Solving General Equilibrium Models with Incomplete Markets and Many Financial Assets

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Martin D. D. Evans[1]
Georgetown University and NBER
Departments of Economics and Finance
Washington DC 20057
Tel: (202) 338-2991
evansm1@georgetown.edu

Viktoria Hnatkovska
UBC
Department of Economics
Vancouver, BC V6T 1Z1
Tel: (604) 822-5941
hnatkovs@interchange.ubc.ca

Abstract

This paper presents a new numerical method for solving stochastic general equilibrium models with dynamic portfolio choice over many financial assets. The method can be applied to models where there are heterogeneous agents, time-varying investment opportunity sets, and incomplete asset markets. We illustrate how the method is used by solving two versions of a two-country general equilibrium model with production and dynamic portfolio choice. We check the accuracy of our method by comparing the numerical solution to a complete markets version of the model against its known analytic properties. We then apply the method to an incomplete markets version where no analytic solution is available. In both models the standard accuracy tests confirm the effectiveness of our method.

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Introduction

This paper presents a new numerical method for solving dynamic stochastic general equilibrium (DSGE) models with dynamic portfolio choice over many financial assets. The method can be applied to models where there are heterogeneous agents, time-varying investment opportunity sets, and incomplete asset markets. As such, our method can be used to solve models that analyze an array of important issues in international macroeconomics and finance. For example, questions concerning the role of revaluation effects in the process of external adjustment cannot be fully addressed without a model that incorporates the dynamic portfolio choices of home and foreign agents across multiple financial assets. Similarly, any theoretical assessment of the implications of greater international financial integration requires a model in which improved access to an array of financial markets has real effects; through capital deepening and/or improved risk sharing (because markets are incomplete). Indeed, there is an emerging consensus among researchers that the class of DSGE models in current use needs to be extended to include dynamic portfolio choice and incomplete markets (see, for example, Obstfeld 2004, and Gourinchas 2006). This paper shows how an accurate approximation to the equilibrium in such models can be derived.

We illustrate the use of our solution method by solving two versions of a canonical two-country DSGE model. The full version of the model includes production, traded and nontraded goods, and an array of equity and bond markets. Households choose between multiple assets as part of their optimal consumption and saving decisions, but only have access to a subset of the world's financial markets. As a result, there is both dynamic portfolio choice and incomplete risk-sharing in the equilibrium. We also study the equilibrium in a simplified version of the model without nontraded goods. Here households still face a dynamic portfolio choice problem but the available array of financial assets is sufficient for complete risk-sharing. We use the two versions of our model to illustrate how well our solution method works in complete and incomplete market settings. In particular, we present several tests to show that our approximations to both sets of equilibrium dynamics are very accurate.

The presence of portfolio choice and incomplete markets in a DSGE model gives rise to a number of problems that must be addressed by any solution method. First, and foremost, the method must address the complex interactions between the real and financial sides of the economy. One the one hand, portfolio decisions affect the degree of risk-sharing which in turn affects equilibrium real allocations. On the other, real allocations affect the behavior of returns via their implications for market-clearing prices, which in turn affect portfolio choices. Second, we need to track the distribution of households' financial wealth in order to account for the wealth effects that arise when risk-sharing is incomplete. This adds to the number of state variables needed to characterize the equilibrium dynamics of the economy and hence increases the complexity of finding the equilibrium. Third, it is well-known that transitory shocks can have very persistent effects on the distribution of financial wealth when markets are incomplete. The presence of such persistence should not impair the accuracy of the proposed approximation to the model's equilibrium. Our solution method addresses all these problems.

The method we propose combines a perturbation technique commonly used in solving macro models with continuous-time approximations common in solving finance models of portfolio choice. In so doing, we contribute to the literature along several dimensions. First, relative to the finance literature, our method delivers optimal portfolios in a discrete-time general equilibrium setting in which returns are endogenously determined. It also enables us to characterize the dynamics of returns and the stochastic investment opportunity set as functions of macroeconomic state variables.[2] Second, relative to the macroeconomics literature, portfolio decisions are derived without assuming complete asset markets or constant returns to scale in production.[3]

Recent papers by Devereux and Sutherland (2006a,b) and Tille and van Wincoop (2006) have proposed an alternative method for solving DSGE models with portfolio choice and incomplete markets.[4] Two key features differentiate their approach from the one we propose. First, their method requires at least third-order approximations to some of the model's equilibrium conditions in order to identify variations in the portfolio holdings. By contrast, we are able to accurately characterize optimal portfolio holdings from second-order approximations of the equilibrium conditions. This difference is important when it comes to solving models with a large state space (i.e. a large number of state variables). We have applied our method to models with 8 state variables and 10 decision variables (see Evans and Hnatkovska 2006). Second, we characterize the consumption and portfolio problem facing households using the approximations developed by John Campbell and his co-authors over the past decade. These approximations differ from those commonly used in solving DSGE models without portfolio choice, but they have proved very useful in characterizing intertemporal financial decision-making (see, for example, Campbell and Viceira, 2002). In particular, they provide simple closed-form expressions for portfolio holdings that are useful in identifying the role of different economic factors. In this sense, our approach can be viewed as an extension of the existing literature on dynamic portfolio choice to a general equilibrium setting.

The paper is structured as follows. Section 1 presents the model we use to illustrate our solution method. Section 2 describes the solution method in detail. Section 3 provides a step-by-step description of how the method is applied to our illustrative model. We present results on the accuracy of the solutions to both versions of our model in Section 4. Section 5 concludes.

1 The Model

This section describes the discrete-time DSGE model we employ to illustrate our solution method. Our starting point is a standard international asset pricing model with production, which we extend to incorporate dynamic portfolio choice over equities and an international bond. A frictionless production world economy in

[2] A number of approximate solution methods have been developed in partial equilibrium frameworks. Kogan and Uppal (2000) approximate portfolio and consumption allocations around the solution for a log-investor. Barberis (2000), Brennan, Schwartz, and Lagnado (1997) use discrete-state approximations. Brandt, Goyal, and Santa-Clara (2001) solve for portfolio policies by applying dynamic programming to an approximated simulated model. Brandt and Santa-Clara (2004) expand the asset space to include asset portfolios and then solve for the optimal portfolio choice in the resulting static model.

[3] Solutions to portfolio problems with complete markets are developed in Heathcote and Perri (2004), Serrat (2001), Kollmann (2005), Baxter, Jermann and King (1998), Uppal (1993), Engel and Matsumoto (2004). Pesenti and van Wincoop (1996) analyze equilibrium portfolios in a partial equilibrium setting with incomplete markets.

[4] Ghironi, Lee and Rebucci (2007) also develop and analyze a model with portfolio choice and incomplete asset markets. To compute the steady state asset allocations they introduce financial transaction fees. In our frictionless model portfolio holdings are derived endogenously using the conditional distributions of asset returns.

this model consists of two symmetric countries, called Home (H) and Foreign (F). Each country is populated by a continuum of identical households who consume and invest in different assets, and a continuum of firms that are split between the traded and nontraded goods' sectors. Firms are infinitely-lived, perfectly competitive, and issue equity claims to their dividend streams.

1.1 Firms

We shall refer to firms in the traded and nontraded sectors as "traded" and "nontraded". A representative traded firm in country H starts period t with a stock of firm-specific capital K_t. Period-t production is $Y_t = Z_t^{\mathrm{T}} K_t^{\theta}$ with $\theta > 0$, and Z_t^{T} denotes the current state of productivity. The output produced by traded firms in country F, $\hat{Y}_t$, is given by an identical production function using firm-specific foreign capital, $\hat{K}_t$, and productivity, $\hat{Z}_t^{\mathrm{T}}$. (Hereafter we use "$\hat{\ }$" to denote foreign variables.) The goods produced by H and F traded firms are identical and can be costlessly transported between countries. Under these conditions, the law of one price prevails in the traded sector to eliminate arbitrage opportunities.

At the beginning of period t, each traded firm observes the productivity realization, produces output, and uses the proceeds to finance investment and to pay dividends to its shareholders. We assume that firms allocate output to maximize the value of the firm to its domestic shareholders every period. If the total number of outstanding shares is normalized to unity, the optimization problem facing a traded firm in country H can be summarized as

$$\max_{I_t} \mathbb{E}_t \sum_{i=0}^{\infty} M_{t+i,t}\ D_{t+i}^{\mathrm{T}}, \tag{1}$$

subject to

$$K_{t+1} = (1-\delta)K_t + I_t, \tag{2}$$

$$D_t^{\mathrm{T}} = Z_t^{\mathrm{T}} K_t^{\theta} - I_t, \tag{3}$$

where D_t^{T} is the dividend per share paid at t, I_t is real investment and $\delta > 0$ is the depreciation rate on physical capital. $\mathbb{E}_t$ denotes expectations conditioned on information at the start of period t. $M_{t+i,t}$ is the intertemporal marginal rate of substitution (IMRS) between consumption of tradables in period t and period $t+i$ of domestic households, and $M_{t,t} = 1$.[5] The representative traded firm in country F solves an analogous problem: It chooses investment, $\hat{I}_t$, to maximize the present discounted value of foreign dividends per share, $\hat{D}_t^{\mathrm{T}}$, using $\hat{M}_{t+i,t}$, the IMRS of F households.

The output of nontraded firms in countries H and F is given by $Y_t^{\mathrm{N}} = \eta Z_t^{\mathrm{N}}$ and $\hat{Y}_t^{\mathrm{N}} = \eta \hat{Z}_t^{\mathrm{N}}$ respectively, where $\eta > 0$ is a constant. Nontraded firms have no investment decisions to make; they simply pass on sales revenue as dividends to their shareholders. Z_t^{N} and $\hat{Z}_t^{\mathrm{N}}$ denote the period-t state of nontradable productivity in countries H and F, respectively.

[5] Although our specification in (1) is straightforward, we note that it can potentially induce home bias in households' traded equity holdings when markets are incomplete. If the array of assets available to households is insufficient for complete risk-sharing (as will be the case in one of the equilibria we study), the IMRS for H and F households will differ. Under these circumstances, households will prefer the dividend stream chosen by domestic traded firms.

Let $z_t \equiv [\ln Z_t^{\mathrm{T}}, \ln \hat{Z}_t^{\mathrm{T}}, \ln Z_t^{\mathrm{N}}, \ln \hat{Z}_t^{\mathrm{N}}]'$ denote the state of productivity in period t. We assume that the productivity vector, z_t, follows an AR(1) process:

$$z_t = a z_{t-1} + S_e^{1/2} e_t, \tag{4}$$

where a is a 4×4 matrix and e_t is a 4×1 vector of i.i.d. mean zero, unit variance shocks. $S_e^{1/2}$ is a 4×4 matrix of scaling parameters.

1.2 Households

Each country is populated by a continuum of households who have identical preferences over the consumption of traded and nontraded goods. The preferences of a representative household in country H are given by

$$\mathbb{E}_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}^{\mathrm{T}}, C_{t+i}^{\mathrm{N}}), \tag{5}$$

where $0 < \beta < 1$ is the discount factor, and $U(.)$ is a concave sub-utility function defined over the consumption of traded and nontraded goods, C_t^{T} and C_t^{N} :

$$U(C^{\mathrm{T}}, C^{\mathrm{N}}) = \frac{1}{\phi} \ln \left[\lambda_{\mathrm{T}}^{1-\phi} \left(C^{\mathrm{T}}\right)^{\phi} + \lambda_{\mathrm{N}}^{1-\phi} \left(C^{\mathrm{N}}\right)^{\phi} \right],$$

with $\phi < 1$. λ_{T} and λ_{N} are the weights that the household assigns to traded and nontraded consumption, respectively. The elasticity of substitution between the two goods is $(1-\phi)^{-1} > 0$. Preferences for households in country F are identically defined in terms of foreign traded and nontraded consumption, $\hat{C}_t^{\mathrm{T}}$ and $\hat{C}_t^{\mathrm{N}}$. Notice that preferences are not separable across the two goods.

Households can save by holding domestic equities (i.e., traded and nontraded), an international bond, and the equity issued by foreign traded firms. They cannot hold equity issued by foreign nontraded firms. This restriction makes markets incomplete. Let $C_t \equiv C_t^{\mathrm{T}} + Q_t^{\mathrm{N}} C_t^{\mathrm{N}}$ denote total consumption *expenditure,* where Q_t^{N} is the relative price of H nontraded in terms of traded goods (our numeraire). The budget constraint of the representative H household can now be written as

$$W_{t+1} = R_{t+1}^{\mathrm{W}} (W_t - C_t), \tag{6}$$

where W_t is financial wealth and R_{t+1}^{W} is the (gross) return on wealth between period t and $t+1$. This return depends on how the household allocates wealth across the available array of financial assets, and on the realized returns on those assets. In particular,

$$R_{t+1}^{\mathrm{W}} \equiv R_t + \alpha_t^{\mathrm{H}} (R_{t+1}^{\mathrm{H}} - R_t) + \alpha_t^{\mathrm{F}} (R_{t+1}^{\mathrm{F}} - R_t) + \alpha_t^{\mathrm{N}} (R_{t+1}^{\mathrm{N}} - R_t), \tag{7}$$

where α_t^i and α_t^{N} respectively denote the shares of wealth allocated in period t by H households into equity issued by $i = \{\mathrm{H}, \mathrm{F}\}$ traded firms and H nontraded firms. R_t is the risk-free return on bonds, R_{t+1}^{H} and R_{t+1}^{F}

are the returns on equity issued by the H and F traded firms, and R^{N}_{t+1} is the return on equity issued by H nontraded firms. These returns are defined as

$$\begin{aligned} R^{\mathrm{H}}_{t+1} &\equiv (P^{\mathrm{T}}_{t+1} + D^{\mathrm{T}}_{t+1})/P^{\mathrm{T}}_{t}, \qquad R^{\mathrm{F}}_{t+1} \equiv (\hat{P}^{\mathrm{T}}_{t+1} + \hat{D}^{\mathrm{T}}_{t+1})/\hat{P}^{\mathrm{T}}_{t}, && \text{(8a)} \\ R^{\mathrm{N}}_{t+1} &\equiv \{(P^{\mathrm{N}}_{t+1} + D^{\mathrm{N}}_{t+1})/P^{\mathrm{N}}_{t}\}\{Q^{\mathrm{N}}_{t+1}/Q^{\mathrm{N}}_{t}\}, && \text{(8b)} \end{aligned}$$

where P^{T}_t and P^{N}_t are period-t prices of equity issued by traded and nontraded firms in country H and D^{N}_t is the period-t flow of dividends from H nontraded firms. P^{N}_t and D^{N}_t are measured in terms of nontradables. The three portfolio shares $\{\alpha^{\mathrm{H}}_t, \alpha^{\mathrm{F}}_t, \alpha^{\mathrm{N}}_t\}$ are related to the corresponding portfolio holdings $\{A^{\mathrm{H}}_t,\ A^{\mathrm{F}}_t,\ A^{\mathrm{N}}_t\}$ by the identities: $P^{\mathrm{T}}_t A^{\mathrm{H}}_t \equiv \alpha^{\mathrm{H}}_t (W_t - C_t)$, $\hat{P}^{\mathrm{T}}_t A^{\mathrm{F}}_t \equiv \alpha^{\mathrm{F}}_t (W_t - C_t)$ and $Q^{\mathrm{N}}_t P^{\mathrm{N}}_t A^{\mathrm{N}}_t \equiv \alpha^{\mathrm{N}}_t (W_t - C_t)$.

The budget constraint for F households is similarly defined as

$$\hat{W}_{t+1} = \hat{R}^{\mathrm{W}}_{t+1}(\hat{W}_t - \hat{C}_t),$$

with $\hat{C}_t \equiv \hat{C}^{\mathrm{T}}_t + \hat{Q}^{\mathrm{N}}_t \hat{C}^{\mathrm{N}}_t$ and

$$\hat{R}^{\mathrm{W}}_{t+1} = R_t + \hat{\alpha}^{\mathrm{H}}_t (R^{\mathrm{H}}_{t+1} - R_t) + \hat{\alpha}^{\mathrm{F}}_t (R^{\mathrm{F}}_{t+1} - R_t) + \hat{\alpha}^{\mathrm{N}}_t (\hat{R}^{\mathrm{N}}_{t+1} - R_t),$$

where $\hat{\alpha}^{\mathrm{H}}_t$, $\hat{\alpha}^{\mathrm{F}}_t$ and $\hat{\alpha}^{\mathrm{N}}_t$ denote the shares of wealth allocated by F households into H and F country traded equities, and F nontraded equity, respectively.

Households in country H choose how much to consume of traded and nontraded goods, and how to allocate their portfolio between equities and the international bond to maximize expected utility (5) subject to (6) and (7), given current equity and goods prices, and the return on bonds. The optimization problem facing F households is analogous.

1.3 Equilibrium

We now summarize the conditions that characterize the equilibrium in our model. The first-order conditions for the representative H household's problem are

$$\begin{aligned} Q^{\mathrm{N}}_t &= \frac{\partial U/\partial C^{\mathrm{N}}_t}{\partial U/\partial C^{\mathrm{T}}_t}, && \text{(9a)} \\ 1 &= \mathbb{E}_t \left[M_{t+1} R_t \right], && \text{(9b)} \\ 1 &= \mathbb{E}_t \left[M_{t+1} R^{\mathrm{H}}_{t+1} \right], && \text{(9c)} \\ 1 &= \mathbb{E}_t \left[M_{t+1} R^{\mathrm{F}}_{t+1} \right], && \text{(9d)} \\ 1 &= \mathbb{E}_t \left[M_{t+1} R^{\mathrm{N}}_{t+1} \right], && \text{(9e)} \end{aligned}$$

where $M_{t+1} \equiv M_{t+1,t} = \beta \left(\partial U / \partial C^{\mathrm{T}}_{t+1} \right) / \left(\partial U / \partial C^{\mathrm{T}}_{t} \right)$ is the IMRS between traded consumption in period t and period $t+1$. The first-order condition associated with the H traded firm's optimization problem is

$$1 = \mathbb{E}_t \left[M_{t+1} R^{\mathrm{K}}_{t+1} \right], \tag{10}$$

where $R^{\mathrm{K}}_{t+1} \equiv \theta Z^{\mathrm{T}}_{t+1} \left(K_{t+1} \right)^{\theta - 1} + (1 - \delta)$ is the return on capital. This condition determines the optimal investment of H traded firms and thus implicitly identifies the level of traded dividends in period t, D^{T}_t, via equation (3). The first-order conditions for households and traded firms in country F take an analogous form.

Solving for the equilibrium in this economy requires finding equity prices $\{P^{\mathrm{T}}_t, \hat{P}^{\mathrm{T}}_t, P^{\mathrm{N}}_t, \hat{P}^{\mathrm{N}}_t\}$, the risk-free return R_t, and goods prices $\{Q^{\mathrm{N}}_t, \hat{Q}^{\mathrm{N}}_t\}$, such that markets clear when households follow optimal consumption, savings and portfolio strategies, and firms make optimal investment decisions. Under the assumption that bonds are in zero net supply, market clearing in the bond market requires

$$0 = B_t + \hat{B}_t. \tag{11}$$

The traded goods market clears globally. In particular, since H and F traded firms produce a single good that can be costlessly transported between countries, the traded goods market clearing condition is

$$C^{\mathrm{T}}_t + \hat{C}^{\mathrm{T}}_t = Y^{\mathrm{T}}_t - I_t + \hat{Y}^{\mathrm{T}}_t - \hat{I}_t = D^{\mathrm{T}}_t + \hat{D}^{\mathrm{T}}_t. \tag{12}$$

Market clearing in the nontraded sector of each country requires that

$$C^{\mathrm{N}}_t = Y^{\mathrm{N}}_t = D^{\mathrm{N}}_t \qquad \text{and} \qquad \hat{C}^{\mathrm{N}}_t = \hat{Y}^{\mathrm{N}}_t = \hat{D}^{\mathrm{N}}_t. \tag{13}$$

Since the equity liabilities of all firms are normalized to unity, the market clearing conditions in the H and F traded equity markets are

$$1 = A^{\mathrm{H}}_t + \hat{A}^{\mathrm{H}}_t \qquad \text{and} \qquad 1 = A^{\mathrm{F}}_t + \hat{A}^{\mathrm{F}}_t. \tag{14}$$

Recall that nontraded equity can only be held by domestic households. Market clearing in these equity markets therefore requires that

$$1 = A^{\mathrm{N}}_t \qquad \text{and} \qquad 1 = \hat{A}^{\mathrm{N}}_t. \tag{15}$$

2 The Solution Method

In this section we discuss the solution to the nonlinear system of stochastic difference equations characterizing the equilibrium of our DSGE model. First, we outline why standard approximation methods (e.g., projections or perturbations) are inapplicable for solving DSGE models with incomplete markets and portfolio choice.

We then provide an overview of our solution method and discuss how it relates to other methods in the literature.

2.1 Market Incompleteness and Portfolio Choice

The model in Section 1 is hard to solve because it combines dynamic portfolio choice with market incompleteness. In our model, markets are incomplete because households do not have access to the complete array of financial assets in the world economy. In particular, households cannot hold the equities issued by foreign nontraded firms. If we lifted this restriction, households would be able to completely share risks internationally (i.e., the H and F IMRS would be equal). In this special case, the problem of finding the equilibrium could be split into two sub-problems: First, we could use the risk-sharing conditions to find the real allocations as the solution to a social planning problem. Second, we could solve for the equilibrium prices and portfolio choices that support these allocations in a decentralized market setting. Examples of this approach include Obstfeld and Rogoff (1996, p. 302), Baxter, Jerman and King (1998), Engel and Matsumoto (2004), and Kollmann (2006).

When markets are incomplete there are complex interactions between the real and financial sides of the economy; interactions that cannot be accommodated by existing solution methods if there are many financial assets. On the one hand, household portfolio decisions determine the degree of international risk-sharing, which in turn affects equilibrium real allocations. On the other, market-clearing prices affect the behavior of equilibrium returns, which in turn influence portfolio choices. We account for this interaction between the real and financial sides of the economy in our solution method by tracking the behavior of financial wealth across all households. More specifically, we track how shocks to the world economy affect the distribution of wealth given optimal portfolio choices (because risk-sharing is incomplete), and how changes in the distribution of wealth affect market-clearing prices. We also track how these distributional effects on prices affect returns and hence the portfolio choices of households.

In order to track the behavior of the world's wealth distribution, we must include the wealth of each household in the state vector; the vector of variables needed to described the complete state of the economy at a point in time. This leads to two technical problems. First, the numerical complexity in solving for an equilibrium in any model increases sharply with the number of variables in the state vector. The state vector for the simple model in Section 1 has 8 variables, but this is too many to apply a solution method based on a discretization of the state space (see, for example, chapter 12 of Judd 1998). We must therefore use projection and/or perturbation methods to solve the model. The second problem relates to the long-run distribution of wealth. In our model, and many others with incomplete markets (see, for example, Obstfeld and Rogoff 1995, Baxter and Crucini 1995, Correia, Neves, and Rebelo 1995), shocks that have no long run effect on real variables have very persistent effects on the wealth of individual households. Our solution method aims to characterize the equilibrium behavior of the economy in a neighborhood around a particular initial wealth distribution. The advantage of this approach is that it does not require an assumption about how the international distribution of wealth is affected by such shocks in the long run. The disadvantage is that our characterization of the equilibrium dynamics will only be accurate while wealth remains close to the initial distribution. This does not appear to be an important limitation in practice. In Section 4 we show that our solution remains very accurate in simulations of 75 years of quarterly data.

The presence of portfolio choice also introduces technical problems. Perturbation solution methods use n'th-order Taylor approximations to the optimality and market-clearing conditions around the unique non-stochastic steady state of the economy. This approach is inapplicable to the household's portfolio choice problem because there is no unique steady state portfolio allocation: There is no risk in the non-stochastic steady state, so all assets have exactly the same (riskless) return. To address this problem, we use a projection method of approximation that does not require the existence of a unique portfolio allocation in the non-stochastic steady state, but instead solves for it endogenously. Our method only requires us to pin down the initial net foreign asset positions. This is derived from our assumption about the initial wealth distribution.

The main methodological innovation in our solution method relates to the behavior of financial returns. Optimal portfolio choices in each period are determined by the conditional distribution of returns. In a partial equilibrium model the distribution of returns is exogenous, but in our general equilibrium setting we must derive the conditional distribution from the properties of the equilibrium asset prices and dividends. Our method does just this. We track how the conditional distribution of equilibrium returns changes with the state of the economy. This aspect of our method highlights an important implication of market incompleteness for portfolio choice. When risk-sharing is incomplete, the distributional effects of shocks on equilibrium asset prices can induce variations in the conditional distribution of returns even when the underlying shocks come from an i.i.d. distribution. Thus, our solution method allows us to examine how time-variation in portfolio choices and risk premia can arise endogenously when markets are incomplete.

2.2 An Overview

Let us provide an overview of our solution method. The set of equations characterizing the equilibrium of a DSGE model with portfolio choice and incomplete markets can conveniently be written in a general form as

$$\begin{aligned} 0 &= \mathbb{E}_t f\left(Y_{t+1}, Y_t, X_{t+1}, X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right), \qquad (16) \\ X_{t+1} &= \mathcal{H}\left(X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right), \end{aligned}$$

where $f(.)$ is a known function. X_t is a vector of state variables and Y_t is a vector of non-predetermined variables. In our model, X_t contains the state of productivity, the capital stocks and households' wealth, while Y_t includes consumption, dividends, asset allocations, prices and the risk-free rate. The function $\mathcal{H}(.,.)$ determines how past states affect the current state. ε_t is a vector of i.i.d. mean zero, unit variance shocks. In our model, ε_t contains the four productivity shocks. $\mathcal{S}^{1/2}(X_t)$ is a state-dependent scaling matrix. The vector of shocks driving the equilibrium dynamics of the model is $U_{t+1} \equiv \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}$. This vector includes exogenous shocks, like the productivity shocks, and innovations to endogenous variables, like the shocks to households' wealth. These shocks have a conditional mean of zero and a conditional covariance equal to

$\mathcal{S}(X_t)$, a function of the current state vector X_t:

$$
\begin{aligned}
\mathbb{E}(U_{t+1}|X_t) &= 0, \\
\mathbb{E}\left(U_{t+1}U_{t+1}'|X_t\right) &= \mathcal{S}^{1/2}(X_t)\,\mathcal{S}^{1/2}(X_t)' = \mathcal{S}(X_t).
\end{aligned} \tag{17}
$$

An important aspect of our formulation is that it explicitly allows for the possibility that shocks driving the equilibrium dynamics are conditionally heteroskedastic. By contrast, standard perturbation methods assume that U_{t+1} follows an i.i.d. process, in which case $\mathcal{S}(X_t)$ would be a constant matrix.

Given our formulation in (16) and (17), a solution to the model is characterized by a decision rule for the non-predetermined variables

$$
Y_t = \mathcal{G}(X_t, \mathcal{S}(X_t)), \tag{18}
$$

that satisfies the equilibrium conditions in (16):

$$
\begin{aligned}
0 = \mathbb{E}_t f\Big(&\mathcal{G}\left(\mathcal{H}\left(X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right), \mathcal{S}\left(\mathcal{H}\left(X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right)\right)\right), \\
&\mathcal{G}(X_t, \mathcal{S}(X_t)), \mathcal{H}\left(X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\right), X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1}\Big).
\end{aligned}
$$

Or, in a more compact notation,

$$
0 = \mathcal{F}(X_t).
$$

The first step in our method follows the perturbation procedure by approximating the policy functions as

$$
\widehat{\mathcal{G}} = \sum\nolimits_i \psi_i \varphi_i(X_t), \qquad \widehat{\mathcal{H}} = \sum\nolimits_i \delta_i \varphi_i(X_t), \qquad \text{and,} \qquad \widehat{\mathcal{S}} = \sum\nolimits_i s_i \varphi_i(X_t),
$$

for some unknown coefficient sequences $\{\psi_i\}$, $\{\delta_i\}$, and $\{s_i\}$. $\varphi_i(X_t)$ are ordinary polynomials in X_t. Next we approximate the function $f(.)$, as $\widehat{f}(.)$. The equations associated with the real side of the economy are approximated using Taylor series expansions, while those pertinent to the portfolio side are approximated using the continuous-time expansions of Campbell, Chan and Viceira (2003). We denote the derivatives in these expansions as $\{\varsigma_i\}$.

Substituting $\widehat{\mathcal{G}}$, $\widehat{\mathcal{H}}$, and $\widehat{\mathcal{S}}$ into $\widehat{f}$ and taking expectations gives us an approximation for $\mathcal{F}$:

$$
\widehat{\mathcal{F}}\left(X_t; \widehat{\mathcal{G}}, \widehat{\mathcal{H}}, \widehat{\mathcal{S}}, \varsigma, \psi, \delta, s\right) = \sum\nolimits_i \zeta_i \varphi_i(X_t),
$$

where $\{\zeta_i\}$ are functions of $\{\varsigma_i\}$, $\{\psi_i\}$, $\{\delta_i\}$, and $\{s_i\}$. $\widehat{\mathcal{F}}$ is our residual function. To solve the model, we find the coefficient vectors ς, ψ, δ, and s that set the residual function equal to zero.[6]

[6]This step is reminiscent of the projection method introduced in Economics by Judd (1992). In its general formulation, the

The main feature of our method that distinguishes it from a standard perturbation approach is the introduction of the function $\mathcal{S}(X_t)$, which identifies the covariance matrix of the shocks driving the state vector. We need to accommodate conditional heteroskedasticity here because it can arise in models that incorporate portfolio choice with incomplete markets. This is true even when the exogenous shocks to the economy are homoskedastic. As we noted above, we need to track the distribution of wealth when markets are incomplete, so X_t must include the wealth of individual households. Now if the conditional distribution of equilibrium returns is time-varying, optimally chosen portfolio shares will also be time-varying as X_t changes. This means that the susceptibility of wealth to period-$(t+1)$ shocks will generally vary with X_t as households change the composition of their period-t portfolios. In sum, the $\mathcal{S}(X_t)$ function is necessary to accommodate the general equilibrium implications of time-varying portfolio choice when markets are incomplete. The $\mathcal{S}(X_t)$ function also allows us to identify the conditional second moments of all the variables in the economy for each value of the state vector X_t. This facilitates finding the equilibrium risk premia and optimal portfolio shares as functions of X_t.

2.3 Related Methods

Our solution method is most closely related to Campbell, Chan and Viceira (2003) (CCV). They developed an approximation for returns on household's wealth which preserves the multiplicative nature of portfolio weighting. Their expression for returns holds exactly in continuous time when asset prices follow diffusions and remains very accurate in discrete time for short time intervals. CCV apply this approximation method to study dynamic portfolio choice in a partial equilibrium setting where returns follow an exogenous process. Our solution method can be viewed as an extension of CCV to DSGE models.

Our approach also builds on the perturbation methods developed and applied in Judd and Guu (1993, 1997), Judd (1998), and further discussed in Collard and Juillard (2001), Jin and Judd (2002), Schmitt-Grohe and Uribe (2004) among others. These methods extend solution techniques relying on linearizations by allowing for second- and higher-order terms in the approximation of the policy functions. Applications of the perturbation technique to the models with portfolio choice have been developed in Devereux and Sutherland (2006a,b) and Tille and van Wincoop (2007). Both approaches are based on Taylor series approximations. Devereux and Sutherland (2006a) use second-order approximations to the portfolio choice conditions and first-order approximations to the other optimality conditions in order to calculate the steady state portfolio allocations in a DSGE model. Equilibrium conditions are approximated around the unknown portfolio, which is then derived endogenously as the one consistent with the approximations. Our method also produces constant portfolio shares in the case where equilibrium returns are i.i.d. because $\mathcal{S}(X_t)$ is a constant matrix.

To study time-variation in portfolio choice, Devereux and Sutherland (2006b) and Tille and van Wincoop (2007) use a method that incorporates third-order approximations of portfolio equations and second-order

technique consists of choosing basis functions over the space of continuous functions and using them to approximate $\mathcal{G}(X_t, \sigma)$ and $\mathcal{H}(X_t, \sigma\varepsilon_{t+1})$. In most applications, families of orthogonal polynomials, like Chebyshev's polynomials, are used to form $\varphi_i(X_t, \sigma)$. Given the chosen order of approximation, the problem of solving the model translates into finding the coefficient vectors ψ and δ that minimize a residual function.

approximations to the rest of the model's equilibrium conditions. This approach delivers a first-order approximation for optimal portfolio holdings that vary with the state of the economy. By contrast, we are able to derive second-order approximations to portfolio holdings from a set of second-order approximations to the equilibrium conditions of the model and covariance matrix $\mathcal{S}(X_t)$. Thus, we avoid the numerical complexity of computing at least third-order approximations in order to study the portfolio-choice dynamics. This aspect of our method will be important in models with larger number of state variables and where agents choose between many assets. The model in Section 1 has 8 state variables and five assets, but was solved without much computational difficulty. We view this as an important practical advantage of our method that will make it particularly useful for solving international DSGE models. By their very nature, even a minimally specified two-country DSGE model will have many state variables and several assets.

3 Implementing the Method

We now provide a detailed, step-by-step description of how the model in Section 1 is solved. We proceed in four steps: In Step 1 we write the system of nonlinear stochastic difference equations summarized in Section 1.3 in log-approximate form. In Step 2 we conjecture the time-series process describing the equilibrium dynamics of the state variables, prices and the risk-free rate. In Step 3 we use the conjecture from Step 2 to characterize the optimal decisions of firms and households. Step 4 combines the aggregate implications of the firms' and households' decisions with the requirements of market clearing to determine the properties of equilibrium prices and returns. We then check that these properties match the conjecture made in Step 2.

Step 1: Log-Approximations

Here we derive the log-approximations to the equations arising from the households' and firms' first-order conditions, budget constraints and market clearing conditions. These approximations are quite standard in both Macro and Finance aside from the point of approximation. Let x_t denote the state vector, where $x_t \equiv [z_t, k_t, \hat{k}_t, w_t, \hat{w}_t]'$, $k_t \equiv \ln(K_t/K)$, $\hat{k}_t \equiv \ln(\hat{K}_t/K)$, $w_t \equiv \ln(W_t/W_0)$ and $\hat{w}_t \equiv \ln(\hat{W}_t/\hat{W}_0)$ with K and $\hat{K}$ as the steady state capital stocks (steady state values have no t subscript). W_0 and $\hat{W}_0$ are the initial levels of H and F households' wealth. Hereafter, lowercase letters denote the log transformations for all other variables in deviations from their steady state or initial levels (e.g., $r_t \equiv \ln R_t - \ln R$, $p_t^{\mathrm{T}} \equiv \ln P_t^{\mathrm{T}} - \ln P^{\mathrm{T}}$, etc.). Appendix A.1 summarizes the approximation point of our economy and lists all equations used in the model's solution. We focus below on the behavior of households and firms in country H; the behavior in country F is characterized in an analogous manner.

Following CCV we use a first-order log-approximation to the budget constraint of the representative H household:

$$\begin{aligned} \Delta w_{t+1} &= \ln(1 - C_t/W_t) + \ln R_{t+1}^{\mathrm{W}}, \\ &= -\tfrac{\mu}{1-\mu}(c_t - w_t) + r_{t+1}^{\mathrm{W}}, \end{aligned} \tag{19}$$

where μ is the steady state consumption expenditure to wealth ratio. In our model, households have log preferences, so the optimal consumption expenditure is a constant fraction of wealth, $C_t \equiv C_t^{\mathrm{T}} + Q_t^{\mathrm{N}} C_t^{\mathrm{N}} = (1-\beta)W_t$. Thus, in this case $c_t - w_t = 0$. r_{t+1}^{W} is the log return on optimally invested wealth which CCV approximate as

$$r_{t+1}^{\mathrm{W}} = r_t + \boldsymbol{\alpha}_t' er_{t+1} + \tfrac{1}{2}\boldsymbol{\alpha}_t' \left(\mathrm{diag}\left(\mathbb{V}_t(er_{t+1})\right) - \mathbb{V}_t(er_{t+1})\boldsymbol{\alpha}_t\right), \tag{20}$$

where $\boldsymbol{\alpha}_t' \equiv \begin{bmatrix} \alpha_t^{\mathrm{H}} & \alpha_t^{\mathrm{F}} & \alpha_t^{\mathrm{N}} \end{bmatrix}$ is the vector of portfolio shares, $er_{t+1}' \equiv \begin{bmatrix} r_{t+1}^{\mathrm{H}} - r_t & r_{t+1}^{\mathrm{F}} - r_t & r_{t+1}^{\mathrm{N}} - r_t \end{bmatrix}$ is a vector of excess log equity returns, and $\mathbb{V}_t(.)$ is the variance conditioned on period-t information. Importantly, we can say something about the accuracy of this approximation. In particular, CCV show that the approximation error associated with the expression in (20) disappears in the limit where asset prices follow continuous–time diffusion processes.

Next, we turn to the first-order conditions in (9). Using standard log-normal approximations, we obtain

$$\begin{aligned} \mathbb{E}_t\left[r_{t+1}^{\varkappa}\right] - r_t + \tfrac{1}{2}\mathbb{V}_t\left(r_{t+1}^{\varkappa}\right) &= -\mathbb{CV}_t\left(m_{t+1}, r_{t+1}^{\varkappa}\right), &\text{(21a)}\\ r_t &= -\mathbb{E}_t\left[m_{t+1}\right] - \tfrac{1}{2}\mathbb{V}_t(m_{t+1}), &\text{(21b)}\end{aligned}$$

where $r_{t+1}^{\varkappa}$ is the log return for equity $\varkappa = \{\mathrm{H}, \mathrm{F}, \mathrm{N}\}$, and $m_{t+1} \equiv \ln M_{t+1} - \ln M$ is the log IMRS. $\mathbb{CV}_t(.,.)$ denotes the covariance conditioned on period-t information. With log utility the IMRS of H households, M_{t+1}, is equal to $\beta W_t / W_{t+1}$, so $m_{t+1} = -\Delta w_{t+1}$. After substituting for log wealth from (19) and (20), equation (21a) can be rewritten in vector form as

$$\mathbb{E}_t\left[er_{t+1}\right] = \mathbb{V}_t(er_{t+1})\boldsymbol{\alpha}_t - \tfrac{1}{2}\mathrm{diag}\left(\mathbb{V}_t(er_{t+1})\right). \tag{22}$$

This equation implicitly identifies the optimal choice of the H household's portfolio shares, $\boldsymbol{\alpha}_t$. Notice that this approximation does not require an assumption about the portfolio shares chosen in the steady state. We will determine those endogenously below. Combining (22) with (19) and (20) gives us a log-approximate version of the H household's budget constraint:

$$\Delta w_{t+1} = -\tfrac{\mu}{1-\mu}\left(c_t - w_t\right) + r_t + \tfrac{1}{2}\boldsymbol{\alpha}_t'\mathbb{V}_t(er_{t+1})\boldsymbol{\alpha}_t + \boldsymbol{\alpha}_t'\left(er_{t+1} - \mathbb{E}_t er_{t+1}\right). \tag{23}$$

This equation shows that the growth in household's wealth between t and $t+1$ depends upon the consumption/wealth ratio in period t (zero in the case of log utility), the period-t risk free rate, r_t, portfolio shares, $\boldsymbol{\alpha}_t$, the variance-covariance matrix of excess returns, $\mathbb{V}_t(er_{t+1})$, and the unexpected return on assets held between t and $t+1$, $\boldsymbol{\alpha}_t'\left(er_{t+1} - \mathbb{E}_t er_{t+1}\right)$. The first three terms on the right comprise the expected growth rate of wealth under the optimal portfolio strategy.

The remaining equations characterizing the model's equilibrium are approximated in a standard way. The consumption of traded and nontraded goods is pinned down by combining (9a) with $C_t^{\mathrm{T}} + Q_t^{\mathrm{N}} C_t^{\mathrm{N}} = (1-\beta)W_t$ to give $Q_t^{\mathrm{N}} C_t^{\mathrm{N}} = (1-\beta)[1+\vartheta(Q_t^{\mathrm{N}})]^{-1}W_t$ and $C_t^{\mathrm{T}} = (1-\beta)\,\vartheta(Q_t^{\mathrm{N}})[1+\vartheta(Q_t^{\mathrm{N}})]^{-1}W_t$, where $\vartheta(Q_t^{\mathrm{N}}) \equiv$

$(\lambda_{\text{T}}/\lambda_{\text{N}})\,(Q_t^{\text{N}})^{\phi/(1-\phi)}$. Log-linearizing these expressions around the initial value of W_t and Q_t^{N} gives

$$c_t^{\text{T}} = w_t + \tfrac{\phi}{(1+\vartheta)(1-\phi)} q_t^{\text{N}}, \qquad \text{and} \qquad c_t^{\text{N}} = w_t - \tfrac{1-\phi+\vartheta}{(1+\vartheta)(1-\phi)} q_t^{\text{N}}, \tag{24}$$

with ϑ denoting the initial value of $\vartheta(Q_t^{\text{N}})$.

Optimal investment by H firms requires that

$$\mathbb{E}_t r_{t+1}^{\text{K}} - r_t + \tfrac{1}{2}\mathbb{V}_t\left(r_{t+1}^{\text{K}}\right) = \mathbb{CV}_t\left(r_{t+1}^{\text{K}}, \Delta w_{t+1}\right), \tag{25}$$

where r_{t+1}^{K} is the log return on capital approximated by

$$r_{t+1}^{\text{K}} = \psi z_{t+1}^{\text{T}} - (1-\theta)\psi k_{t+1}, \tag{26}$$

with $\psi \equiv 1 - \beta(1-\delta) < 1$. The dynamics of the H capital stock are approximated by

$$k_{t+1} = \tfrac{1}{\beta}k_t + \tfrac{\psi}{\beta\theta} z_t^{\text{T}} - \tfrac{\varphi}{\theta\beta} d_t^{\text{T}}, \tag{27}$$

where $\varphi = \psi - \delta\theta\beta > 0$.

We follow Campbell and Shiller (1988) in relating the log returns on equity to the log dividends and the log prices of equity:

$$r_{t+1}^{\text{H}} = \rho^{\text{H}} p_{t+1}^{\text{T}} + (1-\rho^{\text{H}}) d_{t+1}^{\text{T}} - p_t^{\text{T}}, \tag{28a}$$

$$r_{t+1}^{\text{F}} = \rho^{\text{F}} \hat{p}_{t+1}^{\text{T}} + (1-\rho^{\text{F}}) \hat{d}_{t+1}^{\text{T}} - \hat{p}_t^{\text{T}}, \tag{28b}$$

$$r_{t+1}^{\text{N}} = \rho^{\text{N}} p_{t+1}^{\text{N}} + (1-\rho^{\text{N}}) d_{t+1}^{\text{N}} - p_t^{\text{N}}, \tag{28c}$$

where $\rho^{\varkappa}$ is the reciprocal of one plus the dividend-to-price ratio. In the non-stochastic steady state, $\rho^{\varkappa} = \beta$ for $\varkappa = \{\text{H, F, N}\}$. Making this substitution, iterating forward, taking conditional expectations, and imposing $\lim_{j\to\infty}\mathbb{E}_t\beta^j p_{t+j}^{\text{T}} = 0$, we can derive the H traded equity price as

$$p_t^{\text{T}} = \sum_{i=0}^{\infty}\beta^i\left\{(1-\beta)\mathbb{E}_t d_{t+1+i}^{\text{T}} - \mathbb{E}_t r_{t+1+i}^{\text{H}}\right\}. \tag{29}$$

Analogous expressions describe the log prices of F traded equity and nontraded equities.[7]

Finally, the market clearing conditions are approximated as follows. Market clearing in the goods' markets requires $D_t^{\text{N}} = \eta Z_t^{\text{N}}$, $\hat{D}_t^{\text{N}} = \eta\hat{Z}_t^{\text{N}}$ and $D_t^{\text{T}} + \hat{D}_t^{\text{T}} = C_t^{\text{T}} + \hat{C}_t^{\text{T}}$. The first two conditions can be imposed without approximation as $d_t^{\text{N}} = z_t^{\text{N}}$ and $\hat{d}_t^{\text{N}} = \hat{z}_t^{\text{N}}$. We rewrite the condition for traded goods as $d_t^{\text{T}} + \ln(1+\exp(\hat{d}_t^{\text{T}} - d_t^{\text{T}})$ $= c_t^{\text{T}} + \ln(1+\exp(\hat{c}_t^{\text{T}} - c_t^{\text{T}}))$ and take second-order approximations around the initial values for consumption

[7]We confirm that the no-bubbles conditions are satisfied in our model.

and steady state values for dividends:

$$(d_t^{\mathrm{T}} + \hat{d}_t^{\mathrm{T}}) + \tfrac{1}{4}(d_t^{\mathrm{T}} - \hat{d}_t^{\mathrm{T}})^2 = (c_t^{\mathrm{T}} + \hat{c}_t^{\mathrm{T}}) + \tfrac{1}{4}(c_t^{\mathrm{T}} - \hat{c}_t^{\mathrm{T}})^2. \tag{30}$$

Market clearing in traded equity requires $A_t^{\mathrm{H}} + \hat{A}_t^{\mathrm{H}} = 1$ and $A_t^{\mathrm{F}} + \hat{A}_t^{\mathrm{F}} = 1$. Combining these conditions with the definitions for portfolio shares and the fact that the consumption-wealth ratio for all households is equal to $1-\beta$, we obtain $\exp(p_t^{\mathrm{T}} - w_t)/\beta = \alpha_t^{\mathrm{H}} + \hat{\alpha}_t^{\mathrm{H}} \exp(\hat{w}_t - w_t)$ and $\exp(\hat{p}_t^{\mathrm{T}} - \hat{w}_t)/\beta = \hat{\alpha}_t^{\mathrm{F}} + \alpha_t^{\mathrm{F}} \exp(w_t - \hat{w}_t)$. We approximate the left-hand side of these expressions around the steady state values for $P_t^{\mathrm{T}}/W_t\beta$ and $\hat{P}_t^{\mathrm{T}}/\hat{W}_t\beta$ and their right-hand side around the initial wealth ratio $\hat{W}_0/W_0$, which we take to equal one. In this model, it is straightforward to show that the steady state values $P^{\mathrm{T}}/W\beta = [(1-\beta)/\beta]\,(P^{\mathrm{T}}/D^{\mathrm{T}})$ and $\hat{P}^{\mathrm{T}}/\hat{W}\beta = [(1-\beta)/\beta]\,(\hat{P}^{\mathrm{T}}/\hat{D}^{\mathrm{T}})$ equal 1/2, so a second-order approximation to both sides of the market clearing conditions gives

$$\alpha^{\mathrm{H}}\left[1 + p_t^{\mathrm{T}} - w_t + \tfrac{1}{2}(p_t^{\mathrm{T}} - w_t)^2\right] = \alpha_t^{\mathrm{H}} + \hat{\alpha}_t^{\mathrm{H}}\left(1 + \hat{w}_t - w_t + \tfrac{1}{2}\left(\hat{w}_t - w_t\right)^2\right), \tag{31a}$$

$$\alpha^{\mathrm{F}}\left[1 + \hat{p}_t^{\mathrm{T}} - \hat{w}_t + \tfrac{1}{2}(\hat{p}_t^{\mathrm{T}} - \hat{w}_t)^2\right] = \hat{\alpha}_t^{\mathrm{F}} + \alpha_t^{\mathrm{F}}\left(1 + w_t - \hat{w}_t + \tfrac{1}{2}\left(w_t - \hat{w}_t\right)^2\right), \tag{31b}$$

where α^{H} is the initial value of $\alpha_t^{\mathrm{H}} + \hat{\alpha}_t^{\mathrm{H}}$, and α^{F} is the initial value of $\hat{\alpha}_t^{\mathrm{F}} + \alpha_t^{\mathrm{F}}$. These values are pinned down by the steady state share of traded consumption in the total consumption expenditure. When the traded and nontraded sectors are of equal size, as in our model, $\alpha^{\mathrm{H}} = \alpha^{\mathrm{F}} = 1/2$. Market clearing in the nontraded equity (15) requires $\alpha_t^{\mathrm{N}} = \exp(q_t^{\mathrm{N}} + p_t^{\mathrm{N}} - w_t)/\beta$ and $\hat{\alpha}_t^{\mathrm{N}} = \exp(\hat{q}_t^{\mathrm{N}} + \hat{p}_t^{\mathrm{N}} - \hat{w}_t)/\beta$. Using the same approach we obtain

$$\alpha_t^{\mathrm{N}} = \alpha^{\mathrm{N}}\left(1 + q_t^{\mathrm{N}} + p_t^{\mathrm{N}} - w_t + \tfrac{1}{2}(q_t^{\mathrm{N}} + p_t^{\mathrm{N}} - w_t)^2\right), \tag{32a}$$

$$\hat{\alpha}_t^{\mathrm{N}} = \hat{\alpha}^{\mathrm{N}}\left(1 + \hat{q}_t^{\mathrm{N}} + \hat{p}_t^{\mathrm{N}} - \hat{w}_t + \tfrac{1}{2}(\hat{q}_t^{\mathrm{N}} + \hat{p}_t^{\mathrm{N}} - \hat{w}_t)^2\right), \tag{32b}$$

where α^{N} and $\hat{\alpha}^{\mathrm{N}}$ are the initial values of α_t^{N} and $\hat{\alpha}_t^{\mathrm{N}}$; $\alpha^{\mathrm{N}} = \hat{\alpha}^{\mathrm{N}} = 1/2$. All that now remains is the bond market clearing condition: $B_t + \hat{B}_t = 0$. Walras Law implies that this restriction is redundant given the other market clearing conditions and budget constraints.

Step 2: State Variable Dynamics

The key step in our solution procedure is deriving a general yet tractable set of equations that describe the equilibrium dynamics of the state variables. We conjecture that the $l \times 1$ vector of state variables x_t follows

$$x_{t+1} = \Phi_0 + (I - \Phi_1)x_t + \Phi_2\tilde{x}_t + u_{t+1}, \tag{33}$$

where $\tilde{x}_t \equiv vec(x_t x_t')$, Φ_0 is the $l \times 1$ vector of constants, Φ_1 is the $l \times l$ matrix of autoregressive coefficients and Φ_2 is the $l \times l^2$ matrix of coefficients on the second-order terms. u_{t+1} is a vector of innovations with a zero conditional mean, and a conditional covariance that is a function of X_t :

$$\begin{aligned}
\mathbb{E}\left(u_{t+1}|x_t\right) &= 0, \\
\mathbb{E}\left(u_{t+1}u'_{t+1}|x_t\right) &= \Omega(X_t) = \Omega_0 + \Omega_1 x_t x'_t \Omega'_1.
\end{aligned} \tag{34}$$

This conjecture has two notable features: First, it introduces nonlinearity in the process for x_{t+1} by allowing its squares and cross-products in period t to enter the law of motion via Φ_2 matrix. Second, the variance-covariance matrix of x_{t+1} depends on x_t. As we noted above, this conditional heteroskedasticity arises even though the productivity process is homoskedastic because x_t contains w_t and $\hat{w}_t$, and log wealth is endogenously heteroskedastic when asset markets are incomplete.

The period-t information set of our economy consists of x_t and $\tilde{x}_t$, which we conveniently combine in the extended state vector $X_t = [\ 1 \quad x'_t \quad \tilde{x}'_t\ ]'$ with $\mathfrak{L} = 1 + l + l^2$ elements. Our solution method requires that we characterize the dynamics of X_t. In particular, we need to find an equation for the dynamics of $\tilde{x}_t$ consistent with (33) and (34). For this purpose, we first write the vectorized conditional variance of u_t as

$$vec\left(\Omega(X_t)\right) = \begin{bmatrix} \Sigma_0 & 0 & \Sigma_1 \end{bmatrix} \begin{bmatrix} 1 \\ x_t \\ \tilde{x}_t \end{bmatrix} = \Sigma X_t. \tag{35}$$

Next, we consider the continuous time analogue to (33) and derive the dynamics of $\tilde{x}_{t+1}$ via Ito's lemma. Appendix A.2 shows that the resulting process can be approximated in discrete time by

$$\tilde{x}_{t+1} = \tfrac{1}{2}D\Sigma_0 + \left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right)x_t + \left(I - \left(\Phi_1 \otimes I\right) - \left(I \otimes \Phi_1\right) + \tfrac{1}{2}D\Sigma_1\right)\tilde{x}_t + \tilde{u}_{t+1} \tag{36}$$

where

$$\tilde{u}_{t+1} = \left[\left(I \otimes x_t\right) + \left(x_t \otimes I\right)\right]u_{t+1},$$

$$D = \left[\mathbb{U}\left(\frac{\partial x}{\partial x'} \otimes I\right) + \left(\frac{\partial x}{\partial x'} \otimes I\right)\right], \qquad \text{and} \qquad \mathbb{U} = \sum_r \sum_s E_{rs} \otimes E'_{r,s}.$$

$E_{r,s}$ is the elementary matrix which has a unity at the $(r,s)^{th}$ position and zero elsewhere. Equation (36) approximates the dynamics of $\tilde{x}_{t+1}$ because it ignores the role played by cubic and higher order terms involving the elements of x_t. In this sense, (36) represents a second–order approximation to the dynamics of the second–order terms in the state vector. Notice that the variance of u_{t+1} affects the dynamics of $\tilde{x}_{t+1}$ via the D matrix and that $\tilde{u}_{t+1}$ will generally be conditionally heteroskedastic.

We can now combine (33) and (36) into a single equation:

$$\begin{bmatrix} 1 \\ x_{t+1} \\ \tilde{x}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \Phi_0 & I - \Phi_1 & \Phi_2 \\ \frac{1}{2}D\Sigma_0 & \left(\Phi_0 \otimes I\right) + \left(I \otimes \Phi_0\right) & I - \left(\Phi_1 \otimes I\right) - \left(I \otimes \Phi_1\right) + \frac{1}{2}D\Sigma_1 \end{bmatrix} \begin{bmatrix} 1 \\ x_t \\ \tilde{x}_t \end{bmatrix} + \begin{bmatrix} 0 \\ u_{t+1} \\ \tilde{u}_{t+1} \end{bmatrix},$$

or more compactly

$$X_{t+1} = \mathbb{A}X_t + U_{t+1}, \tag{37}$$

with $\mathbb{E}(U_{t+1}|X_t) = 0$ and $\mathbb{E}\left(U_{t+1}U'_{t+1}|X_t\right) \equiv \mathcal{S}(X_t)$. In Appendix A.3 we show that

$$\mathcal{S}(X_t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega(X_t) & \Gamma(X_t) \\ 0 & \Gamma(X_t)' & \Psi(X_t) \end{pmatrix}, \tag{38}$$

where

$$\begin{aligned} vec\left(\Gamma(X_t)\right) &= \Gamma_0 + \Gamma_1 x_t + \Gamma_2 \tilde{x}_t, \\ vec\left(\Gamma(X_t)'\right) &= \Lambda_0 + \Lambda_1 x_t + \Lambda_2 \tilde{x}_t, \\ vec\left(\Psi(X_t)\right) &= \Psi_0 + \Psi_1 x_t + \Psi_2 \tilde{x}_t. \end{aligned}$$

The Γ_i, Λ_i and Ψ_i matrices are functions of the parameters in (33) and (34); their precise form is shown in Appendix A.3.

Our solution procedure expresses all the endogenous variables in the model as linear combinations of X_t. Thus, for any two variables a_t and b_t, we find the vectors π_a and π_b such that $a_t = \pi_a X_t$ and $b_t = \pi_b X_t$. Below we derive restrictions from the optimality and market clearing conditions sufficient to identify the π vectors for all the endogenous variables. As part of this process we will need to compute conditional first and second moments. Appendix A.4 shows that to a second-order approximation, (37) implies

$$\mathbb{E}\left[a_{t+h}|X_t\right] = \pi_a \mathbb{A}^h X_t, \qquad \text{and} \tag{R1}$$

$$\mathbb{CV}\left(a_{t+1}, b_{t+1}|X_t\right) = \mathcal{A}\left(\pi_a, \pi_b\right) X_t. \tag{R2}$$

These expressions show that both the first and second conditional moments are approximately linear in X_t. This is straightforward in the case of $\mathbb{E}\left[a_{t+h}|X_t\right]$, but for $\mathbb{CV}\left(a_{t+1}, b_{t+1}|X_t\right)$ the linear dependence is determined by the $\mathcal{A}(.,.)$ vector which has elements that depend on the vectors π_a, π_b, and the parameters of the X_t process. The product of a_t and b_t can be similarly approximated to second-order by

$$a_t b_t = \mathcal{B}\left(\pi_a, \pi_b\right) X_t, \tag{R3}$$

where $\mathcal{B}(.,.)$ is another vector with elements that depend on π_a, π_b, and the parameters of the X_t process. We use (R1)-(R3) extensively in the steps below. The precise forms for $\mathcal{A}(.,.)$ and $\mathcal{B}(.,.)$ are presented in Appendix A.4.

To this point we have approximated the dynamics of X_t given a conjecture concerning Φ_0, Φ_1, Φ_2, Ω_0, and Ω_1. To complete Step 2, we characterize the behavior of log asset prices and the log risk-free rate. In particular, we conjecture that

$$\begin{aligned} p_t^{\mathrm{T}} &= \pi_p^{\mathrm{T}} X_t, \quad p_t^{\mathrm{N}} = \pi_p^{\mathrm{N}} X_t, \quad q_t^{\mathrm{N}} = \pi_q^{\mathrm{N}} X_t, \\ \hat{p}_t^{\mathrm{T}} &= \pi_{\hat{p}}^{\mathrm{T}} X_t, \quad \hat{p}_t^{\mathrm{N}} = \pi_{\hat{p}}^{\mathrm{N}} X_t, \quad \hat{q}_t^{\mathrm{N}} = \pi_{\hat{q}}^{\mathrm{N}} X_t, \quad \text{and} \qquad r_t = \pi_r X_t, \end{aligned} \tag{39}$$

for some $\pi_{\varkappa}$ vectors of coefficients determined in Steps 3 and 4 below.

Step 3: Non-Predetermined Variables

In this step we use our conjectures for the dynamics of the state variables, prices and the risk-free rate to characterize the equilibrium behavior of firms and households. We begin with the restrictions on the process for dividends, which are determined by the firms' first-order conditions approximated in (25). Combining the expressions for the log capital stock from (27) with the log return on capital from (26), and taking conditional expectation yields

$$\mathbb{E}[r_{t+1}^{\mathrm{K}}|X_t] = \psi \mathbb{E}\left[z_{t+1}^{\mathrm{T}}|X_t\right] - (1-\theta)\psi\left[\tfrac{1}{\beta}k_t + \tfrac{\psi}{\beta\theta}z_t^{\mathrm{T}} - \tfrac{\varphi}{\beta\theta}d_t^{\mathrm{T}}\right].$$

Combining this expression with the firm's first-order conditions in (25), we can solve for dividends as

$$d_t^{\mathrm{T}} = \tfrac{\theta}{\varphi}k_t + \tfrac{\psi}{\varphi}z_t^{\mathrm{T}} - \tfrac{\beta\theta}{(1-\theta)\varphi}\mathbb{E}\left[z_{t+1}^{\mathrm{T}}|X_t\right] + \tfrac{\beta\theta}{(1-\theta)\psi\varphi}\left[r_t - \tfrac{1}{2}\mathbb{V}\left(r_{t+1}^{\mathrm{K}}|X_t\right) + \mathbb{CV}(r_{t+1}^{\mathrm{K}}, \Delta w_{t+1}|X_t)\right].$$

Applying (R1) and (R2) to this expression implies the following restriction on the π_d vector:

$$\pi_d^{\mathrm{T}} = \tfrac{\theta}{\varphi}\imath_k + \tfrac{\psi}{\varphi}\imath_{z^{\mathrm{T}}} - \tfrac{\theta\beta}{(1-\theta)\varphi}\imath_{z^{\mathrm{T}}}\mathbb{A} + \tfrac{\theta\beta}{(1-\theta)\varphi\psi}\left[\pi_r - \tfrac{1}{2}\psi^2\mathcal{A}(\imath_{z^{\mathrm{T}}}, \imath_{z^{\mathrm{T}}}) + \psi\mathcal{A}(\imath_{z^{\mathrm{T}}}, \imath_w)\right], \tag{40}$$

where $\imath_{\varkappa}$ is a vector of zeros and a one that picks out variable $\varkappa$ from X_t (e.g. $z_t^{\mathrm{T}} = \imath_{z^{\mathrm{T}}}X_t$, $1 = \imath_1 X_t$, etc.). The π_d vector characterizes the optimal dynamics of dividends given the process for the state variables and the risk-free rate conjectured in Step 2.

Next we derive the optimal portfolio and consumption decisions of households. Equation (22) implicitly identifies the relation between the optimal portfolio shares and the state vector. Let us first write this relation as

$$\boldsymbol{\alpha}_t \equiv \begin{bmatrix} \alpha_t^{\mathrm{H}} \\ \alpha_t^{\mathrm{F}} \\ \alpha_t^{\mathrm{N}} \end{bmatrix} = \begin{bmatrix} \pi_\alpha^{\mathrm{H}} \\ \pi_\alpha^{\mathrm{F}} \\ \pi_\alpha^{\mathrm{N}} \end{bmatrix} X_t. \tag{41}$$

To find the $\pi_\alpha^{\varkappa}$ vectors we first must derive log excess returns as linear functions of the state variables.

Substituting for equity prices and dividends from (39) into equations in (28) gives

$$\begin{bmatrix} er^{\text{H}}_{t+1} \\ er^{\text{F}}_{t+1} \\ er^{\text{N}}_{t+1} \end{bmatrix} \equiv \begin{bmatrix} r^{\text{H}}_{t+1} - r_t \\ r^{\text{F}}_{t+1} - r_t \\ r^{\text{N}}_{t+1} - r_t \end{bmatrix} = \begin{bmatrix} \beta\pi^{\text{T}}_{p} + (1-\beta)\pi^{\text{T}}_{d} \\ \beta\pi^{\text{T}}_{\hat{p}} + (1-\beta)\pi^{\text{T}}_{\hat{d}} \\ \beta\pi^{\text{N}}_{p} + (1-\beta)\pi^{\text{N}}_{d} \end{bmatrix} X_{t+1} + \begin{bmatrix} -\pi^{\text{T}}_{p} - \pi_r \\ -\pi^{\text{T}}_{\hat{p}} - \pi_r \\ -\pi^{\text{N}}_{p} - \pi_r \end{bmatrix} X_t,$$

or, more compactly,

$$er_{t+1} = \gamma_1 X_{t+1} + \gamma_2 X_t, \tag{42}$$

where $\gamma_i \equiv [\ \gamma^{\text{H}}_i \quad \gamma^{\text{F}}_i \quad \gamma^{\text{N}}_i\ ]'$. Using (R1) and (R2) we can now derive the moments of log excess returns as

$$\mathbb{E}\left[er_{t+1}|X_t\right] = \left(\gamma_1\mathbb{A} + \gamma_2\right)X_t, \qquad \text{and} \qquad \mathbb{V}(er_{t+1}|X_t) = \begin{bmatrix} \mathcal{A}(\gamma^{\text{H}}_1,\gamma^{\text{H}}_1)X_t & \mathcal{A}(\gamma^{\text{H}}_1,\gamma^{\text{F}}_1)X_t & \mathcal{A}(\gamma^{\text{H}}_1,\gamma^{\text{N}}_1)X_t \\ \mathcal{A}(\gamma^{\text{F}}_1,\gamma^{\text{H}}_1)X_t & \mathcal{A}(\gamma^{\text{F}}_1,\gamma^{\text{F}}_1)X_t & \mathcal{A}(\gamma^{\text{F}}_1,\gamma^{\text{N}}_1)X_t \\ \mathcal{A}(\gamma^{\text{N}}_1,\gamma^{\text{H}}_1)X_t & \mathcal{A}(\gamma^{\text{N}}_1,\gamma^{\text{F}}_1)X_t & \mathcal{A}(\gamma^{\text{N}}_1,\gamma^{\text{N}}_1)X_t \end{bmatrix}.$$

Substituting these results into equation (22) and combining the result with (41) gives us the following set of restrictions on the $\pi^{\varkappa}_{\alpha}$ vectors:

$$\gamma^{\varkappa}_1\mathbb{A} + \gamma^{\varkappa}_2 = \mathcal{B}\left(\mathcal{A}(\gamma^{\varkappa}_1,\gamma^{\text{H}}_1),\pi^{\text{H}}_{\alpha}\right) + \mathcal{B}\left(\mathcal{A}(\gamma^{\varkappa}_1,\gamma^{\text{F}}_1),\pi^{\text{F}}_{\alpha}\right) + \mathcal{B}\left(\mathcal{A}(\gamma^{\varkappa}_1,\gamma^{\text{N}}_1),\pi^{\text{N}}_{\alpha}\right) - \tfrac{1}{2}\mathcal{A}(\gamma^{\varkappa}_1,\gamma^{\varkappa}_1) \tag{43}$$

for $\varkappa = \{\text{H, F, N}\}$.

Let $c^{\text{T}}_t = \pi^{\text{T}}_c X_t$ and $c^{\text{N}}_t = \pi^{\text{N}}_c X_t$ represent the optimal choice of traded and nontraded consumption by country H households. Combining (24) with the conjecture for relative prices in (39) gives

$$\pi^{\text{T}}_c = \imath_w + \tfrac{\phi}{(1+\vartheta)(1-\phi)}\pi^{\text{N}}_q, \qquad \text{and} \qquad \pi^{\text{N}}_c = \imath_w - \tfrac{1-\phi+\vartheta}{(1+\vartheta)(1-\phi)}\pi^{\text{N}}_q.$$

The optimal decisions of households and firms in country F can be related to the state vector in a similar manner. More specifically, we can derive an analogous set of equations that pin down the vectors $\{\pi^{\text{T}}_{\hat{d}}, \pi^{\text{T}}_{\hat{c}}, \pi^{\text{N}}_{\hat{c}}, \pi^{\text{H}}_{\hat{\alpha}}, \pi^{\text{F}}_{\hat{\alpha}}, \pi^{\text{N}}_{\hat{\alpha}}\}$, where $\hat{d}^{\text{T}}_t = \pi^{\text{T}}_{\hat{d}} X_t$, $\hat{c}^{\text{T}}_t = \pi^{\text{T}}_{\hat{c}} X_t$, $\hat{c}^{\text{N}}_t = \pi^{\text{N}}_{\hat{c}} X_t$ and $\hat{\alpha}^{\varkappa}_t = \pi^{\varkappa}_{\hat{\alpha}} X_t$ for $\varkappa = \{\text{H, F, }\hat{\text{N}}\}$.

Step 4: Verification

We now verify our conjectures about the state vector $x_t \equiv [z_t, k_t, \hat{k}_t, w_t, \hat{w}_t]$, equilibrium prices and the risk-free rate. In particular, we use the firms' and households' optimal dividend, portfolio and consumption decision rules to make sure our conjectures for the parameters in the process for x_t and vectors π in (39) satisfy the market clearing conditions.

To verify our conjecture concerning the behavior of the state variables in (33), we equate the conditional first and second moments of all the elements in x_t with the moments implied by the firms' and households' decisions derived in Step 3. Equation (33) implies that the expectation of the i'th. element in x_{t+1} conditioned on X_t is given by the i'th. row of $[\ \Phi_0 \quad I-\Phi_1 \quad \Phi_2\ ]X_t$, while the conditional covariance between the i'th. and j'th. elements is equal to $[\ \Omega_0^{i,j} \quad 0 \quad \Omega_1^{j,\cdot} \otimes \Omega_1^{i,\cdot}\ ]X_t$ where $\Omega_0^{i,j}$ denotes the i,j'th. element of Ω_0 and $\Omega_1^{i,\cdot}$ denotes the i'th. row of Ω_1. We now compare these expressions with the moments of equilibrium productivity, capital and wealth.

Recall that the first four rows of x_t comprise the vector of productivities that follow the exogenous AR(1) process in (4) so $\mathbb{E}[z_{t+1}|X_t] = [\ 0 \quad a \quad 0\ ]X_t$ and $\mathbb{V}[z_{t+1}|X_t] = S_e$. Equating moments gives the following restrictions on Φ_i and Ω_i parameters of the x_t process:

$$[\ 0 \quad a \quad 0\ ]^{i,\cdot} = [\ \Phi_0 \quad I-\Phi_1 \quad \Phi_2\ ]^{i,\cdot} \qquad \text{and} \qquad [\ S_e^{i,j} \quad 0 \quad 0\ ] = [\ \Omega_0^{i,j} \quad 0 \quad \Omega_1^{j,\cdot} \otimes \Omega_1^{i,\cdot}\ ],$$

for $i = \{1,2,3,4\}$ and $j = \{1,2,..8\}$.

The next elements in x_t are the log capital stocks in the two countries. From the log-approximated dynamics for k_t in (27) we get $\mathbb{E}[k_{t+1}|X_t] = [\frac{1}{\beta}\imath_k + \frac{\psi}{\beta\theta}\imath_z^{\mathrm{T}} - \frac{\varphi}{\theta\beta}\pi_d^{\mathrm{T}}]X_t$ and $\mathbb{CV}[k_{t+1}, x_{t+1}^{j,\cdot}|X_t] = 0$ for $j = \{1,2,..8\}$. The moment restrictions on the x_t process parameters are therefore

$$[\tfrac{1}{\beta}\imath_k + \tfrac{\psi}{\beta\theta}\imath_z^{\mathrm{T}} - \tfrac{\varphi}{\theta\beta}\pi_d^{\mathrm{T}}] = [\ \Phi_0 \quad I-\Phi_1 \quad \Phi_2\ ]^5 \qquad \text{and} \qquad [\ 0 \quad 0 \quad 0\ ] = [\ \Omega_0^{5,j} \quad 0 \quad \Omega_1^{j,\cdot} \otimes \Omega_1^{5,\cdot}\ ],$$

for $j = \{1,2,..8\}$. The dynamics of the F capital stock imply an analogous set of restrictions.

Deriving the equilibrium restrictions on the dynamics of wealth in (23) is a little more complicated and requires the use of (R2) and (R3). With log utility, equation (23) implies that $\mathbb{E}[w_{t+1}|X_t] = w_t + r_t + \frac{1}{2}\boldsymbol{\alpha}_t'\mathbb{V}(er_{t+1}|X_t)\boldsymbol{\alpha}_t$. Using the expressions for $\mathbb{V}[er_{t+1}|X_t]$ and $\boldsymbol{\alpha}_t$ derived in Step 3, together with (R3), we have

$$\tfrac{1}{2}\boldsymbol{\alpha}_t'\mathbb{V}(er_{t+1}|X_t)\boldsymbol{\alpha}_t = \tfrac{1}{2}\textstyle\sum_{\varkappa'}\mathcal{B}\left(\pi_\alpha^{\varkappa'}, \textstyle\sum_{\varkappa}\mathcal{B}\left(\pi_\alpha^{\varkappa}, \mathcal{A}(\gamma_1^{\varkappa'}, \gamma_1^{\varkappa})\right)\right)X_t,$$

where the $\varkappa$ and $\varkappa'$ indices pick out the three equities {H, F, N} available to H households. The restriction on the x_t process implied by the first conditional moment of H wealth is, therefore,

$$\imath_w + \pi_r + \tfrac{1}{2}\textstyle\sum_{\varkappa'}\mathcal{B}\left(\pi_\alpha^{\varkappa'}, \textstyle\sum_{\varkappa}\mathcal{B}\left(\pi_\alpha^{\varkappa}, \mathcal{A}(\gamma_1^{\varkappa'}, \gamma_1^{\varkappa})\right)\right) = [\ \Phi_0 \quad I-\Phi_1 \quad \Phi_2\ ]^7. \tag{45}$$

Next we consider the implications of the wealth dynamics in (23) for the covariance between w_{t+1} and all the elements of x_{t+1}. According to (23), the conditional covariance between w_{t+1} and the j'th. element of x_{t+1}, $\imath_j X_{t+1}$, is $\sum_\varkappa \alpha_t^\varkappa \mathbb{CV}(er_{t+1}^\varkappa, \imath_j X_{t+1}|X_t)$ for $\varkappa = \{$H, F, N$\}$. After substituting for $er_{t+1}^\varkappa$ and $\alpha_t^\varkappa$ with (42) and (41), and using (R2) and (R3), we can rewrite this covariance as $\sum_\varkappa \mathcal{B}(\pi_\alpha^\varkappa, \mathcal{A}(\gamma_1^\varkappa, \imath_j))X_t$. Now (33) implies that this covariance equals $[\ \Omega_0^{7,j} \quad 0 \quad \Omega_1^{j,\cdot} \otimes \Omega_1^{7,\cdot}\ ]X_t$, so the second moment restrictions on H household's wealth are

$$\sum_\varkappa \mathcal{B}(\pi_\alpha^\varkappa, \mathcal{A}(\gamma_1^\varkappa, \imath_j)) = [\ \Omega_0^{7,j} \quad 0 \quad \Omega_1^{j,\cdot} \otimes \Omega_1^{7,\cdot}\ ], \tag{46}$$

for $j = \{1,2,...8\}$. The dynamics of F wealth imply a further set of moment restrictions analogous to (45) and (46). These restrictions identify the 8'th. row of $\begin{bmatrix} \Phi_0 & I-\Phi_1 & \Phi_2 \end{bmatrix}$ and the corresponding rows of Ω.

We now need to verify the conjecture about equilibrium prices and the risk-free rate in (39). Combining $m_{t+1} = -\Delta w_{t+1}$ with the first-order condition for bonds in (21b) and substituting for the conditional moments using (R1) and (R2) gives

$$r_t = \pi_r X_t = \left[\imath_w \left(\mathbb{A} - I\right) - \tfrac{1}{2}\mathcal{A}\left(\imath_w, \imath_w\right)\right] X_t. \tag{47}$$

The term in brackets identifies the π_r vector that determines the equilibrium log risk-free rate. Turning to equity prices, we first combine (47) with (42) to give us the expected return on equity $\varkappa$:

$$\mathbb{E}[r^{\varkappa}_{t+1}|X_t] \equiv \mathbb{E}[er^{\varkappa}_{t+1}|X_t] + r_t = [\gamma^{\varkappa}_1\mathbb{A} + \gamma^{\varkappa}_2 + \pi_r]X_t,$$

for $\varkappa = \{\text{H, F, N}\}$. We can now compute the present value term in the equations for equilibrium log equity prices. In particular, (29) becomes

$$\begin{aligned} p^{\text{T}}_t &= \sum_{i=0}^{\infty}\beta^i\left\{(1-\beta)\pi^{\text{T}}_d\mathbb{E}[X_{t+1+i}|X_t] - (\gamma^{\text{H}}_1\mathbb{A} + \gamma^{\text{H}}_2 + \pi_r)\mathbb{E}[X_{t+i}|X_t]\right\}, \\ &= \left[\left\{(1-\beta)\pi^{\text{T}}_d\mathbb{A} - (\gamma^{\text{H}}_1\mathbb{A} + \gamma^{\text{H}}_2 + \pi_r)\right\}(I - \beta\mathbb{A})^{-1}\right] X_t. \end{aligned}$$

The term in brackets identifies the π^{T}_p vector. The other vectors relating equilibrium log equity prices to the state (i.e., $\pi^{\text{T}}_{\hat{p}}, \pi^{\text{N}}_p$ and $\pi^{\text{N}}_{\hat{p}}$) are pinned down in an analogous manner.

Finally, we need to verify that equilibrium goods prices satisfy the market clearing conditions. Market clearing in nontraded goods implies that $d^{\text{N}}_t = z^{\text{N}}_t$ and $\hat{d}^{\text{N}}_t = \hat{z}^{\text{N}}_t$, so $\pi^{\text{N}}_d = \imath_{z^{\text{N}}}$ and $\pi^{\text{N}}_{\hat{d}} = \imath_{\hat{z}^{\text{N}}}$. Applying (R3) to the market clearing condition for traded goods in (30) implies the following restriction:

$$\pi^{\text{T}}_d + \pi^{\text{T}}_{\hat{d}} + \tfrac{1}{4}\mathcal{B}(\pi^{\text{T}}_{\hat{d}} - \pi^{\text{T}}_d, \pi^{\text{T}}_{\hat{d}} - \pi^{\text{T}}_d) = \pi^{\text{T}}_c + \pi^{\text{T}}_{\hat{c}} + \tfrac{1}{4}\mathcal{B}(\pi^{\text{T}}_{\hat{c}} - \pi^{\text{T}}_c, \pi^{\text{T}}_{\hat{c}} - \pi^{\text{T}}_c).$$

Similar sets of restrictions come from combining the market clearing conditions for equity in (31) and (32) with the equations for the optimal portfolio shares in (41):

$$\begin{aligned} \alpha^{\text{H}}\left[\iota_1 + \pi^{\text{T}}_p - \iota_w + \tfrac{1}{2}\mathcal{B}(\pi^{\text{T}}_p - \iota_w, \pi^{\text{T}}_p - \iota_w)\right] &= \pi^{\text{H}}_\alpha + \mathcal{B}\left(\pi^{\text{H}}_{\hat{\alpha}}, \left[\iota_1 + \iota_{\hat{w}} - \iota_w + \tfrac{1}{2}\mathcal{B}\left(\iota_{\hat{w}} - \iota_w, \iota_{\hat{w}} - \iota_w\right)\right]\right), \\ \alpha^{\text{F}}\left[\iota_1 + \pi^{\text{T}}_{\hat{p}} - \iota_{\hat{w}} + \tfrac{1}{2}\mathcal{B}(\pi^{\text{T}}_{\hat{p}} - \iota_{\hat{w}}, \pi^{\text{T}}_{\hat{p}} - \iota_{\hat{w}})\right] &= \pi^{\text{F}}_{\hat{\alpha}} + \mathcal{B}\left(\pi^{\text{F}}_\alpha, \left[\iota_1 + \iota_w - \iota_{\hat{w}} + \tfrac{1}{2}\mathcal{B}\left(\iota_w - \iota_{\hat{w}}, \iota_w - \iota_{\hat{w}}\right)\right]\right), \\ \pi^{\text{N}}_\alpha &= \alpha^{\text{N}}\left[\imath_1 + \pi^{\text{N}}_q + \pi^{\text{N}}_p - \imath_w + \tfrac{1}{2}\mathcal{B}(\pi^{\text{N}}_q + \pi^{\text{N}}_p - \imath_w, \pi^{\text{N}}_q + \pi^{\text{N}}_p - \imath_w)\right], \\ \pi^{\text{N}}_{\hat{\alpha}} &= \hat{\alpha}^{\text{N}}\left[\imath_1 + \pi^{\text{N}}_{\hat{q}} + \pi^{\text{N}}_{\hat{p}} - \imath_{\hat{w}} + \tfrac{1}{2}\mathcal{B}(\pi^{\text{N}}_{\hat{q}} + \pi^{\text{N}}_{\hat{p}} - \imath_{\hat{w}}, \pi^{\text{N}}_{\hat{q}} + \pi^{\text{N}}_{\hat{p}} - \imath_{\hat{w}})\right]. \end{aligned}$$

The Numerical Procedure

We have described how the log-approximated equations characterizing the equilibrium of the model are used to derive a set of restrictions on the behavior of the state vector and the non-predetermined variables. A solution to the model requires that we find values for the π vectors and the state process parameters

$\{\Phi_0, \Phi_1, \Phi_2, \Omega_0, \Omega_1\}$ that satisfy these restrictions for a particular calibration of the taste and technology parameters. Let $\widehat{\mathcal{F}}(\Upsilon) = \mathbf{0}$ denote these restrictions where Υ is a vector of all the unknown coefficients in the π's, Φ's and Ω's. Our objective is to find the value for Υ that satisfies this set of equations. To this end our numerical procedure chooses Υ to minimize the least squares projection $||\widehat{\mathcal{F}}\,(.)\,||^2$, where $||\ .\ ||$ denotes the Euclidean norm.

4 Results

In this section we evaluate the accuracy of our solution method. For this purpose we consider two versions of our model: a simplified version with complete markets and the full version with incomplete markets. Results from the simplified model are informative because they can be compared against known analytical properties of the equilibrium. The results from the full model demonstrate the accuracy of our solution method in an application where no analytical characterization of the equilibrium is available.

Our model simplifies considerably if we let $(1-\phi)^{-1} \to \infty$, set $\lambda^{\mathrm{T}} = 1$ and $\lambda^{\mathrm{N}} = 0$ in both countries, and assume that the variance of nontraded productivity shocks equal zero. These restrictions effectively eliminate the nontraded sectors in each country; the supply and demand for nontraded goods is zero, and so too is the price of nontraded equity. The equilibrium properties of the other variables will be identical to those in a world where households have log preferences defined over traded consumption and allocate their portfolios between H and F traded equities and the risk-free bond. In particular, the equilibrium will be characterized by complete risk-sharing if both H and F households start with the same initial level of wealth.

Complete risk-sharing occurs in our simplified setting because all households have the same preferences and investment opportunity sets. We can see why this is so by returning to conditions determining the households' portfolio choices. In particular, combining the log-approximated first-order conditions with the budget constraint in (22) under the assumption of log preferences gives

$$\boldsymbol{\alpha}_t = \Theta_t^{-1}(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}(\Theta_t)) \qquad \text{and} \qquad \hat{\boldsymbol{\alpha}}_t = \Theta_t^{-1}(\mathbb{E}_t er_{t+1} + \tfrac{1}{2}\mathrm{diag}(\Theta_t)), \tag{48}$$

where $\boldsymbol{\alpha}_t' \equiv [\ \alpha_t^{\mathrm{H}} \quad \alpha_t^{\mathrm{F}}\ ]$, $\hat{\boldsymbol{\alpha}}_t' \equiv [\ \hat{\alpha}_t^{\mathrm{H}} \quad \hat{\alpha}_t^{\mathrm{F}}\ ]$, $er_{t+1}' \equiv [\ r_{t+1}^{\mathrm{H}} - r_t \quad r_{t+1}^{\mathrm{F}} - r_t\ ]$, and $\Theta_t \equiv \mathbb{V}_t(er_{t+1})$. The key point to note here is that all households face the same set of returns and have the same information. So the right hand side of both expressions in (48) are identical in equilibrium. H and F households will therefore find it optimal to hold the same portfolio shares. This has a number of implications if the initial distribution of wealth is equal. First, households' wealth will be equalized across countries in all periods. Second, since households with log utility consume a constant fraction of wealth, consumption will also be equalized. This symmetry in consumption implies that $m_{t+1} = \hat{m}_{t+1}$, so risk sharing is complete. It also implies, together with the market clearing conditions, that bond holdings are zero and wealth is equally split between H and F equities (i.e., $A_t^{\mathrm{H}} = \hat{A}_t^{\mathrm{H}} = A_t^{\mathrm{F}} = \hat{A}_t^{\mathrm{F}} = 1/2$). We can use these equilibrium asset holdings as a benchmark for judging the accuracy of our solution technique.

The remainder of this section examines the equilibrium properties of both the complete and incomplete markets versions of the model computed by our solution method.[8] These calculations were performed assuming a discount factor β equal to 0.99, the technology parameter θ equal to 0.36 and a depreciation rate for capital, δ, of 0.02. In the complete markets version, the log of H and F traded productivity, $\ln Z_t$ and $\ln \hat{Z}_t$, are assumed to follow independent AR(1) processes with autocorrelation coefficients, a_{ii}, equal to 0.95 and innovation variance, S_e^{ii}, equal to 0.0001 for $i = \{\text{H}, \text{F}\}$. In the incomplete markets version we set the share parameters, λ^{T} and $\hat{\lambda}^{\text{T}}$, equal to 0.5 and the elasticity of substitution, $(1-\phi)^{-1}$, equal to 0.74. The autocorrelation in traded and nontraded productivity was set to 0.99 and 0.78 respectively, and the innovations variances, S_e^{ii}, were assumed equal to 0.0001, for $i = \{\text{T}, \hat{\text{T}}, \text{N}, \hat{\text{N}}\}$. All of these parameter values are quite standard and were chosen so that each period in the model represents one quarter. Once the model is "solved", we simulate X_t over 300 quarters starting from an equal wealth distribution. The statistics we report are derived from 1200 simulations and so are based on 90,000 years of simulated quarterly data in the neighborhood of the initial wealth distribution.

4.1 Risk-Sharing and Asset Holdings

We begin our assessment of the solution method by considering the equilibrium portfolio holdings. Panel A of Table 1 reports statistics on the equilibrium asset holdings of H households computed from the simulations of the complete markets model. Theoretically speaking, we should see that $B_t = 0$ and $A_t^{\text{H}} = A_t^{\text{F}} = 1/2$. The simulation results conform closely to these predictions. The equity portfolio holdings show no variation and on average are exactly as theory predicts. Average bond holdings, measured as a share of wealth, are similarly close to zero, but show a little more variation. Overall, simulations based on our solution method appear to closely replicate the asset holdings theory predicts with complete risk sharing.

Panel B of Table 1 reports statistics on the asset holdings of H households in the incomplete markets model. Households continue to diversify their holdings between the equity issued by H and F firms producing tradable goods. The table shows that while these holdings are split equally on average, they are far from constant. Both the standard deviation and range of the tradable equity holdings are orders of magnitude larger than the simulated holdings from the complete markets model. The differences between panels A and B are even more pronounced for bond holdings. When markets are incomplete, shocks to productivity in the nontradable sector affect H and F households differently and create incentives for international borrowing and lending. In equilibrium most of this activity takes place via trading in the bond market, so bond holdings display a good deal of volatility in our simulations of the incomplete markets model.

4.2 Accuracy Tests

To assess the performance of our solution method we compute several tests of model accuracy. First, we evaluate the importance of the third-order terms omitted in the model solution. Second, we report the size

[8] Implementation of the solution method for the complete markets version follows the steps described in Section 3, but excludes the restrictions involving the nontraded sectors.

Table 1: Portfolio Holdings

	A_t^{H} (i)	A_t^{F} (ii)	A_t^{N} (iii)	B_t (iv)
A: Complete Markets				
mean	0.5000	0.5000		0.0000%
stdev	0.0000	0.0000		0.0174%
min	0.5000	0.5000		-0.0581%
max	0.5000	0.5000		0.1438%
B: Incomplete Markets				
mean	0.5000	0.5000	1.0000	0.0316%
stdev	0.0015	0.0015	0.0000	0.1415%
min	0.4911	0.4912	1.0000	-0.5611%
max	0.5097	0.5094	1.0000	0.8268%

Note: A_t^{H}, A_t^{F}, and A_t^{N} correspond, respectively, to H household's holdings of equity issued by H, F traded firms, and H nontraded firms. B_t refers to H household's bond holdings as a share of H wealth.

of Euler equation errors. Next, we compute a summary measure of accuracy based on the den Haan and Marcet (1994) χ^2 test. Finally, we examine the accuracy of our solution over various simulation spans.

Third-Order Terms

When we derived the approximate dynamics of the state vector in equation (37) we ignored the impact of third-order terms in x_t. In this way we abstracted from the role of skewness, kurtosis, and higher-order moments of returns for the portfolio decisions of households. We now evaluate the importance of the third-order terms.

Recall that x_t denotes the vector of state variables expressed in log deviations from the steady state or initial distribution and $\tilde{x}_t \equiv vec(x_t x_t')$. To evaluate the importance of third-order terms in the state vector, we compute the maximum, average, and standard deviation for each of the elements in $|vec(x_t \tilde{x}_t')|$ over our simulated data sample. We then report the 90th, 95th and 99th percentiles of the distributions of these summary statistics across the cross-section of elements in $|vec(x_t \tilde{x}_t')|$.

Table 2 reports the percentiles for the third-order terms from the solution to both versions of our model. Panel A shows that in the complete markets model 99% of the largest third-order terms in $|vec(x_t \tilde{x}_t')|$ are smaller than 4.86E-03. Among the average absolute third-order terms, 99% lie to the left of 1.08E-04, while the standard deviation of third-order terms exceeds 3.56E-03 only 1 % of the time. The results in panel B from the incomplete markets model are quite comparable. Overall, there is little evidence in these results to indicate that the omission of third-order terms is significant for the models we are studying.

Table 2. Accuracy: 3rd Order Terms

	90% (i)	95% (ii)	99% (iii)
A: Complete Markets			
max	3.2955E-03	3.9629E-03	4.8562E-03
mean	5.9929E-05	7.5292E-05	1.0820E-04
stdev	2.1558E-03	2.9178E-03	3.5606E-03
B: Incomplete Markets			
max	1.8399E-03	2.8364E-03	9.3801E-03
mean	4.4280E-05	5.7344E-05	2.1923E-04
stdev	1.7294E-03	2.2252E-03	8.3427E-03

Note: max, mean and stdev refer to the corresponding summary statistic calculated for each element in the absolute vector of third-order terms, $|vec(x_t\tilde{x}_t')|$. 90%, 95%, and 99% stand for the respective percentiles of the distributions of these summary statistics across the cross-section of $|vec(x_t\tilde{x}_t')|$.

Euler Equation Errors

Judd (1992) recommends using the size of the errors that households and firms make to assess the accuracy of an approximated solution. Recall from (16) that the Euler equations from the firms' and households' optimizations problems can be expressed as $0 = \mathbb{E}_t f(Y_{t+1}, Y_t, X_{t+1}, X_t, \mathcal{S}^{1/2}(X_t)\,\varepsilon_{t+1})$. To implement Judd's approach we use our approximate solution for the state variable dynamics and the non-predetermined variables to compute the Euler equation errors:

$$\begin{aligned} \xi_{t+1} \quad = \quad & f(\widehat{\mathcal{G}}\left(\widehat{\mathcal{H}}\left(X_t, \widehat{\mathcal{S}}^{1/2}(X_t)\,\varepsilon_{t+1}\right),\ \widehat{\mathcal{S}}\left(\widehat{\mathcal{H}}\left(X_t, \widehat{\mathcal{S}}^{1/2}(X_t)\,\varepsilon_{t+1}\right)\right)\right), \\ & \widehat{\mathcal{G}}\left(X_t, \widehat{\mathcal{S}}(X_t)\right), \widehat{\mathcal{H}}\left(X_t, \widehat{\mathcal{S}}^{1/2}(X_t)\,\varepsilon_{t+1}\right),\ X_t,\ \widehat{\mathcal{S}}^{1/2}(X_t)\,\varepsilon_{t+1}), \end{aligned} \tag{49}$$

where $\widehat{\mathcal{G}}$, $\widehat{\mathcal{H}}$, and $\widehat{\mathcal{S}}$ are the approximate decision rules. In the complete markets model the ξ_{t+1} vector contains four errors for each country: two for equity, one for capital, and one for bonds. For example, the Euler equation errors for H households and firms are given by $\xi_{t+1} = 1 - [M_{t+1} \otimes R^{\varkappa}]$, where $R^{\varkappa} = \{R^{\mathrm{H}}_{t+1}, R^{\mathrm{F}}_{t+1}, R^{\mathrm{K}}_{t+1}, R_t\}$ and $M_{t+1} = \beta W_t / W_{t+1}$. Notice that ξ_{t+1} provides a scale-free measure of the error. In the incomplete markets model there are two more errors associated with the optional choice of nontraded equity holdings.

Table 3 reports the upper percentiles of the distribution for the absolute errors in both versions of the model. Columns (i)-(iii) show percentiles for the errors from H households' Euler equations for H, F and N equity; while columns (iv) and (v) show the percentile from the H capital and bond Euler equations, respectively. Comparing the results in panels A and B, we see that the percentiles of Euler equation errors in the incomplete markets model are similar to those found in the complete markets version. More importantly,

Table 3. Accuracy: Euler Equation Errors

	A_t^{H} (i)	A_t^{F} (ii)	A_t^{N} (iii)	K_t (iv)	B_t (v)
A: Complete Markets					
90^{th} percentile	0.0026	0.0026		0.0029	0.0033
95^{th} percentile	0.0031	0.0031		0.0035	0.0039
99^{th} percentile	0.0040	0.0040		0.0046	0.0051
B: Incomplete Markets					
90^{th} percentile	0.0016	0.0016	0.0015	0.0023	0.0025
95^{th} percentile	0.0019	0.0019	0.0017	0.0028	0.0030
99^{th} percentile	0.0025	0.0025	0.0023	0.0036	0.0039

Note: A_t^{H}, A_t^{F}, and A_t^{N} refer to the absolute errors from the Euler equations for H household's holdings of equity issued by H and F traded firms, and H nontraded firms; K_t and B_t correspond to the absolute errors from capital and bond Euler equations at H.

the results in both panels are comparable to those reported in the accuracy checks for standard growth models without portfolio choice (e.g., Arouba et al. 2005 and Pichler 2005).

The Den Haan and Marcet Test

We can supplement the results in Table 3 with an accuracy test applied to the optimality conditions in the model. The den Haan and Marcet (1994) test of approximation accuracy consists of checking whether Euler equation errors are orthogonal to any function of the state variables describing the information set in period t. Let $\omega(X_t)$ denote any function that converts the $\mathfrak{L}$–dimensional vector of state variables X_t into a q–dimensional sequence of instrumental variables, $\omega : \mathbb{R}^{\mathfrak{L}} \longrightarrow \mathbb{R}^q$. If households form their expectations rationally, the Euler equation errors derived in (49) must satisfy

$$\mathbb{E}[\xi_{t+1} \otimes \omega(X_t)] = 0. \tag{50}$$

The idea behind the test consists of evaluating how closely condition (50) holds for simulated data on X_t and for any function $\omega(.)$. In particular, let *bars* denote simulated data from the model, allowing us to calculate the sample analog of (50) as

$$B_T = \frac{1}{T}\sum^{T} \bar{\xi}_{t+1} \otimes \omega(\bar{X}_t),$$

where T is a simulated sample size. den Haan and Marcet evaluate whether B_T is close to zero by constructing a test-statistic

$$J_T = TB_T' A_T^{-1} B_T, \tag{51}$$

where A_T is a consistent estimate of the matrix

$$\sum_{i=-\infty}^{\infty} \mathbb{E}\left[\left(\xi_{t+1} \otimes \omega(X_t)\right) \left(\xi_{t+1-i} \otimes \omega(X_{t-i})\right)' \right].$$

Under the null that the solution is accurate and if X_t is stationary and ergodic, den Haan and Marcet show that J_T converges to a χ^2 distribution with qg degrees of freedom, where g is the number of Euler equation errors. To implement the test, our vector of instruments for the complete markets model consists of a constant, $\{Z_t^{\mathrm{T}}, \hat{Z}_t^{\mathrm{T}}, K_t, \hat{K}_t, \Delta W_t\}$, and two lags of $\{K_t, \hat{K}_t, \Delta W_t\}$. For the incomplete markets model, we use a constant, $\{Z_t^{\mathrm{T}}, \hat{Z}_t^{\mathrm{T}}, Z_t^{\mathrm{N}}, \hat{Z}_t^{\mathrm{N}}, K_t, \hat{K}_t, \Delta W_t, \Delta\hat{W}_t\}$ and two lags of $\{K_t, \hat{K}_t, \Delta W_t, \Delta\hat{W}_t\}$ as instruments.[9] Estimates of A_T are computed from the standard GMM estimator that allows for heteroskedasticity but no serial correlation in the errors.

Table 4 reports the results of the test applied to both versions of our model. Following den Haan and Marcet, we repeat the test 100 times for different realizations of the stochastic processes and compare the resulting distribution of J_T with its true distribution. The table reports the percentage of realizations of J_T in the lower and upper 5% of a χ^2_{qg} distribution.

Columns (i) - (v) of Table 4 report the results of tests on each Euler equation in country H. These J_T statistics indicate that our method provides a very accurate solution to both versions of the model. The upper and lower percentiles computed from the empirical distribution of the J_T statistics closely correspond to the percentiles from the true χ^2_{qg} distribution. Column (vi) reports the results from joint tests on the Euler equations. Unfortunately, it is impossible to compute accurate J_T statistics for all the Euler equations in each version of the model because the errors from the individual equations are very highly correlated.[10] We therefore report results for the joint accuracy of a subset of the Euler equations. The statistics in panel A are based on the Euler equations for H equity and capital. In panel B they are based on the equations for H and F traded equity and capital. As the table shows, the empirical distribution of the J_T statistics for the joint tests correspond closely to the true χ^2_{qg} distribution for both versions of our model.

Wealth Dynamics and Simulation Spans

Our solution method does not incorporate any assumptions about how shocks affect the international distribution of wealth in the long run. Instead, we characterize the equilibrium dynamics of the model in the

[9] We did not use lagged productivity shocks to reduce the collinearity across the set of instruments. We include the first difference of wealth to insure that our instruments are stationary.

[10] The high degree of correlation makes it impossible to invert the estimate of matrix A_T that enters the test statistic in (51) accurately. Indeed, we find the condition number for the estimate of A_T based on all the Euler equations within each country to be in excess of 10^6.

Table 4. Accuracy: den Haan and Marcet Test

	A_t^{H} (i)	A_t^{F} (ii)	A_t^{N} (iii)	K_t (iv)	B_t (v)	Joint (vi)
A: Complete Markets						
lower 5%	0.07	0.08		0.07	0.06	0.04
upper 5%	0.08	0.08		0.02	0.02	0.04
B: Incomplete Markets						
lower 5%	0.05	0.03	0.06	0.08	0.08	0.02
upper 5%	0.06	0.04	0.06	0.07	0.08	0.06

Note: A_t^{H}, A_t^{F}, and A_t^{N} correspond to the percentiles of the χ^2 test statistics calculated based on the errors from H country Euler equation for H and F tradable equity, and N equity; K_t and B_t refer to the percentiles of the χ^2 test statistics for capital and bond Euler equations at H. The percentiles in column (vi) of panel A are for joint J_T statistics on the Euler equation errors for H equity and capital. In panel B they are for the Euler equation errors for H and F traded equity and capital.

neighborhood of an initial international wealth distribution. This approach broadens the applicability of our solution method but it also has implications for how we simulate solutions to the model.

Recall that the dynamics of H and F wealth are given by

$$\begin{aligned} w_t &= w_{t-1} + \mathbb{E}_{t-1} r_t^{\mathrm{W}} + \boldsymbol{\alpha}'_{t-1} \left(er_t - \mathbb{E}_{t-1} er_t \right), \qquad \text{and} \\ \hat{w}_t &= \hat{w}_{t-1} + \mathbb{E}_{t-1} \hat{r}_t^{\mathrm{W}} + \hat{\boldsymbol{\alpha}}'_{t-1} \left(\widehat{er}_t - \mathbb{E}_{t-1} \widehat{er}_t \right). \end{aligned}$$

These equations show that productivity shocks can affect the household wealth through two channels. First, period-t shocks to productivity produce unexpected capital gains and losses on households' equity holdings that affect wealth via the third terms in each equation. Second, productivity shocks can change expectations regarding future dividends, risk premia and the risk-free rate which in turn affect the expected future return on optimally invested wealth, $\mathbb{E}_t r_{t+i}^{\mathrm{W}}$ and $\mathbb{E}_t \hat{r}_{t+i}^{\mathrm{W}}$ for $i > 0$. Consequently, period-t productivity shocks can affect the expected future growth in wealth, $\mathbb{E}_t \Delta w_{t+i} = \mathbb{E}_t r_{t+i}^{\mathrm{W}}$ and $\mathbb{E}_t \Delta \hat{w}_{t+i} = \mathbb{E}_t \hat{r}_{t+i}^{\mathrm{W}}$ for $i > 0$. Notice that when the second channel is inoperable, productivity shocks will have permanent effects on the level of wealth because the log of period-t wealth appears with a unit coefficient on the right hand side of each equation. Under these circumstances, a productivity shock that results in, say, a capital gain for H households alone, will permanently shift the international distribution of wealth towards country H. Our solution method allows productivity shocks to affect wealth via both channels: we identify how period-t shocks produce capital gains via the $\boldsymbol{\alpha}'_{t-1} \left(er_t - \mathbb{E}_{t-1} er_t \right)$ and $\hat{\boldsymbol{\alpha}}'_{t-1} \left(\widehat{er}_t - \mathbb{E}_{t-1} \widehat{er}_t \right)$ terms, and also how they affect $\mathbb{E}_t r_{t+i}^{\mathrm{W}}$ and $\mathbb{E}_t \hat{r}_{t+i}^{\mathrm{W}}$ for $i > 0$. This approach does not require any assumption about how a productivity shock affects wealth in the long run. It is applicable to models where productivity shocks can have extremely persistent effects on individual wealth and to models where the long run distribution of wealth is easily identified (e.g., models with portfolio adjustment costs, Uzawa-type preferences, or overlapping generations).

Our solution method does require an assumption about the initial wealth distribution. This raises two possible concerns. The first relates to robustness. Our characterization of the equilibrium dynamics is conditioned on a particular initial wealth distribution, so the characterization may materially change if we assume a different initial distribution. In view of the model's complexity, we cannot check for this problem analytically. However, it is straightforward to compare solutions based on different initial distributions. Our experience with this and other models is that the equilibrium dynamics are robust to the choice of initial wealth distribution, but this is something that should be checked on a case-by-case basis. Of course, the long-run wealth distribution is a natural choice for the initial distribution in cases where the former is easily identified.

The second concern relates to simulations of the model's solution. The approximations we use to characterize the solution are only accurate in a neighborhood of the initial wealth distribution. If shocks to productivity push the wealth distribution outside this neighborhood in a few periods with high probability, we will not be able to accurately simulate long time series from the model's equilibrium. This is not a concern for the model in this paper. The accuracy tests reported in Tables 2 - 4 are based on solution simulations that span 75 years of quarterly data, a longer time span than is available for most macroeconomic data series. Nevertheless, it is instructive to consider how the accuracy of the simulated equilibrium dynamics varies with the simulation span. For this purpose we examined the empirical error distributions from bond market clearing for different simulation spans.

Recall that the bond market clearing condition, $B_t + \hat{B}_t = 0$, was not used in our method, so the value of $B_t + \hat{B}_t$ implied by our solution provides a further accuracy check: If there is no approximation error in the equations we use for the other market clearing conditions and budged constraints, $B_t + \hat{B}_t$ should equal zero by Walras Law in our simulations of the model's solution.[11] We examine the accuracy of the simulated equilibrium dynamics by computing the empirical distribution of $(B_t + \hat{B}_t)/(2\beta R_t W_t)$ within a simulation of a given span, and then comparing the distributions across different spans. The scaling allows us to interpret the bond market errors as shares of H household's wealth.

Table 5 reports the mean and percentiles of the bond market error distribution from simulations spanning 50 to 500 quarters. Panel A shows statistics for the error distributions computed from the complete markets version of the model where optimal bond holdings are zero. Here, the dispersion of the error distribution increases with the span of our simulations, but the upper and lower percentiles of the distributions remain a very small percentage of wealth. The bond market errors in this version of the model are economically insignificant. These results are not surprising. The initial wealth distribution used in the simulations is equal to the long run distribution in the complete markets version of our model. Consequently, realizations of equilibrium wealth should never be too far from their initial values even when the span of the simulations is very long.

The statistics derived from the incomplete markets version of the model tell a different story. Panel B

[11] We thank Anna Pavlova for suggesting this accuracy evaluation.

Table 5. Accuracy: Bond Market Clearing

Span	1% (i)	5% (ii)	mean (iii)	95% (iv)	99% (v)
A: Complete Markets					
50	-0.0091%	-0.0058%	0.0000%	0.0073%	0.0152%
100	-0.0133%	-0.0096%	0.0000%	0.0132%	0.0285%
200	-0.0235%	-0.0170%	0.0001%	0.0262%	0.0477%
300	-0.0296%	-0.0225%	-0.0002%	0.0297%	0.0482%
400	-0.0374%	-0.0272%	-0.0006%	0.0333%	0.0503%
500	-0.0451%	-0.0320%	-0.0008%	0.0367%	0.0552%
B: Incomplete markets					
50	-0.0012%	-0.0003%	0.0015%	0.0043%	0.0061%
100	-0.0009%	-0.0001%	0.0034%	0.0090%	0.0129%
200	-0.0005%	0.0003%	0.0111%	0.0332%	0.0477%
300	-0.0003%	0.0006%	0.0284%	0.0928%	0.1340%
400	-0.0001%	0.0009%	0.0672%	0.2438%	0.3586%
500	-0.0001%	0.0012%	0.1560%	0.6267%	0.9325%

Note: 1, 5, 95, 99 and *mean* refer to the corresponding percentiles and mean of the error distribution in the bond market clearing condition. All entries are measured as shares of H household's wealth.

shows that both the location and dispersion of the error distribution shift significantly as the span of our simulations increases. The change in the error distribution is particularly pronounced in the upper percentiles as the span increases beyond 300 quarters. For perspective on these statistics, recall from Table 1 that the estimated bond holdings of country H households range from -0.56% to 0.83% of wealth over simulations spanning 300 quarters. The support of the corresponding bond error distribution is an order of magnitude smaller. Beyond 300 quarters, the support of the distributions approaches the range of variation in the estimated bond holdings. At least some of the bond errors in these simulations are economically significant.

The results in Table 5 have two important implications for the applicability and accuracy of our solution method. First, we can stimulate very long accurate equilibrium time series from models if we can use the known long-run wealth distribution as a point of approximation in our solution method. Second, our method is capable of generating accurate equilibrium time series over empirically relevant time spans in the neighborhood of an assumed initial wealth distribution. For the model studied here, the results in Panel B indicate that the accuracy of the simulated series deteriorates in an economically significant way in spans greater than 300 quarters or 75 years. For this reason all the accuracy statistics reported in Tables 1 - 4 were based on simulations with a span of 300 quarters.

5 Conclusion

We have presented a numerical method for solving general equilibrium models with many financial assets, heterogeneous agents and incomplete markets. Our method builds on the log-approximations of Campbell, Chan and Viceira (2003) and the second-order perturbation and projection techniques developed by Judd (1992) and others. To illustrate its use, we applied our solution method to complete and incomplete markets versions of a two-country general equilibrium model with production. The numerical solution to the complete markets version closely conforms to the predictions of theory and is highly accurate based on a number of standard tests. This gives us confidence in the accuracy of our technique. The power of our method is illustrated by solving the incomplete markets version of the model. The array of assets in this model is insufficient to permit complete risk-sharing among households, so the equilibrium allocations cannot be found by standard analytical techniques. Our accuracy tests show that simulations of our solution to this version of the model are very accurate over spans of 75 years of quarterly data.

Our solution method can be applied to more richly specified models than the one examined here. For example, the method can be applied to solve models with more complex preferences, capital adjustment costs, or portfolio constraints. As a result, we believe that our method will be useful in the future analysis of many models in international macroeconomics and finance.

References

[1] Aruoba, S. B., Jesús Fernández-Villaverde, and Juan F. Rubio-Ramirez (2005). "Comparing Solution Methods for Dynamic Equilibrium Economies", Journal of Economic Dynamics and Control 30(12), 2477-2508.

[2] Barberis, Nicholas (2000). "Investing for the Long Run when Returns are Predictable", Journal of Finance 55(1), 225-264.

[3] Baxter, Marianne and Mario J. Crucini (1995). "Business Cycles and the Asset Structure of Foreign Trade", International Economic Review 36, 821-54.

[4] Baxter, Marianne, Urban J. Jermann, and Robert G. King (1998). "Nontraded Goods, Nontraded Factors and International Non-diversification", Journal of International Economics 44, 211-229.

[5] Brandt, Michael W., Amit Goyal, and Pedro Santa-Clara (2001). "Dynamic Portfolio Choice: A Simulation Approach", Anderson Graduate School of Management Finance Paper 4-01.

[6] Brandt, Michael W. and Pedro Santa-Clara (2004). "Dynamic Portfolio Selection by Augmenting the Asset Space", NBER Working Paper 10372.

[7] Brennan, Michael J., Eduardo S. Schwartz, Ronald Lagnado (1997). "Strategic Asset Allocation", Journal of Economic Dynamics and Control 21, 1377-1403.

[8] Campbell, John Y. and Robert J. Shiller (1988). "The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors", The Review of Financial Studies 1(3), 195-228.

[9] Campbell, John Y. (1994). "Inspecting the Mechanism: An Analytical Approach to the Stochastic Growth Model," Journal of Monetary Economics 33(3), 463-506.

[10] Campbell, John Y., and Luis M. Viceira (2002). "Strategic Asset Allocation", Oxford University Press.

[11] Campbell, John Y., Yeung Lewis Chan, and Luis M. Viceira (2003). "A Multivariate Model of Strategic Asset Allocation", Journal of Financial Economics 67(1), 41-80.

[12] Collard, Fabrice and Michel Juillard (2001). "Accuracy of Stochastic Perturbation Methods: The Case of Asset Pricing Models", Journal of Economic Dynamics and Control 25, 979-999.

[13] Correia, Isabel, Joao C. Neves, and Sergio Rebelo (1995). "Business Cycles in a Small Open Economy", European Economic Review 39, 1089-1113.

[14] Danthine, Jean-Pierre and John B. Donaldson (1994). "Asset Pricing Implications of Real Market Frictions", Columbia - Graduate School of Business, Paper 95-04.

[15] Den Haan, Wouter J. and Marcet, Albert (1994). "Accuracy in Simulations", Review of Economic Studies 61(1), 3-17.

[16] Engel, Charles and Akito Matsumoto (2004). "Home Bias Under New Open Economy Macroeconomics", IMF Working Paper 05/165.

[17] Evans, Martin D. D. and Viktoria V. Hnatkovska (2005). "International Capital Flows, Returns and World Financial Integration", Georgetown University Working Paper and NBER Working Paper 11701.

[18] Ghironi, Fabio, Jaewoo Lee, and Alessandro Rebucci (2007). "The Valuation Channel of External Adjustment", NBER Working Paper 12937.

[19] Gourinchas, Pierre Olivier (2006). "The Research Agenda: Pierre-Olivier Gourinchas on Global Imbalances and Financial Factors", Review of Economic Dynamics Newsletter.

[20] Heathcote, Jonathan and Fabrizio Perri (2004). "The International Diversification Puzzle is not as Bad as You Think", Georgetown University Working Paper.

[21] Hnatkovska, Viktoria (2005). "Home Bias and High Turnover: Dynamic Portfolio Choice with Incomplete Markets", mimeo, Georgetown University.

[22] Jin, He-Hui and Kenneth L. Judd (2002). "Perturbation Methods for General Dynamic Stochastic Models", manuscript, Stanford University.

[23] Judd, Kenneth L. (1992). "Projection Methods for Solving Aggregate Growth Models", Journal of Economics Theory 58, 410-452.

[24] ——— (1998). "Numerical Methods in Economics", MIT Press, Cambridge, Massachusetts.

[25] Judd, Kenneth L. and Sy-Ming Guu (1993). "Perturbation Solution Methods for Economic Growth Models", in Economic and Financial Modeling with Mathematica, Hal Varian, ed., Springer-Verlag: New York.

[26] ——— (1997). "Asymptotic Methods for Aggregate Growth Models", Journal of Economic Dynamics and Control 21(6), 1025-1042.

[27] ——— (2000). "The Economic Effects of New Assets: An Asymptotic Approach", Hoover Institution, Stanford University Working Paper.

[28] Kogan, Leonid and Raman Uppal (2000). "Risk Aversion and Optimal Portfolio Policies in Partial and General Equilibrium Economies", NBER Working Paper 8609.

[29] Kollmann, Robert (2005). "A Dynamic Equilibrium Model of International Portfolio Holdings: Comment", University of Paris XII Working Paper.

[30] Obstfeld, Maurice and Kenneth Rogoff (1995). "Exchange Rate Dynamics Redux", Journal of Political Economy 103, 624-60.

[31] Obstfeld, Maurice and Kenneth Rogoff (1996). "Foundations of International Macroeconomics", The MIT Press, Cambridge, Massachusetts.

[32] Obstfeld, Maurice (2004). "External Adjustment", Review of World Economics, 140(4), 541-568.

[33] Pesenti, Paolo and Eric van Wincoop (1996). "Do Nontraded Goods Explain the Home Bias Puzzle?" NBER Working Paper 5784.

[34] Pichler, Paul (2005). "Evaluating Approximate Equilibria of Dynamic Economic Models", University of Vienna, Department of Economics Paper 0510.

[35] Serrat, Angel (2001). "A Dynamic Equilibrium Model of International Portfolio Holdings", Econometrica 69(6), 1467-1489.

[36] Schmitt-Grohe, Stephanie and Martin Uribe (2004). "Solving Dynamic General Equilibrium Models Using a Second-order Approximation to the Policy Function", Journal of Economic Dynamics and Control 28, 755-775.

[37] Uppal, Raman (1993). "A General Equilibrium Model of International Portfolio Choice", The Journal of Finance 48(2), 529-553.

A Appendix:

A.1 Model equations and the approximation point

The system of equations characterizing the equilibrium of our model consists of

1. Process for productivity

$$z_t = az_{t-1} + S_e^{1/2} e_t$$

2. H and F budget constraints

$$\begin{aligned} W_{t+1} &= R^{\text{W}}_{t+1} \left(W_t - C^{\text{T}}_t - Q^{\text{N}}_t C^{\text{N}}_t \right) \\ R^{\text{W}}_{t+1} &= R_t + \alpha^{\text{H}}_t (R^{\text{H}}_{t+1} - R_t) + \alpha^{\text{F}}_t (R^{\text{F}}_{t+1} - R_t) + \alpha^{\text{N}}_t (R^{\text{N}}_{t+1} - R_t) \end{aligned}$$

and

$$\begin{aligned} \hat{W}_{t+1} &= \hat{R}^{\text{W}}_{t+1} (\hat{W}_t - \hat{C}^{\text{T}}_t - \hat{Q}^{\text{N}}_t \hat{C}^{\text{N}}_t) \\ \hat{R}^{\text{W}}_{t+1} &= R_t + \hat{\alpha}^{\text{H}}_t (R^{\text{H}}_{t+1} - R_t) + \hat{\alpha}^{\text{F}}_t (R^{\text{F}}_{t+1} - R_t) + \hat{\alpha}^{\text{N}}_t (\hat{R}^{\text{N}}_{t+1} - R_t). \end{aligned}$$

3. H and F bond and equity Euler equations

$$\begin{aligned} 1 = \mathbb{E}_t \left[M_{t+1} R_t \right], &\qquad 1 = \mathbb{E}_t \left[\hat{M}_{t+1} R_t \right], \\ 1 = \mathbb{E}_t \left[M_{t+1} R^{\text{H}}_{t+1} \right], &\qquad 1 = \mathbb{E}_t \left[\hat{M}_{t+1} R^{\text{H}}_{t+1} \right], \\ 1 = \mathbb{E}_t \left[M_{t+1} R^{\text{F}}_{t+1} \right], &\qquad 1 = \mathbb{E}_t \left[\hat{M}_{t+1} R^{\text{F}}_{t+1} \right], \\ 1 = \mathbb{E}_t \left[M_{t+1} R^{\text{N}}_{t+1} \right], &\qquad 1 = \mathbb{E}_t \left[\hat{M}_{t+1} \hat{R}^{\text{N}}_{t+1} \right], \end{aligned}$$

where $M_{t+1} = \beta \left(C^{\text{T}}_t + Q^{\text{N}}_t C^{\text{N}}_t \right) / \left(C^{\text{T}}_{t+1} + Q^{\text{N}}_{t+1} C^{\text{N}}_{t+1} \right) = \beta W_t / W_{t+1}$ and $\hat{M}_{t+1} = \beta \hat{W}_t / \hat{W}_{t+1}$.

4. H and F optimality conditions determining relative goods prices

$$Q^{\text{N}}_t = \left(\frac{\lambda_{\text{N}}}{\lambda_{\text{T}}} \right)^{1-\phi} \left(\frac{C^{\text{N}}_t}{C^{\text{T}}_t} \right)^{\phi-1}, \qquad \text{and} \qquad \hat{Q}^{\text{N}}_t = \left(\frac{\hat{\lambda}_{\text{N}}}{\hat{\lambda}_{\text{T}}} \right)^{1-\phi} \left(\frac{\hat{C}^{\text{N}}_t}{\hat{C}^{\text{T}}_t} \right)^{\phi-1}.$$

5. Capital Euler equation at H and F

$$\begin{aligned} 1 &= \mathbb{E}_t \left[M_{t+1} R^{\text{K}}_{t+1} \right], \qquad \text{with } R^{\text{K}}_{t+1} \equiv \theta Z^{\text{T}}_{t+1} \left(K_{t+1} \right)^{\theta-1} + (1-\delta) \\ 1 &= \mathbb{E}_t \left[\hat{M}_{t+1} \hat{R}^{\text{K}}_{t+1} \right], \qquad \text{with } \hat{R}^{\text{K}}_{t+1} \equiv \theta \hat{Z}^{\text{T}}_{t+1} (\hat{K}_{t+1})^{\theta-1} + (1-\delta) \end{aligned}$$

6. Market clearing conditions

(a) traded goods

$$C_t^{\mathrm{T}} + \hat{C}_t^{\mathrm{T}} = D_t^{\mathrm{T}} + \hat{D}_t^{\mathrm{T}}$$

(b) nontraded goods

$$C_t^{\mathrm{N}} = Y_t^{\mathrm{N}} = D_t^{\mathrm{N}} \qquad \text{and} \qquad \hat{C}_t^{\mathrm{N}} = \hat{Y}_t^{\mathrm{N}} = \hat{D}_t^{\mathrm{N}}.$$

(c) bond

$$0 = B_t + \hat{B}_t.$$

(d) traded equity

$$1 = A_t^{\mathrm{H}} + \hat{A}_t^{\mathrm{H}} \qquad \text{and} \qquad 1 = A_t^{\mathrm{F}} + \hat{A}_t^{\mathrm{F}},$$

which can equivalently be written as

$$\begin{aligned} P_t^{\mathrm{T}} &= \alpha_t^{\mathrm{H}} \beta W_t + \hat{\alpha}_t^{\mathrm{H}} \beta \hat{W}_t \\ \hat{P}_t^{\mathrm{T}} &= \alpha_t^{\mathrm{F}} \beta W_t + \hat{\alpha}_t^{\mathrm{F}} \beta \hat{W}_t \end{aligned}$$

(e) nontraded equity

$$1 = A_t^{\mathrm{N}} \qquad 1 = \hat{A}_t^{\mathrm{N}},$$

which is equivalent to

$$\alpha_t^{\mathrm{N}} = Q_t^{\mathrm{N}} P_t^{\mathrm{N}} / \beta W_t \qquad \hat{\alpha}_t^{\mathrm{N}} = \hat{Q}_t^{\mathrm{N}} \hat{P}_t^{\mathrm{N}} / \beta \hat{W}_t.$$

The approximation point is given by $R = R^{\mathrm{K}} = \hat{R}^{\mathrm{K}} = R^{\mathrm{H}} = R^{\mathrm{F}} = R^{\mathrm{N}} = \hat{R}^{\mathrm{N}} = R^{\mathrm{W}} = \hat{R}^{\mathrm{W}} = \frac{1}{\beta}$. $K = \hat{K} = (\beta\theta)^{1/(1-\theta)} (1 - \beta + \beta\delta)^{1/(\theta-1)}$, $D^{\mathrm{T}} = \hat{D}^{\mathrm{T}} = K^{\theta} - \delta K$, $P^{\mathrm{T}} = \hat{P}^{\mathrm{T}} = \beta D^{\mathrm{T}}/(1-\beta)$. $D^{\mathrm{N}} = \hat{D}^{\mathrm{N}} = \eta$, so that $C^{\mathrm{N}} = \hat{C}^{\mathrm{N}} = \eta$ and $P^{\mathrm{N}} = \hat{P}^{\mathrm{N}} = \beta\eta/(1-\beta)$. Wealth at H and F is approximated around an initial level, W_0 and $\hat{W}_0$. When $W_0 = \hat{W}_0$, then $C_0^{\mathrm{T}} = \hat{C}_0^{\mathrm{T}} = D^{\mathrm{T}}$. Then portfolios are approximated around $\alpha^{\mathrm{N}} = \hat{\alpha}^{\mathrm{N}} = \lambda_{\mathrm{N}}^{1-\phi} (C_0^{\mathrm{N}}/C_0)^{\phi}$ and $\alpha^{\mathrm{H}} = \alpha^{\mathrm{F}} = \lambda_{\mathrm{T}}^{1-\phi} (C_0^{\mathrm{T}}/C_0)^{\phi}$, where α^{H} and α^{F} denote the initial values of $\left(\alpha_t^{\mathrm{H}} + \hat{\alpha}_t^{\mathrm{H}}\right)$ and $\left(\hat{\alpha}_t^{\mathrm{F}} + \alpha_t^{\mathrm{F}}\right)$, respectively, as before.

A.2 Derivation of Equation (36)

We start with quadratic and cross-product terms, $\tilde{x}_t$ and approximate their laws of motion using Ito's lemma. In continuous time, the discrete process for x_{t+1} in (33) becomes

$$dx_t = [\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t]\, dt + \Omega(\tilde{x}_t)^{1/2} dW_t$$

Then by Ito's lemma:

$$\begin{aligned}
dvec(x_t x_t') &= [(I \otimes x_t) + (x_t \otimes I)] \left([\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t]\, dt + \Omega(\tilde{x}_t)^{1/2} dW_t \right) \\
&\quad + \tfrac{1}{2} \left[(I \otimes U) \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right] d\,[x, x]_t \\
&= [(I \otimes x_t) + (x_t \otimes I)] \left([\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t]\, dt + \Omega(\tilde{x}_t)^{1/2} dW_t \right) \\
&\quad + \tfrac{1}{2} \left[\mathbb{U} \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right] vec\left\{ \Omega(\tilde{x}_t) \right\} dt \\
&= [(I \otimes x_t) + (x_t \otimes I)] \left([\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t]\, dt + \Omega(\tilde{x}_t)^{1/2} dW_t \right) + \tfrac{1}{2} D vec\left\{ \Omega(\tilde{x}_t) \right\} dt, \qquad \text{(A1)}
\end{aligned}$$

where

$$D = \left[\mathbb{U} \left(\frac{\partial x}{\partial x'} \otimes I \right) + \left(\frac{\partial x}{\partial x'} \otimes I \right) \right], \qquad \mathbb{U} = \sum_r \sum_s E_{rs} \otimes E_{r,s}',$$

and $E_{r,s}$ is the elementary matrix which has a unity at the $(r,s)^{th}$ position and zero elsewhere. The law of motion for the quadratic states in (A1) can be rewritten in discrete time as

$$\begin{aligned}
\tilde{x}_{t+1} &\cong \tilde{x}_t + [(I \otimes x_t) + (x_t \otimes I)] [\Phi_0 - \Phi_1 x_t + \Phi_2 \tilde{x}_t] + \tfrac{1}{2} D vec\left(\Omega(\tilde{x}_t) \right) \\
&\quad + [(I \otimes x_t) + (x_t \otimes I)] \varepsilon_{t+1}, \\
&\cong \tfrac{1}{2} D \Sigma_0 + [(\Phi_0 \otimes I) + (I \otimes \Phi_0)] x_t + \left[I - (\Phi_1 \otimes I) - (I \otimes \Phi_1) + \tfrac{1}{2} D \Sigma_1 \right] \tilde{x}_t + \tilde{\varepsilon}_{t+1},
\end{aligned}$$

where $\tilde{\varepsilon}_{t+1} \equiv [(I \otimes x_t) + (x_t \otimes I)] \varepsilon_{t+1}$. The last equality is obtained by using an expression for $vec\left(\Omega(X_t) \right)$ in (35), where $\Sigma_0 = vec(\Omega_0)$ and $\Sigma_1 = \Omega_1 \otimes \Omega_1$, and by combining together the corresponding coefficients on a constant, linear and second-order terms.

A.3 Derivation of Equation (38)

Recall that $U_{t+1} = [\begin{array}{ccc} 0 & \varepsilon_{t+1} & \tilde{\varepsilon}_{t+1} \end{array}]'$, so $\mathbb{E}\left(U_{t+1} | X_t \right) = 0$ and

$$\mathbb{E}\left(U_{t+1} U_{t+1}' | X_t \right) \equiv \mathcal{S}\left(X_t \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Omega\left(X_t \right) & \Gamma\left(X_t \right) \\ 0 & \Gamma\left(X_t \right)' & \Psi(X_t) \end{pmatrix}$$

To evaluate the covariance matrix, we assume that $vec(x_{t+1}\tilde{x}'_{t+1}) \cong 0$ and define:

$$
\begin{aligned}
\Gamma\left(X_t\right) &\equiv \mathbb{E}_t\varepsilon_{t+1}\tilde{\varepsilon}'_{t+1}, \\
&= \mathbb{E}_t x_{t+1}\tilde{x}'_{t+1} - \mathbb{E}_t x_{t+1}\mathbb{E}_t\tilde{x}'_{t+1}, \\
&= \mathbb{E}_t x_{t+1}\tilde{x}'_{t+1} - (\Phi_0 + (I-\Phi_1)x_t + \Phi_2\tilde{x}_t) \\
&\quad \times \left(\tfrac{1}{2}\Sigma'_0 D' + x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]' + \tilde{x}'_t\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]'\right), \\
&\cong -\Phi_0\left(\tfrac{1}{2}\Sigma'_0 D' + x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]' + \tilde{x}'_t\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]'\right) \\
&\quad -(I-\Phi_1)x_t\left(\tfrac{1}{2}\Sigma'_0 D' + x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]'\right) - \tfrac{1}{2}\Phi_2\tilde{x}_t\Sigma'_0 D', \\
&= -\tfrac{1}{2}\Phi_0\Sigma'_0 D' - \Phi_0 x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]' - \tfrac{1}{2}(I-\Phi_1)x_t\Sigma'_0 D' \\
&\quad -\Phi_0\tilde{x}'_t\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]' - (I-\Phi_1)x_t x'_t\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]' - \tfrac{1}{2}\Phi_2\tilde{x}_t\Sigma'_0 D'.
\end{aligned}
$$

Hence

$$
\begin{aligned}
vec\left(\Gamma\left(X_t\right)\right) &= \Gamma_0 + \Gamma_1 x_t + \Gamma_2\tilde{x}_t, \\
\Gamma_0 &= -\frac{1}{2}\left(D\Sigma_0\otimes\Phi_0\right)vec(I), \\
\Gamma_1 &= -\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]\otimes\Phi_0 + \tfrac{1}{2}\left(D\Sigma_0\otimes(I-\Phi_1)\right), \\
\Gamma_2 &= -\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]\otimes\Phi_0 - \tfrac{1}{2}\left(D\Sigma_0\otimes\Phi_2\right) \\
&\quad -\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]\otimes(I-\Phi_1).
\end{aligned}
$$

Note also from above that

$$
\begin{aligned}
\Gamma\left(X_t\right)' &= -\tfrac{1}{2}D\Sigma_0\Phi'_0 - \left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]x_t\Phi'_0 - \Sigma_0 x'_t(I-\Phi_1)' \\
&\quad -\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]\tilde{x}_t\Phi'_0 - \left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]x_t x'_t(I-\Phi_1)' - \tfrac{1}{2}D\Sigma_0\tilde{x}'_t\Phi'_2.
\end{aligned}
$$

So

$$
\begin{aligned}
vec\left(\Gamma\left(X_t\right)'\right) &= \Lambda_0 + \Lambda_1 x_t + \Lambda_2\tilde{x}_t, \\
\Lambda_0 &= -\tfrac{1}{2}\left(\Phi_0\otimes D\Sigma_0\right)vec(I), \\
\Lambda_1 &= -\left(\Phi_0\otimes\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]\right) + \tfrac{1}{2}\left((I-\Phi_1)\otimes D\Sigma_0\right), \\
\Lambda_2 &= -\left(\Phi_0\otimes\left[I - ((\Phi_1\otimes I) + (I\otimes\Phi_1)) + \tfrac{1}{2}D\Sigma_1\right]\right) - \tfrac{1}{2}\left(\Phi_2\otimes D\Sigma_0\right) \\
&\quad -\left((I-\Phi_1)\otimes\left[(\Phi_0\otimes I) + (I\otimes\Phi_0)\right]\right).
\end{aligned}
$$

Next, consider the variance of $\tilde{\varepsilon}_{t+1}$:

$$
\begin{aligned}
\Psi\left(X_t\right) &\equiv \mathbb{E}_t \tilde{\varepsilon}_{t+1} \tilde{\varepsilon}_{t+1}^{\prime}=\mathbb{E}_t \tilde{x}_{t+1} \tilde{x}_{t+1}^{\prime}-\mathbb{E}_t \tilde{x}_{t+1} \mathbb{E}_t \tilde{x}_{t+1}^{\prime}, \\
&= \mathbb{E}_t \tilde{x}_{t+1} \tilde{x}_{t+1}^{\prime}-\left(\tfrac{1}{2} D \Sigma_0+\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right] x_t+\left[I-\left(\left(\Phi_1 \otimes I\right)+\left(I \otimes \Phi_1\right)\right)+\tfrac{1}{2} D \Sigma_1\right] \tilde{x}_t\right) \\
&\quad \times\left(\tfrac{1}{2} \Sigma_0^{\prime} D^{\prime}+x_t^{\prime}\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right]^{\prime}+\tilde{x}_t^{\prime}\left[I-\left(\left(\Phi_1 \otimes I\right)+\left(I \otimes \Phi_1\right)\right)+\tfrac{1}{2} D \Sigma_1\right]^{\prime}\right), \\
&\cong -\tfrac{1}{2} D \Sigma_0\left(\tfrac{1}{2} \Sigma_0^{\prime} D^{\prime}+x_t^{\prime}\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right]^{\prime}+\tilde{x}_t^{\prime}\left[I-\left(\left(\Phi_1 \otimes I\right)+\left(I \otimes \Phi_1\right)\right)+\tfrac{1}{2} D \Sigma_1\right]^{\prime}\right) \\
&\quad -\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right] x_t\left(\tfrac{1}{2} \Sigma_0^{\prime} D^{\prime}+x_t^{\prime}\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right]^{\prime}\right) \\
&\quad -\left[I-\left(\left(\Phi_1 \otimes I\right)+\left(I \otimes \Phi_1\right)\right)+\tfrac{1}{2} \mathbb{D} \Sigma_1\right] \tilde{x}_t \tfrac{1}{2} \Sigma_0^{\prime} D^{\prime}, \\
&= -\tfrac{1}{4} D \Sigma_0 \Sigma_0^{\prime} D^{\prime}-\tfrac{1}{2} D \Sigma_0 x_t^{\prime}\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right]^{\prime}-\tfrac{1}{2}\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right] x_t \Sigma_0^{\prime} D^{\prime} \\
&\quad -\tfrac{1}{2} D \Sigma_0 \tilde{x}_t^{\prime}\left[I-\left(\left(\Phi_1 \otimes I\right)+\left(I \otimes \Phi_1\right)\right)+\tfrac{1}{2} D \Sigma_1\right]^{\prime}-\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right] x_t x_t^{\prime}\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right]^{\prime} \\
&\quad -\tfrac{1}{2}\left[I-\left(\left(\Phi_1 \otimes I\right)+\left(I \otimes \Phi_1\right)\right)+\tfrac{1}{2} D \Sigma_1\right] \tilde{x}_t \Sigma_0^{\prime} D^{\prime}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
vec\left(\Psi\left(X_t\right)\right) &= \Psi_0+\Psi_1 x_t+\Psi_2 \tilde{x}_t, \\
\Psi_0 &= -\tfrac{1}{4}\left(D \Sigma_0 \otimes D \Sigma_0\right) vec(I), \\
\Psi_1 &= -\tfrac{1}{2}\left(\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right] \otimes D \Sigma_0\right)-\tfrac{1}{2}\left(D \Sigma_0 \otimes\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right]\right), \\
\Psi_2 &= -\tfrac{1}{2}\left[I-\left(\left(\Phi_1 \otimes I\right)+\left(I \otimes \Phi_1\right)\right)+\tfrac{1}{2} D \Sigma_1\right] \otimes D \Sigma_0-\tfrac{1}{2}\left(D \Sigma_0 \otimes\left[I-\left(\left(\Phi_1 \otimes I\right)+\left(I \otimes \Phi_1\right)\right)+\tfrac{1}{2} D \Sigma_1\right]\right) \\
&\quad -\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right] \otimes\left[\left(\Phi_0 \otimes I\right)+\left(I \otimes \Phi_0\right)\right].
\end{aligned}
$$

A.4 Derivation of Results R2 and R3

Let $m_t=\pi_m X_t$ and $n_t=\pi_n X_t$ for two variables m_t and n_t.We want to find the conditional covariance between the two:

$$
\begin{aligned}
\mathbb{CV}_t\left(m_{t+1}, n_{t+1}\right) &= \left[\begin{array}{lll}
\pi_m^0 & \pi_m^1 & \pi_m^2
\end{array}\right]\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & \Omega(X_t) & \Gamma\left(X_t\right) \\
0 & \Gamma\left(X_t\right)^{\prime} & \Psi(X_t)
\end{array}\right]\left[\begin{array}{c}
\pi_n^{0 \prime} \\
\pi_n^{1 \prime} \\
\pi_n^{2 \prime}
\end{array}\right], \\
&= \pi_m^1 \Omega(X_t) \pi_n^{1 \prime}+\pi_m^2 \Gamma\left(X_t\right)^{\prime} \pi_n^{1 \prime}+\pi_m^1 \Gamma\left(X_t\right) \pi_n^{2 \prime}+\pi_m^2 \Psi(X_t) \pi_n^{2 \prime}, \\
&= \left(\pi_n^1 \otimes \pi_m^1\right) vec\left(\Omega(X_t)\right)+\left(\pi_n^1 \otimes \pi_m^2\right) vec\left(\Gamma\left(X_t\right)^{\prime}\right) \\
&\quad +\left(\pi_n^2 \otimes \pi_m^1\right) vec\left(\Gamma\left(X_t\right)\right)+\left(\pi_n^2 \otimes \pi_m^2\right) vec\left(\Psi\left(X_t\right)\right), \\
&= \left(\pi_n^1 \otimes \pi_m^1\right) \Sigma_0+\left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_0+\left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_0+\left(\pi_n^2 \otimes \pi_m^2\right) \Psi_0 \\
&\quad +\left(\left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_1+\left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_1+\left(\pi_n^2 \otimes \pi_m^2\right) \Psi_1\right) x_t \\
&\quad +\left(\left(\pi_n^1 \otimes \pi_m^1\right) \Sigma_1+\left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_2+\left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_2+\left(\pi_n^2 \otimes \pi_m^2\right) \Psi_2\right) \tilde{x}_t.
\end{aligned}
$$

So, to summarize,

$$
\begin{aligned}
\mathbb{CV}_t\left(m_{t+1}, n_{t+1}\right) &= \mathcal{A}\left(\pi_m, \pi_n\right) X_t, \\
\mathcal{A}\left(\pi_m, \pi_n\right) &= \left[\begin{array}{lll} \mathcal{A}_{m,n}^0 & \mathcal{A}_{m,n}^1 & \mathcal{A}_{m,n}^2 \end{array}\right], \\
\mathcal{A}_{m,n}^0 &= \left(\pi_n^1 \otimes \pi_m^1\right) \Sigma_0 + \left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_0 + \left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_0 + \left(\pi_n^2 \otimes \pi_m^2\right) \Psi_0, \\
\mathcal{A}_{m,n}^1 &= \left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_1 + \left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_1 + \left(\pi_n^2 \otimes \pi_m^2\right) \Psi_1, \\
\mathcal{A}_{m,n}^2 &= \left(\pi_n^1 \otimes \pi_m^1\right) \Sigma_1 + \left(\pi_n^1 \otimes \pi_m^2\right) \Lambda_2 + \left(\pi_n^2 \otimes \pi_m^1\right) \Gamma_2 + \left(\pi_n^2 \otimes \pi_m^2\right) \Psi_2.
\end{aligned}
$$

To obtain the products of vectors involving the state vector X_t, we note that

$$
\begin{aligned}
\pi_m X_t X_t' \pi_n' &= \left[\begin{array}{lll} \pi_m^0 & \pi_m^1 & \pi_m^2 \end{array}\right] \left[\begin{array}{ccc} 1 & x_t' & \tilde{x}_t' \\ x_t & x_t x_t' & 0 \\ \tilde{x}_t & 0 & 0 \end{array}\right] \left[\begin{array}{c} \pi_n^{0\prime} \\ \pi_n^{1\prime} \\ \pi_n^{2\prime} \end{array}\right], \\
&= \left(\pi_m^0 + \pi_m^1 x_t + \pi_m^2 \tilde{x}_t\right) \pi_n^{0\prime} + \left(\pi_m^0 x_t' + \pi_m^1 x_t x_t'\right) \pi_n^{1\prime} + \pi_m^0 \tilde{x}_t' \pi_n^{2\prime}, \\
&= \left(\pi_n^0 \otimes \pi_m^0\right) + \left(\pi_n^0 \otimes \pi_m^1\right) x_t + \left(\pi_n^0 \otimes \pi_m^2\right) \tilde{x}_t + \left(\pi_n^1 \otimes \pi_m^0\right) x_t \\
&\quad + \left(\pi_n^1 \otimes \pi_m^1\right) \tilde{x}_t + \left(\pi_n^2 \otimes \pi_m^0\right) \tilde{x}_t.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\pi_m X_t X_t' \pi_n' &= \mathcal{B}\left(\pi_m, \pi_n\right) X_t, \\
\mathcal{B}\left(\pi_m, \pi_n\right) &= \left[\begin{array}{lll} \mathcal{B}_{m,n}^0 & \mathcal{B}_{m,n}^1 & \mathcal{B}_{m,n}^2 \end{array}\right], \\
\mathcal{B}_{m,n}^0 &= \left(\pi_n^0 \otimes \pi_m^0\right) vec(I) = vec(\pi_n^0 * \pi_m^0), \\
\mathcal{B}_{m,n}^1 &= \left(\pi_n^0 \otimes \pi_m^1\right) + \left(\pi_n^1 \otimes \pi_m^0\right), \\
\mathcal{B}_{m,n}^2 &= \left(\pi_n^0 \otimes \pi_m^2\right) + \left(\pi_n^1 \otimes \pi_m^1\right) + \left(\pi_n^2 \otimes \pi_m^0\right).
\end{aligned}
$$