THE UNIVERSITY of York

Discussion Papers in Economics

No. 2000/15

Game Theory Via Revealed Preferences

by

Indrajit Ray and Lin Zhou

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

Game Theory via Revealed Preferences*

Indrajit Ray

Department of Economics, University of York, Heslington, York YO10 5DD, UK.
E-mail: ir2@york.ac.uk

and

Lin Zhou

Department of Economics, Duke University, Durham, NC 27708-0097, USA.
E-mail: linzhou@econ.duke.edu; EFZHOU@cityu.edu.hk

January 2000

*We would like to thank the seminar and conference participants at Brunel, CORE, Duke-UNC, Essex, Montreal, Warwick, Washington and York for valuable discussions and particularly, an anonymous referee and an associate editor of this journal for their careful reading of the earlier version of the paper and extremely helpful suggestions.

Revealed Preferences

The author for correspondences:

Indrajit Ray

Department of Economics
University of York
Heslington, York YO10 5DD
United Kingdom

E-mail : ir2@york.ac.uk
Fax : 44-1904-433759

Abstract

We investigate equilibrium notions in game theory from the revealed preference approach. For extensive game forms with complete information, we derive a set of independent necessary and sufficient conditions for the observed outcomes to be rationalized by subgame perfect Nash equilibrium.

Journal of Economic Literature Classification Number: C72, C92.

1. INTRODUCTION

In the past two decades we have witnessed a rapid development in non-cooperative game theory. Many new solution concepts, particularly refinements of Nash equilibrium, have been introduced and successfully applied to many fields in economics, as well as other social sciences. Though game theory has been applied to many fields, one important issue has not yet received much attention in the literature. To model any situation as a game, we first specify individuals' preferences and then apply certain solution concept to predict the outcome. However, what if the theoretical prediction differs from the actual outcome? There could be two possible explanations: either individuals do not play according to the theory, or individuals' preferences are not correctly specified. In fact, even when the theoretical prediction coincides with the actual outcome, we should not rule out the possibility that individuals do not play according to the theory if we are not sure about the specification of individuals' preferences. The question therefore is: can we, by any means, test whether individuals play according to the game theoretic prediction, solely on the basis of the observations of individuals' actual moves? If we could not find such a mean of testing, the empirical content of game theory would virtually be void. Surprisingly, the existing literature hardly contains anything on this issue. The purpose of this paper is therefore to take a first step to address this issue.

We, in this paper, instead of criticising the predictive power of the solution concepts, try to rationalize the actual outcomes. We would ask what the *necessary and sufficient* conditions for the observed outcomes to be rationalized by the equilibrium solution concept are. In a sense, our approach is similar to the classical revealed preference theory initiated by Samuelson (1938).

The general problem under consideration here takes the following form. Let Γ be a collection of related game forms that several individuals play. Suppose that we can observe, for every game form G in Γ an outcome $O(G)$. What conditions must the observed outcomes satisfy so that $O(G)$ is the equilibrium outcome of a game associated with G, for every G? Moreover, what are the *necessary and* *sufficient* conditions for the observed outcomes to be rationalized by the equilibrium solution concept?

Notice that in our framework, only outcomes are observed, but not the preferences of individuals. This is the important distinction between our work and the studies in experimental game theory. In experimental studies, players' preferences are assumed to be known to the researchers and hence, the theoretical equilibrium outcome is known as well. Then the individuals' actual plays are compared with the theoretical equilibrium outcome. In our model, as in any other model of revealed preference, only outcomes are observed. From these observations, we have to construct preferences of individuals so that the observed outcomes can be rationalized by the equilibrium notion with these constructed preferences. Also, note that we are imposing no restriction on the preferences (except that they must be orderings). In many contexts, it is natural to assume some a priori restrictions, for example, monotonicity in payoffs when the outcomes are monetary payoffs to the individuals. Any such restriction would indeed imply further conditions for rationalizability of solutions.

Since in strategic situations players can move simultaneously or sequentially, and in each case there are a variety of theoretical equilibrium concepts, we have a rich class of models to investigate. Clearly, different structures of strategic interactions and different solution concepts will have different implications on the observed outcomes.

In a recent and independent paper, Sprumont (1999) has taken up exactly the same issue for normal form games. Sprumont considers finite sets of actions, A_i, one for each player, i; the product set, A, is called the set of *joint actions*. A *joint choice function f*, assigns to every possible subset B, of the set of joint actions a nonempty set. A *joint choice function* is Nash rationalizable if there exist preference orderings on A such that for every B, $f(B)$ coincides with the set of Nash equilibria for the game defined by the set of actions B with the preferences. Sprumont provides a couple of necessary and sufficient conditions (*Persistence under Expansion* and *Persistence under Contraction*) for a joint choice function to be Nash-rationalizable.

In this paper, we consider situations in which the players move sequentially with perfect information. Our paper is therefore a perfect complement to that of Sprumont. We fix an extensive game form (tree) G with complete information so that each player knows all the previous moves whenever he has to move. A *reduced tree*, G', is obtained from G by deleting some branches of G. Suppose that we observe the players' actual moves for all reduced trees. If players always play subgame perfect Nash equilibria, what are the conditions that these observed outcomes must satisfy? We can derive three necessary conditions (*acyclicity of the revealed base relation*, *internal consistency* and *subgame consistency*) and also prove that these three (independent) conditions together are sufficient for subgame perfection rationalisation. It should be emphasised that we do not claim here that intelligent players must, in practice, play according to these conditions. These conditions are just logical consequences of subgame perfection. It is possible that these consistency conditions are violated in real life situations.

2. EXTENSIVE GAME FORMS WITH PERFECT INFORMATION

Let us first describe an extensive game form G with perfect information.[1]

An extensive game form G is a rooted tree which consists of a set of nodes, X, with a distinct initial node x_0, and a *precedence function* p: $X \setminus \{x_0\} \to X$. For any node $x \in X$, the node $p(x)$ is the *immediate predecessor* of x. In turn, x is an *immediate successor* of the node $p(x)$. Formally, a node z is an immediate successor of a node x if $p(z) = x$.

A node y is a *predecessor* of x if $y = p^k(x)$, for some k. In return, x is called a *successor* of y. For any $x \in X$, let $s(x)$ and $S(x)$ denote, respectively, the set of immediate successors and the set of all

[1] The description is the same as that of an extensive form game as in most textbooks; for example, see Fudenberg and Tirole (1991) or Myerson (1991).

successors of x. A node z is called a *terminal* node if there exists no $x \in X$ such that $p(x) = z$. The set of all terminal nodes is denoted by Z.

The set of players is $N = \{1, 2, \dots, n\}$. The set of non-terminal nodes, $X \setminus Z$, are partitioned into n subsets, $X_1, \dots, X_n$, where X_i is the set of nodes at which player i moves; at any $x \in X_i$, player i has to choose an immediate successor $y \in s(x)$.

A *path* $\wp$ is a finite sequence of nodes $\{x_k: k = 0, .., m\}$ where $x_k = p(x_{k+1})$, for each k and x_m is a terminal node. Since G is a tree, any path $\wp$ can also be identified with the terminal node $z(\wp)$ at which it ends. In this paper, we consider game forms that always end, i.e., game forms that do not have any infinite sequence of nodes $\{x_k\}$ with $x_k = p(x_{k+1})$ for each k.

Suppose that we let all the players play an extensive game form G. As G always ends, we can observe, at its completion, a path $\wp$ leading to a terminal node $z(\wp)$. Nothing else, particularly, players' intended moves off the observed path $\wp$ can be observed. But additional information is revealed by the players' choices when they play different reduced game forms of G.

Definition 1. A *reduced game form* G' of an extensive game form G is a game form consisting of (i) the set of terminal nodes Z', a subset of Z, and (ii) all the non-terminal nodes that belong to path leading to any terminal node in Z'. Let us denote the set of all nodes of G' by X'.

Any standard *subgame form* selects all the terminal nodes that are successors of a particular non-terminal node. Note that it may not start at x_0, however, it can always be associated trivially with a reduced game form, consisting of the unique "path" from x_0 to the starting point of the subgame and the subgame form itself. Obviously, not all reduced game forms are subgame forms.

Let Γ be the set of all possible reduced game forms of an extensive game form G. Suppose that we can observe the *outcome* for each reduced game form $G' \in Z'$ so that we have an *outcome function* O: $\Gamma \to Z$ with $O(G') \in Z'$ for every $G' \in \Gamma$.

We now consider several necessary conditions for an outcome function O to be rationalized by subgame perfect Nash equilibrium. So let us assume that each player i has a strict ordering Q_i over all the terminal nodes and, therefore, $O(G')$ is the unique subgame perfect Nash equilibrium outcome for every reduced game G', with players' preferences $(Q_i)_{i \in N}$ (which we do not know), restricted to Z'.

Given an outcome function O, let us first construct, for each player i, the *revealed base relation*, P_i, over the terminal nodes. If a player i has to play differently in a game form to reach two different terminal nodes u and v, his preferences over u and v can be revealed via the reduced game form G' which has only two terminal nodes, u and v.

Formally, for any $u, v \in Z$, let x be the node at which the paths to u and v diverge. If $x \in X_i$, then uP_iv if and only if $u = O(G')$, where G' is the reduced game form which has only two terminal nodes, u and v.

Although the revealed base relation, P_i, is generally incomplete, it should be acyclic as it must coincide with Q_i whenever it is defined. Thus, our first necessary condition is:

Condition 1. (Acyclicity of the Revealed Base Relation) For each player i, the revealed base relation, P_i, is acyclic.

Later, we will show that when Condition 1 holds, P_i can actually be extended, via Zorn's Lemma, to a strict preference ordering on Z which is both complete and transitive. It is important to note that all P_i's are constructed through the outcome function O only, and hence, Condition 1 is a well-defined condition on O.

For any reduced game form $G' \in \Gamma$, and any node $x \in X' \setminus Z'$, let G_x' denote the reduced game form which has $Z_x' = Z' \cap S(x)$ as the set of terminal nodes. (This is just the standard subgame form of G' beginning at x.) If a terminal node u is the unique (subgame perfect Nash equilibrium) outcome for a reduced game G' and u is a successor of x, then u should also be the (subgame perfect Nash equilibrium) outcome for any subgame G_x' of G'. Thus, our second necessary condition is:

Condition 2. (Internal Consistency) For any $G' \in \Gamma$, if x is a predecessor of $O(G')$, then $O(G'_x) = O(G')$.

Finally, if u is the unique (subgame perfect Nash equilibrium) outcome of a game, then at each node x on the path leading to u, the player who moves at x should prefer u to any other terminal node that could have been reached from x had he moved otherwise. Thus, our third necessary condition is:

Condition 3. (Subgame Consistency) Consider any $G' \in \Gamma$, and $x \in X' \setminus Z'$. Let $u = O(G_x')$. Take any $y \in s(x)$ such that y is not on the path to u. Let $v = O(G_y')$. Then, $O(G'') = u$, where $G'' \in \Gamma$ is the reduced game form which has only two terminal nodes, u and v.

We have obtained the above three conditions as consequences of subgame perfect rationalization. If players play according to subgame perfect Nash equilibrium, then the observed outcomes must satisfy these conditions. Of course, there may be other necessary conditions. However, all other possible conditions must follow from Conditions 1-3 as Conditions 1-3 together turn out to be sufficient for subgame perfection rationalization.

Theorem 1. *Assume that G is an extensive game form and* $O(.)$ *is an outcome function on* Γ*. There exist strict preference orderings* Q_i*'s over Z, for all players, such that* $O(G')$ *coincides with the unique subgame perfect Nash equilibrium outcome of the game* $(G', (Q_i)_{i\in N})$ *for every* $G' \in \Gamma$ *if and only if* $O(.)$ *satisfies Conditions* 1-3.

Proof. The necessity of the three conditions is straightforward and hence the proof is devoted to showing the sufficiency.

We first show that the strict partial preference relation, P_i, as defined above can be extended to a strict ordering on Z which is complete and acyclic (equivalently, transitive, for a complete ordering) for each player i. Suppose that P_i is not complete. For any pair u and v that are not ranked by P_i, let $uP_i'v$ if there exists a sequence $\{u_k\}_{1\leq k\leq K}$ with $u = u_1$, $v = u_K$, and $u_kP_iu_{k+1}$ for all k; and $vP_i'u$ otherwise. By adding exactly one of these two relations to the original P_i, we obtain an extension of P_i. Suppose that this extension has a cycle C. Since the original P_i is acyclic (by Condition 1), C must contain $uP_i'v$, or $vP_i'u$, as one of the links in C. If $uP_i'v$, then there exists a chain from u to v via P_i-domination. But then, replacing $uP_i'v$ in C by this chain yields a cycle for P_i, which contradicts Condition 1. On the other hand, if C contains $vP_i'u$, then the remaining links in C from u to v implies $uP_i'v$ (by the definition of $uP_i'v$); again a contradiction. Hence, the extension is also acyclic. Then a routine argument using Zorn's Lemma shows that we can extend P_i to a strict preference relation Q_i on Z which is complete and acyclic (cf. Theorem 1, Richter, 1966).

We now prove that for each reduced game form G', the observed outcome $O(G')$ coincides with the unique subgame perfect Nash equilibrium outcome of the game $(G', (Q_i)_{i\in N})$. First, we construct a subgame perfect Nash equilibrium that yields $O(G')$ as the outcome for every G'. Consider the following strategy for every player i. At each node $x \in X_i$, i chooses the immediate successor that leads to $O(G')$. If every player follows this strategy, then the outcome is $O(G')$. Let us show that these strategies indeed constitute a Nash equilibrium for every G_x' where $x \in X' \setminus Z'$.

Suppose this is not true. Then there must exist a node x such that these strategies do not constitute a Nash equilibrium for G_x', but they do for G_y', for all successors y of x. For, if such an x does not exist, we would be able to find an infinite sequence of nodes $\{x_k\}$ with $x_k = p(x_{k+1})$, for each k, which contradicts the assumption that the game always ends. Since these strategies do not form a Nash equilibrium for G_x', there exists a player i who can gain by deviating in G_x'. Obviously, i has to deviate at some node on $\wp$, the path that leads to $u = O(G_x')$. Let y be the first node on $\wp$ after x at which player i deviates. If $y \neq x$, then $O(G_y') = O(G_x')$ by Condition 2 and $O(G_y')$ is the Nash equilibrium outcome for G_y', by assumption; so, i cannot gain. Thus $y = x$, i.e., i has to deviate at x in order to gain. Now consider z, the immediate successor of x under i's deviation. Let $v = O(G_z') = \text{SPNE}(G_z', (Q_i)_{i \in N})$. Suppose G'' is the reduced game form which has only two terminal nodes, u and v. By Condition 3, $u = O(G'')$. This implies uP_iv (i.e., uQ_iv), by the definition of P_i, contradicting the hypothesis that player i can deviate and gain[2].

The uniqueness of the subgame prefect Nash equilibrium outcome for $(G', (Q_i)_{i \in N})$ can be proved in a similar fashion. ■

The three conditions in Theorem 1 are logically independent. Let us present three examples in which the respective outcome functions satisfy different pairs of the three conditions but not all of them, and thus no subgame perfect rationalisation can be found for these outcome functions.

Example 1. Consider the game tree G as in Figure 1. G has three proper reduced game forms, G_1, G_2, and G_3.

[Insert Figure 1 here]

[2]The last part of the proof rather shows the *one deviation property* (as in Lemma 98.2 of Osborne and Rubinstein, 1994) which is a necessary and sufficient condition for subgame perfect equilibrium.

Consider the outcome function O_1, as shown in Figure 1: $O_1(G) = B$, $O_1(G_1) = (A, L)$, $O_1(G_2) = (A, L)$, and $O_1(G_3) = B$. It is easy to verify that O_1 satisfies acyclicity of the revealed base relation and internal consistency. However, consider the condition of subgame consistency with $G' = G$ and $x = x_0$. The outcome, $O_1(G)$ is B. Now consider the subgame form G_1, the outcome of which is (A, L). But the outcome of G_2 which consists of B and (A, L) only is (A, L), violating subgame consistency.

Example 2. Consider the same game tree as in Example 1 with a different outcome function O_2, as shown in Figure 2: $O_2(G) = (A, L)$, $O_2(G_1) = (A, R)$, $O_2(G_2) = (A, L)$, and $O_2(G_3) = B$.

[Insert Figure 2 here]

It is easy to verify that O_2 satisfies acyclicity of the revealed base relation and subgame consistency. However, consider the condition of internal consistency with $G' = G$ and the node x where player 2 moves. Note that $O_2(G)$ is (A, L) but $O_2(G_1)$ is (A, R), violating internal consistency.

Example 3. This is a variation of the game tree in Example 1 in which only player 1 moves at both the non-terminal nodes. Consider the outcome function O_3, as shown in Figure 3: $O_3(G) = B$, $O_3(G_1) = (A, R)$, $O_3(G_2) = (A, L)$, and $O_3(G_3) = B$.

[Insert Figure 3 here]

It is easy to verify that O_3 satisfies internal and subgame consistency. However, consider the outcomes of the three reduced games forms, G_1, G_2 and G_3. These outcomes clearly violate the condition of acyclicity of the revealed base relation.

It is worth mentioning that Conditions 1-3 are not sufficient for the existence of *one* single preference relation (a social preference relation) P over all terminal nodes of G such that $O(G')$ is the best element of P among terminal nodes of G' for every $G' \in \Gamma$. The following outcome function has a subgame perfect rationalization, but does not have a representative single agent rationalization.

Example 4. Consider the same game tree as in Example 1 and the outcome function O_4, as shown in Figure 4: $O_4(G) = B$, $O_4(G_1) = (A, L)$, $O_4(G_2) = B$, and $O_4(G_3) = (A, R)$.

[Insert Figure 4 here]

It is easy to verify that O_4 satisfies all the three conditions. However this outcome function could not be rationalised had this been a choice of one individual only as the last three outcomes form a cycle for one individual.

4. CONCLUDING REMARKS

In this paper we have studied equilibrium notions for games from the revealed preference perspective. For game trees with perfect information we have derived necessary and sufficient conditions for the observed outcomes to be rationalized by subgame perfect Nash equilibrium. This result extends

standard result of revealed preferences to multi-agent decision making situations. However, in contrast to the classical result in the individual revealed preference theory, the preference relations of agents in a game context cannot be uniquely determined by the observed choices. If an agent cannot change the outcome from a to b by a unilateral change of his action, his preferences over a and b can never be recovered from the observed actions.

Our result in this paper deals with a simple case only and we have not imposed any restrictions on the outcomes or the preferences. In more concrete problems, one can think of introducing additional assumptions on the preferences. For example, if the outcome is just a division of money, then it may be reasonable to assume that agents' preferences are monotonic in their own monetary payoffs. Or, if each agent's actions can be represented by a convex set in a Euclidean space, it is often assumed (in the standard game theory literature) that preferences are continuous and convex (particularly, in the context of normal form games). We would, eventually, want to implement our study through some form of experiments. Since all meaningful experiments are conducted in concrete contexts, we need to establish results that are sensible to the specifics of the experiments. Hence, our paper is just the beginning of a broader research programme, which will extend the basic result here to cover other structures of strategic interactions and game theoretic solution concepts.

We should mention that other authors such as Peleg and Tijs (1996) have previously considered consistency conditions to characterise Nash equilibrium[3]. For any well-specified (normal form) game, in which the players' payoff functions are already given, they show that the players must play Nash equilibria if (i) each player is a utility maximiser and (ii) all players choose strategies consistently. Their viewpoint is clearly different from ours. In their work, consistency is used as a normative criterion that forces players to adopt Nash equilibrium behaviour. We, however, consider weak consistency as an empirically verifiable condition on the observed choices for rationalization by strict Nash equilibrium. We make no claim that the players always choose or should choose strategies in such a consistent fashion.

For extensive game forms, we require that the outcomes of all possible reduced trees be observable. There are many simple and interesting game situations such as dividing-a-dollar, in which the players have infinitely many strategies but, practically, only finitely many observations can be made. Hence, one would like to know whether our results can be extended to such cases with only

[3]Peleg and Tijs (1996) actually call this condition *independence of irrelevant strategies*. The term *consistency* in their paper refers to a narrower condition.

limited observations[4]. It is easy to see that our conditions are still necessary for subgame prefect rationalisation, but sufficiency is far from obvious.

[4]This is parallel to Afriat's (1967) approach to the theory of revealed preferences, vis-a-vis Samuelson's.

REFERENCES

1. Afriat, S. (1967). "The Construction of a Utility Function from Demand Data," *International Economic Review* **8**, 67-77.

2. Fudenberg, D., and Tirole, J. (1991). *Game Theory*. Cambridge, MA: MIT Press.

3. Myerson, R. B. (1991). *Game Theory*: *Analysis of Conflict*. Cambridge, MA: Harvard University Press.

4. Osborne, M. J., and Rubinstein, A. (1994). *A Course in Game Theory*. Cambridge, MA: MIT Press.

5. Peleg, B., and Tijs, S. (1996). "The Consistency Principle for Games in Strategic Form," *International Journal of Game Theory* **25**, 13-34.

6. Richter, M. (1966). "Revealed Preference Theory, " *Econometrica* **34**, 635-645.

7. Samuelson, P. A. (1938). "A note on the Pure Theory of Consumers' Behavior," *Economica* **5**, 61-71.

8. Sprumont, Y. (1999). "On the Testable Implications of Collective Choice Theories," Mimeo, University of Montreal.

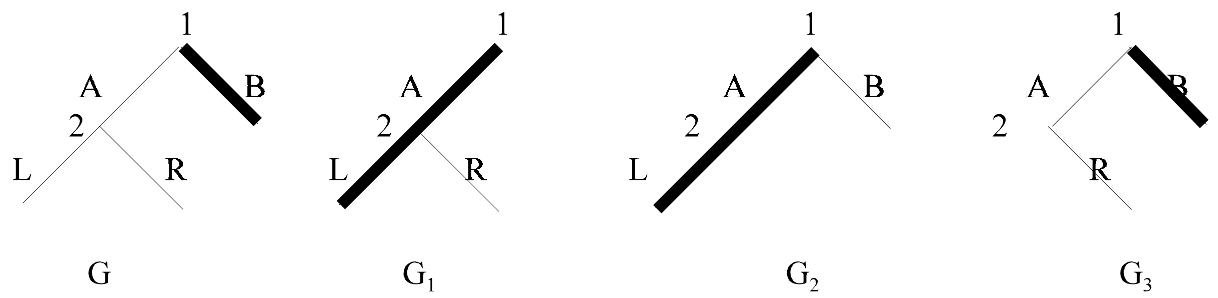

Figure 1

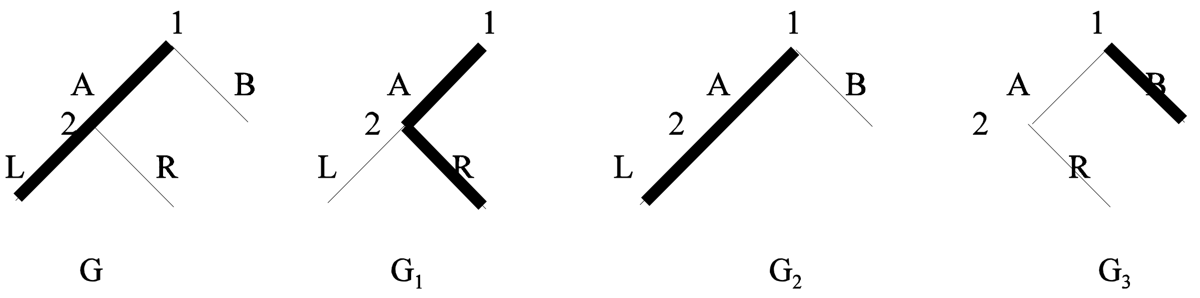

Figure 2

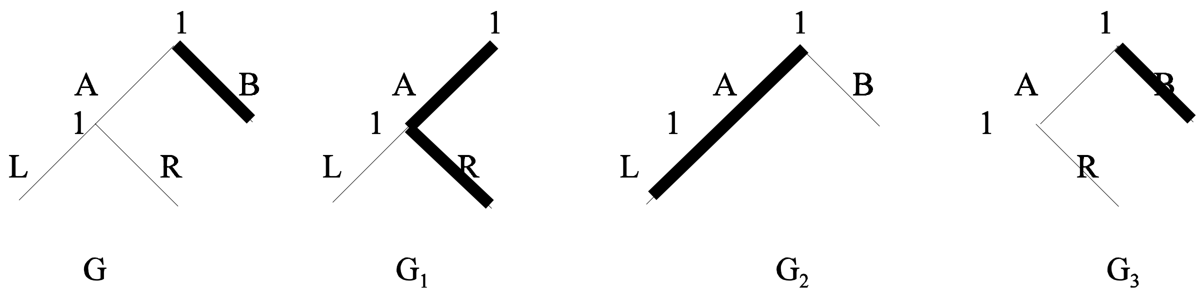

Figure 3

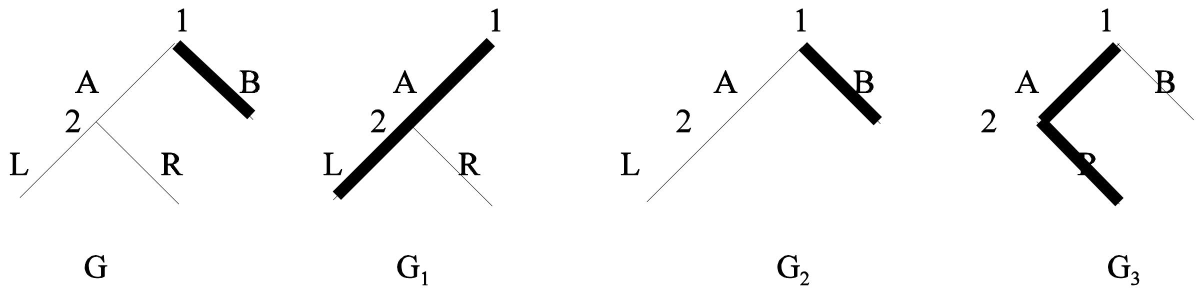

Figure 4