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Saddlepoint Approximations in Non-Stationary Time Series

by

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Abstract

Saddlepoint approximations for the finite sample density and distribution of the estimate of the autoregressive parameter in an AR(1) model are derived. We directly extend the results of Phillips (1978) and Lieberman (1994), to nonstationary cases, by allowing unit and explosive roots and the presence of deterministic components, in particular time trends and drifts. A tractable algorithm is presented, leading to a general form which is unchanged whether the model is stationary, non-stationary or contains deterministic components. The accuracy of the approximation is demonstrated, in particular in comparison with established asymptotic results.

Key Words: Autoregression, trend and difference stationarity, saddlepoint approximation.

1 Introduction

Expressions for the density and distribution, whether approximate or exact, of the estimator of the autoregressive parameter have been extensively sought throughout the time series literature. For example, in the simplest zero mean, stationary $AR(1)$ model asymptotic normality has been established and Edgeworth type corrections to the asymptotic density presented, for example in Phillips (1977). However, in unit root and explosive cases asymptotic densities are functionals of standard Wiener processes (see White (1958) and Phillips (1986)). The situation is further complicated since introducing exogenous regressors, for example a drift or time trend, into unit root models, re-establishes asymptotic normality (see Evans and Savin (1984) or Hamilton (1994)). As a consequence, asymptotic analysis of the properties of the autoregressive estimator in some general dynamic regression model is somewhat of a piecemeal process since the asymptotic distribution depends upon; i) whether the model is stationary, and ii) if it is not stationary, whether trends and drifts are included.

In this paper we present a simple unified approximation for the density and distribution of the estimator, whose functional form is independent of whether the model is stationary and/or if exogenous regressors are included. The approximation technique is a variant on the saddlepoint approximation introduced by Daniels (1954), and which is becoming established as excellent approximations to the finite sample density and distribution of various estimators and tests. The statistical applications of the technique seem to fall into one of two categories. Approximations for likelihood based statistics may be constructed through exponential tilting of the likelihood, for example see Durbin (1980) or Reid's (1988) review. Alternately, approximations may be constructed through the exploitation of the properties of the particular statistic itself, for example ratios of quadratic forms, as in Marsh (1998). For a comprehensive exposition of the technique, examples and the seminal references, see Jensen (1995).

This paper presents a saddlepoint algorithm for approximating the density and distribution of the estimate of the autoregressive parameter in simple non-stationary time series, including both difference and trend stationary processes. This directly generalises similar algorithms presented and investigated by Daniels (1956), Phillips (1978), Wang (1992) and Lieberman (1994) for the simplest, stationary $AR(1)$. Those papers found the approximation a significant improvement on the first-order asymptotic result and the competing higher-order Edgeworth expansion. This paper finds that such accuracy is preserved, even when the nature of the autoregression is more complicated, in particular when we allow unit roots with drifts or time trends. Since it can also be established that the estimate itself forms a basis for exact inference, i.e. invariance with respect to nuisance parameters, any improved distribution theory, in such widely applied models, seems worthwhile.

Moreover, since we are approximating the finite sample density and distribution, through approximation of their exact inversion formulae, the general form of the approximation is seen to be the same for stationary and non-stationary models, whether regressors are included or not. Of more relevance is the performance of the approximate distribution, that is how well it approximates the distribution of the estimator in moderate sample sizes. While asymptotic representations in the unit root case, Abadir (1993) perform well, in stationary models with autoregressive parameter approaching unity and unit root models with drift, the asymptotic distribution is inadequate in this regard. Some insight as to why this occurs is detailed in the studies of the curvature of autoregressive models by Ravishanker, Melnick and Tsai (1990) and van Garderen (1999). Performance of the saddlepoint approximation, in terms of both accuracy and implementation, will be demonstrated to be far less case sensitive than the respective limiting asymptotic approximations.

The plan for the paper is as follows. The following section details the particular model under consideration, the estimator itself and consequently presents the main result, the leading term saddlepoint approximation. Section 3 presents an analysis of the approximation, in particular comparisons between the 'exact' density, obtained through Monte Carlo simulation, the limiting asymptotic result, and the saddlepoint, for particular configurations of the model. Section 4 concludes, and an appendix details the precise form of the algorithm, implemented through the symbolic package *Mathematica*, in order to make the procedure much more transparent.

2 Model and Approximations

Formally this paper is concerned with inference on the autoregressive parameter in the following general model

$$y_i = z_i'\beta + \rho y_{i-1} + \varepsilon_i \ ; \ \ i = 1, ..., N \ , \tag{1}$$

where $y_0 = 0$, $\varepsilon_i \sim NID(0, \sigma^2)$ and the z_i are $k \times 1$ vectors containing any relevant deterministic (or exogenous) variables. Extension to the case $y_0 \neq 0$ is trivial on application of the transformation in Evans and Savin (1984, Section 2). Defining the sample vector as $y = \{y_1, ..., y_N\}'$, it is easily seen that

$$y \sim N\left(T_\rho^{-1} Z\beta, \sigma^2 \Sigma(\rho)\right), \tag{2}$$

where

$$Z = \begin{pmatrix} z_1' \\ z_2' \\ \vdots \\ z_N' \end{pmatrix}, \; T_\rho = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\rho & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & -\rho & 1 \end{pmatrix} \text{ and } \Sigma^{-1}(\rho) = T_\rho' T_\rho,$$

and hence the likelihood is

$$L(\rho, \beta, \sigma^2 \mid y) = (2\pi\sigma^2)^{-N/2} \exp\left\{-\frac{1}{2\sigma^2}(y - T_\rho^{-1}Z'\beta)'T_\rho'T_\rho(y - T_\rho^{-1}Z'\beta)\right\}. \tag{3}$$

Further, estimating β and σ^2, temporarily supposing ρ fixed, the profile likelihood is

$$\begin{aligned} L_P(\rho) &= (2\pi\hat{\sigma}^2)^{-N/2} \exp\left\{-\frac{1}{2\hat{\sigma}^2}(y'T_\rho'T_\rho y - \hat{\beta}'Z'T_\rho y)\right\}, \quad (4) \\ \hat{\beta} &= \hat{\beta}(\rho) = (Z'Z)^{-1}Z'T_\rho y \quad \text{and} \\ \hat{\sigma}^2 &= \hat{\sigma}^2(\rho) = \frac{(y - \hat{\beta}'T_\rho^{-1}Z)'T_\rho'T_\rho(y - \hat{\beta}'T_\rho^{-1}Z)}{N}. \end{aligned}$$

Hence we obtain the usual maximum likelihood estimator (MLE) for ρ

$$\hat{\rho} = \frac{y'M_Z By}{y'B'M_Z By},$$

where $M_Z = I - Z(Z'Z)^{-1}Z'$ and B is the lag-operator matrix with ones on the upper off diagonal and zeros elsewhere.

Hypothesis tests on ρ may be performed, for example, either simply using the MLE or through the profile log-likelihood ratio

$$w(\rho_0) = 2(\ln[L_P(\hat{\rho})] - \ln[L_P(\rho_0)]).$$

In particular, invariance to parameter transforms of the form

$$\begin{pmatrix} \beta \\ \sigma^2 \end{pmatrix} \to \begin{pmatrix} a + b\beta \\ c\sigma^2 \end{pmatrix},$$

$a, b \in \mathcal{R}$, $c \in \mathcal{R}^+$, is readily shown for both $\hat{\rho}$ and $w(\rho_0)$, see for example Dufour and Kiviet (1997). Moreover, at no stage will we place any restrictions upon the range of ρ in the context of the general approximation. Now, both $\hat{\rho}$ and $w(\rho_0)$ may be seen to be ratios of quadratic forms in y. In this paper we present the approximation only for $\hat{\rho}$, in order to extend the analysis of Phillips (1978) and Lieberman (1994), although the algorithm may easily be adapted for the density and distribution of $w(\rho_0)$.

In order to implement the saddlepoint approximation for $\hat{\rho}$, for any β, ρ and σ^2, we write

$$\hat{\rho} = \frac{\frac{1}{2}v'(T_\rho^{-1})'(M_Z B + B'M_Z)T_\rho^{-1}v}{v'(T_\rho^{-1})'B'M_Z BT_\rho^{-1}v} = \frac{v'A_1(\rho)v}{v'A_2(\rho)v}, \tag{5}$$

where now $v \sim N(Z\beta, I_N)$. As a consequence of Gurland (1948), the density and distribution, at any point q, of the estimate are

$$\begin{aligned} f_{\hat{\rho}}(q) &= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \left.\frac{\partial\chi(\theta_1,\theta_2)}{\partial\theta_1}\right|_{\theta_2=-q\theta_1} d\theta_1, \\ F_{\hat{\rho}}(q) &= \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} \frac{\chi(\theta_1,-q\theta_1)}{\theta_1} d\theta_1, \end{aligned} \tag{6}$$

where $\chi(\theta_1,\theta_2)$ is the joint characteristic function of the quadratic forms $v'A_1(\rho)v$ and $v'A_2(\rho)v$. For the estimator in (5) we have

$$\begin{aligned} \left.\frac{\partial\chi(\theta_1,\theta_2)}{\partial\theta_1}\right|_{\theta_2=-q\theta_1} &= e^{-\frac{\beta'Z'Z\beta}{2}} \exp\left\{\frac{1}{2}\left(Tr[G^{-1}Z\beta\beta'Z'] - \ln|G|\right)\right\} \times \\ & \quad Tr[G^{-1}A_2(\rho)(G^{-1}Z\beta\beta'Z' + I)], \\ \chi(\theta_1,-q\theta_1) &= e^{-\frac{\beta'Z'Z\beta}{2}} \exp\left\{\frac{1}{2}\left(Tr[G^{-1}Z\beta\beta'Z'] - \ln|G|\right)\right\}, \end{aligned} \tag{7}$$

where $G = G(\theta_1) = I - 2\theta_1 F$, and $F = A_1(\rho) - qA_2(\rho)$. Upon substitution of (7) into inversions (6), application of the saddlepoint technique of Daniels (1954) is straightforward.

Theorem 1 *Under the conditions of the process described in (1) the leading term saddlepoint approximations take the form*

$$\hat{f}_{\hat{\rho}}(q) = \frac{e^{-\frac{\beta' Z' Z\beta}{2}} \exp\left\{Tr[\hat{G}^{-1}Z\beta\beta' Z']\right\} Tr\left[\hat{G}^{-1}A_2(\rho)(\hat{G}^{-1}Z\beta\beta' Z' + I)\right]}{|\hat{G}|^{1/2}\sqrt{4\pi Tr\left[(\hat{G}^{-1}F)^2(2\hat{G}^{-1}Z\beta\beta' Z' + I\right]}}, \quad \text{(8a)}$$

$$\hat{F}_{\hat{\rho}}(q) = (1 - e^{-\frac{\beta' Z' Z\beta}{2}}) + e^{-\frac{\beta' Z' Z\beta}{2}} \Phi\left(\hat{p} - \frac{1}{\hat{p}}\ln\left(\frac{\hat{p}}{\hat{r}}\right)\right), \quad \text{(8b)}$$

where $\hat{G} = G(\hat{\theta}_1)$,

$$\hat{p} = sign(\hat{\theta}_1)\sqrt{Tr[\hat{G}^{-1}Z\beta\beta' Z'] - \ln|\hat{G}|},$$
$$\hat{r} = 2\hat{\theta}_1\sqrt{Tr\left[(\hat{G}^{-1}F)^2(2\hat{G}^{-1}Z\beta\beta' Z' + I\right]},$$

and the saddlepoint $\hat{\theta}_1$ *solves the saddlepoint defining equation*

$$Tr\left[G^{-1}F(G^{-1}Z\beta\beta' Z' + I)\right] = 0. \quad \text{(9)}$$

The derivation of the approximations follows that of Marsh (1998), sections 3 and 4, see also Jensen (1995). In particular, note that we may write the saddlepoint defining equation as

$$\sum\nolimits_{j=1}^{N}\left[\frac{f_j}{1 - 2\theta_1 f_j}\left(1 + \frac{u_j^2}{1 - 2\theta_1 f_j}\right)\right] = 0, \quad (10)$$

where the f_j are the ordered eigenvalues of F and the u_j are elements of $u = RZ\beta$, where the $N \times N$ matrix R diagonalises F. The eigenvalues are continuous in q and (10) is a polynomial of degree $2N$ in θ, hence choosing the unique solution

$$\hat{\theta}_1 \in \left(\frac{1}{2f_N}, \frac{1}{2f_1}\right),$$

see Daniels (1954), ensures that first the approximations exist and second, are continuous in q, by the implicit function theorem.

If $\beta = 0$ and $|\rho| < 1$, then the approximations correspond to those given by Phillips (1978) and Lieberman (1994), except of course that here y_0 is fixed. Here, we do not focus upon the asymptotic nature of the approximation, although Marsh (1998) demonstrates that in the far tails the order of error of the first correction to (8a) is $O(N^{-1})$ in general. Of more interest is the numerical accuracy of (8a) and (8b), and the following section contains an analysis of the finite sample performance of the approximations.

3 Analysis of the Approximations.

This paper is specifically concerned with inference in non-stationary time-series. In order to assess the accuracy of the approximation we consider three specific parameter configurations for the general model given in (1). In particular we characterise these by

$$\begin{aligned} \mathrm{M}_1 &: \beta = 0 \\ \mathrm{M}_2 &: \beta \neq 0 \text{ and } z_i = 1 \text{ for all } i \\ \mathrm{M}_3 &: |\rho| < 1 \text{ , } \beta \neq 0 \text{ and } z_i = i \text{ for all } i. \end{aligned}$$

M_1 encompasses the pure stationary, random walk and explosive processes, M_2 the non-zero mean stationary, random walk with drift and directed explosive processes and M_3 trend nonstationary processes. The analysis can be extended to more general models, for instance including both a constant and trend, or any other set of relevant deterministic (or ancillary) variables, by simply including such terms in the z_i.

Consider first $\mathrm{M}_1 : y_i = y_{i-1} + \varepsilon_i$, the pure random walk. It is well known (White

(1959)) that the standardised MLE converges in distribution

$$\frac{N}{\sqrt{2}}(\hat{\rho}-1) \rightarrow_d \frac{W(1)^2-1}{\sqrt{8}\int_0^1 W^2(s)ds},$$

where $W(.)$ denotes the standard Weiner process on $[0,1]$. Moreover, Abadir[1] (1993) derives an explicit closed form for the density and distribution, his equations (2.14) and (2.15) respectively.

For model $\mathrm{M}_2 : y_i = \beta + y_{i-1} + \varepsilon_i$ the corresponding limiting result is (see Hamilton (1994), Chapter 17)

$$\frac{N^{3/2}}{\sqrt{2}}(\hat{\rho}-1) \rightarrow_d N\left(0, \frac{3\sigma^2}{\beta^2}\right),$$

and for the sake of comparison, approximation (8a) may be transformed such that the approximate density of $\kappa = \frac{N^j}{\sqrt{2}}(\hat{\rho}-1)$ at a point k is

$$\hat{f}_\kappa(k) = f_{\hat{\rho}}(\hat{\rho})\frac{\sqrt{2}}{N^j},$$

where $j = 1, 3/2$ for $\beta = 0$ and $\beta \neq 0$ respectively. Implementation of the saddlepoint algorithm for the particular statistics was performed using the symbolic algebra package *Mathematica*. The Appendix contains the approximation generating programme itself. Tables 1 through 6 then give the relevant quantiles obtained from 250000 Monte Carlo replications, the saddlepoint and limiting approximations (with $\sigma^2 = 1$).

Some immediate points arise. Standardisation was for comparative purposes only, however for an arbitrary $AR(1)$ process asymptotic results for the MLE depend on the particular parameter configurations. The form of the saddlepoint approximation is invariant to such changes. Equally, what constitutes a 'large' or a 'small' sample size is also dependent upon the parameter configuration. As a consequence in the

[1] The author is indebted to Karim Abadir for making available the *Gauss* code for enumeration of the limiting approximation.

random walk model convergence is obtained much more quickly than in the drift case (and also stationary models). Differing sample sizes were thus taken for the basis of comparison. The saddlepoint seems, in an absolute sense, to perform well in small samples, and exceptionally so in a relative (to the limiting) sense.

For the case $M_3 : y_i = \beta i + \rho y_{i-1} + \varepsilon_i$, figures 1 through 4, plot the saddlepoint and the simulated densities for the non-standardised MLE. Again 250000 replications were performed in the Monte Carlo study. We consider cases where $\beta = 1/4$ and $\rho = 0, 1/3, 1/2, 2/3$, for a sample size of 10 and again, the approximation performs well in comparison with the simulated, although the performance tends to suffer when ρ becomes large.

Tables

Each table contains the quantiles of the 'exact' distribution, obtained via simulation, and the approximations from the Saddlepoint (equation (8b)) and Limiting distributions, given by equation (11) for model M_1 and by equation (12) for model M_2.

Table 1: M_1; $N = 5$

Quantile	1	2.5	5	10	50	90	95	97.5	99
Exact	-5.56	-4.69	-3.89	-3.03	-0.45	1.07	1.55	2.08	2.85
Saddlepoint	-5.57	-4.59	-3.78	-2.86	-0.31	1.14	1.63	2.20	3.08
Limiting	-9.67	-7.37	-5.67	-4.02	-0.62	0.68	0.93	1.13	1.43

Table 2: M_1; $N = 10$

Quantile	1	2.5	5	10	50	90	95	97.5	99
Exact	-7.57	-6.12	-4.92	-3.63	-0.57	0.76	1.05	1.34	1.70
Saddlepoint	-8.01	-6.46	-5.22	-3.83	-0.81	0.81	0.99	1.20	1.57
Limiting	-9.67	-7.37	-5.67	-4.02	-0.62	0.68	0.93	1.13	1.43

Table 3: M_2; $\beta = 1/4$ & $N = 10$

Quantile	1	2.5	5	10	50	90	95	97.5	99
Exact	-26.85	-23.70	-20.67	-17.28	-6.81	-0.17	1.44	2.94	4.71
Saddlepoint	-25.04	-22.58	-20.32	-17.66	-8.49	-2.01	-0.67	0.44	2.01
Limiting	-16.12	-13.58	-11.40	-8.88	0.00	8.88	11.40	13.58	16.12

Table 4: M_2; $\beta = 1/4$ & $N = 20$

Quantile	1	2.5	5	10	50	90	95	97.5	99
Exact	-44.69	-37.76	-32.08	-25.97	-8.60	-0.05	1.72	3.26	4.78
Saddlepoint	-48.38	-41.42	-35.10	-28.14	-9.80	-0.12	1.69	3.27	4.84
Limiting	-16.12	-13.58	-11.40	-8.88	0.00	8.88	11.40	13.58	16.12

Table 5: M_2; $\beta = 1/2$ & $N = 10$

Quantile	1	2.5	5	10	50	90	95	97.5	99
Exact	-24.41	-20.41	-17.00	-13.22	-3.70	1.18	2.59	3.92	5.78
Saddlepoint	-22.58	-20.34	-18.33	-15.87	-7.37	-1.34	0.44	1.34	2.90
Limiting	-11.39	-9.60	-8.06	-6.28	0.00	6.28	8.06	9.60	11.39

Table 6: M_2; $\beta = 1/2$ & $N = 20$

Quantile	1	2.5	5	10	50	90	95	97.5	99
Exact	-32.06	-24.10	-18.28	-13.42	-3.30	1.92	3.25	4.48	6.17
Saddlepoint	-32.86	-28.71	-23.82	-19.30	-5.34	2.46	3.69	5.34	6.57
Limiting	-11.39	-9.60	-8.06	-6.28	0.00	6.28	8.06	9.60	11.39

Figures

These figures plot the 'exact' density function of $\hat{\rho}$, obtained via simulation and represented by the solid line, while the Saddlepoint approximation, equation (8a) is represented by the dotted line. In each case the sample size was fixed and $N = 10$.

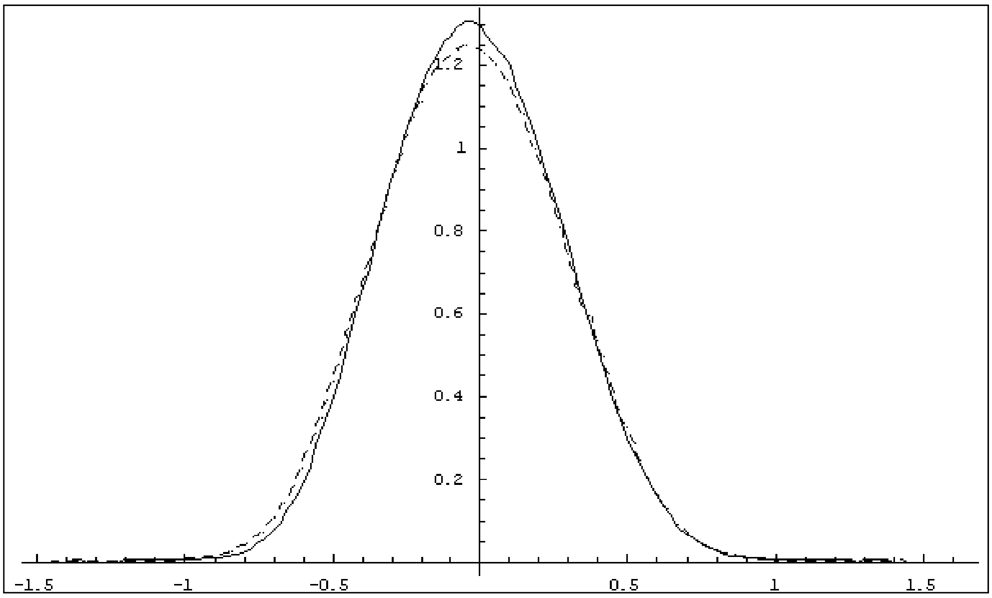

Fig. 1: M_3 with $\rho = 0$

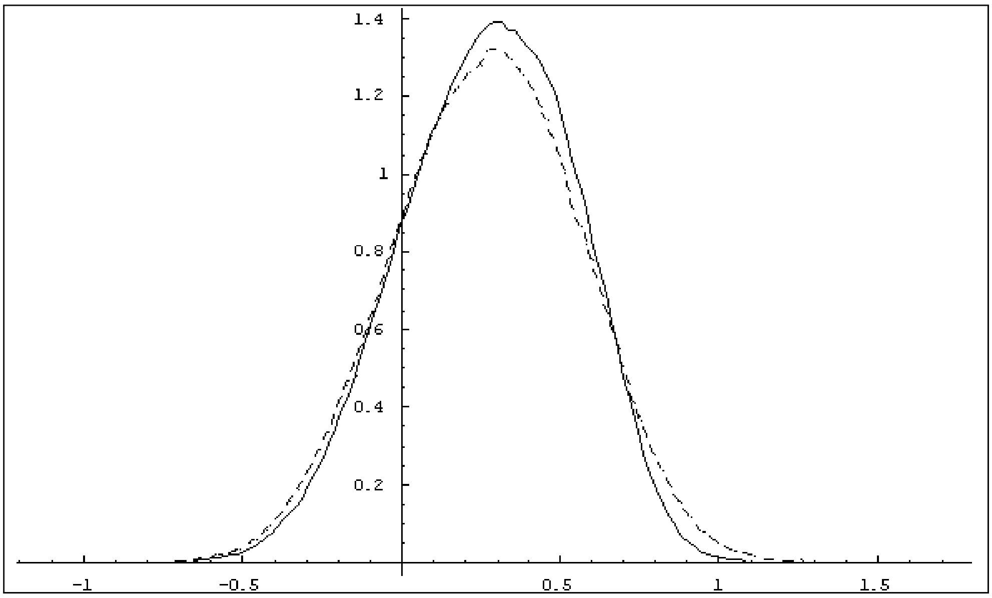

Fig. 2: M_3 with $\rho = 1/3$

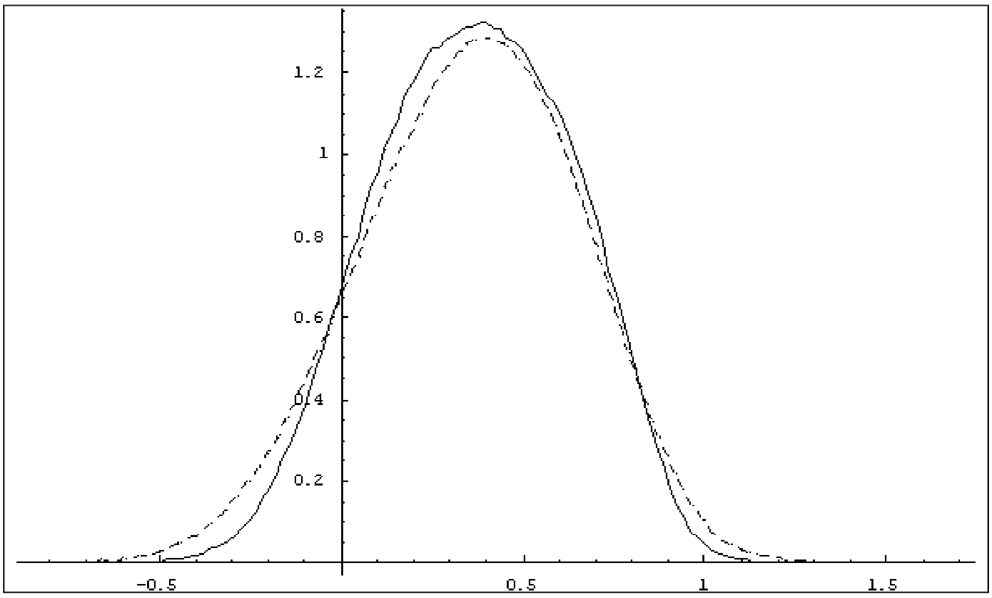

Fig. 3: M_3 with $\rho = 1/2$

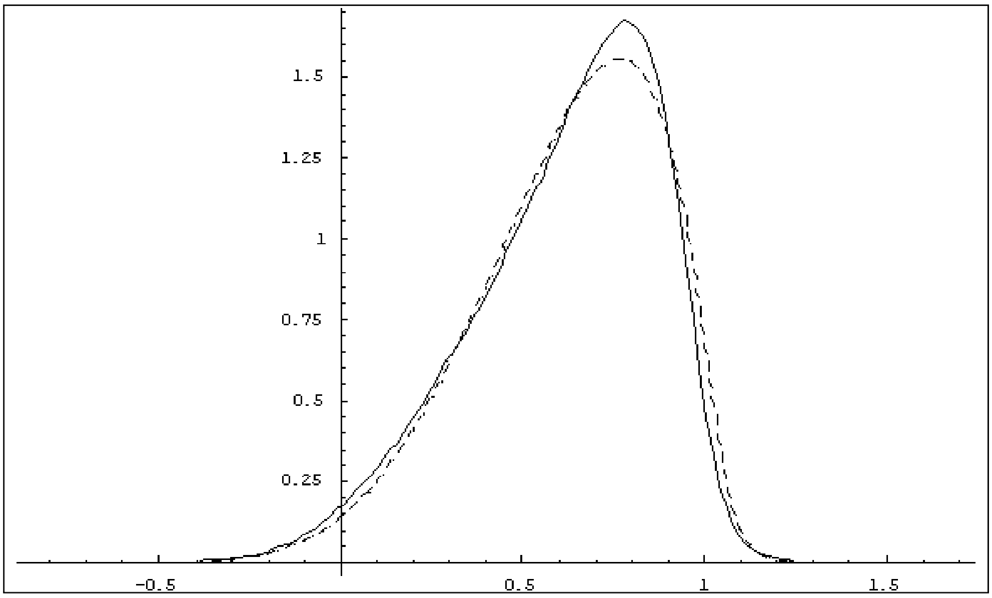

Fig. 4: M_3 with $\rho = 2/3$

4 Conclusions

The saddlepoint technique appears to offer a convenient, tractable and accurate approximation to the finite sample density and distribution of many estimators and tests. In this case an arbitrary $AR(1)$ process was analysed and the saddlepoint approximation for the estimate of the autoregressive parameter was derived. The numerical accuracy found by Phillips (1978) and Lieberman (1994) is seen to hold even in the nonstationary and non-central cases. Not only is the performance good with respect to both the 'exact' and competing limiting approximations, but computation itself is swifter than Monte Carlo studies, and comparable to limiting representations, such as derived in Abadir (1993).

Appendix: (Mathematica Code)

The following is the code for the implementation of the general saddlepoint approximation, written for *Mathematica 3.0*.

```
<<Statistics‘ContinuousDistributions‘

nn = Input[”Sample Size”];

al = Input[”AR parameter”];

be = Input[”Constant/Trend coefficient”];

zz = Table[N[be * i], {i,1,nn}][

Mx = IdentityMatrix[nn] - Outer[Times, zz,zz]/(zz.zz);

B = Table[Switch[i-j,-1,1,0,0,1,0,_,0], {i,1,nn}, {j,1,nn}];

A = Table[Switch[i-j,-1,1,1/2,0,0,1/2,_,0], {i,1,nn}, {j,1,nn}];

T = Table[Switch[i-j,-1,0,0,1,1,-al,_,0], {i,1,nn}, {j,1,nn}];
```

```
T1 = Inverse[T]; T2 = Transpose[T1];
B1 = Mx.B; B2 = Transpose[B].Mx.B;
A1 = (T2.(B1+Transpose[B1]).T1)/2;
A2 = T2.B2.T1;
F[q_] = Chop[A1 - q A2];
S = Outer[Times,zz,zz] ; lambda = -zz.zz/2;

Do[
{r[j] , s[j]} = SchurDecomposition[N[F[j]]];
u[j] = Transpose[r[j]].zz;
fi[j] = Sort[Table[s[j][[i,i]], {i,1,nn}]];
p[t_,j] = (1/2) * (Sum[u[j][[i]]^2 * (1 - 2 t fi[j][[i]])^(-1), {i,1,nn}] - Sum[Log[1 - 2 t fi[j][[i]]], {i,1,nn}]);
p1[t_,j] = D[p[t,j], t];
e1[j] = 1/(2 Max[fi[j]]) ; e2[j] = 1/(2 Min[fi[j]]);
sap1[j] = FindRoot[p1[t,j], {t,1/2 * (e1[j] + e2[j])}];
sap[j] = t /. sap1[j]; ,   {j,-1,1.5, 0.01}]

Do[
pp1[j] = (1/2) * (Sum[u[j][[i]]^2 / (1 - 2 sap[j] fi[j][[i]]), {i,1,nn}] - Sum[Log[1 - 2 sap[j] fi[j][[i]]], {i,1,nn}]);
pp2[j] = 2 * Sum[(u[j][[i]]^2 * fi[j][[i]]^2) / (1 - 2 sap[j] fi[j][[i]])^3, {i,1,nn}] + Sum[fi[j][[i]]^2 / (1 - 2 sap[j] fi[j][[i]]])^2, {i,1,nn}];
G[j] = IdentityMatrix[nn] - 2 sap[j] F[j];
M1[j] = Inverse[G[j]].A2;
M2[j] = Inverse[G[j]].S + IdentityMatrix[nn];
```

```
M3[j] = M1[j].M2[j];
q[j] = Sum[M3[j][[i,i]], {i,1,nn}];
dens[j] = (Exp[lambda] * Exp[pp1[j]] * q[j]) / (Sqrt[4 Pi pp2[j]]);,  {j,-1,1.5,0.01}]

norm = Sum[dens[j], {j, -1,1.5, 0.01}]]
density = Table[{j, dens[j]/(norm * 0.01), {j,-1,1.5,0.01}];
ListPlot[density, PlotJoined->True, PlotStyle->Dashing[{0.01}]]
```

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