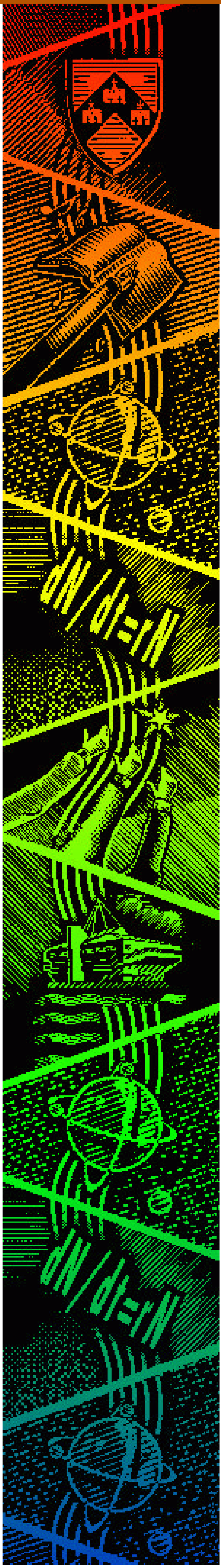

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The Covariance Structure of Mixed ARMA Models

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THE COVARIANCE STRUCTURE OF MIXED ARMA MODELS

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Abstract

The purpose of this paper is to examine the covariance structure of mixed ARMA models, as discussed in Granger and Morris (1976). The method we use to obtain the autocovariances is based on the Wold representation of an ARMA model as it is given in Pandit (1973) or in Karanasos (2000). We give two examples to illustrate our general results: (i) two ARMA(2,2) processes with identical autoregressive polynomials and different moving average ones, and (ii) two ARMA(2,1) processes with different autoregressive and moving average polynomials.

Key Words: Autocovariance Structure, Mixed ARMA Models, Wold Representation.

JEL Classification: C22

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1 Introduction

Ever since their introduction by Yule (1921, 1927) the autoregressive and moving average models have been greatly favoured by time series analysts. The purpose of this paper is to examine the covariance structure of mixed autoregressive and moving average (ARMA) time series models.

The theoretical autocovariance function (acf) of the ARMA model is an important tool in time series analysis. The calculation of the theoretical autocovariances is used (i) to estimate ARMA models with conventional exact maximum likelihood procedures (ii) to analyze the distribution of estimated ARMA parameters (Hannan, 1970) and (iii) to initialize simulations with ARMA models. The theoretical acf of the univariate ARMA(p,q) model has been already derived in the literature. The autocovariances are obtained in terms of the roots of the autoregressive (AR) polynomial and the parameters of the moving average (MA) one.

Pandit (1973, pp. 100, 141) and Pandit and Wu[1] (1983, pp. 105, 129-130), hereafter PW, derived the acf of the ARMA(p,q) model (when the roots of the AR polynomial are distinct) by using the infinite moving average (ima) or Wold representation[2] of the aforementioned model. Nerlove, Grether and Carvalho (1979, pp. 39, 78-85) used the canonical factorization (cf) of the autocovariance generating function (agf[3]) to derive a general expression for the acf of the ARMA(p,q) process (As an illustration they presented the acf of the MA(q), AR(p), and ARMA(1,1) models).

Zinde-Walsh (1988) obtained the acf of the ARMA(p,q) model by using the standard (see for example, Priestley, 1981[4]) spectral representation for the acf of a stationary ARMA process. She derived expressions for both cases of simple and multiple roots of the AR polynomial. Karanasos (1998), hereafter K, derived the acf of the ARMA(p,q) model (when the roots of the AR polynomial are distinct) by expressing the ARMA(p,q) model as an AR(1) process with an ARMA(p-1,q) error.

Algorithms for computing the theoretical autocovariances for univariate ARMA processes have been suggested in McLeod (1975, 1977) and Tunnicliffe Wilson (1979).

This paper contributes to the above literature by deriving the acf of the sum of ARMA processes when the roots of the AR polynomials are distinct. Granger and Morris (1976)

[1]PW only examine the case where $q < p$.

[2]They called the coefficient function in this expansion "Green's function" (see Miller (1968)). What PW have called Green's function is also referred to in the literature as weighting function (see Wiener (1949) and Pugachev (1957)) or as ψ weights (see Box and Jenkins (1970)).

For a proof of Wold's theorem see Wold (1938, pp. 75-89), Hannan (1970, pp.136-137), Anderson (1971, pp. 420-421), Sargent (1979, pp.257-262) or Brockwell and Davis (1987, pp. 180-182), hereafter BD; also see Hannan (1970, pp. 157-158) for the vector case. Wold's theorem has also been discussed by, for example, Nerlove, Grether and Carvalho (1979, pp. 30-36), hereafter NGC, PW (1983, pp. 87-89), and Reinsel (1993, p. 7), hereafter R.

The ima representation of the ARMA model has been discussed by, for example, O.D. Anderson (1976, p 44), PW (1983, p 108), Granger and Newbold (1986, pp. 25-26), hereafter GN, BD (1987, pp. 87-89), Wei (1989, p 56), Hamilton (1994, pp. 59-60), Gourieroux and Monfort (1997, pp 160-162); see also Mittnik (1987), R (1993, pp. 33-34), Lutkepohl (1993, pp. 220-221), hereafter L, or Gourieroux and Monfort (1997, p 252) for the vector case.

[3]The cf of the agf has also been discussed by, for example, O.D. Anderson (1976, p 129), GN (1986, pp. 26-27), BD (1987, pp. 102-103), Wei (1989, pp. 242-243) or Hamilton (1994, pp. 61-63); see also BD (1987, p 410) or R(1993, pp. 33-34), for the vector case.

[4]For the important subject of spectral analysis see also the excellent books by Jenkins and Watts (1968), and Hannan (1967).

consider such mixed ARMA models. As most economic series are both aggregates and are measured with error it follows that these mixed models are often found in practice. We examine three distinct cases, depending on whether the processes have identical or different autoregressive parts and uncorrelated or correlated errors. We obtain our results by using the ima representation of the ARMA(p,q) model, as it is given in Pandit (1973) or in K (2000)[5] of the ARMA(p,q) model. The autocovariances are expressed in terms of the roots of the AR polynomials and the parameters of the MA ones. In the case of distinct roots Zinde-Walsh (1988) or K(1998) results are special cases of our Theorem 1 (see Proposition 1a).

2 MIXED ARMA MODELS

The following Proposition presents the autocovariances of an ARMA(p,q) model, when the roots of the AR polynomial are distinct.

Let y_t be an ARMA(p,q) process given by:

$$\Phi(L)y_t = \mu + \Theta(L)\epsilon_t, \;\; \epsilon_t \sim IID(0, \sigma_\epsilon^2), \tag{2.1}$$

$$\Theta(L) = \sum_{k=0}^{q} \theta_k L^k, \;\; \Phi(L) = -\sum_{k=0}^{p} \phi_k' L^k = \prod_{k=1}^{p}(1-\phi_k L), \;\; \phi_0' = 1, \;\; \phi_k \neq \phi_l \tag{2.1a}$$

Assumption 1. All the roots of the autoregressive polynomial $[\Phi(L)]$ lie outside the unit circle (Stationarity condition).

Assumption 2. The polynomials $\Phi(L)$ and $\Theta(L)$ are left coprime. In other words the representation $\frac{\Phi(L)}{\Theta(L)}$ is irreducible.

Proposition 1a. Under assumptions 1-2 the autocovariances of y_t (γ_m) are given by[6]

$$\gamma_m = \sum_{l=1}^{p} \zeta_{lm} \lambda_{l,min(m,q)} \sigma_\epsilon^2, \;\; \zeta_{lm} = \frac{\phi_l^{p-1+m}}{\prod_{k=1}^{p}(1-\phi_l\phi_k)\prod_{\substack{k=1\\k\neq l}}^{p}(\phi_l-\phi_k)}, \tag{2.2}$$

$$\lambda_{lm} = \sum_{k=0}^{q}\theta_k^2 + \sum_{d=1}^{m}\sum_{k=0}^{q-d}\theta_k\theta_{k+d}(\phi_l^d + \phi_l^{-d}) + \sum_{d=m+1}^{q}\sum_{k=0}^{q-d}\theta_k\theta_{k+d}(\phi_l^d + \phi_l^{d-2m}) \tag{2.2a}$$

The proof of Proposition 1a is given in Appendix A.

By considering the model generating the sum of two or more series, Granger and Morris (1976) have shown that the mixed ARMA model is the one most likely to occur.

[5] K has also used the ima representation in the context of the GARCH and GARCH in mean models.

[6] Zinde Victoria Walsh (1988) derived the above result (eqs 2a and 2b) by using the standard (see for example, Priestley (1981)) spectral representation for the acf of a stationary ARMA process whereas K(1998) derived the same result by expressing the ARMA(p,q) model as an AR(1) process with an ARMA(p-1,q) error. Zinde-Walsh's formula is general as it is not restricted to the case of simple roots.

They mentioned that "two situations where series are added together are of particular interpretational importance. The first is where series are aggregated to form some total, and the second one is where the observed series is the sum of the true process plus an observational error. Most of the macroeconomic series, such as GNP, unemployment or exports are aggregates. Virtually any macroeconomic series, other than certain prices or interest rates, contain important observation errors." So speaking the following analysis of the autocovariance function of the sum of ARMA processes can be a useful tool for the applied economist.

Let y_t be equal to the sum of n uncorrelated ARMA(p, q_i) processes that have identical autoregressive parts:

$$y_t = \sum_{i=1}^{n} y_{it}, \;\; \Phi(L)y_{it} = \Theta_i(L)\epsilon_{it}, \;\; \Theta_i(L) = \sum_{k=0}^{q_i} \theta_{ik}L^k, \tag{2.3}$$

$$\bar{\epsilon}_t = \begin{bmatrix} \epsilon_{1t} \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_{nt} \end{bmatrix}, \;\; \bar{\epsilon}_t \sim IID(0, \bar{\sigma}_\epsilon^2), \;\; \bar{\sigma}_\epsilon^2 = \begin{bmatrix} \sigma_{11,\epsilon} & \dots & 0 \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ 0 & \dots & \sigma_{nn,\epsilon} \end{bmatrix} \tag{2.3a}$$

where $\Phi(L)$is defined by $(2.1a)$.

Assumption 3. The polynomials $\Phi(L)$ and $\Theta_i(L)$ are left coprime. In other words the representation $\frac{\Phi(L)}{\Theta_i(L)}$ is irreducible.

Proposition 1b. Under assumptions 1 and 3 the autocovariances of the preceding ARMA process are given by

$$\gamma_m = \sum_{i=1}^{n}\sum_{l=1}^{p} \zeta_{lm}\lambda^i_{l,min(m,q_i)}\sigma_{ii,\epsilon}, \tag{2.4}$$

$$\lambda^i_{lm} = \sum_{d=0}^{q_i}\theta_{id}^2 + \sum_{d=1}^{m}\sum_{r=0}^{q_i-d}\theta_{ir}\theta_{ir+d}(\phi_l^d + \phi_l^{-d}) + \sum_{d=m+1}^{q_i}\sum_{r=0}^{q_i-d}\theta_{ir}\theta_{ir+d}(\phi_l^d + \phi_l^{d-2m}) \tag{2.4a}$$

where the ζ_{lm} is defined in (2.2).

The Proof of Proposition 1b is given in Appendix A.

In the Proposition that follows the assumption of independence is weakened to allow for contemporaneous correlation between the noise series.

Let y_t be equal to the sum of n correlated ARMA(p, q_i) processes as defined by (2.3), where $\bar{\sigma}_\epsilon^2$ is given by

$$\bar{\sigma}_\epsilon^2 = \begin{bmatrix} \sigma_{11,\epsilon} & \dots & \sigma_{1n,\epsilon} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \sigma_{n1,\epsilon} & \dots & \sigma_{nn,\epsilon} \end{bmatrix} \tag{2.5}$$

Proposition 1c. Under assumptions 1 and 3 the autocovariances of y_t are given by

$$\gamma_m = \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{l=1}^{p} \zeta_{lm} \lambda_{l,m}^{ij} \sigma_{ij,\epsilon}, \tag{2.6}$$

$$\lambda_{l,m}^{ij} = \sum_{c=0}^{q_j} \sum_{d=0}^{q_i'} \theta_{id}\theta_{j,d+c}\phi_l^c + \sum_{c=1}^{m\star} \sum_{d=0}^{q_j'} \theta_{jd}\theta_{id+c}\phi_l^{-c} + \sum_{c=m^\star+1}^{q_i} \sum_{d=0}^{q_j'} \theta_{jd}\theta_{id+c}\phi_l^{c-2m} \tag{2.6a}$$

where

$$\begin{aligned} q_i' &= min(q_i, q_j - c) \qquad q_j' = min(q_j, q_i - c) \\ m^\star &= min(m, q_i) \end{aligned} \tag{2.6b}$$

and ζ_{lm} is defined in (2.2). The proof of Proposition 1c is given in Appendix A.

Example 1. Consider two ARMA(2,2) processes $(y_{1t},\ y_{2t})$ with identical autoregressive polynomials and different moving average ones

$$(1-\phi_1 L)(1-\phi_2 L)y_{1t} = (1+\theta_{11}L+\theta_{12}L^2)\epsilon_{1t},\ \ \phi_1 \neq \phi_2,\ \ |\phi_1|, |\phi_2| < 1$$

$$(1-\phi_1 L)(1-\phi_2 L)y_{2t} = (1+\theta_{21}L+\theta_{22}L^2)\epsilon_{2t},$$

$$\begin{bmatrix}\epsilon_{1t}\\ \epsilon_{2t}\end{bmatrix} \sim IID(0, \bar{\sigma}_\epsilon^2),\ \ \bar{\sigma}_\epsilon^2 = \begin{bmatrix} var(\epsilon_{1t}) & cov(\epsilon_{1t},\epsilon_{2t}) \\ cov(\epsilon_{1t},\epsilon_{2t}) & var(\epsilon_{2t})\end{bmatrix} = \begin{bmatrix}\sigma_{11,\epsilon} & \sigma_{12,\epsilon} \\ \sigma_{12,\epsilon} & \sigma_{22,\epsilon}\end{bmatrix}$$

The cross covariances between the two processes are

$$cov(y_{1t}, y_{2t}) = \sigma_{12,\epsilon} \times$$

$$\{\frac{\phi_1[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22}+\theta_{11}+\theta_{21}\theta_{12})\phi_1+(\theta_{12}+\theta_{22})\phi_1^2]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)} +$$

$$\frac{\phi_2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22}+\theta_{11}+\theta_{21}\theta_{12})\phi_2+(\theta_{12}+\theta_{22})\phi_2^2]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\}$$

$$cov(y_{1t}, y_{2,t-1}) = \sigma_{12,\epsilon} \times$$

$$\{\frac{\phi_1^2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22})\phi_1+(\theta_{11}+\theta_{21}\theta_{12})\phi_1^{-1}+\theta_{22}\phi_1^2+\theta_{12}\phi_1^0]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)}$$

$$+\ \frac{\phi_2^2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22})\phi_2+(\theta_{11}+\theta_{21}\theta_{12})\phi_2^{-1}+\theta_{22}\phi_2^2+\theta_{12}\phi_2^0]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\}$$

$$cov(y_{2t}, y_{1,t-1}) = \sigma_{12,\epsilon} \times$$

$$\{\frac{\phi_1^2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{11}+\theta_{21}\theta_{12})\phi_1+(\theta_{21}+\theta_{11}\theta_{22})\phi_1^{-1}+\theta_{12}\phi_1^2+\theta_{22}\phi_1^0]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)}$$
$$+\frac{\phi_2^2[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{11}+\theta_{21}\theta_{12})\phi_2+(\theta_{21}+\theta_{11}\theta_{22})\phi_2^{-1}+\theta_{12}\phi_2^2+\theta_{22}\phi_2^0]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\}$$

$$cov(y_{1t}, y_{2,t-\underset{m\geq 2}{m}}) = \sigma_{12,\epsilon} \times$$

$$\{\frac{\phi_1^{1+m}[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22})\phi_1+(\theta_{11}+\theta_{21}\theta_{12})\phi_1^{-1}+\theta_{22}\phi_1^2+\theta_{12}\phi_1^{-2}]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)}$$

$$+\frac{\phi_2^{1+m}[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{21}+\theta_{11}\theta_{22})\phi_2+(\theta_{11}+\theta_{21}\theta_{12})\phi_2^{-1}+\theta_{22}\phi_2^2+\theta_{12}\phi_2^{-2}]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\}$$

$$cov(y_{2t}, y_{1,t-\underset{m\geq 2}{m}}) = \sigma_{12,\epsilon} \times$$

$$\{\frac{\phi_1^{1+m}[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{11}+\theta_{21}\theta_{12})\phi_1+(\theta_{21}+\theta_{11}\theta_{22})\phi_1^{-1}+\theta_{12}\phi_1^2+\theta_{22}\phi_1^{-2}]}{(1-\phi_1\phi_2)(1-\phi_1^2)(\phi_1-\phi_2)}$$

$$+\frac{\phi_2^{1+m}[1+\theta_{11}\theta_{21}+\theta_{12}\theta_{22}+(\theta_{11}+\theta_{21}\theta_{12})\phi_2+(\theta_{21}+\theta_{11}\theta_{22})\phi_2^{-1}+\theta_{12}\phi_2^2+\theta_{22}\phi_2^{-2}]}{(1-\phi_1\phi_2)(1-\phi_2^2)(\phi_2-\phi_1)}\}$$

Note, that when the two moving average polynomials and the two error terms are identical ($y_{1t} = y_{2t} = y_t$) the above expressions give the autocovariances of the y_t process: $cov(y_{1t}, y_{2,t-m}) = cov(y_{2t}, y_{1,t-m}) = cov_m(y_t)$.

In the following Theorem the assumption of identical autoregressive parts is relaxed.

Let y_t be equal to the sum of n correlated ARMA(p_i, q_i) processes, where the roots of the AR polynomials are distinct:

$$y_t = \sum_{i=1}^{n} y_{it},\;\; \Phi_i(L)y_{it} = \Theta_i(L)\epsilon_{it},\;\; \Phi_i(L) = -\sum_{k=0}^{p_i}\phi'_{ik}L^k = \prod_{k=1}^{p_i}(1-\phi_{ik}L),\;\; \phi_{ik} \neq \phi_{il} \tag{2.7}$$

where $\Theta_i(L)$, $\bar{\epsilon}_t$, are defined by (2.3) and $\bar{\sigma}_\epsilon^2$ is defined by (2.5).

Assumption 4. All the roots of the autoregressive polynomials [$\Phi_i(L)$] lie outside the unit circle (Stationarity conditions).

Assumption 5. The polynomials $\Phi_i(L)$ and $\Theta_i(L)$ are left coprime. In other words the representation $\frac{\Phi_i(L)}{\Theta_i(L)}$ is irreducible.

Theorem 1. Under assumptions 4 and 5 the autocovariances of the preceding process are given by

$$\gamma_m = \sum_{j=1}^{n}\sum_{i=1}^{n}(\sum_{l=1}^{p_i}\zeta_{il,j}^{m}\lambda_{l,m}^{ij} + \sum_{k=1}^{p_j}\zeta_{jk,i}^{m}\lambda_{ij}^{k,m})\sigma_{ij,\epsilon}, \tag{2.8}$$

$$\zeta_{il,j}^{m} = \frac{\zeta_{il}^{m}}{\prod_{k=1}^{p_j}(1-\phi_{il}\phi_{jk})}, \ \ \zeta_{il}^{m} = \frac{\phi_{il}^{p_i-1+m}}{\prod_{\substack{k=1\\k\neq l}}^{p_i}(\phi_{il}-\phi_{ik})}, \tag{2.8a}$$

$$\lambda_{l,m}^{ij} = \sum_{c=0}^{q_j}\sum_{d=0}^{q_i'}\theta_{id}\theta_{j,d+c}\phi_{il}^{c} + \sum_{c=1}^{m^\star}\sum_{d=0}^{q_j'}\theta_{jd}\theta_{id+c}\phi_{il}^{-c}, \ \ \lambda_{ij}^{k,m} = \sum_{c=m^\star+1}^{q_i}\sum_{d=0}^{q_j'}\theta_{jd}\theta_{id+c}\phi_{jk}^{c-2m} \tag{2.8b}$$

The proof of Theorem 1 is given in Appendix B.

Example 2. Consider two ARMA(2,1) processes $(y_{1t},\ y_{2t})$ with different autoregressive and moving average polynomials

$$(1-\phi_{11}L)(1-\phi_{12}L)y_{1t} = (1+\theta_{11}L)\epsilon_{1t}, \ \ \phi_{11}\neq\phi_{12}, \ \ |\phi_{11}|,|\phi_{12}|<1$$

$$(1-\phi_{21}L)(1-\phi_{22}L)y_{2t} = (1+\theta_{21}L)\epsilon_{2t}, \ \ \phi_{21}\neq\phi_{22}, \ \ |\phi_{21}|,|\phi_{22}|<1$$

$$\begin{bmatrix}\epsilon_{1t}\\ \epsilon_{2t}\end{bmatrix} \sim IID(0,\bar{\sigma}_\epsilon^2), \ \ \bar{\sigma}_\epsilon^2 = \begin{bmatrix} var(\epsilon_{1t}) & cov(\epsilon_{1t},\epsilon_{2t})\\ cov(\epsilon_{1t},\epsilon_{2t}) & var(\epsilon_{2t})\end{bmatrix} = \begin{bmatrix}\sigma_{11,\epsilon} & \sigma_{12,\epsilon}\\ \sigma_{12,\epsilon} & \sigma_{22,\epsilon}\end{bmatrix}$$

The cross covariances between the two processes are

$$cov(y_{1t},y_{2t}) = \sigma_{12,\epsilon}\times$$

$$\{\frac{\phi_{11}[1+\theta_{11}\theta_{21}+\theta_{21}\phi_{11}]}{(\phi_{11}-\phi_{12})(1-\phi_{11}\phi_{21})(1-\phi_{11}\phi_{22})} + \frac{\phi_{12}[1+\theta_{11}\theta_{21}+\theta_{21}\phi_{12}]}{(\phi_{12}-\phi_{11})(1-\phi_{12}\phi_{21})(1-\phi_{12}\phi_{22})} +$$

$$+ \frac{\phi_{21}^2\theta_{11}}{(\phi_{21}-\phi_{22})(1-\phi_{21}\phi_{11})(1-\phi_{21}\phi_{12})} + \frac{\phi_{22}^2\theta_{11}}{(\phi_{22}-\phi_{21})(1-\phi_{22}\phi_{11})(1-\phi_{22}\phi_{12})}\}$$

$$cov(y_{1t},y_{2,t-m}) = \{\frac{\phi_{11}^{1+m}[1+\theta_{11}\theta_{21}+\theta_{21}\phi_{11}+\theta_{11}\phi_{11}^{-1}]}{(\phi_{11}-\phi_{12})(1-\phi_{11}\phi_{21})(1-\phi_{11}\phi_{22})} +$$

$$+ \frac{\phi_{12}^{1+m}[1+\theta_{11}\theta_{21}+\theta_{21}\phi_{12}+\theta_{11}\phi_{12}^{-1}]}{(\phi_{12}-\phi_{11})(1-\phi_{12}\phi_{21})(1-\phi_{12}\phi_{22})}\}\sigma_{12,\epsilon}, \ \ m\geq 1$$

$$cov(y_{2t}, y_{1,t-m}) = \{\frac{\phi_{21}^{1+m}[1+\theta_{11}\theta_{21}+\theta_{11}\phi_{21}+\theta_{21}\phi_{21}^{-1}]}{(\phi_{21}-\phi_{22})(1-\phi_{21}\phi_{11})(1-\phi_{21}\phi_{12})} +$$

$$+ \frac{\phi_{22}^{1+m}[1+\theta_{11}\theta_{21}+\theta_{11}\phi_{22}+\theta_{21}\phi_{22}^{-1}]}{(\phi_{22}-\phi_{21})(1-\phi_{22}\phi_{11})(1-\phi_{22}\phi_{12})}\}\sigma_{12,\epsilon}, \;\; m \geq 1$$

Note that when the autoregressive polynomials of the two processes are identical ($\phi_{21} = \phi_{11}$, $\phi_{22} = \phi_{12}$) the above expressions reduce to those given in Example 1 with $\theta_{12} = \theta_{22} = 0$.

3 Concluding Remarks

Since the publication of the book by Box and Jenkins, almost 30 years ago, the interest by time series analysts and econometricians in more complicated even if more parsimonious linear time series models has greatly increased. The purpose of this paper was to examine the covariance structure of mixed ARMA models. In Section 2 we presented the autocovariance function of the sum of n ARMA(p_i, q_i) processes using the infinite moving average representation of an ARMA model. The autocovariances were expressed in terms of the roots of the AR polynomials and the parameters of the MA polynomials. In this paper we examined only the case of distinct roots of the autoregressive polynomials-the case of equal roots is left for future research. The ima representation technique can also be applied to non linear multivariate time series models (e.g. GARCH or Markov Switching) to derive their covariance structure[7].

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Appendix

A *PROOF OF PROPOSITION 1a*

The impulse response function (IRF) of y_t is given by[8]

$$y_t = \sum_{r=0}^{\infty} e^r \epsilon_{t-r}, \text{ where } e^r = \sum_{l=1}^{p} \sum_{k=0}^{min(r,q)} \zeta_l^{r-k}\theta_k, \;\; \zeta_l^m = \frac{\phi_l^{p-1+m}}{\prod\limits_{\substack{k=1\\k\neq l}}^{p}(\phi_l - \phi_k)} \tag{A.1}$$

From the preceding equation we have that

$$\gamma_m = cov_m(y_t) = \sum_{r=0}^{\infty} e^r e^{r+m}\sigma_\epsilon^2 \tag{A.1a}$$

After some algebra we get

$$\gamma_m = \sum_{l=1}^{p} \zeta_l^m s_l^0 \lambda_{lm}\sigma_\epsilon^2, \text{ where } s_l^0 = \sum_{k=1}^{p} \frac{\zeta_k^0}{(1-\phi_l\phi_k)} \tag{A.1b}$$

Subsequently, using

$$s_l^0 = \frac{1}{\prod\limits_{k=1}^{p}(1-\phi_l\phi_k)} \tag{A.1c}$$

we get (2.2)-(2.2a). ■

PROOF OF PROPOSITION 1b

From (2.3) and using Proposition 1a we get

$$\gamma_m = \sum_{i=1}^{n} cov_m(y_{it}) = \sum_{i=1}^{n}\sum_{l=1}^{p} \zeta_{lm}\lambda^i_{l,min(m,q_i)}\sigma_{ii,\epsilon} \tag{A.2}$$

■

[8] It is given in Pandit (1973) or K (2000).

PROOF OF PROPOSITION 1c

The IRF of y_{it} is given by

$$y_{it} = \sum_{r=0}^{\infty} e_i^r \epsilon_{it-r}, \text{ where } e_i^r = \sum_{l=1}^{p} \sum_{k=0}^{min(r,q_i)} \zeta_l^{r-k} \theta_{ik} \tag{A.3}$$

From the preceding equation we get

$$cov(y_{it}, y_{jt-m}) = \sum_{r=0}^{\infty} e_j^r e_i^{r+m} \sigma_{ij,\epsilon} \tag{A.3a}$$

After some algebra we have that

$$cov(y_{it}, y_{jt-m}) = \sum_{l=1}^{p} \zeta_m^l s_l^0 \lambda_{l,m}^{ij} \sigma_{ij,\epsilon} = \sum_{l=1}^{p} \zeta_{lm} \lambda_{l,m}^{ij} \sigma_{ij,\epsilon} \tag{A.3b}$$

Using the preceding equation, together with equations (2.3) and (2.5) we get

$$\gamma_m = \sum_{j=1}^{n} \sum_{i=1}^{n} cov(y_{it}, y_{jt-m}) = \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{l=1}^{p} \zeta_{lm} \zeta_{l,m}^{ij} \sigma_{ij,\epsilon} \tag{A.3c}$$

■

B *PROOF OF THEOREM 1*

The IRF of y_{it} is given by

$$y_{it} = \sum_{r=0}^{\infty} e_i^r \epsilon_{it-r}, \text{ where } e_i^r = \sum_{l=1}^{p_i} \sum_{k=0}^{min(r,q_i)} \zeta_{il}^{r-k} \theta_{ik}, \; \zeta_{il}^m = \frac{\phi_{il}^{p_i-1+m}}{\prod_{\substack{k=1 \\ k \neq l}}^{p_i} (\phi_{il} - \phi_{ik})} \tag{B.1}$$

From the preceding equation we have

$$cov(y_{it}, y_{j,t-m}) = \sum_{r=0}^{\infty} e_j^r e_i^{r+m} \sigma_{ij,\epsilon} \tag{B.1a}$$

After some algebra we get

$$cov(y_{it}, y_{j,t-m}) = (\sum_{l=1}^{p_i} \zeta_{il}^{m} s_{j,il}^{0} \lambda_{l,m}^{ij} + \sum_{k=1}^{p_j} \zeta_{jk}^{m} s_{i,jk}^{0} \lambda_{ij}^{km})\sigma_{ij,\epsilon}, \text{ where } s_{j,il}^{0} = \sum_{k=1}^{p_j} \zeta_{jk}^{0,il}, \tag{B.1b}$$

$$\zeta_{jk}^{0,il} = \frac{\zeta_{jk}^{0}}{(1-\phi_{il}\phi_{jk})}, \ \lambda_{l,m}^{ij} = \sum_{c=0}^{q_j} \sum_{d=0}^{q'_i} \theta_{id}\theta_{j,d+c}\phi_{il}^{c} + \sum_{c=1}^{m^\star} \sum_{d=0}^{q'_j} \theta_{jd}\theta_{i,d+c}\phi_{il}^{-c} \tag{B.1c}$$

$$\lambda_{ij}^{km} = \sum_{c=m+1}^{q_i} \sum_{d=0}^{q'_j} \theta_{jd}\theta_{id+c}\phi_{jk}^{c-2m}, \tag{B.1d}$$

and $q'_i = min(q_i, q_j - c)$, $q'_j = min(q_j, q_i - c)$, $m^\star = min(m, q_i)$.
In the preceding equation, we use

$$s_{j,il}^{0} = \frac{1}{\prod_{k=1}^{p_j} (1-\phi_{il}\phi_{jk})} \tag{B.1e}$$

to get

$$cov(y_{it}, y_{j,t-m}) = (\sum_{l=1}^{p_i} \zeta_{il,j}^{m} \lambda_{l,m}^{ij} + \sum_{k=1}^{p_j} \zeta_{jk,i}^{m} \lambda_{ij}^{km})\sigma_{ij,\epsilon}, \text{ where } \zeta_{il,j}^{m} = \frac{\zeta_{il}^{m}}{\prod_{k=1}^{p_j} (1-\phi_{il}\phi_{jk})} \tag{B.1f}$$

Thus, we have

$$y_t = \sum_{i=1}^{n} y_{it} \Rightarrow cov_m(y_t) = \sum_{j=1}^{n} \sum_{i=1}^{n} cov(y_{it}, y_{j,t-m})$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} (\sum_{l=1}^{p_i} \zeta_{il,j}^{m} \lambda_{l,m}^{ij} + \sum_{k=1}^{p_j} \zeta_{jk,i}^{m} \lambda_{ij}^{km})\sigma_{ij,\epsilon} \tag{B.1g}$$

■