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AN EQUILIBRIUM APPROACH

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ABSTRACT

This paper proposes that equilibrium valuation is a powerful method to generate endogenous jumps in asset prices, which provides a structural alternative to traditional reduced-form specifications with exogenous discontinuities. We specify an economy with continuous consumption and dividend paths, in which endogenous price jumps originate from the market impact of regime-switches in the drifts and volatilities of fundamentals. We parsimoniously incorporate shocks of heterogeneous durations in consumption and dividends while keeping constant the number of parameters. Equilibrium valuation creates an endogenous relation between a shock's persistence and the magnitude of the induced price jump. As the number of frequencies driving fundamentals goes to infinity, the price process converges to a novel stochastic process, which we call a multifractal jump-diffusion.

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1. Introduction

In continuous-time settings, jumps in financial prices seem necessary to account for thick tails in asset returns, and the corresponding implied volatility smiles in near-maturity options.[1] In a seminal contribution, Merton (1976) assumes that the stock price follows an exogenous jump-diffusion with constant volatility. Subsequent research considers econometric refinements such as: stochastic volatility, priced jumps, jumps in volatility, correlation between jumps in returns and volatility, and infinite activity.[2,3] For example, Bakshi, Cao, and Chen (1997) and Bates (2000) consider price processes with exogenous jumps and stochastic volatility, concluding that additional discontinuities in volatility are necessary to match option valuations. Duffie, Pan, and Singleton (2000) consequently analyze an extension with discrete volatility changes while exogenously specifying the relation between volatility and returns. A related line of research (e.g., Madan, Carr, and Chang, 1998) advocates in favor of pure jump processes, which permit infinite activity with many small events and fewer large discontinuities. In all of this literature, jumps in valuations, and their relation to volatility, are exogenously specified.[4]

In this paper, we propose that equilibrium valuation is a powerful method to generate endogenous discontinuities in asset prices, and provides a structural alternative to ad hoc jump specifications. We consider an exchange economy with regime-switching fundamentals and endogenously obtain a number of return characteristics that prior literature specifies exogenously and emphasizes as appealing. Our approach builds on a standard consumption-based asset-pricing economy with homogeneous investors, where dividends and consumption may be identical as in Lucas (1978), or correlated but not identical as in Campbell and Cochrane (1999). We specify consumption and dividends as continuous diffusions, but permit discrete changes in their drift rates and volatilities through a stationary regime-switching Markov state vector. This setup produces endogenous jumps in stock prices, and equilibrium characterizes the feedback between volatility fluctuations and price discontinuities. Our paper therefore bridges a gap

[1] Numerous studies provide evidence for jumps in stock or other financial returns, based either directly on returns or on the prices of derivative assets. See, for example, Andersen, Benzoni, and Lund (2002), Ball and Torous (1985), Bates (1996), Carr, Geman, Madan, and Yor (2002), Carr and Wu (2003), Eraker, Johannes, and Polson (2003), Jarrow and Rosenfeld (1984), Jorion (1988), Maheu and McCurdy (2004), and Press (1967).

[2] Jump processes are classified as having finite or infinite activity depending on whether the number of jumps in a bounded time interval is finite or infinite.

[3] Examples include Barndorff-Nielsen (1998), Bates (1996, 2000), Carr and Wu (2004), Carr, Geman, Madan, and Yor (2002), Duffie, Pan, and Singleton (2000), Eberlein, Keller, and Prause (1998), Liu, Pan, and Wang (2005), Naik and Lee (1990), and Pan (2002).

[4] Endogenous jumps arise in models of real investment with non-convex adjustment costs (e.g., Casassus, Collin-Dufresne and Routledge, 2004). The exercise of a real option is the source of the discontinuity in this earlier literature, which thus tends to focus on lower frequency dynamics.

between jump-diffusions and the extensive discrete-time literature relating exogenous movements in dividend volatility to endogenous "feedback" effects in returns (e.g., Abel, 1988; Barsky, 1989; Calvet and Fisher, 2005; Campbell and Hentschel, 1992).

We incorporate shocks of heterogeneous durations into our economy by adopting the Markov-Switching Multifractal (MSM) of Calvet and Fisher (2001). Under this assumption, dividend volatility is the product of a vector of state components that follow independent regime-switching processes. The components are assumed to have identical marginal distributions but heterogeneous durations. Thus, some may switch on average only once every several years or decades, while others can have average durations measured in days or less. This multifactor volatility specification is highly parsimonious and requires only a few parameters regardless of the size of the state vector. Previous research shows that MSM is consistent with the slowly declining autocovariograms, fat tails, and power variations of financial series. It further provides a closed form likelihood, and substantially outperforms standard benchmarks volatility models both in- and out-of-sample.[5]

The present paper now embeds an MSM specification for dividends and consumption within a continuous-time equilibrium and explores the consequences for endogenous prices. We find that the asset value then displays jumps of heterogeneous frequencies, and the largest jump sizes are endogenously triggered by the most persistent volatility shocks. The model thus produces many small jumps and fewer large jumps, which has been emphasized as appealing by Madan, Carr, and Chang (1998), Carr, Geman, Madan, and Yor (2002), and others. Our equilibrium contributes to this literature by endogenizing the heterogeneity of jump sizes and the association between jump-size and frequency.

We then consider the impact of allowing the number of volatility state variables to become countably infinite. Under mild conditions, the dividend process weakly converges to a multifractal diffusion,[6] building on results from Calvet and Fisher (2001). Even more striking, the equilibrium price:dividend ratio converges to an infinite intensity pure jump process with heterogeneous frequencies. Prices are then conveniently decomposed into the continuous multifractal diffusion and the infinite intensity pure jump process, creating a new stochastic process that we accordingly call a *multifractal jump-diffusion*. A jump in stock prices occurs in the neighborhood of any instant, but the process is continuous almost everywhere.

[5] MSM has been shown to outperform GARCH, Markov-Switching GARCH and Fractionally Integrated GARCH (Calvet and Fisher, 2002, 2004; Calvet, Fisher, and Thompson, 2006; Lux, 2006).

[6] One definition of multifractality relates to the local behavior of sample paths. For example, Itô diffusions have local variations of the order $(dt)^{1/2}$, but multifractals permit variations of the order $(dt)^{\beta(t)}$, where the local scale $\beta(t)$ takes a continuum of values in any finite interval. This rich infinitesimal behavior is intertwined with appealing properties over finite intervals, as made clear by the empirical contributions discussed above.

For simplicity, the majority of the paper focuses on time-separable preferences. The stochastic discount factor is then continuous, and endogenous jumps in stock valuations are "unpriced" in the sense that they do not affect expected excess returns (e.g., Merton, 1976). In the final section, we show that it is straightforward to obtain priced jumps by considering non-separable preferences. Specifically, previous work in discrete time (Calvet and Fisher, 2005) uses Epstein-Zin utility to permit that switches in the state vector simultaneously impact the SDF and P/D ratio, and hence are priced. The equilibrium impact of non-separable recursive utility easily extends to continuous-time, providing additional flexibility in structural modelling of endogenous jump-diffusions.

We emphasize that the main goal of paper is to show that equilibrium conditions can help to generate a parsimonious but rich model of asset prices, including jumps in valuations, multifrequency volatility shocks, negative correlation between jumps and volatility, infinite activity, and priced jumps. For expositional clarity, the model is not presented at the highest level of generality. We anticipate that extensions to heterogeneous investors, incomplete markets, and more general preferences will broaden the applicability of our approach. These topics deserve further research.

The remainder of the paper is structured as follows: Section Two sets out the general consumption-based model with regime-switching dividends and endogenous price jumps. Section Three introduces the MSM dividend specification and derives the equilibrium price process with multifrequency jumps. Section Four considers the weak limit as the number of frequencies goes to infinity, and derives the multifractal jump-diffusion. Section Five extends the approach to stochastic differential utility and endogenous jump premia. All proofs are in an Appendix unless stated otherwise.

2. An Equilibrium Model with Endogenous Price Jumps

This section develops a continuous-time equilibrium model with regime-shifts in the mean and volatility of consumption and dividend growth.[7]

2.1. Preferences, Information and Income

We consider an exchange economy with a single consumption good defined on the set of instants $t \in [0, \infty)$. The information structure is represented by a filtration $\{\mathcal{F}_t\}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

[7] Following the seminal contribution of Hamilton (1989), several authors have considered discrete-time settings with regime-shifts in the drifts and volatilities of consumption and/or dividends (e.g., Calvet and Fisher, 2005; Cechetti, Lam, and Mark, 1990; Garcia, Luger, and Renault, 2003; Lettau, Ludvigson, and Wachter, 2004). In continuous time, Veronesi (1999,2000) and David and Veronesi (2002) investigate the impact of investor learning about Markov-switches in the drift rate of a Lucas economy and an IID consumption economy. In the settings they consider, information diffuses slowly, and hence beliefs and prices have continuous sample paths.

The economy is specified by two independent stochastic processes: a bivariate Brownian motion $Z_t = (Z_Y(t), Z_D(t)) \in \mathbb{R}^2$ and a random state vector $M_t \in \mathbb{R}_+^{\bar{k}}$, where $\bar{k}$ is a finite integer. The processes Z and M are adapted to the filtration $\{\mathcal{F}_t\}$ and mutually independent. The bivariate Brownian Z has zero mean and covariance matrix

$$\begin{pmatrix} 1 & \rho_{Y,D} \\ \rho_{Y,D} & 1 \end{pmatrix},$$

where the correlation coefficient $\rho_{Y,D} = Cov(dZ_Y, dZ_D)/dt$ is strictly positive. The vector M_t is a stationary Markov process with right-continuous sample paths. We make no other regularity assumptions in the rest of the Section. The process M_t can have either a discrete or continuous support, and its sample paths may be discontinuous.

The economy is populated by a finite set of identical investors $h \in \{1, .., H\}$, who have homogeneous information, preferences and endowment. Investors know at t the realization of the processes Z and M up to this date, and have information set $I_t = \{(Z_s, M_s); s \leq t\}$. The common utility is given by

$$U_t = \mathbb{E}\left[\int_0^{+\infty} e^{-\delta s} u(c_{t+s}) ds \middle| I_t\right],$$

where the discount rate is a positive constant: $\delta \in (0, \infty)$. The Bernoulli utility $u(\cdot)$ is twice continuously differentiable, and satisfies the usual monotonicity and concavity conditions: $u' > 0$ and $u'' < 0$. We also assume that the Inada conditions hold: $\lim_{c \to 0} u'(c) = +\infty$ and $\lim_{c \to +\infty} u'(c) = 0$.

Agents continuously receive an exogenous endowment stream $Y_t \in (0, \infty)$. The rate of income flow Y_t follows a geometric Brownian motion with stochastic drift $g_Y(M_t)$ and volatility $\sigma_Y(M_t)$, where $g_Y(\cdot)$ and $\sigma_Y(\cdot)$ are deterministic measurable functions defined on $\mathbb{R}_+^{\bar{k}}$ and taking values on the real line. Specifically,

Assumption 1 (Endowment process). *The stochastic drift and volatility satisfy* $\mathbb{E}\left[\int_0^t |g_Y(M_s)| \, ds\right] < \infty$ *and* $\mathbb{E}\left[\int_0^t \sigma_Y^2(M_s) ds\right] < \infty$ *for all* t*. The exogenous income stream is given by*

$$\ln(Y_t) \equiv \ln(Y_0) + \int_0^t \left[g_Y(M_s) - \frac{\sigma_Y^2(M_s)}{2}\right] ds + \int_0^t \sigma_Y(M_s) dZ_Y(s)$$

at every instant $t \in [0, \infty)$.

The moment conditions guarantee that the stochastic integrals are well-defined. By Ito's lemma, the income flow satisfies the stochastic differential equation

$$\frac{dY_t}{Y_t} = g_Y(M_t)dt + \sigma_Y(M_t)dZ_Y(t). \tag{2.1}$$

Note that Y_t is sometimes called *cumulative* income process in the literature.

2.2. Financial Markets and Equilibrium

Agents can trade two financial assets: a bond and a stock. The bond has an instantaneous rate of return $r_f(t)$, which is endogenously determined in equilibrium. Its net supply is equal to zero.

The stock is a claim on the stochastic dividend stream $\{D_t\}_{t\geq 0}$, which is specified by

Assumption 2 (Dividend process). *The dividend stream is given by*

$$\ln(D_t) \equiv \ln(D_0) + \int_0^t \left[g_D(M_s) - \frac{\sigma_D^2(M_s)}{2} \right] ds + \int_0^t \sigma_D(M_s) dZ_D(s),$$

where $g_D(\cdot)$ and $\sigma_D(\cdot)$ are measurable functions defined on $\mathbb{R}_+^{\bar{k}}$ and valued in $\mathbb{R}$ such that $\mathbb{E}\left[\int_0^t |g_D(M_s)|\, ds\right] < \infty$ and $\mathbb{E}\left[\int_0^t \sigma_D^2(M_s) ds\right] < \infty$ for all t.

By Ito's lemma, dividend growth satisfies:

$$\frac{dD_t}{D_t} = g_D(M_t)dt + \sigma_D(M_t)dZ_D(t). \tag{2.2}$$

We leave the exact specification of the drift $g_D(\cdot)$ and volatility $\sigma_D(\cdot)$ fully general in the rest of Section 2. The dividend process has continuous sample paths, but its drift and volatility can exhibit discontinuities.

Let N_s denote the per-capita net supply of the stock. The choice of N_s has been widely discussed in the literature (e.g. Anderson and Raimondo, 2006; Santos and Veronesi, 2005), because the constraint $D_t \leq C_t$ is difficult to verify when C_t and D_t are imperfectly correlated diffusions. The resolution of this modeling issue is beyond the scope of this paper. We simply assume that $N_s = 1$ when $D_t = Y_t$ (Lucas tree economy), or $N_s = 0$ otherwise. The latter is a simplifying assumption that has often been used in the continuous-time equilibrium literature (e.g. Duffie and Zame, 1989; Huang, 1987; Karatzas and Shreve, 1998).

Each agent selects a consumption-portfolio strategy (c^h, N^h, B^h) defined on $\Omega \times [0, \infty)$ and taking values on $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$, where $c^h(\omega, t)$, $N^h(\omega, t)$ and $B^h(\omega, t)$ respectively denote consumption, stockholdings and bondholdings in every date-event (ω, t). A strategy is called *admissible* if it is adapted and satisfies the usual individual budget contraint. We assume that financial markets are frictionless.

Definition (General Equilibrium). *An* equilibrium *consists of a stock price process P, an interest rate process r_f, and a collection of individual admissible consumption-portfolio plans $(c^h, N^h, B^h)_{1\leq h\leq H}$, such that: (1) for every h, the plan (c^h, N^h, B^h)*

maximizes utility given the budget constraint; (2) securities and good markets clear:

$$\frac{1}{H}\sum_{h=1}^{H} N^h(t,\omega) = N_s, \quad \frac{1}{H}\sum_{h=1}^{H} B^h(t,\omega) = 0, \quad \text{and} \quad \frac{1}{H}\sum_{h} c^h(t,\omega) = Y(t,\omega)$$

for almost all (t,ω).

The proof of equilibrium existence in continuous-time economies has been the object of a wide literature.[8] We focus here on the simple case of an autarkic equilibrium, in which individual consumption equals individual income: $c^h(t,\omega) = Y(t,\omega)$ for every h,t,ω.

To reflect the identity between consumption and income, let $C_t \equiv Y_t$, $Z_C \equiv Z_Y$, $g_C(.) \equiv g_Y(.)$, $\sigma_C(.) \equiv \sigma_Y(.)$, and $\rho_{C,D} \equiv \rho_{Y,D}$. The special case where g_C and σ_C are constant implies IID consumption growth - a standard assumption in asset pricing that is broadly consistent with postwar US data (e.g. Campbell, 2003). Recently, however, Bansal and Yaron (2004), Hansen, Heaton and Li (2005), Lettau, Ludvigson and Wachter (2004) and others have argued that consumption may contain small, highly persistent componenents with large price impacts. The more general dynamics (2.1) accommodate this possibility.

The stochastic discount factor (SDF) is equal to instantaneous marginal utility:

$$\Lambda_t = e^{-\delta t} u'(C_t).$$

It satisfies the stochastic differential equation:

$$\frac{d\Lambda_t}{\Lambda_t} = -r_f(M_t)dt - \alpha(C_t)\sigma_C(M_t)dZ_C(t),$$

where $\alpha(c) \equiv -cu''(c)/u'(c)$ denotes the coefficient of relative risk aversion and $\pi(c) \equiv -cu'''(c)/u''(c)$ is the coefficient of relative risk prudence. The instantaneous interest rate is

$$r_f(M_t) = \delta + \alpha(C_t)g_C(M_t) - \alpha(C_t)\pi(C_t)\sigma_C^2(M_t)/2. \tag{2.3}$$

It increases with investor impatience and the growth rate of the economy, and is reduced by the precautionary motive.

In equilibrium, the stock price P_t is given by

$$\frac{P_t}{D_t} = \mathbb{E}\left[\int_0^{+\infty} e^{-\delta s}\frac{u'(C_{t+s})}{u'(C_t)}\frac{D_{t+s}}{D_t}ds \,\middle|\, I_t\right].$$

[8] Bick (1990), Cox, Ingersoll and Ross (1985), Duffie and Skiadas (1994), Duffie and Zame (1989), He and Leland (1993) and Raimondo (2005) establish the existence of equilibrium in single-agent economies. The case of heterogeneous investors is considered by Anderson and Raimondo (2006), Bank and Riedel (2001) and Mas-Colell and Richard (1991).

The joint distribution of $(C_{t+s}; D_{t+s}/D_t)$ depends on the state M_t and the consumption level C_t, but not on the initial dividend level D_t. The valuation ratio is therefore a deterministic function of M_t and C_t, which will henceforth be denoted by $Q(M_t, C_t)$. Shifts in the state M_t induce discontinuous changes in the P/D ratio and the stock price. We find it convenient to use lower cases for the log of prices, dividends and P/D.

Proposition 1 (Equilibrium stock price). *The stock price follows a jump diffusion, which can be written in logs as the sum of the continuous dividend process and the price:dividend ratio:*

$$p_t = d_t + q(M_t, C_t).$$

A price jump occurs when there is a discontinuous change in the Markov state M_t driving the continuous dividend and consumption processes.

The endogenous price jumps contrast with the continuity of the fundamentals and the SDF.

2.3. Equilibrium Dynamics under Isoelastic Utility

As noted by Campbell (2003), consumption and wealth have increased manyfold over the past two centuries, but real interest rates, risk premia and valuation ratios have not consistently trended up or down. To capture the apparent stationarity of aggregate stock returns and the P:D ratio in a representative agent setting, power utility is often useful because of its scale invariance.[9] We correspondingly assume that every investor has the same constant relative risk aversion $\alpha \in (0, \infty)$, i.e.

$$u(c) \equiv \begin{cases} c^{1-\alpha}/(1-\alpha) & \text{if } \alpha \neq 1, \\ \ln(c) & \text{if } \alpha = 1. \end{cases}$$

We easily show:

Proposition 2 (Equilibrium with isoelastic utility). *The price:dividend ratio is a deterministic function of the Markov state:*

$$q(M_t) = \ln \mathbb{E}_t \left(\int_0^{+\infty} e^{-\int_0^s \left[r_f(M_{t+h}) - g_D(M_{t+h}) + \alpha \sigma_C(M_{t+h}) \sigma_D(M_{t+h}) \rho_{C,D} \right] dh} ds \right), \tag{2.4}$$

where $\mathbb{E}_t$ denotes the conditional expectation given M_t.

[9] There is of course abundant evidence that individual investors do not have isoelastic utility. For instance, richer agents are widely known to invest a higher share in risky assets than poorer agents (e.g. Carroll, 2002). The implications of such wealth effects are central to finance and economics and well-deserving of further research, but lie outside the scope of the present paper.

The P/D ratio increases with the anticipation of high growth and low volatility of future dividends (for a given distribution of consumption drift and volatility).

Over an infinitesimal time interval, the stock price changes by

$$d(p_t) = d(d_t) + \Delta(q_t),$$

where $\Delta(q_t) \equiv q_t - q_{t^-}$ denotes the finite variation of the price:dividend ratio in case of a regime change. Consider the effect of a Markov switch that increases the volatility of current and future dividends (without impacting consumption). The P/D ratio falls and induces a negative realization of $\Delta(q_t)$. Market pricing can thus generate an endogenous negative correlation between volatility changes and price jumps. This contrasts with earlier jump models where the relation between discontinuities and volatility is exogenously postulated (e.g. Duffie, Pan and Singleton, 2000; Carr and Wu, 2004).

The conditional excess return

$$-\frac{1}{dt}\mathbb{E}_t\left(\frac{d\Lambda_t}{\Lambda_t}\frac{dP_t}{P_t}\right) = \alpha\sigma_C(M_t)\sigma_D(M_t)\rho_{C,D}$$

varies through time with the volatilities of the SDF and dividends. Early work on derivative valuation considered the case where jump risk is diversifiable and therefore "unpriced" in the sense that required returns are not affected (e.g., Merton, 1976). In our above model, the SDF Λ_t is continuous due to the assumptions of time-separable utility and continuous consumption paths. Hence, the possibility of a price discontinuity in the next instant does not contribute to the conditional risk premium, and jumps are "unpriced." To obtain priced jumps is nonetheless straightforward. For example, Calvet and Fisher (2005) in discrete time use Epstein-Zin utility to obtain an equity risk premium associated with the switches in M_t. Section 5 thus considers non-separable preferences, which can generate discontinuities in the SDF and priced jumps that impact the equity risk premium. For expositional simplicity and to emphasize other issues, we focus in Sections 2-4 on separable isoelastic utility.

We readily acknowledge that the economy in this paper is quite specific and does not have the level of generality sometimes considered in continuous-time general equilibrium theory (e.g. Anderson and Raimondo, 2006; Duffie and Zame, 1989; Huang, 1987; Karatzas and Shreve, 1998; Raimondo, 2005). Our objective is somewhat different and consists of proposing a novel and simple class of Markov-switching economies that endogenously generate equilibrium jumps. Our results are robust to some degree of investor heterogeneity if markets are complete. Consider for instance an Arrow-Debreu economy in which agents have heterogeneous coefficients of relative risk aversion α_h and homogeneous discount rates $\delta > 0$. Huang (1987) shows that equilibrium asset prices are then supported by an isoelastic representative investor. Financial incompleteness would seem to be a natural assumption, however, in the Markov-switching environments we consider. Earlier work suggests that rich price dynamics can then arise (e.g.

Constantinides and Duffie, 1996; Calvet, 2001). Extensions to incomplete markets and investor heterogeneity seem likely to broaden the applicability of our approach, and are well-deserving of further research.

3. A Multifrequency Jump-Diffusion for Equilibrium Stock Prices

We now show how to parsimoniously incorporate multifrequency shocks into the economy. We specify in Section 3.1 the dynamics of the dividend process D_t with a finite number of volatility frequencies. We then discuss in Section 3.2 how to select the consumption process and thus complete the construction of the multifrequency exchange economy. The outcome of these specifications is an endogenous multifrequency jump-diffusion for prices, investigated in Section 3.3.

3.1. Dividends with Multifrequency Volatility

We introduce shocks of multiple frequencies by assuming that dividends follow an MSM process (Calvet and Fisher, 2001, 2004), as is now explained. The basic principle of the construction is that the Markov state vector

$$M_t = (M_{1,t}; M_{2,t}; ...; M_{\bar{k},t}) \in \mathbb{R}_+^{\bar{k}}$$

has components with heterogeneous durations. Persistence is highest for the first component, and progressively diminishes with the component index k.

We assume that each component $M_{k,t}$ is itself a Markov process. For parsimony, these components are mutually independent: $M_{k,t}$ and $M_{k',t'}$ are statistically independent if $k \neq k'$. Given the Markov state M_t at date t, its dynamics over an infinitesimal interval are conveniently defined as follows. For each $k \in \{1, .., \bar{k}\}$, the change in $M_{k,t}$ is triggered by a Poisson arrival with intensity γ_k. The component $M_{k,t+dt}$ is drawn from a fixed distribution M if there is an arrival, and otherwise remains at its current value: $M_{k,t+dt} = M_{k,t}$. The construction can be summarized as:

$$\begin{array}{ll} M_{k,t+dt} \text{ drawn from distribution } M & \text{with probability } \gamma_k dt \\ M_{k,t+dt} = M_{k,t} & \text{with probability } 1 - \gamma_k dt. \end{array}$$

The Poisson arrivals and new draws from M are independent across k and t.

We observe that the components $M_{k,t}$ differ in their transition probabilities but not in their marginal distribution M. Each component therefore follows a Markov process that is identical except for time scale. These features greatly contribute to the parsimony of the model. As with any process driven by Poisson arrivals, the sample paths of a component $M_{k,t}$ are right-continuous and have a limit point to the left of any instant, i.e. are 'cadlag' functions.[10]

The construction can accommodate any distribution M with positive support. We normalize the distribution by imposing that it has a unit mean: $\mathbb{E}(M) = 1$. The parameter $\bar{\sigma}_D$ is then the unconditional standard deviation of the dividend growth process: $Var(dD_t/D_t) = \bar{\sigma}_D^2 dt$. For parsimony, we henceforth consider that components are drawn from a family of distributions specified by a single parameter $m_0 \in \mathbb{R}$. We also tightly parameterize the intensities of arrival by assuming

$$\gamma_k = \gamma_1 b^{k-1}, \qquad k \in \{1, .., \bar{k}\}. \tag{3.1}$$

The parameter γ_1 determines the persistence of the lowest frequency component, and b the spacing between component frequencies.

We consider in the remainder of this paper that the state M_t only affects volatility. Specifically, dividends have a constant growth rate

$$g_D(M_t) \equiv \bar{g}_D,$$

and a stochastic volatility equal to the renormalized product

$$\sigma_D(M_t) \equiv \bar{\sigma}_D \left(\prod_{k=1}^{\bar{k}} M_{k,t} \right)^{1/2}, \tag{3.2}$$

where $\bar{\sigma}_D$ is a positive constant.[11] The components of the state vector interact multiplicatively, and for this reason are called *multipliers*. These conditions conclude the specification of the Markov-Switching Multifractal (MSM) volatility process. Calvet and Fisher (2001) introduce MSM, and subsequent work demonstrates its empirical validity in financial data (Calvet and Fisher, 2004, 2005; Calvet, Fisher, and Thompson, 2006). We summarize the structure of the dividend process in

Assumption 3 (Multifrequency dynamics). *The dividend process has a constant drift and an MSM volatility with a finite number $\bar{k}$ of frequencies.*

Figure 1 illustrates the construction of MSM. We assume for simplicity that the distribution M is a binomial that can take the high value $m_0 \in [1, 2)$ or the low value

[10] Cadlag is a French acronym for *continue à droite, limites à gauche.* We refer the reader to Billingsley (1999) for further details.

[11] The conditions $\mathbb{E}\left[\int_0^t |g_D(M_s)|\, ds\right] = |\bar{g}_D| t < \infty$ and $\mathbb{E}\left[\int_0^t \sigma_D^2(M_s) ds\right] = \bar{\sigma}_D^2 t < \infty$ are then trivially satisfied.

$m_1 \in (0, 1]$ with equal probability.[12] The normalization $\mathbb{E}(M) = 1$ implies that $m_1 = 2 - m_0$. The top three panels represent the sample path of the volatility components $M_{k,t}$ for k varying from 1 to 3. We see that the number of switches tends to increase with k due to the geometric progression of arrival intensities (3.1). The bottom panel represents the dividend variance $\sigma_D^2(M_t) \equiv \bar{\sigma}_D^2 \; M_{1,t}M_{2,t}M_{3,t}$, where $\bar{\sigma}_D = 1$. The dividend variance exhibits peaks and troughs, which helps to capture the changing variability of dividend news. The construction generates cycles of different frequencies, consistent with the economic intuition that there are volatile decades and less volatile decades, volatile years and less volatile years, and so on.

Figure 2 then shows the complete construction of a dividend process with eight frequencies. The first panel shows the volatility $\sigma_D^2(M_t)$, where for comparison the first three components are identical to those reported in Figure 1. In contrast to the previous three-stage construction, this panel now shows much greater detail with more pronounced peaks and intermittent bursts of volatility. The larger number of volatility components accommodates a broad range of long-run, medium-run, and short-run dynamics. The second panel illustrates the impact of these various frequencies on dividend growth. Finally, the last panel reports the dividend process D_t. These last two panels confirm that MSM generates both short and long-swings in volatility and thick tails in the dividend growth series, while by design there are no jumps in dividends themselves.

We finally emphasize two theoretical properties of our approach. First, MSM permits the state space to be very large. For instance with a binomial distribution M, the number of states is equal to $2^{\bar{k}}$. The example considered in Figure 2 is based on 2^8 or 256 states. Second, MSM is very parsimonious. In a general Markov chain, the size of the transition matrix is equal to the square of the number of states. For instance a general Markov chain with 2^8 states generally needs to be parametrized by $2^{16} = 65,536$ elements. In comparison, the MSM dividend dynamics are fully characterized by

$$(\bar{g}_D, \bar{\sigma}_D, m_0, \gamma_1, b) \in \mathbb{R}^5,$$

where $\bar{g}_D$ and $\bar{\sigma}_D$ quantify the mean and standard deviations of dividend growth, m_0 parameterizes the distribution M, γ_1 is the intensity of the most persistent component, and b quantifies the growth rates of intensities. The five-parameter specification accommodates an arbitrary number of frequencies.

3.2. Multifrequency Economies

The previous section specified the dividend process. We now turn to the specification of aggregate consumption, which will close the description of the exchange economy.

[12] Earlier research shows the empirical usefulness of the binomial distribution in MSM (e.g. Calvet and Fisher, 2004).

Case 1: Lucas tree economy. In a general equilibrium model, a natural specification is to consider that the stock is a claim on aggregate consumption: $D_t = C_t$. The seven parameters $(\bar{g}_D, \bar{\sigma}_D, m_0, \gamma_1, b, \alpha, \delta)$ then fully specify the jump-diffusion price process. By Proposition 2, the P/D ratio is given by

$$\mathbb{E}_t \left(\int_0^{+\infty} e^{-[\delta-(1-\alpha)\bar{g}_D]s - \frac{\alpha(1-\alpha)}{2} \int_0^s \sigma_D^2(M_{t+h})dh} ds \right). \tag{3.3}$$

An increase in volatility reduces the price:dividend ratio only if $\alpha < 1$, which is consistent with earlier research in discrete time (e.g., Barsky, 1989; Abel, 1988).[13]

Case 2: IID consumption. We can alternatively assume that consumption has a constant drift and volatility. The interest rate (2.3) is then constant through time. The equilibrium model is specified by the dividend parameters $(\bar{g}_D, \bar{\sigma}_D, m_0, \gamma_1, b)$, the utility coefficients (α, δ), the consumption parameters $(\bar{g}_C, \bar{\sigma}_C)$, and the correlation $\rho_{C,D}$.

By Proposition 2, the price:dividend ratio is equal to

$$\mathbb{E}_t \left(\int_0^{+\infty} e^{-(r_f - \bar{g}_D)s - \alpha\rho_{C,D}\bar{\sigma}_C \int_0^s \sigma_D(M_{t+h})dh} ds \right).$$

High volatility feeds into low asset prices for any choices of relative risk aversion α. This approach fits well with the equilibrium/volatility feedback literature, which suggests that aggregate stock prices decrease with the volatility of dividend news (e.g. Bekaert and Wu, 2000; Calvet and Fisher, 2005; Campbell and Hentschel, 1992; French, Schwert and Stambaugh, 1987; Pindyck, 1984).

Case 3: Multivariate MSM. We develop in the Appendix a multivariate extension of MSM that permits more flexible specifications of consumption and the SDF. This approach helps to construct SDF models with a stochastic volatility only partially correlated to the stochastic volatility of dividends. While this construction is appealing for empirical applications, we choose for expositional simplicity to focus on Cases 1 and 2 in the remainder of the paper.

3.3. The Equilibrium Stock Price

Jumps in our model are triggered by regime-changes in the volatility components. Since heterogeneous components switch at a range of frequencies, the model avoids the difficult

[13] When future consumption becomes riskier, the ratio is affected by two opposite effects. First, the covariances become more negative and *reduce* the price:dividend ratio. Second, the precautionary motive increases the expected marginal utility of future consumption, which lowers interest rates and tends to *increase* P/D.

choice of a unique frequency and size for "rare events," which is a common issue in specifying traditional jump-diffusions.[14]

We can analytically quantify the link between jump frequency and size when the multipliers exhibit small variations around unity. More formally, we consider the parametric family of state processes $M_t(\varepsilon) = 1 + \varepsilon(\nu_t - 1)$, $t \in \mathbb{R}_+$, $\varepsilon \in [0, 1)$, where ν is itself a fixed MSM state vector. The components of $M_t(\varepsilon)$ are equal to unity at every instant if $\varepsilon = 0$. For any given ν_t, we show in the Appendix how to linearize the P/D ratio around $\varepsilon = 0$.

Proposition 3 (First-order expansion of P/D). *The log of the P/D ratio is approximated around $\varepsilon = 0$ by the first-order Taylor expansion:*

$$q[M_t(\varepsilon)] = \bar{q} - q_1 \sum_{k=1}^{\bar{k}} \frac{M_{k,t}(\varepsilon) - 1}{\delta' + \gamma_k} + o(\varepsilon). \tag{3.4}$$

In the Lucas tree economy, we have $\delta' = \delta - (1-\alpha)\bar{g}_D + \alpha(1-\alpha)\bar{\sigma}_D^2/2$, $\bar{q} = -\ln(\delta')$ and $q_1 = \alpha(1-\alpha)\bar{\sigma}_D^2/2$. When consumption is IID, we instead have $\delta' = r_f - \bar{g}_D + \alpha\rho_{C,D}\bar{\sigma}_C\bar{\sigma}_D$, $\bar{q} = -\ln(\delta')$ and $q_1 = \alpha\rho_{C,D}\bar{\sigma}_C\bar{\sigma}_D/2$.

When the distribution M is close to unity, the P/D ratio is approximated by a persistence-weighed sum of the volatility components. Low-frequency multipliers deliver persistent and discrete switches, which have a large impact on the P/D ratio. By contrast, higher frequency components have no noticeable effect on prices, but give additional outliers in returns through their direct effect on the tails of the dividend process. The price process is thus characterized by a large number of small jumps (high frequency $M_{k,t}$), a moderate number of moderate jumps (intermediate frequency $M_{k,t}$), and a small number of very large jumps. Intuition and earlier empirical research suggest that this is a good characterization of the dynamics of stock returns.

We illustrate in Figure 3 the endogenous multifrequency pricing dynamics of the model, in the case where consumption is IID. The top two panels present a simulated dividend process, in growth rates and in logarithms of the level respectively. The middle two panels then display the corresponding stock returns and log prices. We observe that the price series exhibits much larger movements than dividends, due to the presence of endogenous jumps in the P/D ratio. To see this clearly, the bottom two panels

[14] In the simplest exogenously specified jump-diffusions, it is often possible that discontinuities of heterogeneous but fixed sizes and different frequencies can be aggregated into a single collective jump process with an intensity equal to the sum of all the individual jumps, and a random distribution of sizes. A comparable analogy can be made for the state vector M_t in our model, but due to the equilibrium linkages between jump size and the duration of volatility shocks, and the state dependence of price jumps, no such reduction to a single aggregated frequency is possible for the equilibrium stock price.

represent consecutively: 1) the "feedback" effects, defined as the difference between log stock returns and log dividend growth, and 2) the price:dividend ratio. Consistent with Proposition 3, we observe that endogenous market pricing causes a few infrequent but large jumps in prices, with smaller but more numerous small discontinuities. The simulation demonstrates that the difference between stock returns and dividend growth can be large, even when variations in the P/D ratio are relatively modest and quite realistic, varying between 26 and 33.

The pricing model thus captures multifrequency stochastic volatility, endogenous multifrequency jumps in returns, and endogenous correlation between volatility and return innovations. We find it appealing that many of these features are derived not from exogenous econometric assumptions but from equilibrium conditions.

4. Price Dynamics with an Infinity of Frequencies

We now investigate how the price diffusion evolves as $\bar{k} \to \infty$, i.e. as components of increasingly high frequency are added into the state vector. This can help guide our judgement about the number of components that are useful in empirical applications. Two apparently contradictory observation can be made. On the one hand, Figures 1 and 2 suggest that the volatility process $\sigma_D(M_t)$ exhibits increasingly extreme behavior as $\bar{k}$ increases. On the other hand, the equilibrium jump-diffusion for prices seems to be quite insensitive to higher frequency components. We show in this Section how these two observations can be reconciled by deriving the limit behavior of the price dynamics.

4.1. Time Deformation

We begin by reviewing the limit behavior of MSM when the number of high-frequency components $\overline{k}$ goes to infinity. The parameters $(\bar{g}_D, \bar{\sigma}_D, m_0, \gamma_1, b)$ are fixed. Let $M_t = (M_{k,t})_{k=1}^{\infty} \in \mathbb{R}_+^{\infty}$ denote an MSM Markov state process with countably many frequencies. The process M_t is defined for $t \in [0, \infty)$, has mutually independent components, and each component $M_{k,t}$ is characterized by the transition probability $\gamma_k = \gamma_1 b^{k-1}$. For a finite $\bar{k}$, stochastic volatility is defined as the product of the first $\overline{k}$ components of the state vector: $\sigma_{D,\bar{k}}(M_t) \equiv \bar{\sigma}_D \ (M_{1,t} M_{2,t} ... M_{\overline{k},t})^{1/2}$.

Since instantaneous volatility $\sigma_{D,\bar{k}}(M_t)$ depends on an increasing number of components, the differential representation (2.2) becomes unwieldy as $\bar{k} \to \infty$. We instead find it convenient to characterize the dividend dynamics in terms of the time deformation

$$\theta_{\bar{k}}(t) \equiv \int_0^t \sigma^2_{D,\bar{k}}(M_s) ds.$$

Given a fixed instant t, the sequence $\{\theta_{\bar{k}}(t)\}_{\bar{k}=1}^{\infty}$ is a positive martingale with bounded expectation; by the martingale convergence theorem, the random variable $\theta_{\bar{k}}(t)$ converges to a limit distribution when $\bar{k} \to \infty$. A similar argument applies to any vector sequence $\{\theta_{\bar{k}}(t_1); ...; \theta_{\bar{k}}(t_d)\}$, guaranteeing that the stochastic process $\theta_{\bar{k}}$ has at most one limit point. We verify that a limit process does indeed exist by checking that the sequence $(\theta_{\bar{k}})_{\bar{k}}$ is tight.[15] Intuitively, tightness prevents the process from oscillating too wildly as $\bar{k} \to \infty$. As shown in Calvet and Fisher (2001), the tightness property holds on a *bounded* time interval $[0, T]$ under the following sufficient condition.

Assumption 4. $\mathbb{E}(M^2) < b$

This inequality restricts the fluctuations exhibited by the time deformation process by requiring that volatility shocks be sufficiently small or that intensities grow sufficiently fast. When T is finite, the sequence $\theta_{\bar{k}}$ then weakly converges to a limit process θ_∞, which generates continuous sample paths (Calvet and Fisher, 2001).

We now check that the same results hold when the time domain is unbounded. Consider the space $D[0, \infty)$ of cadlag functions defined on $[0, \infty)$, and let d_∞° denote the Skohorod distance. We show in the Appendix:

Proposition 4 (Time deformation with countably many frequencies). *Under Assumption 4, the sequence $(\theta_{\bar{k}})_{\bar{k}}$ weakly converges as $\bar{k} \to \infty$ to a measure θ_∞ defined on the metric space $(D[0, \infty), d_\infty^\circ)$. Furthermore, the sample paths of θ_∞ are continuous almost surely.*

The limiting process has a Markov structure analogous to MSM with a finite $\bar{k}$. We interpret $M_t = (M_{k,t})_{k=1}^\infty$ as the state vector of the limiting time deformation θ_∞.

Using the time-deformation approach, the dividend process for a finite $\bar{k}$ can be represented by

$$d_{\bar{k}}(t) \equiv d_0 + \bar{g}_D t - \theta_{\bar{k}}(t)/2 + B[\theta_{\bar{k}}(t)],$$

where B is a standard Brownian. By Proposition 4, $d_{\bar{k}}(t)$ therefore converges to

$$d_\infty(t) \equiv d_0 + \bar{g}_D t - \theta_\infty(t)/2 + B[\theta_\infty(t)]$$

as $\bar{k} \to \infty$.

The dividend process converges even though volatility $\sigma_{D,\bar{k}}(M_t)$ seems to have a degenerate behavior as $\bar{k} \to \infty$. This apparent contradiction is best understood by examining the local properties of the limit dividend process. The local variability of a sample path is characterized by the local Hölder exponent

$$\beta(t) = \max\{\beta \in \mathbb{R}_+ \text{ s.t. } |d(t + \Delta t) - d(t)| = O(|\Delta t|^\beta)\}.$$

[15] We refer the reader to Billingsley (1999) for a detailed exposition of weak convergence in function spaces.

The local Hölder exponent quantifies the order of variation of the process around instant t. In jump diffusions, the coefficient $\beta(t)$ is equal to 0 at points of discontinuity, and to 1/2 otherwise. In contrast, the continuous dividend process with countably many frequencies implies that $\beta(t)$ takes a *continuum* of values in any time interval, which is a defining property of a multifractal diffusion.[16]

4.2. Limiting Equilibrium Price Process

We now examine the equilibrium impact of increasingly many frequencies in the volatility of dividends. A particularly striking example is provided by the Lucas tree economies discussed in Section 3.2. We consider

Assumption 5. $\alpha \leq 1$ and $\rho = \delta - (1-\alpha)\bar{g}_D > 0$.

For finite $\bar{k}$, the equilibrium price:dividend ratio is given by (3.3), or equivalently

$$q_{\bar{k}}(t) = \ln \mathbb{E}\left[\int_0^{+\infty} e^{-\rho s - \frac{\alpha(1-\alpha)}{2}[\theta_{\bar{k}}(t+s)-\theta_{\bar{k}}(t)]} ds \,\middle|\, (M_{k,t})_{k=1}^{\bar{k}}\right]. \tag{4.1}$$

The price process has therefore the same distribution as

$$p_{\bar{k}}(t) \equiv d_{\bar{k}}(t) + q_{\bar{k}}(t).$$

When the number of frequencies goes to infinity, the dividend process has a well-defined limit. We check in the Appendix that the P/D ratio (4.1) is a positive submartingale, which also converges to a limit as $\bar{k} \to \infty$. These observations help to establish:

Proposition 5 (Jump diffusion with countably many frequencies). *Consider the maintained Assumptions $1-5$. When the number of frequencies goes to infinity, the log-price process weakly converges to*

$$p_\infty(t) \equiv d_\infty(t) + q_\infty(t),$$

where

$$q_\infty(t) = \ln \mathbb{E}\left[\int_0^{+\infty} e^{-\rho s - \frac{\alpha(1-\alpha)}{2}[\theta_\infty(t+s)-\theta_\infty(t)]} ds \,\middle|\, (M_{k,t})_{k=1}^{\infty}\right]$$

is a pure jump process. The limiting price is thus a jump diffusion with countably many frequencies.

[16] Multifractal diffusions were introduced in Calvet, Fisher and Mandelbrot (1997) and Calvet and Fisher (2002). We refer the reader to this earlier work for a more detailed discussion of local properties.

In an economy with countably many frequencies, the log-price process is the sum of (1) the continuous multifractal diffusion $d_\infty(t)$; and (2) the pure jump process $q_\infty(t)$. We correspondingly call $p_\infty(t)$ a *multifractal jump-diffusion.*

We observe that when $\bar{k} = \infty$, the state space is a continuum while the Lucas tree economy is still specified by the seven parameters $(\bar{g}_D, \bar{\sigma}_D, m_0, \gamma_1, b, \alpha, \delta)$. The equilibrium P/D ratio $q_\infty(t)$ exhibits rich dynamic properties. Within any bounded time interval, there exists almost surely (a.s.) at least one multiplier $M_{k,t}$ that switches and triggers a jump in the stock price. This property implies that a jump in price occurs a.s. in the neighborhood of any instant. The number of switches is also countable a.s. within any bounded time interval, implying that the process has infinite activity and is continuous almost everywhere. Equilibrium valuation therefore generates a limit P/D ratio that follows an infinite intensity pure jump process.

We illustrate in Figure 4 the convergence of the equilibrium price processes as $\bar{k}$ becomes large. The first panel shows a simulation with $\bar{k} = 2$ volatility components, and the following panels consecutively add higher frequency components to obtain paths with $\bar{k} = 4$, $\bar{k} = 6$, and $\bar{k} = 8$ components. Consistent with the theoretical construction, the figure is obtained by randomly drawing a trajectory of the Brownian motion Z_C in stage $\bar{k} = 0$, which is therafter taken as fixed. Similarly, each multiplier $M_{k,t}$ is drawn only once, so that $(M_{k,t})_{k=1}^{\bar{k}}$ does not vary when we move from stage $\bar{k}$ to stage $\bar{k}+1$. The figure suggests that the price process becomes progressively insensitive to the addition of new high-frequency components, and the sample path of the price process stabilizes. This illustrates the main result of Proposition 5: For low $\bar{k}$, adding components has a significant impact, and as $\bar{k}$ increases the process converges.

The results of this section provide useful guidance on the choice of the number of frequencies in theoretical and empirical applications. On the one hand, the convergence of the price process implies that the marginal contribution of additional components is likely to be small in applications concerned with fitting the price or return series. It is then convenient to consider a number of frequencies $\bar{k}$ that is sufficiently large to capture the heteroskedasticity of financial series, but sufficiently small to remain tractable. On the other hand, countably many frequencies might prove useful in more theoretical contexts, in which the local behavior of the price process needs to be carefully understood. Examples could include the construction of learning models or the design of dynamic hedging strategies.

5. Recursive Utility and Priced Jumps

In the previous sections, we have focused for simplicity on the case of time-separable preferences, which imply a continuous SDF Λ_t in our representative agent economy with continuous consumption. Jumps in stock valuations are then "unpriced" in the sense that they do not contribute to expected excess returns. Previous work in discrete time (Calvet and Fisher, 2005) uses Epstein-Zin utility to permit that switches in the state vector M_t impact the SDF and are priced in equilibrium. The recursive preference approach easily generalizes to continuous-time and gives priced jumps, as we now show.

We assume that agents have a stochastic differential utility V_t (Duffie and Epstein, 1992), which is specified by a normalized aggregator $f(c,v)$ and satisfies the fixed point equation

$$V_t = \mathbb{E}_t \left[\int_0^T f(C_{t+s}, V_{t+s}) ds + V_T \right] \tag{5.1}$$

for any instants $T \geq t \geq 0$. We consider for simplicity the standard aggregator

$$f(c,v) \equiv \frac{\delta}{1-\psi^{-1}} \frac{c^{1-\psi^{-1}} - [(1-\alpha)v]^{\theta}}{[(1-\alpha)v]^{\theta-1}},$$

where α is the coefficient of relative risk aversion, ψ is the elasticity of intertemporal substitution, and $\theta = (1-\psi^{-1})/(1-\alpha)$. The case where $\theta = 1$ corresponds to isoelastic utility as considered previously.

Under the consumption process in Assumption 1, the recursive utility has functional form $V(c, M_t) = \varphi(M_t) c^{1-\alpha}/(1-\alpha)$.[17] The stochastic discount factor is then $\Lambda_t = \frac{1}{\delta} \exp\left[\int_0^t f_v(C_s, V_s) ds \right] f_c(C_t, V_t)$ (Duffie and Epstein, 1992; Duffie and Skiadas, 1994), or equivalently

$$\Lambda_t = [\varphi(M_t)]^{1-\theta} C_t^{-\alpha} e^{-\frac{\delta}{\theta}t + \delta\left(1-\frac{1}{\theta}\right) \int_0^t [\varphi(M_s)]^{-\theta} ds}.$$

The exponential expression contains smooth terms and an integral and is therefore continuous, as is consumption. On the other hand, the first factor in the equation is a function of $\varphi(M_t)$, which switches with the current state and is discontinuous. In the simplifying case where $\theta = 1$ (power utility), the first factor $[\varphi(M_t)]^{1-\theta}$ drops out and the SDF has continuous sample paths, reducing to $\Lambda_t = e^{-\delta t} C_t^{-\alpha}$. Thus, with power utility the marginal utility of consumption is not state dependent. On the other hand, if $\theta \neq 1$, the term $[\varphi(M_t)]^{1-\theta}$ is discontinuous and the marginal utility of consumption depends on the current state. Switches in the state vector M_t thus cause jumps in the SDF.

[17] The fixed point equation (5.1) can be written as $f(C_t, V_t)dt + \mathbb{E}_t(dV_t) = 0$. Let $\varphi_1, ..., \varphi_d$ denote the value of φ in all possible states $m^1, ..., m^d$. The fixed point equation is then

$$\frac{\delta}{\theta}(\varphi_i^{1-\theta} - \varphi_i) + \varphi_i \left[(1-\alpha) g_C(m^i) + \frac{\alpha(\alpha-1)}{2} \sigma_C^2(m^i) \right] + \sum_{j \neq i} a_{i,j} (\varphi_j - \varphi_i) = 0,$$

where $a_{i,j} = \mathbb{P}\left(M_{t+dt} = m^j | M_t = m^i \right)/dt$. Existence and uniqueness can then be analyzed using standard methods.

Since switches in M_t trigger simultaneous jumps in the stochastic discount factor and the P:D ratio, we anticipate an impact to expected returns. Let γ denote the probability that a change occurs; for instance $\gamma = \sum_{k=1}^{\bar{k}} \gamma_k$ when M_t is an MSM state variable. The conditional equity premium is

$$-\frac{1}{dt}\mathbb{E}_t\left(\frac{d\Lambda_t}{\Lambda_t}\frac{dP_t}{P_t}\right) = \alpha\sigma_C(M_t)\sigma_D(M_t)\rho_{C,D} + \gamma\mathbb{E}\left(-\frac{\Delta\Lambda}{\Lambda}\ \frac{\Delta P}{P}\right).$$

When $\theta \neq 1$ the final term is generally non-zero, confirming that the occurrence of a jump is priced in equilibrium.

The ability of our framework to accommodate priced jumps is potentially useful for empirical applications. For example, in discrete-time Calvet and Fisher (2005) use non-separable preferences to obtain priced switches in a calibration that simultaneously fits, with reasonable levels of risk-aversion, the equity premium, equity volatility, and the drifts and volatilities of consumption and dividends. Further, in a recent contribution, Bhamra, Kuehn, and Strebulaev (2006) extend our framework by considering levered claims on the priced asset. They find that the ability to capture priced jumps is empirically important in simultaneously reconciling the equity premium, default spreads, and empirically observed default rates. We anticipate that future work will use our structural approach to modelling priced jumps in other applications, including, for example, pricing derivative assets such as options.

6. Conclusion

We specify a continuous-time asset-pricing economy with endogenous multifrequency jumps in stock prices. Equilibrium valuation gives a number of appealing features that are often assumed exogenously in previous literature, including: 1) heterogeneous jump sizes with many and frequent small jumps and few large jumps; and 2) endogenous correlation between jumps in prices and volatility. Further, jumps may be priced in equilibrium, in which case the equity premium is larger.

We consider the weak limit of our economic equilibrium as the number of components driving fundamentals becomes large. Under appropriate conditions, the stock price converges to a new mathematical object called a multifractal jump-diffusion. The equity value can be decomposed into: 1) a multifractal diffusion related to the exogenous dividend process; and 2) an infinite-intensity pure jump process corresponding to endogenous variations in the price:dividend ratio. Stock price jumps occur in the neighborhood of any instant, but sample paths are continuous almost everywhere.

Our results focus on two special cases of consumption-based asset pricing economies: IID consumption growth and a Lucas tree. In an appendix, we show how the economy may be generalized to accommodate intermediate cases where consumption and dividend growth state variables are correlated but not identical. Future work may further develop these cases.

7. Appendix A - Proofs

Proof of Proposition 1. The price dividend ratio satisfies

$$Q(M_t) = \mathbb{E}\left(\int_0^{+\infty} \frac{\Lambda_{t+s}}{\Lambda_t} \frac{D_{t+s}}{D_t} ds \middle| M_t \right).$$

Since

$$\begin{aligned} d \ln \Lambda_t &= \left[-r_f(M_t) - \alpha^2 \sigma_C^2(M_t)/2\right] dt - \alpha \sigma_C(M_t) dZ_C(t), \\ d \ln D_t &= \left[g_D(M_t) - \sigma_D^2(M_t)/2\right] dt + \sigma_D(M_t) dZ_D(t), \end{aligned}$$

we infer that

$$\begin{aligned} \ln \frac{\Lambda_{t+s}}{\Lambda_t} + \ln \frac{D_{t+s}}{D_t} &= \int_0^s \left[g_D(M_{t+h}) - r_f(M_{t+h}) - \frac{\sigma_D^2(M_{t+h}) + \alpha^2 \sigma_C^2(M_{t+h})}{2} \right] dh \\ &+ \int_0^s \left[\sigma_D(M_{t+h}) dZ_D(t+h) - \alpha \sigma_C(M_{t+h}) dZ_C(t+h) \right] \end{aligned}$$

is conditionally Gaussian with mean $\int_0^s \left[g_D(M_{t+h}) - r_f(M_{t+h}) - \frac{\sigma_D^2(M_{t+h}) + \alpha^2 \sigma_C^2(M_{t+h})}{2} \right] dh$ and variance $\int_0^s [\alpha^2 \sigma_C^2(M_{t+h}) + \sigma_D^2(M_{t+h}) - 2\alpha \rho_{C,D} \sigma_C(M_{t+h}) \sigma_D(M_{t+h})] dh$. We then easily check that

$$\mathbb{E}\left(\frac{\Lambda_{t+s}}{\Lambda_t} \frac{D_{t+s}}{D_t} \middle| M_t \right) = e^{\int_0^s \left[g_D(M_{t+h}) - r_f(M_{t+h}) - \alpha \rho_{C,D} \sigma_C(M_{t+h}) \sigma_D(M_{t+h}) \right] dh},$$

and conclude that equation (2.4) holds.

Proof of Proposition 3. Given an initial state ν_t, the P/D ratio of the Lucas tree economy with random state $(M_s(\varepsilon))_{s \geq 0}$ can be written as

$$Q(\varepsilon) = \mathbb{E}\left(\int_0^{+\infty} e^{-\delta' s - \frac{\alpha(1-\alpha)}{2} \int_0^s \left(\sigma_D^2[M_{t+h}(\varepsilon)] - \bar{\sigma}_D^2 \right) dh} ds \middle| \nu_t \right),$$

We note that $Q(0) = 1/\delta'$. By the dominated convergence theorem, the function Q is differentiable and

$$Q'(0) = -q_1 \mathbb{E}\left\{ \int_0^{+\infty} e^{-\delta' s} \left[\int_0^s \sum_{k=1}^{\bar{k}} (\nu_{k,t+h} - 1) dh \right] ds \middle| \nu_t \right\}.$$

Since $\mathbb{E}_t(\nu_{k,t+h}-1) = e^{-\gamma_k h}(\nu_{k,t}-1)$, we infer that

$$\begin{aligned} Q'(0) &= -q_1 \sum_{k=1}^{\bar{k}} (\nu_{k,t}-1)\mathbb{E}\left(\int_0^{+\infty} e^{-\delta' s}\int_0^s e^{-\gamma_k h} dh\, ds\right) \\ &= -q_1 \sum_{k=1}^{\bar{k}} \frac{\nu_{k,t}-1}{\delta'(\delta'+\gamma_k)}. \end{aligned}$$

Hence

$$Q(\varepsilon) = Q(0)\left(1 - q_1 \sum_{k=1}^{\bar{k}} \frac{\nu_{k,t}-1}{\delta'+\gamma_k}\varepsilon\right) + o(\varepsilon).$$

We take the log and conclude that (3.4) holds. A similar argument holds in the IID consumption case.

Proof of Proposition 4. We showed in Calvet and Fisher (2001) that the restriction of $(\theta_{\bar{k}})$ on any bounded subinterval $[0, T]$ is uniformly equicontinuous and has a continuous limiting process. Theorem 16.8 in Billingsley (1999) implies that the sequence $\theta_{\bar{k}}$ is also tight on $D[0, \infty)$. We conclude that the sequence $\theta_{\bar{k}}$ converges in $D[0, \infty)$ to a limit process θ_∞ with continuous sample paths.

Proof of Proposition 5. Consider

$$Q_{\bar{k}}(t) \equiv \mathbb{E}\left[\int_0^{+\infty} e^{-\rho s} e^{-\lambda[\theta_{\bar{k}}(t+s)-\theta_{\bar{k}}(t)]} ds \,\middle|\, M_t\right],$$

where $\lambda = \alpha(1-\alpha)/2 > 0$. We easily check that $Q_{\bar{k}}(t)$ is a positive and bounded submartingale:

$$Q_{\bar{k}}(t) \leq \mathbb{E}_{\bar{k}}\left[Q_{\bar{k}+1}(t)\right] \leq 1/\rho.$$

The P/D ratio $Q_{\bar{k}}(t)$ therefore converges to a limit distribution, which we now easily characterize.

Consider the function $\Phi : D[0, \infty) \to D[0, \infty)$ defined for every cadlag function f by the integral transform

$$(\Phi f)(t) = \int_0^{+\infty} \exp\left\{-\rho s - \lambda[f(t+s) - f(t)]\right\} ds.$$

The function Φ is bounded with respect to the Skohorod distance since $(\Phi f)(t) \in [0, 1/\rho]$ for all t. We also check that it is continuous. Since $\theta_k \to \theta_\infty$, we infer that $\Phi\theta_k$ weakly converges to $\Phi\theta_\infty$. Hence $Q_{\bar{k}}(t) \to Q_\infty(t)$, and the proposition holds.

8. Appendix B - Multivariate Extensions

The asset pricing models in Section 3 are based on univariate MSM, and assume either IID consumption or Lucas tree economies. We now introduce an extension of MSM that permit intermediate comovements of consumption and dividends.

Allowing the SDF and the dividend process to have correlated regime-shifts is important for several reasons. First, the recent asset-pricing literature shows that consumption may be exposed to a slowly varying component (e.g. Bansal and Yaron, 2004). Second, if the arrivals are correlated with the SDF, then they are priced and can therefore help to explain the equity premium.

8.1. Bivariate MSM in Continuous Time

We now generalize to a continuous-time setting the multivariate discrete-time specification of MSM (Calvet, Fisher and Thompson, 2006). We consider two economic processes α and β, which could for instance correspond to consumption and dividends. For every frequency k, the processes have volatility components

$$M_{k,t} = \begin{bmatrix} M_{k,t}^{\alpha} \\ M_{k,t}^{\beta} \end{bmatrix} \in \mathbb{R}_+^2.$$

The period-t volatility column vectors $M_{k,t}$ are stacked into the $2 \times \bar{k}$ matrix

$$M_t = (M_{1,t}; M_{2,t}; ...; M_{\bar{k},t}).$$

As in univariate MSM, we assume that $M_{1,t}, M_{2,t}...M_{\bar{k},t}$ at a given time t are statistically independent. The main task is to choose appropriate dynamics for each vector $M_{k,t}$.

Economic intuition suggests that volatility arrivals are correlated but not necessarily simultaneous across economic series. For this reason, we allow arrivals across series to be characterized by a correlation coefficient $\rho^* \in [0, 1]$. Assume that the volatility vector $M_{k,s}$ has been constructed up to date t. Over the following interval of infinitesimal length dt, each series $c \in \{\alpha, \beta\}$ is hit by an arrival with probability $\gamma_k dt$. The probability of an arrival on β conditional on an arrival on α is $[(1-\rho^*)\gamma_k + \rho^*]dt$. Symmetrically, the probability of no arrival on β conditional on no arrival on α is also given by $[(1-\rho^*)(1-\gamma_k) + \rho^*]dt$.

The construction of the volatility components $M_{k,t}$ is then based on a bivariate distribution $M = (M^{\alpha}, M^{\beta}) \in \mathbb{R}_+^2$.[18] If arrivals hit both series, the state vector $M_{k,t+dt}$ is drawn from M. If only series $c \in \{\alpha, \beta\}$ receives an arrival, the new component

[18] We can for instance choose a bivariate binomial. See Calvet, Fisher and Thompson (2006) for further details.

$M^c_{k,t+dt}$ is sampled from the marginal M^c of the bivariate distribution M. Finally, $M_{k,t+dt} = M_{k,t}$ if there is no arrival.

As in the univariate case, the transition probabilities $\left(\gamma_1, \gamma_2, ..., \gamma_{\bar{k}}\right)$ are defined as

$$\gamma_k = \gamma_1 b^{k-1}, \tag{8.1}$$

where $\gamma_1 > 0$ and $b \in (1, \infty)$. This completes the specification of bivariate MSM in continuous time.

8.2. Extension of the Asset Pricing Results

We assume that the SDF and dividend processes have constant drifts but stochastic volatilities

$$\begin{aligned}
\sigma_\Lambda(M_t) &= \bar{\sigma}_\Lambda(M^\alpha_{1,t} M^\alpha_{2,t} ... M^\alpha_{\bar{k},t})^{1/2}, \\
\sigma_D(M_t) &= \bar{\sigma}_D(M^\beta_{1,t} M^\beta_{2,t} ... M^\beta_{\bar{k},t})^{1/2}.
\end{aligned}$$

The construction thus permits correlation in volatility across series through the bivariate distribution M, and correlation in returns through the Brownian motions Z_Λ and Z_D. This flexible specification permits to construct a more general class of jump diffusions for stock prices.

The generalized model might also be useful for option pricing. In our environment, the price of a European option $f(P_T)$ is therefore given by[19]

$$f_0 = \mathbb{E}_0 \left[\frac{\Lambda_T}{\Lambda_0} f(P_T) \right]$$

As in Hull and White (1987), let $f[(M_t)_{t\in[0,T]}] = \mathbb{E}_0 \left[\Lambda_T f(P_T)/\Lambda_0 | \, (M_t)_{t\in[0,T]} \right]$ denote the option price conditional on the state history. The law of iterated expectations implies

$$f_0 = \mathbb{E}_0 f\left((M_t)_{t\in[0,T]} \right).$$

If consumption or the SDF is IID, jumps are not priced and the standard results derived in Hull and White (1987) will hold. Jumps are priced, on the other hand, in multivariate MSM settings, which can lead to richer characterizations of option prices.

[19] See Anderson and Raimondo (2005), David and Veronesi (2002), Garcia, Luger and Renault (2003) and Garleanu, Pedersen and Poteshman (2006) for recent work on consumption-based option pricing.

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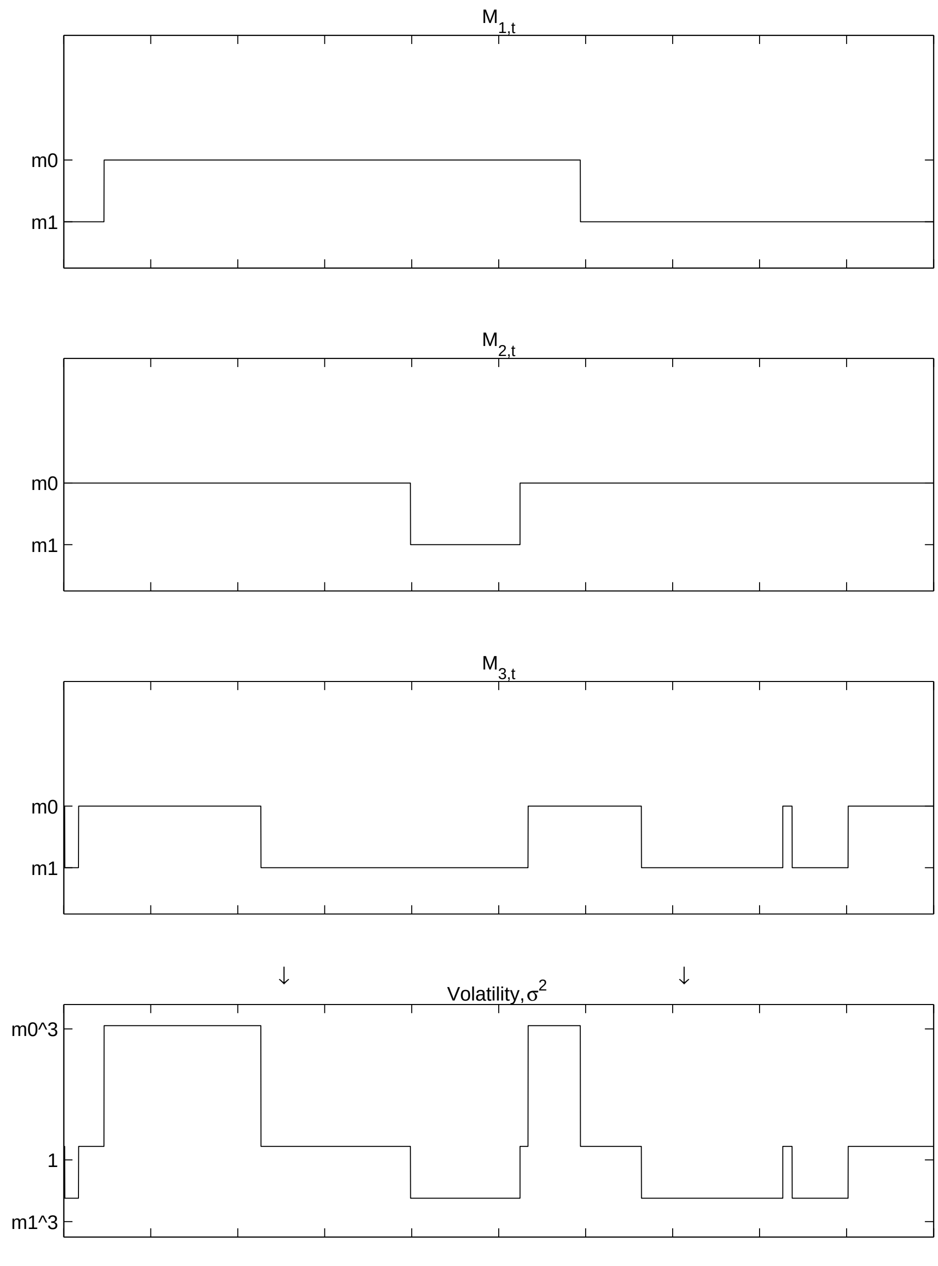

Figure 1: Construction of Multifractal Volatility. This figure illustrates the construction of multifractal volatility with three volatility components and $T = 10,000$ periods. The first panel shows the randomly drawn values of the lowest frequency component $M_{1,t}$ over time. The second and third panels respectively show the middle frequency component $M_{2,t}$ and the high frequency component $M_{3,t}$. The last panel gives the variance $\sigma_t^2 = \bar{\sigma}_D^2 M_{1,t} M_{2,t} M_{3,t}$, where we set $\bar{\sigma}_D = 1$ so that the variance equals the product of the components displayed in the top three panels. The simulation uses the binomial MSM construction with $m_0 = 1.4$, $b = 2$, and $\gamma_1 = .0002$.

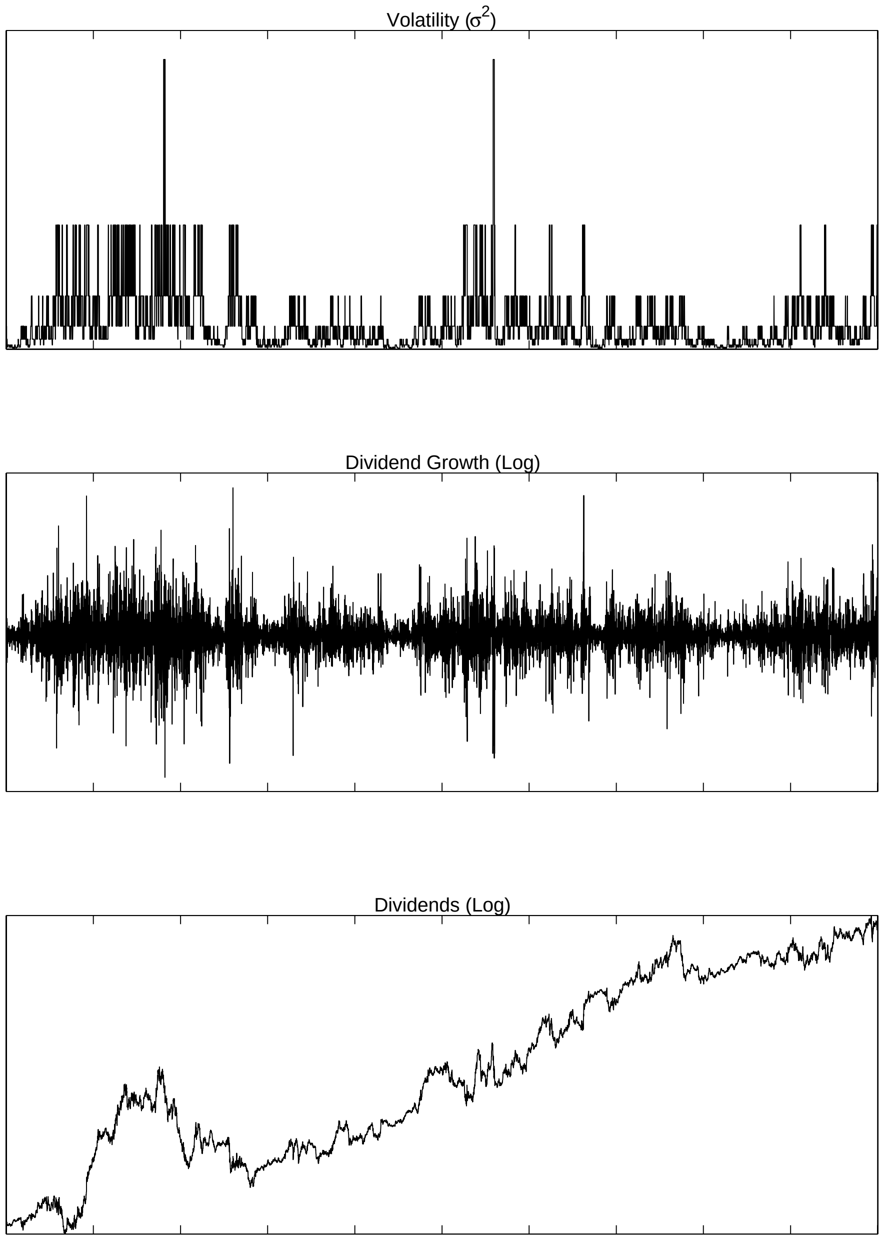

Figure 2: The Multifractal Dividend Process. This figure illustrates the construction of a multifractal dividend path over $T = 10,000$ periods. The first panel shows a simulation of multifractal volatility with $\bar{k} = 8$ volatility components. The volatility parameters $m_0 = 1.4$, $b = 2$ and $\gamma_1 = 0.0002$ are identical to Figure 1, and $\bar{\sigma}_D = 0.01$. The random draws used for the first three components $M_{1,t}$, $M_{2,t}$, and $M_{3,t}$ are also identical to Figure 1. Hence, the displayed volatility in the first panel is the outcome of following the construction in Figure 1 to a higher level of $\bar{k}$ and rescaling for a different $\bar{\sigma}_D$. The second and third panels then show how the volatility process maps into dividend growth and dividends. The second panel displays dividend growth, $\Delta d_t = \left(\bar{g}_D - \sigma_t^2/2\right)\Delta + \sigma_t \epsilon_t$, where ϵ_t are standard iid normals, $\Delta = 1$, and $\bar{g}_D = 0.0001$. The third panel shows the logarithm of dividends, i.e., $d_t = d_0 + \sum_{s=1}^{t} \Delta d_s$.

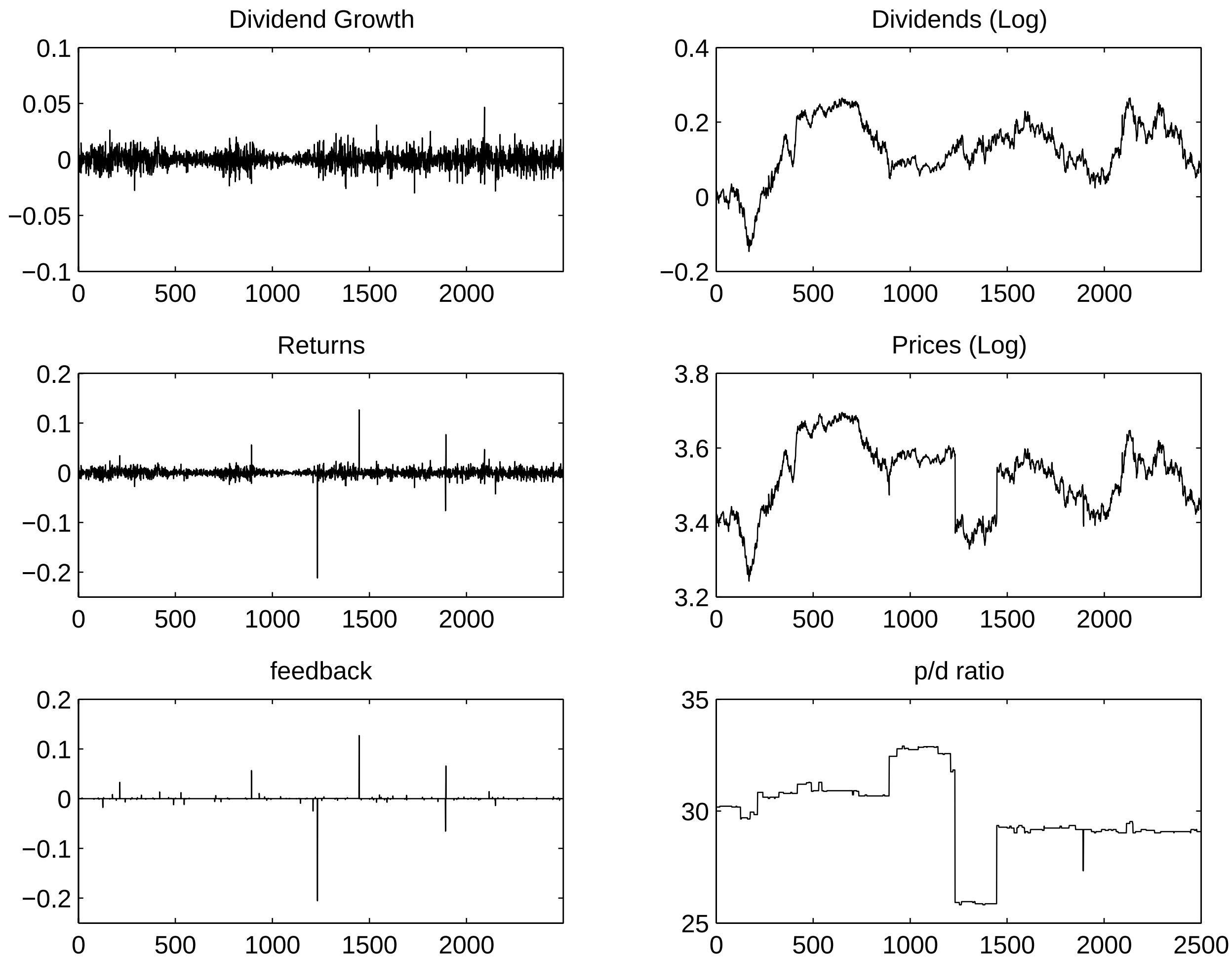

Figure 3: Equilibrium Price and Return Dynamics. This figure illustrates the relation between exogenous dividends and equilibrium prices when consumption is iid. The top two panels display simulated dividend growth rates and dividend levels, constructed in the same manner as Figure 2. The parameters used in the specification are $m_0 = 1.35$, $\bar{\sigma}_D = 0.7$, $b = 2.2$, and $\bar{g}_D = 0.0001$. The middle two panels demonstrate the result of equilibrium pricing. In these panels we use the preference and consumption parameters $\alpha = 25$, $\delta = 0.00005$, $\bar{g}_C = 0.00005$, $\rho_{C,D}\sigma_D = 0.0012$. The left-hand side displays returns, and the right side shows the log price realization. Both show more variability, and in particular jumps, relative to the dividend processs. To isolate the endogenous pricing effects in returns and prices, the bottom left panel shows the volatility "feedback" effect, defined as the difference between log returns and log dividend growth, i.e., $\Delta p_t - \Delta d_t$, or the difference between the middle left and top left panels. To show the same endogenous pricing effects in levels, the bottom right hand panel shows the price:dividend ratio.

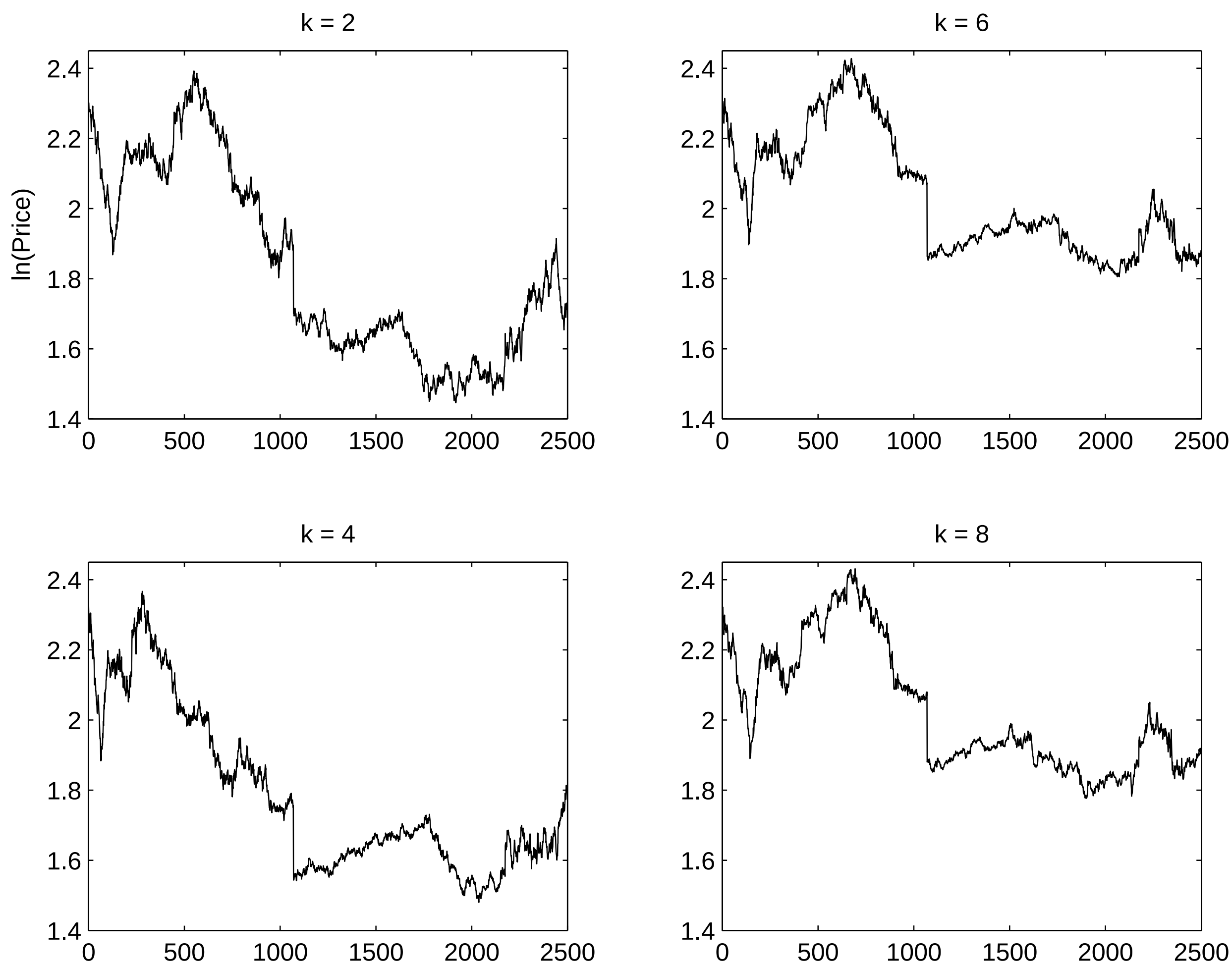

Figure 4: Convergence to Multifractal Jump-Diffusion. This figure illustrates convergence of the equilibrium price process as the number of high-frequency volatility components becomes large. The panels show consecutively simulations of the log price process $p_{\bar{k}}(t) = d_0 + \bar{g}_D t - \theta_{\bar{k}}(t)/2 + B\left[\theta_{\bar{k}}(t)\right] + q_{\bar{k}}(t)$ for $\bar{k} = 2, 4, 6, 8$. All panels hold constant the Brownian $B(t)$. The multipliers $M_{k,t}$ are also drawn only once, and then held constant as higher level multipliers are added. The construction is thus recursive in $\bar{k}$, with each increment requiring the previously drawn non-deformed dividends and multipliers from the preceding level, plus new random draws for the next set of (higher frequency) multipliers being incorporated. We observe large differences between the panels corresponding to $\bar{k} = 2$ and $\bar{k} = 4$, more moderate changes between $\bar{k} = 4$ and $\bar{k} = 6$, and only modest differences between $\bar{k} = 6$ and $\bar{k} = 8$. In this set of simulations, we use the Lucas economy specification with $T = 2,500$, $m_0 = 1.4$, $b = 3.25$, $\gamma_1 = 0.25b^7 \approx 0.0001$, $\bar{\sigma}_C = 0.0125$, $\bar{g}_D = 0.00008$, $\delta = 0.00003$, and $\alpha = 0.5$. The results are consistent with Proposition 5, which ensures that as the number of frequencies $\bar{k}$ grows, the log price $p_{\bar{k}}(t)$ weakly converges to a multifractal jump-diffusion.