of the set of fixed points is a closed subset of $[0, n]^2$ and so is compact. Hence, the maximal fixed point exists.

We now show that the MAC p^* coincides with $\hat{p}$ such that $n\hat{p}$ is the maximal fixed point of Ψ. Let $P^* = np^*$ and $\hat{P} = n\hat{p}$. We show that $P^* = \hat{P}$. Suppose first that $P^* < \hat{P}$. But since $\hat{P} = \Psi(\hat{P})$, there exist $\hat{p}_1, \ldots, \hat{p}_n$ such that $\hat{P} = \sum_i \hat{p}_i$ and $Q(\hat{p}_i, \hat{P}; \delta_i) \geq 0$. Hence, $\hat{p}_1, \ldots, \hat{p}_n$ all satisfy incentive constraints and achieve a higher aggregate (hence average) level of cooperation. But this contradicts that fact that p^* is the MAC.

Consider, instead, that $P^* > \hat{P}$. Since $\hat{P} < n$ we assert that $\Psi(P) < P$ for all $P > \hat{P}$. Suppose this were not the case. Then $\Psi(P') \geq P'$ for some $P' > P$. But since then the map Ψ restricted to $[P', n]$ satisfies the assumptions of Brouwer's Theorem, there is a fixed point $P'' \in [P', n]$ which contradicts the maximality of $\hat{P}$. Hence, $\Psi(P) < P$ for all $P > \hat{P}$, and, in particular, $\Psi(P^*) < P^*$. Then it holds that for at least one individual i, $\psi_i(P^*) < p_i^*$. Given the definition of ψ_i, either $\psi_i(P^*)$ is the root of $Q(\cdot, P^*; \delta_i) = 0$ or $\psi_i(P^*) = 1$ and no root in $[0, 1]$ exists. In the case of the former, since Q is decreasing in p_i, we have $Q(\psi(P^*), P^*; \delta_i) = 0 > Q(p_i^*, P^*; \delta_i)$. But this implies that p_i^* violates the incentive constraint, a clear contradiction. In the case of the latter, $\psi_i(P^*) = 1 < p_i^*$. This is a contradiction: since no root exists, p_i^* violates incentive constraints in this case as well.

We conclude that $P^* = \hat{P}$. ∎

8.3 Proof of Theorem 2

Write $\Psi(P, n; \delta)$ to express the explicit dependence on size n and structure, δ. By standard results for second order stochastic dominance, it suffices to show that for each i, ψ is increasing and concave in δ_i.

Fix i. We now show that ψ_i is increasing and weakly concave in δ_i for each P. In the range of δ_i such that $\delta_i \geq \underline{\delta}(P)$, then $\psi_i(P, \delta_i) = 1$, and so weak concavity holds trivially.

Consider then the range of δ_i such that $0 \leq \delta_i < \underline{\delta}(P)$. In this case, $\psi(P; \delta_i) = R(P, \delta_i)$ where, as before, $R(P, \delta_i)$ is the p_i-root of $Q(p_i, P; \delta_i) = 0$ which is known by previous arguments to uniquely exist in this range. That is,

$$Q(\psi(P,\delta_i),P;\delta_i) = -(1-\delta_i)\pi(0,P-\psi(P,\delta_i)) + \pi(\psi(P,\delta_i),P) = 0$$

Consider $P > 0$ which, combined with the supposition $0 \leq \delta_i < \underline{\delta}(P)$ implies $0 < \psi(P,\delta_i) < \min\{1,P\}$. One can solve for δ_i in this case, yielding

$$\delta_i = 1 - \frac{\pi(\psi(P,\delta_i),P)}{\pi(0,P-\psi(P,\delta_i))}$$

Implicitly differentiating this equation with respect to δ_i yields

$$1 = -\frac{\pi(0,P-\psi(P,\delta_i))D_1\pi(\psi(P,\delta_i),P)\frac{\partial\psi}{\partial\delta_i} + \pi(\psi(P,\delta_i),P)D_2\pi(0,P-\psi(P,\delta_i))\frac{\partial\psi}{\partial\delta_i}}{[\pi(0,P-\psi(P,\delta_i))]^2}$$

where $D_k\pi$ is the partial derivative with respect to the $k = 1,2$ argument. Solving for $\frac{\partial\psi}{\partial\delta_i}$ yields

$$\frac{\partial\psi}{\partial\delta_i} = \frac{-[\pi(0,P-\psi(P,\delta_i))]^2}{\pi(0,P-\psi(P,\delta_i))D_1\pi(\psi(P,\delta_i),P) + \pi(\psi(P,\delta_i),P)D_2\pi(0,P-\psi(P,\delta_i))} \tag{24}$$

The denominator satisfies

$$\begin{aligned} & \pi(0,P-\psi(P,\delta_i))D_1\pi(\psi(P,\delta_i),P) + \pi(\psi(P,\delta_i),P)D_2\pi(0,P-\psi(P,\delta_i)) \\ < \; & \pi(0,P-\psi(P,\delta_i))D_1\pi(0,P-\psi(P,\delta_i)) + \pi(0,P-\psi(P,\delta_i))D_2\pi(0,P-\psi(P,\delta_i)) \\ < \; & 0 \end{aligned}$$

where, the first inequality follows by Concavity and Submodularity; the second follows by Monotonicity. Hence, $\frac{\partial\psi}{\partial\delta_i} > 0$.

Next, we show that $\frac{\partial\psi}{\partial\delta_i}$ is decreasing in δ_i. Since ψ_i is increasing in δ_i, the Inverse Function Theorem implies (using Equation (24)),

$$\frac{\partial\delta_i}{\partial p_i} = \frac{-[\pi(0,P-p_i)D_1\pi(p_i,P) + \pi(p_i,P)D_2\pi(0,P-p_i)]}{[\pi(0,P-p_i)]^2} \tag{25}$$

Hence, it suffices to show $\frac{\partial\delta_i}{\partial p_i}$ is increasing in p_i (i.e., the inverse function $\delta_i(\cdot)$ is convex). Now rewrite Equation (25) as

$$-\frac{D_1\pi(p_i,P)}{\pi(0,P-p_i)} \quad + \quad \frac{\pi(p_i,P)}{\pi(0,P-p_i)}\,\frac{d}{dp_i}\log\pi(0,P-p_i) \tag{26}$$

Let

$$F(p_i) = -\frac{D_1\pi(p_i, P)}{\pi(0, P-p_i)}, \quad G(p_i) = \frac{\pi(p_i, P)}{\pi(0, P-p_i)}, \quad \text{and} \quad H(p_i) = \frac{d}{dp_i}\log\pi(0, P-p_i).$$

We must show

$$F' + GH' + G'H > 0.$$

Observe that $F'(p_i) > 0$ iff $\pi(0, P-p_i)D_{11}\pi(p_i, P) + D_2\pi(0, P-p_i)D_1\pi(p_i, P) < 0$. But the latter is easily verified using the Concavity and Monotonicity Axioms.

Next, observe that $G(p_i) > 0$ and, by Concavity, $H'(p_i) > 0$. By monotonicity of the $\log\pi$ in its second argument, $H(p_i) < 0$. Hence, it suffices to show $G'(p_i) < 0$, or, in other words, $G(p_i) = \frac{\pi(p_i,P)}{\pi(0,P-p_i)}$ is decreasing in p_i. But this holds directly by the Monotonicity Axiom. Hence, we have established that ψ_i is increasing and concave in δ_i. ∎

8.4 Proof of Theorem 3

The incentive constraint may be expressed in this case as

$$Q_i(p_i) \;\equiv\; (\mu-\gamma)\,p_i^2 \;-\; (P(\mu-\gamma)+\gamma(n-1)+\delta_i)\,p_i \;+\; P\delta_i \;\geq\; 0 \tag{27}$$

where $\mu = \frac{d-c}{d}$ and $\gamma = \frac{\ell}{d}$. The following Lemma will prove helpful for the remainder of the proof.

Lemma 3 *For each i, ψ is given by*

$$\psi(P;\delta_i) = p_i = \begin{cases} R(P;\delta_i) & if \quad \delta_i \;<\; \mu + \gamma\frac{n-P}{P-1} \text{ or if } P < 1 \\ 1 & if \qquad\qquad otherwise \end{cases} \tag{28}$$

where $R(P;\delta_i)$ is one of the two real roots of the quadratic function Q_i in Expression (27) and is given by

$$R(P;\delta_i) = \frac{P(\mu-\gamma)+\gamma(n-1)+\delta_i - \sqrt{(P(\mu-\gamma)+\gamma(n-1)+\delta_i)^2 - 4(\mu-\gamma)P\delta_i}}{2(\mu-\gamma)}$$

Proof of Lemma 3 Fix any $P > 0$. Observe that the function Q is a quadratic in p_i of the form,

$$Ap_i^2 + Bp_i + C \geq 0 \tag{29}$$

where $A = \mu - \gamma$, $B = -[P(\mu - \gamma) + \gamma(n-1) + \delta_i]$ and $C = P\delta_i > 0$. It is easily verified that the roots of (29) are real.[15]

Without loss of generality, consider the case in which $p_i > 0$, it follows that $P > 0$, and so $C > 0$. Then $Q(0, P; \delta_i) = C > 0$. To begin, let $P > 1$.

Since $\mu > \gamma$ we have $A > 0$ and $B < 0$. $Q'(p_i, P; \delta_i) = 2Ap_i + B$ and therefore $Q'(0, P; \delta_i) < 0$. Since $A > 0$ we have two positive real roots. We also know by the previous results that p_i must be the maximum value in $[0, 1]$ consistent with $Q(p_i) \geq 0$. That is,

$$p_i = \max\{p'_i \in [0,1] : \ Q(p'_i, P; \delta_i) \geq 0\}$$

Then $p_i < 1$ is a solution iff the lower root of Q is less than 1 while the greater root of Q is greater than 1. In other words:

$$\frac{-B - \sqrt{B^2 - 4AC}}{2A} < 1 < \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

Since $A > 0$, we can write this with some algebra as $A + B + C < 0$, or, equivalently, $(\mu - \gamma) - (P(\mu - \gamma) + \gamma(n-1) + \delta_i) + P\delta_i < 0$. Solving for δ_i, $\delta_i < \mu - \gamma + \gamma\frac{n-1}{P-1}$, and, hence, $\delta_i < \frac{d-c}{d} + \frac{\ell}{d}\frac{n-P}{P-1}$. Therefore, $p_i = R(P; \delta_i)$ whenever $\delta_i < \frac{d-c}{d} + \frac{\ell}{d}\frac{n-P}{P-1}$. This concludes the case of $P > 1$.

Finally, we note that when $P \leq 1$, the constraint that binds is no longer that $p_i \leq 1$, but rather that $p_i \leq P$. ∎

The Rest of the Proof From Lemma 3, $\psi(\alpha P^*_n, \alpha n; \delta_i) = 1$ iff $\delta_i \geq \mu + \gamma\frac{\alpha(n-P^*_n)}{\alpha P^*_n - 1}$.

Notice that $\mu + \gamma\frac{\alpha(n-P^*_n)}{\alpha P^*_n - 1} > \frac{P^*_n}{n}\mu + (1 - \frac{P^*_n}{n})\gamma$. Hence, it is without loss of generality if we consider δ_i such that

$$\frac{P^*_n}{n}\mu + (1 - \frac{P^*_n}{n})\gamma \ < \ \delta_i < \mu + \gamma\frac{\alpha(n - P^*_n)}{\alpha P^*_n - 1}$$

Then $\psi(\alpha P^*_n, \alpha n, \delta_i) < 1$ so that

$$\frac{\partial \psi(\alpha P^*_n, \alpha n; \delta_i)}{\partial \alpha} = \frac{\partial R(\alpha P^*_n, \alpha n; \delta_i)}{\partial \alpha}$$

[15] See the Working paper version, Lagunoff and Haag (2005b) for details.

where $R(\cdot)$ is the quadratic root defined above. Holding fixed δ_i, we have, for any P,

$$\begin{aligned} R^*(P,n,\alpha) &\equiv R(\alpha P_n, \alpha n; \delta_i) \\ &= \frac{\alpha P(\mu-\gamma)+\gamma(\alpha n-1)+\delta_i-\sqrt{(\alpha P(\mu-\gamma)+\gamma(\alpha n-1)+\delta_i)^2-4(\mu-\gamma)\alpha P\delta_i}}{2(\mu-\gamma)} \\ &\equiv \tfrac{-B(\alpha)}{A}-\tfrac{\sqrt{B(\alpha)^2-4AC(\alpha)}}{A} \end{aligned}$$

Substituting for A, B, and C and differentiating with respect to α, one can verify that $\frac{\partial R^*}{\partial \alpha}>0$ iff

$$\delta_i > \frac{P}{n}(\mu-\gamma)+\gamma$$

which is the requisite threshold condition whenever P satisfies $\frac{P}{n}=p^*(n,\delta)$. ∎

8.5 Proof of Theorem 4

The proof is in several steps. First, we show that $\frac{\partial\psi^3(\alpha P^n,\alpha n;\delta_i)}{\partial\alpha\partial\delta_i^2}>0, \forall\alpha>1$. In other words, the marginal response $\partial\psi/\partial\alpha$ is convex in δ_i. Then, using Jensen's inequality, we can show that if the conditional average of discount factors is above the threshold given, there are returns to scale. Using the definitions of A, B, and C in previous proofs, then $\frac{\partial B}{\partial\delta_i}=-1$. Hence,

$$\begin{aligned} \frac{\partial^3\psi}{\partial\delta_i^2\partial\alpha} &= -3(B^2-4AC)^{-\frac{5}{2}}\left(2B\tfrac{\partial B}{\partial\alpha}-4A\tfrac{\partial C}{\partial\alpha}\right)\left[B\tfrac{\partial C}{\partial\delta_i}+A\left(\tfrac{\partial C}{\partial\delta_i}\right)^2+C\right] \\ &\quad +2(B^2-4AC)^{-\frac{3}{2}}\left[\frac{\partial B}{\partial\alpha}\frac{\partial C}{\partial\delta_i}+B\frac{\partial C}{\partial\delta_i\partial\alpha}+2A\frac{\partial C}{\partial\delta_i}\frac{\partial}{\partial\delta_i\partial\alpha}+\frac{\partial C}{\partial\alpha}\right] \end{aligned} \tag{30}$$

By Young's Theorem, we know that $\dfrac{\partial^3\psi}{\partial\delta^2\partial\alpha}=\dfrac{\partial^3\psi}{\partial\alpha\partial\delta_i^2}$. We wish to verify conditions under which $\dfrac{\partial^3\psi}{\partial\alpha\partial\delta_i^2}>0$. Using Equation (30), $\dfrac{\partial^3\psi}{\partial\alpha\partial\delta_i^2}>0$ iff

$$\begin{aligned} &(B^2-4AC)\left[\frac{\partial B}{\partial\alpha}\frac{\partial C}{\partial\delta_i}+B\frac{\partial C}{\partial\delta_i\partial\alpha}+2A\frac{\partial C}{\partial\delta_i}\frac{\partial}{\partial\delta_i\partial\alpha}+\frac{\partial C}{\partial\alpha}\right] \\ > \;& 3\left(B\frac{\partial B}{\partial\alpha}-2A\frac{\partial C}{\partial\alpha}\right)\left[B\frac{\partial C}{\partial\delta_i}+A\left(\frac{\partial C}{\partial\delta_i}\right)^2+C\right] \end{aligned}$$

After substituting in for $A = \mu - \gamma$, $B = -(\alpha P(\mu - \gamma) + \alpha\gamma n + \delta_i - \gamma)$, $C = \alpha P \delta_i$ and $\frac{\partial B}{\partial \alpha} = -(P(\mu - \gamma) + \gamma n)$, $\frac{\partial C}{\partial \delta_i} = \alpha P$, $\frac{\partial C}{\partial \delta \partial \alpha} = P$, $\frac{\partial C}{\partial \alpha} = P\delta_i$, and noting that $(\delta_i - \gamma)^2 < \gamma^2$ and $\mu > \gamma$ we derive, after some algebra,[16]

$$\gamma > \frac{\alpha n}{(\alpha n - 1)^2}$$

The derivative with the right-hand side with respect to α is negative for $\alpha \geq 1$, thus we can take as a sufficient condition that $\alpha = 1$ and thus $\dfrac{\partial \psi}{\partial \alpha}$ is convex if $\gamma > \dfrac{n}{(n-1)^2}$.

Using Jensen's Inequality and the threshold conditional average condition, it follows that

$$\begin{aligned}
\frac{\partial \Psi(\alpha P_n^*, \alpha n, \delta(n))}{\partial \alpha} &\equiv \frac{1}{n}\sum_{i=1}^{n} \frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha} \\
= \frac{1}{n}\left[\sum_{i=1}^{i^*} \frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha} + \sum_{i=i^*+1}^{n} \frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha}\right] &= \frac{1}{n}\sum_{i=1}^{i^*} \frac{\partial \psi(\alpha P_n^*, \alpha n, \delta_i)}{\partial \alpha} + 0 \\
> \frac{\partial \psi(\alpha P_n^*, \alpha n, \frac{1}{n}\sum_{i=1}^{n} \delta_i)}{\partial \alpha} &> 0
\end{aligned} \tag{31}$$

The first inequality is a straightforward application of Jensen's Inequality. The second follows from Theorem 3. Consequently, $\frac{\partial \Psi(\alpha P_n^*, \alpha n; \delta(n))}{\partial \alpha} > 0$, and because Ψ is increasing in α, it follows that

$$\Psi(\alpha P_n^*, \alpha n; \delta(n)) > \Psi(\alpha P_n^*, \alpha n, \delta(n)) = P_n^*$$

Now letting α be an integer,

$$\begin{aligned}
0 < \alpha\left(\Psi(\alpha P_n^*, \alpha n, \delta(n)) - P_n^*\right) &= \alpha \Psi(\alpha P_n^*, \alpha n, \delta(n)) - \alpha P_n^* \\
= \alpha \sum_{i=1}^{n} \psi(\alpha P_n^*, \alpha n, \delta_i) - \alpha P_n^* &= \sum_{j=1}^{\alpha n} \psi(\alpha P_n^*, \alpha n, \delta_j) - \alpha P_n^* \\
= \Psi(\alpha P_n^*, \alpha n, \delta(\alpha n)) - \alpha P_n^*
\end{aligned}$$

Now since $P_{\alpha n}^*$ is the maximal aggregate of the αn-group, it follows that $\Psi(P_{\alpha n}^*, \alpha n, \delta(\alpha n)) = P_{\alpha n}^*$. Since Ψ is increasing and concave in P (see Proof of Theorem 1), the difference $\Psi(P, \alpha n, \delta(\alpha n)) - P$ must be decreasing in P. Therefore, $P_{\alpha n}^* > \alpha P_n^*$ which means that the MAC exhibits increasing returns. We conclude the proof. ∎

[16] Again, see Haag and Lagunoff (2005b) for details.

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