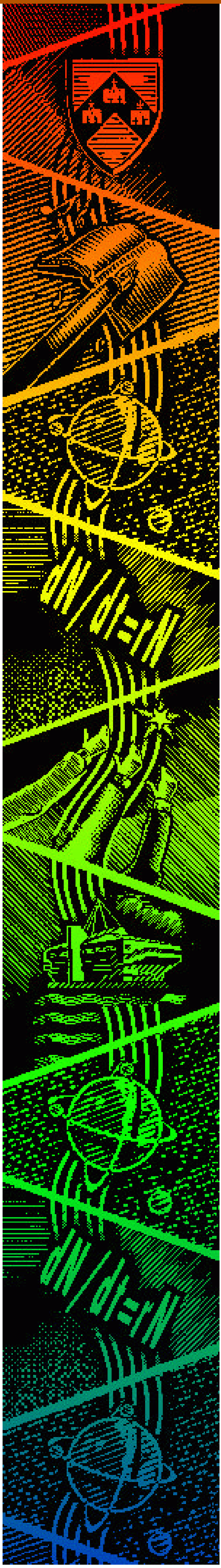

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Buying and Selling in Strategic Market Games

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Buying and Selling in Markets

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Abstract

Players may be allowed to both buy and sell the same commodity in a strategic market game. Any outcome of such a game can also be obtained as an outcome of a game in which players either buy or sell. However, an equilibrium allocation of the buy and sell game may not be so in the corresponding buy or sell game as the set of achievable allocations for any player, given others' strategies, is different in these two games.

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Key Words : Strategic market game, Buy and sell, Buy or Sell.

1. INTRODUCTION

Shapley and Shubik (Shapley, 1976; Shapley and Shubik, 1977) introduced and developed strategic market games as a method of making a non-cooperative game out of an exchange economy. Strategic market games are quite popular among theorists as it offers a model to study game theoretic analogs of well-known results in economic theory, particularly, general equilibrium theory. In a market game, players are allowed to both buy and sell the same commodity. Such a *buy and sell* mechanism has mostly been used in the literature. Buy and sell mechanism allows each player a lot more strategic freedom; however, it is intuitively difficult to justify why agents would sell and buy back the same commodity from the market. There are other variations of strategic market game models : *buy or sell*, and *sell all*. Buy or sell mechanism is clearly an appealing mechanism; an agent in such a market either buys a commodity or sells it off.

The purpose of the paper is to understand precisely the difference or the equivalence, if any, between buy and sell, buy or sell mechanisms. We therefore concentrate on the set of (equilibrium) outcomes of a buy and sell game and that of the corresponding buy or sell game. We first prove that any outcome of a buy and sell game can be obtained as an outcome of the corresponding buy or sell game. Thus these two games are *payoff equivalent* (Ray, 1999).

Two payoff equivalent games may not have the same set of equilibria. Indeed, in this context, the games have considerably different set of equilibria. As Shapley (1976, p.168) noted, any Nash equilibrium of any buy or sell game is also a Nash equilibrium for the corresponding buy and sell game. Conversely, a Nash equilibrium of the buy and sell game that happens to satisfy the buy or sell restriction is obviously a Nash equilibrium of the buy or sell game. However, the hold-back option of the buy and sell mechanism makes the set of solutions usually infinite (Shapley and Shubik, 1977; p. 964).

Consider for example, any (non-zero) Nash equilibrium of a buy and sell market game with only two players. The corresponding strategy profile of the buy or sell game is however not an equilibrium as the only Nash equilibrium of the buy or sell game with two players is no trade. This is because in the buy or sell game, for any player, the opponent's strategy is not an *interior point* and hence a player can grab everything on the market by putting an infinitesimal amount.

In fact, even when we impose *strictly positive opponents' bid* (SPOB) condition, the set of achievable allocations for any player in the buy or sell game, given others' strategies, is not the same as that in the original buy and sell game. Hence the games, buy and sell, buy or sell, are not *individual decision equivalent* (Ray, 1999). This is precisely the reason why the equilibrium set of the buy and sell game is considerably larger than that of the corresponding buy or sell game, although the games are payoff equivalent.

We construct an example (Example 2) to illustrate all the above remarks explicitly. This example itself should be of interest as we do not have many clarifying examples in the market game literature.

2. THE GAME

In this section, we present an exchange economy as a non-cooperative game, namely, strategic market game. We closely follow Dubey and Shubik (1978) in this section.

Consider an exchange economy with n agents and (l + 1) goods. Agents are indexed by i; i = 1,..., n. First l goods are indexed by j; j = 1,..., l. The last good is *money* and is indexed by m. Endowment of agent i is denoted by $w_i = (w_{i1},...,w_{il}, w_{im})$ and the utility function of individual i is given by $U_i : \mathbb{R}^{l+1} \rightarrow \mathbb{R}$. Let us make standard assumptions that the endowments are strictly positive ($w_i >> 0$ for all i) and U_i is continuously differentiable, strictly increasing in all coordinates and strictly concave, for all i.

To describe the strategic game, the agents in the economy are the players in the game. In a market where the players are allowed to both buy and sell, the pure strategy set of player i is given by $S_i = \{(q_i,$

b_i) : $0 \le q_{ij} \le w_{ij}$, $0 \le b_{ij}$ for j = 1, ..., l and $\sum_{j=1}^{l} b_{ij} \le w_{im}$}. If the players only buy or sell goods, the pure strategy set of player i is given by S_i = {(q_i, b_i) : $q_{ij} \,.\, b_{ij} = 0$, $0 \le q_{ij} \le w_{ij}$, $0 \le b_{ij}$ for j = 1, ..., l and $\sum_{j=1}^{l} b_{ij} \le w_{im}$}. Given a strategy profile, (q, b), prices are formed following the mechanism :

p_j = total bid / total supply = $\sum_i b_{ij} / \sum_i q_{ij}$ for j = 1, ..., l (= 0 if $\sum_i q_{ij} = 0$). Money is the numeraire.

The net allocations resulting from the strategy profile (q, b) are as follows. For j = 1, ..., l,

$$x_{ij}(q, b) = w_{ij} - q_{ij} + b_{ij} / p_j \qquad \text{if } p_j > 0$$

$$= w_{ij} - q_{ij} \qquad \text{if } p_j = 0$$

and, $x_{im}(q, b) = w_{im} - \sum_{j=1}^{l} b_{ij} + \sum_{j=1}^{l} p_j q_{ij}$.

To complete the game, the payoff of player i is $u_i(q, b) = U_i(x_i(q, b))$.

A non-cooperative equilibrium is a strategy profile, (q^*, b^*), such that for all i, for all $(q_i, b_i) \in S_i$, $u_i(q^*, b^*) \ge u_i((q_{-i}^*, b_{-i}^*), (q_i, b_i))$. This is nothing but a Nash equilibrium in *pure* strategies.

2. 1. The Set of Achievable Allocations

In this framework, one can easily characterise the set of achievable allocations for player i, given the strategies of other players, (q_{-i}, b_{-i}). If player i plays a strategy (q_i, b_i) against (q_{-i}, b_{-i}), the price of good j would be $p_j = ((B_{-i})_j + b_{ij}) / ((Q_{-i})_j + q_{ij})$, where, $(B_{-i})_j = \sum_{-i} b_{ij}$ and $(Q_{-i})_j = \sum_{-i} q_{ij}$.

For a strictly positive price, the final allocation for player i would then be $x_{ij} = w_{ij} - q_{ij} + b_{ij} / p_j$ and $x_{im} = w_{im} - \sum_{j=1}^{l} b_{ij} + \sum_{j=1}^{l} p_j q_{ij}$. Observe that $x_{ij} = w_{ij} + (Q_{-i})_j - (Q_{-i})_j - q_{ij} + b_{ij} / p_j = w_{ij} + (Q_{-i})_j - ((B_{-i})_j + b_{ij})/p_j + b_{ij} / p_j = w_{ij} + (Q_{-i})_j - (B_{-i})_j/p_j$. Similarly, $x_{im} = w_{im} + \sum_{j=1}^{l} (B_{-i})_j - \sum_{j=1}^{l} (Q_{-i})_j p_j$. Thus, given others' strategies (q_{-i}, b_{-i}) and hence, $(Q_{-i})_j$ and $(B_{-i})_j$, the final allocation of player i depends only on the price, p. So, the final allocation remains same as long as p is constant.

2. 2. Two Goods

If there are only two goods, the strategy set of an individual i, having endowment $w_i = (w_{i1}, w_{i2})$, can be shown easily. Consider Figure 1. For buy and sell model, the pure strategy set of player i is simply the rectangle OMWG constructed by the point W and the axes. For buy or sell model, the pure strategy set of this player is given by the L-shaped part of the axes (MO U OG).

The final allocation for any player, (x, y), is same when p = (B + b) / (Q + q) = constant, which is an equation of a straight line passing through the point (-Q, -B). The set of achievable allocations depends only on the total offers and bids made by the opponents, (Q, B). If both Q and B are strictly positive, the set of achievable allocations, i.e, the locus of the point (x, y) is given by the quadratic equation: $xy - (w_2 + B)x - (w_1 + Q)y + (w_1B + w_2Q + w_1w_2) = 0$, which is a hyperbola[1], as in Figure 1.

[Insert Figure 1 here]

Note that in a buy and sell game, any point (x, y) on this hyperbola can be obtained by a whole straight line passing through the point (-Q, -B). Any such straight line must cut either of the axes and this point on the axis generates allocation (x, y) in a buy or sell game.

If either Q or B is zero, then the set of achievable allocations is considerably different. For example, if B is zero, then a player can achieve the endowment point (by offering and bidding nothing), the good-axis (by offering something but bidding nothing and thereby losing the offer as the market would not open) and a straight line, parallel to the money-axes (by any other strategy, grabbing all the goods offered by others), characterised by $x = w_1 + Q$ and $y = w_2 - Qb/(Q + q)$. All the above observations can be translated to the multi-good case.[2]

It is now easy to find the best response of player i against others' strategies. The optimum allocation would be the point of tangency between an indifference curve and the set of achievable allocations. Thus for buy and sell game, the optimum allocation is unique however the best response

[1] $x = w_1$ and $y = w_2$ satisfy the above equation implying that the hyperbola passes through the endowment point, (w_1, w_2). One can also check that the hyperbola cuts the axes outside the rectangle OMWG as shown in Figure 1.

[2] Some of these observations and Figure 1 can also be found in other articles such as Dubey and Shubik (1978), Rogawski and Shubik (1986).

is a set, the straight line that generates the optimum allocation.

3. RESULTS

The first question we are asking is whether or not an outcome of a buy and sell game can be achieved as an outcome of a buy or sell game. Consider an outcome of a buy and sell game generated by a strategy profile (q, b). Let us first observe the following.

Observation 1. *For any good j, the sum of the expression,* $(B_{-i})_j - p_j(Q_{-i})_j$ *over all i is zero.*

Proof. $\sum_i \{(B_{-i})_j - p_j(Q_{-i})_j\} = (n-1)\sum_i b_{ij} - p_j \,.\, (n-1)\sum_i q_{ij} = 0.$ ■

The above is not surprising at all. The price formation mechanism is such that the market clears automatically. The net supply of good j for any player i is given by the amount $((B_{-i})_j - p_j(Q_{-i})_j)/p_j$. Total net supply for any good j is therefore zero.

Given any strategy profile, (q, b), for any good j, if the expression $(B_{-i})_j - p_j(Q_{-i})_j$ is non-zero (positive or negative) for one player, then there must exist another player for whom the expression is also non-zero (negative or positive). Let us now consider the following condition.

Market-Opening (MO) : *For every good j, the expression* $(B_{-i})_j - p_j(Q_{-i})_j$ *is non zero for at least one player, where,* $p_j = \sum_i b_{ij} / \sum_i q_{ij}$, $(B_{-i})_j = \sum_{-i} b_{ij}$ *and* $(Q_{-i})_j = \sum_{-i} q_{ij}.$

For any strategy profile satisfying the above condition, by Observation 1, there must be at least one player with a net supply of good j and one at least one with a net demand of good j. For any such strategy profile, the market for each good j opens in the corresponding buy or sell game; hence, we call the above condition "market opening".

Proposition 1. *Any outcome of a buy and sell game generated by a strategy profile (q, b) can be obtained as an outcome of a buy or sell game if the profile satisfies MO.*

Proof. Take any strategy profile (q, b) in a buy and sell game. Consider the following strategies for a buy or sell game which generates the same outcome. Consider player i. For each good j, compute the expression $(B_{-i})_j - p_j(Q_{-i})_j$. If it is positive, then player i only sells the good j and $q_{ij}' = ((B_{-i})_j - p_j(Q_{-i})_j)/p_j$. If not, then individual i only buys good j with $b_{ij}' = p_j(Q_{-i})_j - (B_{-i})_j$. If it is zero, then the player neither buys nor sells good j.

If the strategy profile (q, b) satisfies MO, then, for every good j, there exists at least one player with $q_{ij}' > 0$ and one with $b_{ij}' > 0$ and hence, the market for good j opens. Note that in this buy or sell game the price of good j is $p_j' = \sum b_{ij}' / \sum q_{ij}' = p_j$ (by Observation 1). It is now obvious to check that these strategies for buy or sell game generate the same allocation we have started with.

If the strategy profile (q, b) does not satisfy MO, then, nobody would buy or sell some of the goods and those market would not open. Still, the above mentioned strategies for buy or sell game would generate the same allocation we have started with. ■

The above proposition proves that the two games, buy and sell, buy or sell are *payoff equivalent* (Ray, 1999). The following example will illustrate the above the result.

Example 1. Consider a market game with two players and two commodities only. Suppose the agents have endowments $W_1 = (2, 5)$ and $W_2 = (5, 2)$ and any suitable utility function.

Example 1a. Consider the strategies ($q_1 = 1$, $b_1 = 2$) and ($q_2 = 2$, $b_2 = 1$) in the buy and sell game. The price of the good is 1. The expression B - Qp turns out to be -1 and 1 respectively. Therefore, the strategy profile ((0, 1), (1, 0)) of the corresponding buy or sell game would generate the same outcome.

Example 1b. Consider the strategies ($q_1 = 1$, $b_1 = 1$) and ($q_2 = 1$, $b_2 = 1$) for the buy and sell game. The price is 1 in this case as well. This strategy profile does not satisfy MO and the outcome is generated by the no-bid strategy, (0, 0), in the corresponding buy or sell game in which the market does not open.

We are now interested in any individual's achievable outcomes. As we described in the previous section, the set of achievable allocations of any player depends crucially on Q and B, the total bid and offer made by the opponents. We therefore consider the following condition for a strategy profile (q, b) of a buy and sell game.

Strictly Positive Opponents' Bid (SPOB) : For *every player i, for every good j,* $Q_{-ij}' = \sum_{k \notin i} q_{kj}'$ *and* $B_{-ij}' = \sum_{k \notin i} b_{kj}'$ *are strictly positive, where* q_{ij}' *and* b_{ij}' *are the corresponding strategies in the buy or sell game, as characterised in the proof of proposition 1.*

Remark 1. If a strategy profile in a buy and sell game does not satisfy MO, then it obviously does not satisfy SPOB. For example, the strategy profile in Example 1.b. does not satisfy MO and thereby does not satisfy SPOB. A profile satisfying MO, may also not satisfy SPOB. For example, the strategy profile in Example 1.a does not satisfy SPOB. In fact, any strategy profile for any buy and sell game with 2 or 3 players would not satisfy SPOB.

Remark 2. Any (non-zero) strategy profile that does not satisfy SPOB would never be an equilibrium of the buy or sell game as a player for which the condition is violated, can put an infinitesimal amount to grab the all on the other side of the market.

We are now going to show that even when we impose SPOB, the equilibrium strategy profile of any buy and sell game may not correspond to an equilibrium in the corresponding buy or sell game, because the set of achievable allocations for any player may be different. The following example which uses a two-fold replica of the economy described in Example 1, clarifies all our remarks.

Example 2. Consider a market game with four players and two goods only. The first two agents are identical in their endowments and preferences and the last two also have identical endowments and preferences. Suppose the endowments are $W_1 = W_2 = (2, 5)$, $W_3 = W_4 = (5, 2)$ and the utility functions are $u_1(x, y) = u_2(x, y) = (15/16) \ln x + \ln y$ and $u_3(x, y) = u_4(x, y) = (2/3) \ln x + \ln y$.

Consider the strategies $(q_1 = 1, b_1 = 2)$, $(q_2 = 1, b_2 = 2)$, $(q_3 = 2, b_3 = 1)$ and $(q_4 = 2, b_4 = 1)$ in this buy and sell game. It is easy to prove that this profile forms a Nash equilibrium. The strategy profile indeed satisfies MO and SPOB. In the corresponding buy or sell game, the strategy profile ((0, 1), (0, 1), (1, 0), (1, 0)) generates the same outcome. However, it is not an equilibrium as either player can deviate and be better off. The following table gathers all the characteristics of the example.

	Player 1	Player 2	Player 3	Player 4
Endowment : W	2, 5	2, 5	5, 2	5, 2
Utility : u(x, y)	(15/16)ln x + ln y	(15/16)ln x + ln y	(2/3) ln x + ln y	(2/3) ln x + ln y
Buy and Sell Game				
Strategy : (q, b)	1, 2	1, 2	2, 1	2, 1
Allocation : (x, y)	3, 4	3, 4	4, 3	4, 3
Price = 1				
Others' : (Q, B)	5, 4	5, 4	4, 5	4, 5
Achievable set	xy-9x-7y+43 = 0	xy-9x-7y+43 = 0	xy-7x-9y+43 = 0	xy-7x-9y+43=0
Best allocation	3, 4	3, 4	4, 3	4, 3
Buy or Sell Game				
Strategy : (q′, b′)	0, 1	0, 1	1, 0	1, 0
Allocation : (x′, y′)	3, 4	3, 4	4, 3	4, 3
Price = 1				
Others' : (Q′, B′)	2, 1	2, 1	1, 2	1, 2

Achievable set	xy-6x-4y+22 = 0	xy-6x-4y+22 = 0	xy-4x-6y+22 = 0	xy-4x-6y+22=0
Best allocation	2.8, 4.3	2.8, 4.3 [3]	4, 3	4, 3

The above table clearly explains all the features of the relationship between the two games. Any outcome, in particular, an equilibrium outcome, of the buy and sell game can be obtained as an outcome of the buy or sell game; but the equilibrium outcome may not be an equilibrium in the buy or sell game, because the individual's set of achievable allocations may be different.

The two sets of achievable allocations, ie, the two hyperbolas, must have (at least) two allocations in common, the endowment point and the outcome in question. For example the hyperbolas xy-9x-7y+43 = 0 and xy - 6x - 4y + 22 = 0 cut each other at points (2, 5) and (3, 4).

The equilibrium allocation is therefore still achievable by any player in the buy or sell game but it may not be the best allocation for some of them. In the above example, for all four players, the two sets of achievable allocations, ie, the two hyperbolas, are different; still, for players 3 and 4, the allocation in question, (4, 3), is the best allocation they can achieve. However, for players 1 and 2, the allocation in question, (3, 4), is not the best allocation for them; they can, for example, play the strategy (0, 0.7) to achieve a better allocation for them, given others' strategies.

[3]The best allocations for players 1 and 2 shown in the table are approximate values of the exact solutions which can be obtained by solving the optimization problem : max (15/16)ln x + ln y subject to xy - 6x - 4y + 22 = 0.

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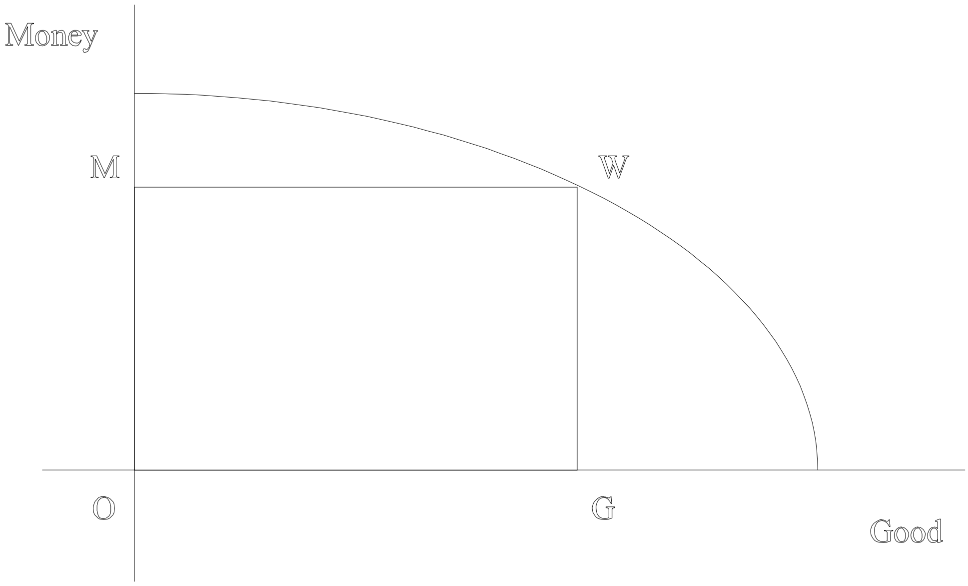

Figure 1