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Abstract

This paper extents Karanasos (1999a) results for the n Component GARCH(1,1) and the two Component GARCH(2,2) models and it further examines the n Component GARCH(n,n) model. In particular, we present the GARCH(n^2,n^2) representation of the aggregate variance and we give the condition for the existence of the fourth moment of the errors. In addition, we use the canonical factorization of the autocovariance generating function for the univariate ARMA representations of the component variances, the aggregate variance and the squared errors to obtain their autocovariances and cross covariances. Finally, we illustrate our general results giving three examples: the three component GARCH(1,1), the two component GARCH(2,2) and the three component GARCH(2,2) models.

Key Words: Persistence in Volatility, Component-GARCH, ARMA Representations, Autocovariance Generating Function.

JEL Classification: C22

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1 INTRODUCTION

Since the beginning of the eighties the ARCH[1]model and its various generalizations have been used extensively in the modeling of the conditional volatility of financial time series. Within this class of models, it is almost a "stylized fact" that the sum of the estimated coefficients in the conditional variance function is insignificantly different from unity, especially for high-frequency financial data. These models were called by Engle and Bollerslev(1986) Integrated GARCH(IGARCH) and have the characteristic that shocks to the conditional variance are persistent in the sense that current information remains important for long-term volatility forecasts. This non-stationary behaviour is important both from a theoretical point of view and for the construction of long-horizon volatility forecasts which are essential in many asset-pricing models (see, for example, Poterba and Summers, 1986).

However, Ding and Granger (1996), hereafter DG, proved that the autocorrelation function for an IGARCH(1,1) process is exponentially decreasing and is very different from the sample autocorrelation function found for several long speculative asset return series (e.g. stock and exchange rate returns). As DG(1996, p199) wrote: "It is quite clear from the sample autocorrelation (of the various speculative returns) that there are different volatility components that will dominate different time periods. Some volatility components may have a very big short-run effect, but die out very quickly. Some of them may have a relatively smaller short-run effect, but they last for a long time period".

Motivated by this empirical result they introduced the N-component GARCH(1,1) (C-GARCH) model[2]. In this model the aggregate conditional variance, hereafter av, of the errors (h_t) is a weighted sum of n component variances, hereafter cv, (h_{it}), $(i = 1, \cdots, n)$ with w_i $(i = 1, \cdots, n)$ as weights, respectively. Each component is a GARCH(1,1)-type specification. DG also mentioned that the n component model corresponds to a GARCH(n,n) model[3]. This GARCH(n,n) representation together with the autocovariance function of the squared errors[4],hereafter se, is obtained in Karanasos(1999a), hereafter K[5]. In addition, K(1999a) derived the GARCH(2n,2n) representation of the two component GARCH(n,n) model and the autocovariance function of the squared errors for this model.

The goal of this article is to provide a comprehensive methodology for the analysis of the general n component GARCH(n,n) model. First, it derives the VARMA representation of the cv and it shows that they follow a n-th order VARMA(n,1) model. Second,

[1]The ARCH model was originally proposed by Engle(1982), whereas Bollerlsev(1986) presented the GARCH model. The existence of the huge literature which uses these processes in modelling conditional volatility in high frequency financial assets demonstrates the popularity of the various GARCH models [see, for example, the surveys by Palm(1996), Shepard(1996) and Pagan(1996); see also the book by Gourieroux(1997) for a detail discussion of the GARCH models and financial applications].

[2]Karanasos, Psaradakis and Sola (1999), hereafter KPS, derive the ARMA-GARCH representation that linear aggregates of ARMA processes with multivariate-GARCH errors (the S-GARCH model) admit, and establish conditions under which persistence in volatility of the aggregate series is higher than persistence in the volatility of the individual series. They illustrated empirically the practical implications of their results in the context of an option pricing exercise. KPS(1999) also show that the C-GARCH model is a special case of the S-GARCH model. The issue of contemporaneous aggregation of GARCH processes has also been examined in K(1999b) and Zaffaroni (1999).

[3]Moreover, they derived the GARCH(2,2) representation of the two-component GARCH(1,1) model.

[4]The autocovariance function of the squared errors of the simple GARCH(p,q) model is given in Karanasos(1999a) (see also He and Terasvirta, 1999).

[5]K(1999a) derived the GARCH(n,n) representation using the Ding and Granger's (1996) method.

it provides the univariate ARMA representations of the cv, the av and the se and it shows that they can be represented as an ARMA(n^2, n^2) model. Third, it gives the GARCH(n^2, n^2) representation of the av. Finally, it uses the canonical factorization of the autocovariance generating function of a stationary stochastic process to obtain: (i) the autocovariances of the av, the cv and the se, (ii) the cross covariances between the cv, and (iii) the cross covariances between the av and the cv, and between the av and the se. It should be noted that we only examine the case of distinct roots in the autoregressive (AR) polynomial of the univariate ARMA representations and we express the autocovariances in terms of the roots of the AR polynomial and the parameters of the moving average polynomials of the univariate ARMA representations.

Section 2 provides the results for the general n component GARCH(n,n) model. Because of the highly complicated nature of the algebraic derivation involved and in order to familiariaze the reader with the notation used, we start by presenting the results of two special cases: the n component GARCH(1,1) model and the two component GARCH(n,n) model. In addition, for illustrative purposes, we give three examples: the three component GARCH(1,1) model, the two component GARCH(2,2) model, and the three component GARCH(2,2) model. Finally, Section 3 concludes.

2 COMPONENT GARCH MODELS

2.1 N Component GARCH(1,1) Model

In what follows we will examine the N component GARCH(1,1) model. In this model the conditional variance of the errors (h_t) is a weighted sum of N components $(h_{it}, i = 1, \cdots, n)$ with $(w_i,\ i = 1, \cdots, n)$ as weights, respectively. Each component is a GARCH(1,1)-type specification:

$$\epsilon_t/\Omega_{t-1} \sim D(0, h_t),\ \ h_t = \sum_{i=1}^{n} w_i h_{it},\ \ \sum_{i=1}^{n} w_i = 1, \tag{2.1}$$

$$h_{it} = \delta_i \omega_1 + a_i \epsilon_{t-1}^2 + \beta_i h_{i,t-1},\ \ \delta_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

Proposition 1a. The univariate ARMA representations of $h_{it},\ i = 1, \cdots, n$ are given by

$$B(L) h_{it} = \omega_i^\star + A_i(L) v_t,\ \ B(L) = 1 + \sum_{l=1}^{n} B_l L^l = \prod_{j=1}^{n} (1 - B_j^\circ L),\ \ A_i(L) = \sum_{l=1}^{n} A_{il} L^l, \tag{2.3}$$

$$B_l = \beta_{1l} + \beta_{2l},\ \ \beta_{1l} = \prod_{k=1}^{l} [\sum_{f_k = f_{k-1}+1}^{n-(l-k)}] \prod_{k=1}^{l} \beta_{f_k} (-1)^l,\ \ f_0 = 0,\ \ \beta_{21} = -\sum_{r=1}^{n} a_r w_r, \tag{2.3a}$$

$$\beta_{2l} = \prod_{k=1}^{l-1} [\sum_{f_k = f_{k-1}+1}^{n-[(l-1)-k]}] \prod_{k=1}^{l-1} \beta_{f_k} (-1)^l \times \sum_{\substack{r=1 \\ r \neq f_k}}^{n} a_r w_r,\ \ A_{il} = a_i \prod_{k=1}^{l-1} [\sum_{\substack{f_k = f_{k-1}+1 \\ f_k \neq i}}^{n-(l-1-k)}] \prod_{k=1}^{l-1} \beta_{f_k} (-1)^{l-1}, \tag{2.3b}$$

$$A_{i1} = a_i,\ \ \omega_i^\star = \begin{cases} \omega_1 [1 + \sum_{l=1}^{n-1} B_l^1] & \text{if } i = 1 \\ a_i \omega_1 w_1 [1 + \sum_{l=1}^{n-2} \beta_{1l}^i] & \text{otherwise} \end{cases},\ \ v_t = \epsilon_t^2 - h_t, \tag{2.3c}$$

$$B_l^i = \beta_{1l}^i + \beta_{2l}^i,\ \ \beta_{1l}^i = \prod_{k=1}^{l} [\sum_{\substack{f_k = f_{k-1}+1 \\ f_k \neq i}}^{n-(l-k)}] \prod_{k=1}^{l} \beta_{f_k} (-1)^l,\ \ f_0 = 1, \beta_{21}^i = -\sum_{\substack{r=1 \\ r \neq i}}^{n} a_r w_r,$$

$$\beta_{2l}^i = \prod_{k=1}^{l-1} [\sum_{\substack{f_k = f_{k-1}+1 \\ f_k \neq i}}^{n-[(l-1)-k]}] \prod_{k=1}^{l-1} \beta_{f_k} (-1)^{l-1} \times \sum_{\substack{r=1 \\ r \neq f_k, i}}^{n} a_r w_r,\ f_0 = 1 \tag{2.3d}$$

The proof of Proposition 1a is given in Appendix A.

Example 1: For the three component GARCH(1,1) model the univariate ARMA representation for the first component conditional variance (h_{1t}) is

$$\{1-(\beta_1+\beta_2+\beta_3+w_1a_1+w_2a_2+w_3a_3)L+[\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3+w_1a_1(\beta_2+\beta_3)+$$

$$+w_2a_2(\beta_1+\beta_3)+w_3a_3(\beta_1+\beta_2)]L^2-[\beta_1\beta_2\beta_3+w_1a_1(\beta_2\beta_3)+w_2a_2(\beta_1\beta_3)+$$

$$+w_3a_3(\beta_1\beta_2)]L^3\}h_{1t}=\omega_1^\star+[a_1L-a_1(\beta_2+\beta_3)L^2+a_1\beta_2\beta_3L^3]v_t \tag{2.4}$$

Corrolary 1a. The ARMA(n,n) representation of h_t is given by

$$B(L)h_t=\omega^\star+A(L)v_t,\ \ A(L)=\sum_{l=1}^{n}A_lL^l,\ \ \omega^\star=\omega_1w_1[1+\sum_{l=1}^{n-1}\beta_{1l}^1],\ \ A_l=-\beta_{2l} \tag{2.5}$$

Moreover, the GARCH(n,n) representation of h_t is given by

$$B^\star(L)h_t=\omega^\star+A(L)\epsilon_t^2,\ \ B^\star(L)=1+\sum_{l=1}^{n}\beta_{1l}L^l \tag{2.6}$$

Proof. The proof of equation (2.5) is given in Appendix A. The proof of equation (2.6) follows immediately from (2.5), using $v_t=\epsilon_t^2-h_t$.

Example 1: For the 3 component GARCH(1,1) model the ARMA(3,3) representation of the aggregate conditional variance is

$$\{1-(\beta_1+\beta_2+\beta_3+w_1a_1+w_2a_2+w_3a_3)L+[\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3+w_1a_1(\beta_2+\beta_3)+$$

$$+w_2a_2(\beta_1+\beta_3)+w_3a_3(\beta_1+\beta_2)]L^2-[\beta_1\beta_2\beta_3+w_1a_1(\beta_2\beta_3)+w_2a_2(\beta_1\beta_3)+$$

$$+w_3a_3(\beta_1\beta_2)]L^3\}h_t=\{(w_1a_1+w_2a_2+w_3a_3)L-[w_1a_1(\beta_2+\beta_3)+w_2a_2(\beta_1+\beta_3)+$$

$$+w_3a_3(\beta_1+\beta_2)]L^2+[w_1a_1(\beta_2\beta_3)+w_2a_2(\beta_1\beta_3)+w_3a_3(\beta_1\beta_2)]L^3\}v_t+\omega^\star \tag{2.7}$$

In addition, the GARCH(3,3) representation of the aggregate conditional variance is

$$\{1-(\beta_1+\beta_2+\beta_3)L+(\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3)L^2-(\beta_1\beta_2\beta_3)L^3\}h_t=$$

$$\{(w_1a_1+w_2a_2+w_3a_3)L-[w_1a_1(\beta_2+\beta_3)+w_2a_2(\beta_1+\beta_3)+w_3a_3(\beta_1+\beta_2)]L^2+$$

$$+w_1a_1(\beta_2\beta_3)+w_2a_2(\beta_1\beta_3)+w_3a_3(\beta_1\beta_2)]L^3\}\epsilon_t^2+\omega^\star \tag{2.8}$$

Assumption 1a. All the roots of the autorergressive polynomial $B(L)$ are lie outside the unit circle (Stationarity Condition).

Assumption 1b. The polynomials B(L) and $A_i(L)$ $(i = 1, \cdots, n)$, A(L) are left coprime. In other words the representations $\frac{B(L)}{A_i(L)}$ and $\frac{B(L)}{A(L)}$ are irreducible.

In what folows we only examine the case where the roots of the autoregressive polynomial [B(L)] are distinct.

Proposition 1b. Under Assumptions 1a and 1b the cross-covariances between the h_{it} *and the* $h_{j,t-m}$ *components are given by*

$$\gamma_{i,jm} = cov(h_{it}, h_{j,t-m}) = \begin{cases} \sum_{r=1}^{n} \zeta_{r,m} \lambda_{r,m}^{ij} \sigma_v^2, & \text{if } m > 0 \\ \sum_{r=1}^{n} \zeta_{r,m} \lambda_{r,m}^{ji} \sigma_v^2, & \text{if } m < 0 \end{cases}, \tag{2.9}$$

$$\zeta_{rm} = \frac{(B_r^\circ)^{n-1+m}}{\prod_{k=1}^{n} (1 - B_r^\circ B_k^\circ) \prod_{\substack{k=1 \\ k \neq r}}^{n} (B_r^\circ - B_k^\circ)}, \tag{2.9a}$$

$$\lambda_{r,m}^{ij} = \sum_{c=0}^{n-1} \sum_{d=1}^{n-c} A_{id} A_{j,d+c} (B_r^\circ)^c + \sum_{c=1}^{m^\star} \sum_{d=1}^{n-c} A_{jd} A_{i,d+c} (B_r^\circ)^{-c} + \sum_{c=m+1}^{n-1} \sum_{d=1}^{n-c} A_{jd} A_{i,d+c} (B_r^\circ)^{c-2m} \tag{2.9b}$$

where $m^\star = min(n-1, m)$ *and* $\sigma_v^2 = \frac{2}{3} E(\epsilon_t^4)$ *(under conditional normality) and is given below. When* $i = j$ *the above formula gives the autocovariance function of* h_{it}*.*

Moreover, the cross-covariances between h_t *and* $h_{j,t-m}$ *are given by*

$$\gamma_{jm} = cov(h_t, h_{j,t-m}) = \begin{cases} \sum_{r=1}^{n} \zeta_{r,m} \lambda_{r,m}^{j+} \sigma_v^2 & \text{if } m > 0 \\ \sum_{r=1}^{n} \zeta_{r,m} \lambda_{r,m}^{j-} \sigma_v^2 & \text{if } m < 0 \end{cases}, \tag{2.10}$$

$$\lambda_{r,m}^{j+} = \sum_{c=0}^{n-1} \sum_{d=1}^{n-c} A_d A_{j,d+c} (B_r^\circ)^c + \sum_{c=1}^{m^\star} \sum_{d=1}^{n-c} A_{jd} A_{d+c} (B_r^\circ)^{-c} + \sum_{c=m+1}^{n-1} \sum_{d=1}^{n-c} A_{jd} A_{d+c} (B_r^\circ)^{c-2m}, \tag{2.10a}$$

$$\lambda_{r,m}^{j-} = \sum_{c=0}^{n-1} \sum_{d=1}^{n-c} A_{j,d} A_{d+c} (B_r^\circ)^c + \sum_{c=1}^{m^\star} \sum_{d=1}^{n-c} A_d A_{j,d+c} (B_r^\circ)^{-c} + \sum_{c=m+1}^{n-1} \sum_{d=1}^{n-c} A_d A_{j,d+c} (B_r^\circ)^{c-2m} \tag{2.10b}$$

When $h_{jt} = h_t$*,* $A_{j,d+c} = A_{d+c}$*,* $\lambda_{rm}^{j-} = \lambda_{rm}^{j+} = \lambda_{rm}$ *the above formula gives the autocovariance function of* h_t*.*

The proof of Proposition 1b is given in Appendix A.

Proposition 1c. The condition for the existence of the fourth moment of the errors (under conditional normality) is

$$\gamma_0 < \frac{1}{2}, \ \ \gamma_0 = \sum_{r=1}^{n} \zeta_{r0}\lambda_{r0} \tag{2.11}$$

Furthermore, the univariate ARMA(n,n) representation of the squared errors ϵ_t^2 *is given by*

$$B(L)\epsilon_t^2 = \omega^\star + A^e(L)v_t, \ \ A^e(L) = \sum_{l=0}^{n} A_l^e L^l = [B(L) + A(L)], \ \ A_0^e = 1 \tag{2.12}$$

Assumption 1c. The polynomials B(L) and $A^e(L)$ are left coprime.

Under assumptions 1a and 1c the autocovariance function of the squared errors is given by

$$\gamma_m^e = cov(\epsilon_t^2, \epsilon_{t-m}^2) = \sum_{r=1}^{n} \zeta_{r,m}\lambda_{r,m}^e \sigma_v^2, \tag{2.13}$$

$$\lambda_{r,m}^e = \sum_{d=0}^{n} (A_d^e)^2 + \sum_{c=1}^{m}\sum_{d=0}^{n-c} A_d^e A_{d+c}^e[(B_r^\circ)^c + (B_r^\circ)^{-c}] + \sum_{c=m+1}^{n}\sum_{d=0}^{n-c} A_d^e A_{d+c}^e[(B_r^\circ)^c + (B_r^\circ)^{c-2m}] \tag{2.13a}$$

Finally, the cross covariances between the squared errors and the aggregate conditional variance are given by

$$cov(\epsilon_t^2, h_{t-m}) = cov(h_t, h_{t-m}), \ \ cov(h_t, \epsilon_{t-m}^2) = cov(\epsilon_t^2, \epsilon_{t-m}^2) \tag{2.14}$$

Proof. Using the form for the variance of h_t and $var(h_t) = \frac{1}{3}E(\epsilon_t^4) - [E(\epsilon_t^2)]^2$, $\sigma_v^2 = \frac{2}{3}E(\epsilon_t^4)$ we get equation (2.11) The proof of (2.12) follows from (2.5) on rearanging terms. The proof of (2.13) is given in Appendix A. Equation (2.14) follows from the law of iterated expectations.

2.2 2 Component GARCH(n,n) Model

In this subsection we will examine the two component GARCH(n,n) model. In this model the conditional variance of the errors (h_t) is a weighted sum of two components $(h_{it}, i = 1, 2)$ with w_i, $i = 1, 2$ as weights, respectively. Each component is a GARCH(n,n)-type specification:

$$\epsilon_t|\Omega_{t-1} \sim D(0,h_t),\ \ h_t = w_1 h_{1t} + w_2 h_{2t},\ \ w_1 + w_2 = 1, \tag{2.15}$$

$$B_i(L)h_{it} = \delta_i \omega_1 + A_i^e(L)\epsilon_t^2,\ \ i=1,2,\ \ \delta_i = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{if } i=2 \end{cases}, \tag{2.15a}$$

$$B_i(L) = -\sum_{l=0}^{n} \beta_i^l L^l,\ \ \beta_i^0 = -1,\ \ A_i^e(L) = \sum_{l=1}^{n} a_i^l L^l \tag{2.15b}$$

Proposition 2a. The univariate representations of h_{it} $(i=1,2)$ are given by

$$B(L)h_{it} = \omega_i^\star + A_i(L)v_t,\ \ v_t = \epsilon_t^2 - h_t,\ \ B(L) = 1 + \sum_{l=1}^{2n} B_l L^l = \prod_{l=1}^{2n}(1 - B_l^\circ L^l), \tag{2.16}$$

$$A_i(L) = \sum_{l=1}^{2n} A_{il}L^l,\ \ A_{il} = \Re'_{1l,n} a_i - \Re'_{2l,n} a_i \beta_{3-i},\ \ i=1,2, \tag{2.16a}$$

$$\omega_i^\star = \begin{cases} \omega_1[B_2(1) - w_2 A_2^e(1)] & \text{if } i=1 \\ \omega_1 w_1 A_2^e(1) & \text{if } i=2 \end{cases}$$

$$B_l = -\Re'_{1l,n}[\beta_1 + \beta_2 + w_1 a_1 + w_2 a_2] + \Re'_{2l,n}[\beta_1\beta_2 + w_1 a_1 \beta_2 + w_2 a_2 \beta_1] \tag{2.16b}$$

where $\Re'_{ml,n}$ is given by

$$\Re'_{ml,n} = \begin{cases} \Re_{ml,n} & \text{if } l = m, \cdots, m\times n \\ 0 & \text{otherwise} \end{cases},\ \ m=1,2$$

$\Re_{ml,n}$ denotes the set of all the combinations of m numbers taking values from 1 to n and adding to l. As an example consider, the case where $n=3$ and $m=2$.

$$\Re'_{ml,n} = \Re'_{2l,3} = \begin{cases} \Re_{2l,3} & \text{if } l = 2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases},\ \ \Re_{22,3} = 11,\ \ \Re_{23,3} = 12,\ 21, \cdots, \Re_{26,3} = 33$$

When for example we multiply $\beta_1\beta_2 + w_1 a_1 \beta_2 + w_2 a_2 \beta_1$ by $\Re_{23,3}$ we get

$$(\beta_1^1\beta_2^2 + w_1 a_1^1 \beta_2^2 + w_2 a_2^1 \beta_1^2) + (\beta_1^2\beta_2^1 + w_1 a_1^2 \beta_2^1 + w_2 a_2^2 \beta_1^1)$$

In addition the ARMA(2n,2n) representation of the aggregate conditional variance is

$$B(L)h_t = \omega^\star + A(L)v_t,\ \ A(L) = \sum_{i=1}^{n} w_i A_i(L) = \sum_{l=1}^{2n} A_l L^l, \tag{2.17}$$

$$A_l = \Re'_{1l,n}(w_1 a_1 + w_2 a_2) - \Re'_{2l,n}(w_1 a_1 \beta_2 + w_2 a_2 \beta_1),\ \ \omega^\star = w_1\omega_1^\star + w_2\omega_2^\star$$

Finally, the GARCH(2n,2n) representation of the aggregate conditional variance is

$$B^\star(L)h_t = \omega^\star + A(L)\epsilon_t^2,\;\; B^\star(L) = 1 + \sum_{l=1}^{2n} B_l^\star L^l,\;\; B_l^\star = -\Re'_{1l,n}(\beta_1+\beta_2) + \Re'_{2l,n}\beta_1\beta_2 \tag{2.18}$$

Proof. The proof of equation (2.16) is given in Appendix A. The proof of (2.17) follows immediately from (2.15) and (2.16). The proof of equation (2.18) follows immediately from (2.17) using $v_t = \epsilon_t^2 - h_t$.

Example 2. For the two component GARCH(2,2) model the univariate ARMA(4,4) representation of the first component conditional variance is

$$\begin{aligned}&\{1-(\beta_1^1+\beta_2^1+w_1a_1^1+w_2a_2^1)L + [-(\beta_1^2+\beta_2^2+w_1a_1^2+w_2a_2^2)+\\ &+(\beta_1^1\beta_2^1+w_1a_1^1\beta_2^1+w_2a_2^1\beta_1^1)]L^2 + [(\beta_1^1\beta_2^2+w_1a_1^1\beta_2^2+w_2a_2^1\beta_1^2)+\\ &+(\beta_1^2\beta_2^1+w_1a_1^2\beta_2^1+w_2a_2^2\beta_1^1)]L^3 + (\beta_1^2\beta_2^2+w_1a_1^2\beta_2^2+w_2a_2^2\beta_1^2)L^4\}h_{1t} =\\ &= \omega_1^\star + \{a_1^1L + (a_1^2 - a_1^1\beta_2^1)L^2 - (a_1^1\beta_2^2 + a_1^2\beta_2^1)L^3 - a_1^2\beta_2^2L^4\}v_t\end{aligned} \tag{2.19}$$

Moreover, the ARMA(4,4) representation of the aggregate conditional variance is

$$\begin{aligned}B(L)h_t = \omega^\star + \{&(w_1a_1^1+w_2a_2^1)L + [(w_1a_1^2+w_2a_2^2) - (w_1a_1^1\beta_2^1+w_2a_2^1\beta_1^1)]L^2\\ &-(w_1a_1^1\beta_2^2+w_2a_2^1\beta_1^2+w_1a_1^2\beta_2^1+w_2a_2^2\beta_1^1)L^3 - (w_1a_1^2\beta_2^2+w_2a_2^2\beta_1^2)L^4\}v_t\end{aligned} \tag{2.20}$$

where the autoregressive polynomial is the same with that of h_{1t} in eq (2.19).

Finally, the GARCH(4,4) representation of the aggregate conditional variance is

$$\begin{aligned}\{1-(\beta_1^1+\beta_2^1)L + [-(\beta_1^2+\beta_2^2)+(\beta_1^1\beta_2^1)]L^2 + (\beta_1^1\beta_2^2 +\\ \beta_1^2\beta_2^1)L^3 + (\beta_1^2\beta_2^2)L^4\}h_t = \omega^\star + A(L)\epsilon_t^2\end{aligned} \tag{2.21}$$

where the ARCH polynomial is the same with the moving average polynomial in equation (2.20).

Proposition 2b. Under assumptions 1a and 1b the cross-covariances between the h_{1t} and h_{2t} components are given by (2.9) where now $i=1$, $j=2$ and n is replaced by 2n.

Moreover, the cross covariances between h_t and $h_{j,t-m}$, $j=1,2$ are given by (2.10) where now n is replaced by 2n. The proof is similar to that of Proposition 1b.

The condition for the existence of the fourth moment of the errors is given by (2.11) where now n is replaced by 2n.

Furthermore, the univariate ARMA(2n,2n) representation of the squared errors ϵ_t^2 is given by (2.12) where now n is replaced by 2n. The proof follows from (2.17) on rearanging terms.

Under assumptions 1a and 1c the autocovariance function of the squared errors is given by (2.13) where now n is replaced by 2n. Finally, the covariances between the squared errors and the conditional variance are given by (2.14). The proof is similar to that of Proposition 1c.

2.3 N Component GARCH(n,n) Model

In what follows we will examine the N component GARCH(n,n) model. In this model the conditional variance of the errors (h_t) is a weighted sum of N components ($h_{it}, i = 1, \cdots, n$) with (w_i, $i = 1, \cdots, n$) as weights, respectively. Each component is a GARCH(n,n)-type specification:

$$\epsilon_t/\Omega_{t-1} \sim D(0, h_t), \;\; h_t = \sum_{i=1}^{n} w_i h_{it}, \;\; \sum_{i=1}^{n} w_i = 1 \tag{2.22}$$

where

$$B_i(L)h_{it} = \delta_i \omega_1 + A_i^e(L)\epsilon_t^2, \;\; i = 1, \cdots, n, \;\; \delta_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}, \tag{2.23}$$

$$B_i(L) = -\sum_{l=0}^{n} \beta_i^l L^l, \;\; \beta_i^0 = -1, \;\; A_i^e(L) = \sum_{l=1}^{n} a_i^l L^l \tag{2.23a}$$

Theorem 1a. The univariate ARMA representations of h_{it} $(i = 1, \cdots, n)$ are given by

$$B(L)h_{it} = \omega_i^\star + A_i(L)v_t, \;\; \omega_i^\star = \begin{cases} \omega_1[1 + \sum_{l=1}^{n-1} \Re'_{lm,n} B_l^1] & \text{if } i = 1 \\ \omega_1 w_1[A_i^e(1) + \sum_{l=1}^{n-2} \Re'_{(l+1)m,n} \beta_{1l}^i] & \text{otherwise} \end{cases}, \tag{2.24}$$

$$B(L) = 1 + \sum_{l=1}^{n^2} B_l L^l = \prod_{l=1}^{n^2}(1 - B_l^\circ L), \;\; A_i(L) = \sum_{l=1}^{n^2} A_{il} L^l \tag{2.24a}$$

In addition, the ARMA (n^2, n^2) representation of h_t is given by

$$B(L)h_t = \omega^\star + A(L)v_t, \;\; A(L) = \sum_{l=1}^{n^2} A_l L^l = \sum_{i=1}^{n} w_i A_i(L), \tag{2.25}$$

$$\omega^\star = \sum_{i=1}^{n} w_i \omega_i^\star$$

Moreover, the GARCH(n^2, n^2) representation of h_t is given by

$$B^\star(L)h_t = \omega^\star + A(L)\epsilon_t^2,\;\; B^\star(L) = 1 + \sum_{l=1}^{n^2} B_l^\star L^l, \tag{2.26}$$

Proof. The proof of equation (2.24) together with the B_l's, the A_{il}'s and the $\Re'_{(l+1)m,n}$ are given in Appendix B. The B_l^1 and β_{1l}^i are defined in Proposition 1a. The proof of equation (2.25) follows immediately from (2.22) and (2.24). The proof of equation (2.26) follows immediately from (2.25), using $v_t = \epsilon_t^2 - h_t$. The $B_l^\star$'s are given in Appendix B.

Example 3. For the three component GARCH(2,2) model the univariate ARMA(6,6) representation for the first component conditional variance is:

$$\begin{aligned}
&\{1 - (\beta_1^1 + \beta_2^1 + \beta_3^1 + w_1 a_1^1 + w_2 a_2^1 + w_3 a_3^1)L + \\
&\{-(\beta_1^2 + \beta_2^2 + \beta_3^2 + w_1 a_1^2 + w_2 a_2^2 + w_3 a_3^2) + [\beta_1^1\beta_2^1 + \beta_1^1\beta_3^1 + \beta_2^1\beta_3^1 \\
&+ w_1 a_1^1(\beta_2^1 + \beta_3^1) + w_2 a_2^1(\beta_1^1 + \beta_3^1) + w_3 a_3^1(\beta_1^1 + \beta_2^1)]\}L^2 + \\
&\{-(\beta_1^3 + \beta_2^3 + \beta_3^3 + w_1 a_1^3 + w_2 a_2^3 + w_3 a_3^3) + [\beta_1^1\beta_2^2 + \beta_1^2\beta_2^1 + \beta_1^1\beta_3^2 + \beta_1^2\beta_3^1 + \beta_2^1\beta_3^2 + \beta_2^2\beta_3^1 \\
&+ w_1 a_1^1(\beta_2^2 + \beta_3^2) + w_1 a_1^2(\beta_2^1 + \beta_3^1) + w_2 a_2^1(\beta_1^2 + \beta_3^2) + w_2 a_2^2(\beta_1^1 + \beta_3^1) + w_3 a_3^1(\beta_1^2 + \beta_2^2) + \\
&w_3 a_3^2(\beta_1^1 + \beta_2^1] - [\beta_1^1\beta_2^1\beta_3^1 + w_1 a_1^1(\beta_2^1\beta_3^1) + w_2 a_2^1(\beta_1^1\beta_3^1) + w_3 a_3^1(\beta_1^1\beta_2^1)]\}L^3 + \\
&\{+\beta_1^2\beta_2^2 + \beta_1^2\beta_3^2 + \beta_2^2\beta_3^2 + w_1 a_1^2(\beta_2^2 + \beta_3^2) + w_2 a_2^2(\beta_1^2 + \beta_3^2) + w_3 a_3^2(\beta_1^2 + \beta_2^2) - \\
&[\beta_1^1\beta_2^1\beta_3^2 + \beta_1^1\beta_2^2\beta_3^1 + \beta_1^2\beta_2^1\beta_3^1 + w_1 a_1^1(\beta_2^1\beta_3^2) + w_1 a_1^1(\beta_2^2\beta_3^1) + w_1 a_1^2(\beta_2^1\beta_3^1) + \\
&w_2 a_2^1(\beta_1^1\beta_3^2) + w_2 a_2^1(\beta_1^2\beta_3^1) + w_2 a_2^2(\beta_1^1\beta_3^1) + w_3 a_3^1(\beta_1^1\beta_2^2) + w_3 a_3^1(\beta_1^2\beta_2^1) + w_3 a_3^2(\beta_1^1\beta_2^1)]\}L^4 \\
&- [\beta_1^2\beta_2^2\beta_3^1 + \beta_1^2\beta_2^1\beta_3^2 + \beta_1^1\beta_2^2\beta_3^2 + w_1 a_1^2(\beta_2^2\beta_3^1) + w_1 a_1^2(\beta_2^1\beta_3^2) + w_1 a_1^1(\beta_2^2\beta_3^2) + \\
&w_2 a_2^2(\beta_1^2\beta_3^1) + w_2 a_2^2(\beta_1^1\beta_3^2) + w_2 a_2^1(\beta_1^2\beta_3^2) + w_3 a_3^2(\beta_1^2\beta_2^1) + w_3 a_3^2(\beta_1^1\beta_2^2) + w_3 a_3^1(\beta_1^2\beta_2^2)]L^5 \\
&- [\beta_1^2\beta_2^2\beta_3^2 + w_1 a_1^2(\beta_2^2\beta_3^2) + w_2 a_2^2(\beta_1^2\beta_3^2) + w_3 a_3^2(\beta_1^2\beta_2^2)]L^6\}h_{1t} = \omega_1^\star + \\
&\{a_1^1 L + [a_1^2 - a_1^1(\beta_2^1 + \beta_3^1)]L^2 + \{-[a_1^1(\beta_2^2 + \beta_3^2) + a_1^2(\beta_2^1 + \beta_3^1)] \\
&+ a_1^1\beta_2^1\beta_3^1\}L^3 + [-a_1^2(\beta_2^2 + \beta_3^2) + a_1^1\beta_2^1\beta_3^2 + a_1^1\beta_2^2\beta_3^1 + a_1^2\beta_2^1\beta_3^1]L^4 + \\
&+ [a_1^1\beta_2^2\beta_3^2 + a_1^2\beta_2^1\beta_3^2 + a_1^2\beta_2^2\beta_3^1]L^5 + a_1^2\beta_2^2\beta_3^2 L^6\}v_t
\end{aligned} \tag{2.27}$$

In addition the ARMA(6,6) representation for the aggregate conditional variance is

$$\begin{aligned}
&B(L)h_t = \omega^\star + \{(w_1 a_1^1 + w_2 a_2^1 + w_3 a_3^1)L + \{(w_1 a_1^2 + w_2 a_2^2 + w_3 a_3^2) - \\
&- [w_1 a_1^1(\beta_2^1 + \beta_3^1) + w_2 a_2^1(\beta_1^1 + \beta_3^1) + w_3 a_3^1(\beta_1^1 + \beta_2^1)]\}L^2 + \{(w_1 a_1^3 + w_2 a_2^3 + w_3 a_3^3) - \\
&- [w_1 a_1^1(\beta_2^2 + \beta_3^2) + w_1 a_1^2(\beta_2^1 + \beta_3^1) + w_2 a_2^1(\beta_1^2 + \beta_3^2) + w_2 a_2^2(\beta_1^1 + \beta_3^1) + w_3 a_3^1(\beta_1^2 + \beta_2^2) + \\
&w_3 a_3^2(\beta_1^1 + \beta_2^1] + [w_1 a_1^1(\beta_2^1\beta_3^1) + w_2 a_2^1(\beta_1^1\beta_3^1) + w_3 a_3^1(\beta_1^1\beta_2^1)]\}L^3 + \{-[w_1 a_1^2(\beta_2^2 + \beta_3^2) + \\
&w_2 a_2^2(\beta_1^2 + \beta_3^2) + w_3 a_3^2(\beta_1^2 + \beta_2^2)] + [w_1 a_1^1(\beta_2^1\beta_3^2) + w_1 a_1^1(\beta_2^2\beta_3^1) + w_1 a_1^2(\beta_2^1\beta_3^1) + \\
&w_2 a_2^1(\beta_1^1\beta_3^2) + w_2 a_2^1(\beta_1^2\beta_3^1) + w_2 a_2^2(\beta_1^1\beta_3^1) + w_3 a_3^1(\beta_1^1\beta_2^2) + w_3 a_3^1(\beta_1^2\beta_2^1) + w_3 a_3^2(\beta_1^1\beta_2^1)]\}L^4 \\
&+ \{w_1 a_1^2(\beta_2^2\beta_3^1) + w_1 a_1^2(\beta_2^1\beta_3^2) + w_1 a_1^1(\beta_2^2\beta_3^2) + w_2 a_2^2(\beta_1^2\beta_3^1) + w_2 a_2^2(\beta_1^1\beta_3^2) + w_2 a_2^1(\beta_1^2\beta_3^2) + \\
&w_3 a_3^2(\beta_1^2\beta_2^1) + w_3 a_3^2(\beta_1^1\beta_2^2) + w_3 a_3^1(\beta_1^2\beta_2^2)\}L^5 + \\
&+ [w_1 a_1^2(\beta_2^2\beta_3^2) + w_2 a_2^2(\beta_1^2\beta_3^2) + w_3 a_3^2(\beta_1^2\beta_2^2)]L^6\}v_t
\end{aligned} \tag{2.28}$$

where the autoregressive polynomial is the same with that of h_{1t} in eq (2.27). Finally, the GARCH(6,6) representation for the aggregate conditional variance is

$$\{1-(\beta_1^1+\beta_2^1+\beta_3^1)L+[-(\beta_1^2+\beta_2^2+\beta_3^2)+\beta_1^1\beta_2^1+\beta_1^1\beta_3^1+\beta_2^1\beta_3^1]L^2+$$
$$[-(\beta_1^3+\beta_2^3+\beta_3^3)+\beta_1^1\beta_2^2+\beta_1^2\beta_2^1+\beta_1^1\beta_3^2+\beta_1^2\beta_3^1+\beta_2^1\beta_3^2+\beta_2^2\beta_3^1-\beta_1^1\beta_2^1\beta_3^1]L^3+$$
$$+[\beta_1^2\beta_2^2+\beta_1^2\beta_3^2+\beta_2^2\beta_3^2-(\beta_1^1\beta_2^1\beta_3^2+\beta_1^1\beta_2^2\beta_3^1+\beta_1^2\beta_2^1\beta_3^1)]L^4-$$
$$-[\beta_1^2\beta_2^2\beta_3^1+\beta_1^2\beta_2^1\beta_3^2+\beta_1^1\beta_2^2\beta_3^2]L^5-\beta_1^2\beta_2^2\beta_3^2L^6\}h_t=\omega^\star+A(L)\epsilon_t^2 \tag{2.29}$$

where the ARCH polynomial is the same with the moving average polynomial in equation (2.28).

Theorem 1b. Under assumptions 1a and 1b the cross-covariances between the h_{it} and the $h_{j,t-m}$ components are given by

$$\gamma_{i,jm}=cov(h_{it},h_{j,t-m})=\begin{cases}\sum\limits_{r=1}^{n^2}\zeta_{r,m}\lambda_{r,m}^{ij}\sigma_v^2, & \text{if } m>0\\ \sum\limits_{r=1}^{n^2}\zeta_{rm}^{ji}\sigma_v^2, & \text{if } m<0\end{cases}, \tag{2.30}$$

$$\zeta_{rm}=\frac{(B_r^\circ)^{n^2-1+m}}{\prod\limits_{k=1}^{n^2}(1-B_r^\circ B_k^\circ)\prod\limits_{\substack{k=1\\k\neq r}}^{n^2}(B_r^\circ-B_k^\circ)}, \tag{2.30a}$$

$$\lambda_{r,m}^{ij}=\sum_{c=0}^{n^2-1}\sum_{d=1}^{n^2-c}A_{id}A_{j,d+c}(B_r^\circ)^c+\sum_{c=1}^{m^\star}\sum_{d=1}^{n^2-c}A_{jd}A_{i,d+c}(B_r^\circ)^{-c}+\sum_{c=m+1}^{n^2-1}\sum_{d=1}^{n^2-c}A_{jd}A_{i,d+c}(B_r^\circ)^{c-2m} \tag{2.30b}$$

where $m^\star=min(n^2-1,m)$ and $\sigma_v^2=\frac{2}{3}E(\epsilon_t^4)$ (under conditional normality) and is given below. When $i=j$ the above formula gives the autocovariances of h_{it}. Moreover, the cross-covariances between the h_t and $h_{j,t-m}$ (γ_{jm}) are as (2.30) where now h_{it} is replaced by h_t, A_{id} is replaced by A_d, λ_{rm}^{ij} is replaced by λ_{rm}^{j+}, and λ_{rm}^{ji} is replaced by λ_{rm}^{j-}. In addition, when h_{jt}, and h_{it} are replaced by h_t, A_{id} and A_{jd} are replaced by A_d and λ_{rm}^{ij} is replaced by λ_{rm} the above formula gives the autocovariances of h_t. The proof is similar to that of Proposition 1a.

Proposition 3a. The condition for the existence of the fourth moment of the errors for this model is $\gamma_0<\frac{1}{2}$, $\gamma_0=\sum\limits_{r=1}^{n^2}\zeta_{r0}\lambda_{r0}$.

Moreover, the univariate ARMA (n^2,n^2) representations of the squared errors ϵ_t^2 is given by

$$B(L)\epsilon_t^2=\omega^\star+A^e(L)v_t,\ \ A^e(L)=\sum_{l=0}^{n^2}A_l^eL^l=[B(L)+A(L)]v_t,\ \ A_0^e=1 \tag{2.31}$$

In addition, the autocovariance function of the squared errors is given by (2.30) where now h_{it} and h_{jt} are replaced by ϵ_t^2, A_{id} and $A_{j,d+c}$ are replaced by A_d^e and λ_{rm}^{ij} is replaced by λ_{rm}^e.

Finally, the covariances between the squared errors and the conditional variance are given by $cov(\epsilon_t^2, h_{t-m}) = cov(h_t, h_{t-m})$, $cov(h_t, \epsilon_{t-m}^2) = cov(\epsilon_t^2, \epsilon_{t-m}^2)$.

Proof. The derivation of the condition for the existence of the fourth moment is similar to that of Proposition 1c. The proof of equation (2.31) follows from (2.25) on rearranging terms. The equalities above follow from the law of iterated expectations.

3 Conclusions

This paper extendend K(1999a) results for the n Component GARCH(1,1) and the two Component GARCH(2,2) models and it further examined the n Component GARCH(n,n) model. First, we derived the VARMA representation of the component variances. Next, we used these VARMA representations to obtain the univariate ARMA representations of all the component variances, of the aggregate variance, and of the squared errors. In addition, we presented the GARCH(n^2, n^2) representation of the aggregate variance and we gave the condition for the existence of the fourth moment of the errors. Moreover, we used the canonical factorization of the autocovariance generating function of the above univariate ARMA representations to obtain (i) the autocovariances of the component variances, the aggregate variance and the squared errors, (ii) the cross covariances between the component variances, and (iii) the cross covariances between the aggregate variance and the component variances, and between the aggregate variance and the squared errors. Finally, we illustrated our general results using three examples: the three component GARCH(1,1), the two component GARCH(2,2) and the three component GARCH(2,2) models.

The potential generalisations of the simple Component GARCH model are numerous. To state a few: The Component Exponential GARCH(C-EGARCH), the Component GARCH-in-mean-level (C-GARCH-M-L), the Assymetric Power Component ARCH (C-APGARCH), the Fractional Integrated Component GARCH (C-FIGARCH), and finally the Multivariate Component GARCH (C-MGARCH) models[6]. Since this study only examined the case where the roots of the autoregressive polynomial are distinct,one potentially important issue relates to the effect of equal roots.

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[6]The EGARCH, GARCH-M-L, APGARCH, FIGARCH were introduced by Nelson (1991), Longstaff and Schwartz (1992), Ding, Granger and Engle(1993), and Baillie, Bollerlsev and Mikkelsen (1996), respectively. The autocovariance function of the conditional variance for the GARCH-in-mean-level model is given in Fountas, Karanasos and Karanassou (2000).

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A PROOF OF CORROLARY 1a, PROPOSITIONS 1a, 1b, 2a

Proof of Proposition 1a

We add and subtract $a_i h_{t-1}$ in (2.2) and we use (2.1) to get

$$h_{it} = \delta_i \omega_i + a_i \sum_{\substack{j=1 \\ j \neq i}}^{n} w_j h_{j,t-1} + (\beta_i + a_i w_i) h_{i,t-1} + a_i v_{t-1}, \;\; i = 1, \cdots, n, \tag{A.1}$$

$$v_t = \epsilon_t^2 - h_t, \;\; E(v_t) = 0, \;\; cov(v_t, v_{t-k}) = 0 \tag{A.1a}$$

Rewriting the system in a VARMA form we have

$$\tilde{B}\tilde{h}_t = \tilde{\omega} + \tilde{a} v_{t-1} \tag{A.2}$$

where $\tilde{B}$ is a $n \times n$ matrix. It's ijth element is $b_{ij} = \begin{cases} -a_i w_j & \text{if } i \neq j \\ 1 - a_i w_i - \beta_i & \text{if } i = j \end{cases}$. $\tilde{a}$ is a $n \times 1$ column vector. It's i1th element is a_i. $\tilde{h}_t$ is the $n \times 1$ column vector of the n components. $\tilde{\omega}$ is a $n \times 1$ column vector. Its i1th element is $\delta_i \omega_i$.

The univariate ARMA representations of (A.1) are given by (in what follows $\bar{B}$ denotes determinant)[7]

$$\sum_{l=0}^{n} \bar{B}_l L^l h_{it} = \omega_i^{\star} + \sum_{l=1}^{n} {}^{i_1}\bar{A}_l \, L^l v_t \tag{A.3}$$

$$\bar{B}_l = \prod_{k=1}^{l} [\sum_{f_k = f_{k-1}+1}^{n-(l-k)}] \prod_{k=1}^{l} (\bar{B}_{f_k}^{f_k}) (-1)^l, \;\; f_0 = 0, \;\; B_0 = 1 \tag{A.3a}$$

$\bar{B}_l$ denotes the sum of the determinants of all the $(l \times l)$ submatrices of the $(n \times n)$ matrix $\tilde{B}$. As an example, consider the case where $n = 3$ and $l = 2$:

$$\bar{B}_l = \bar{B}_2 = \bar{B}_{12}^{12} + \bar{B}_{13}^{13} + \bar{B}_{23}^{23} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}$$

$${}^{i_1}\bar{A}_l = \prod_{k=1}^{l} [\sum_{f_k = f_{k-1}+1}^{n-(l-k)}] \prod_{k=1}^{l} ({}^{i_1}\bar{A}_{1 f_k}^{1 f_k}) (-1)^{l-1}, \;\; f_0 = 0, \;\; ,{}^{i_1}A_0 = a_i \tag{A.3b}$$

[7]The proof is similar to the one used in K(1999b)

${}^{i}\tilde{A}$ denotes an $(n \times n)$ matrix. It is obtained from matrix $\tilde{B}$ by substituting its ith column with the column vector $\tilde{a}$. As an example consider the case where $n = 3$, $i = 2$:

$$ {}^{2}\tilde{A} = \begin{bmatrix} b_{11} & a_1 & b_{13} \\ b_{21} & a_2 & b_{23} \\ b_{31} & a_3 & b_{33} \end{bmatrix} $$

${}^{i_1}\tilde{A}$ denotes an $(n \times n)$matrix. It is obtained from matrix ${}^{i}\tilde{A}$ by moving the ith row (column) into the first row (column). As an example, consider the case where $n = 3$, $i = 2$:

$$ {}^{2_1}\tilde{A} = \begin{bmatrix} a_2 & b_{21} & b_{23} \\ a_1 & b_{11} & b_{13} \\ a_3 & b_{31} & b_{33} \end{bmatrix} $$

${}^{i_1}\tilde{A}_l$ denotes the sum of the $(l+1) \times (l+1)$ submatrices of the $(n \times n)$ matrix ${}^{i_1}\tilde{A}$ which include elements of its first row and column. As an example, consider the case where $n = 3$, $i = 2$, and $l = 1$:

$$ {}^{2_1}\tilde{A}_1 = \begin{bmatrix} a_2 & b_{21} \\ a_1 & b_{11} \end{bmatrix} + \begin{bmatrix} a_2 & b_{23} \\ a_3 & b_{33} \end{bmatrix} $$

From (A.3), (A.3a) and (A.3b) after some algebra we get (2.3).

Proof of Corrolary 1a

Multiplying (2.1) by $B(L)$ and using (2.3) we obtain

$$ B(L)h_t = \sum_{i=1}^{n} w_i \omega_i^{\star} + \sum_{i=1}^{n} w_i \sum_{l=1}^{n} A_{il} v_{t-l} \tag{A.4} $$

or alternatively (2.5). ■

An alternative derivation of the above result is given by K (1999a) (He derived it by using the DG, 1996 technique).

Proof of Proposition 1b

From (2.3a) we get

$$ \frac{1}{B(z)B(z^{-1})} = \sum_{l=1}^{n} \frac{(B_l^{\circ})^{n-1}}{(1 - B_l^{\circ} z)(1 - B_l^{\circ} z^{-1}) \prod\limits_{\substack{k=1 \\ k \neq l}}^{n} (B_l^{\circ} - B_k^{\circ})(1 - B_l^{\circ} B_k^{\circ})} \tag{A.5} $$

Moreover, after some algebra, we can show that

$$\frac{A_i(z)A_i(z^{-1})}{(1-B_l^\circ z)(1-B_l^\circ z^{-1})} = \frac{1}{1-(B_l^\circ)^2}\sum_{m=0}^{\infty}(\lambda_{l,m}^{ij}z^m + \lambda_{l,m}^{ji}z^{-m})(B_l^\circ)^m \tag{A.5a}$$

From (A.5) and (A.5a), after some algebra, we get the cross-covariance generating function $g_{ij}(z)$

$$g_{ij}(z) = \frac{A_i(z)A_j(z^{-1})}{B(z)B(z^{-1})}\sigma_v^2 = \sum_{l=1}^{n}\sum_{m=0}^{\infty} f_m\zeta_{lm}(\lambda_{l,m}^{ij}z^m + \lambda_{l,m}^{ji}z^{-m})\sigma_v^2, \tag{A.5b}$$

where $f_m = \begin{cases} .5 & \text{if } m=0 \\ 1 & \text{otherwise} \end{cases}$. Thus,

$$\gamma_{i,jm} = cov(h_{it}, h_{j,t-m}) = \begin{cases} \sum_{r=1}^{n}\zeta_{rm}\lambda_{rm}^{ij}\sigma_v^2, & \text{if } m>0 \\ \sum_{r=1}^{n}\zeta_{rm}\lambda_{rm}^{ji}\sigma_v^2, & \text{if } m<0 \end{cases} \tag{A.5c}$$

The proofs for the cross-covariances between h_t and h_{it}, and the autocovariances of the squared errors are similar. ■

Proof of Proposition 2a

In (2.15a) we add and subtract $A_i^e(L)h_t$ and we use (2.15) to get

$$B_i(L)h_{it} = \delta_i\omega_i + A_i^e(L)v_t + w_1A_i^e(L)h_{1t} + w_2A_i^e(L)h_{2t} \Rightarrow$$

$$[B_i(L) - w_iA_i^e(L)]h_{it} = \delta_i\omega_i + A_i^e(L)v_t + w_{3-i}A_i^e(L)h_{3-i,t}, \ \ i=1,2 \tag{A.6}$$

We multiply the above equation by $B_{3-i}(L) - w_{3-i}A_{3-i}^e(L)$ and after some algebra we get (2.16). ■

B PROOF OF *Theorem 1a*

Proof of Theorem 1a

Adding and subtracting $\sum_{l=1}^{n}a_i^lh_{t-l}$, $(i=1,\cdots,n)$ in (2.23), using $v_t = \epsilon_t^2 - h_t$, and writing the system in a VARMA representation form we get

$$\tilde{h}_t = \tilde{\omega} + \sum_{l=1}^{n} \tilde{B}_l \tilde{h}_{t-l} + \sum_{l=1}^{n} \tilde{a}_l v_{t-l} \tag{B.1}$$

$$\tilde{h}_t = \begin{bmatrix} h_{1t} \\ . \\ . \\ . \\ h_{nt} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nn} \end{bmatrix}, \quad b_{ij} = a_i w_j + \delta_i \beta_i, \quad \tilde{\omega} = \begin{bmatrix} \omega_1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} \tag{B.1a}$$

$$\tilde{B}_l = \begin{bmatrix} b^l_{11} & \dots & b^l_{1n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ b^l_{n1} & \dots & b^l_{nn} \end{bmatrix}, \quad \delta_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad \tilde{a} = \begin{bmatrix} a_1 \\ . \\ . \\ . \\ a_n \end{bmatrix}, \quad \tilde{a}_l = \begin{bmatrix} a^l_1 \\ . \\ . \\ . \\ a^l_n \end{bmatrix} \tag{B.1b}$$

The univariate ARMA representations of (B.1) are given by (in what follows $\bar{B}$ denotes determinant)[8]

$$\sum_{l=0}^{n^2} \bar{B}_{lm}\, L^l h_{it} = \omega_i^\star + \sum_{l=1}^{n^2} {}^{i_1}\bar{A}_{lm}\, L^l v_t, \quad \bar{B}_{0m} = 1, \quad i = 1, \cdots, n \tag{B.2}$$

where

$$\bar{B}_{lm} = \sum_{m=1}^{n} \Re'_{ml,n}\, \bar{B}_m, \quad \bar{B}_m = \prod_{k=1}^{m} \Big(\sum_{f_k = f_{k-1}+1}^{n-(m-k)} \Big) \prod_{k=1}^{m} (\bar{B}^{f_k}_{f_k})(-1)^m, \quad f_0 = 0 \tag{B.2a}$$

where $\bar{B}_m$ denotes the sum of the determinants of all the $(m \times m)$ submatrices of the $(n \times n)$ matrix $\tilde{B}$. As an example, consider the case where $n = 3$ and $m = 2$:

$$\bar{B}_m = \bar{B}_2 = \bar{B}^{12}_{12} + \bar{B}^{13}_{13} + \bar{B}^{23}_{23} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}$$

$$\Re'_{ml,n} = \begin{cases} \Re_{ml,n} & \text{if } l = m, \cdots, m \times n \\ 0 & \text{otherwise} \end{cases}, \quad \Re_{ml,n} = \prod_{k=1}^{m} \mho_{g_k = max[1, l - [(m-k)n + \sum_{t=1}^{k-1} g_t]]}^{min[l - \sum_{t=1}^{k-1} g_t - (m-k), n]} g_k$$

where $\Re_{ml,n}$ denotes the set of all the combinations of m numbers taking values from 1 to n and adding to l. As an example, consider the case where $n = 2$ and $m = 2$:

$$\Re'_{ml,n} = \Re'_{2l,2} = \begin{cases} \Re_{2l,2} & \text{if } l = 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}, \quad \Re_{22,2} = 11, \quad \Re_{23,2} = 12, 21, \quad \Re_{24,2} = 22$$

[8]The proof is similar to the one used in K(1999b)

${}_{k_1k_2\cdots k_m}\bar{B}_m$ $(k_i = 1, \cdots, n)$ denotes the $\bar{B}_m$ sum of determinants where now the b's in the ith column $(i = 1, \cdots, m)$ are taken from the $\tilde{B}_l$ matrix. As an example, consider the case where $n = 3$ and $m = 3$:

$$ {}_{121}\bar{B}_3 = -\begin{vmatrix} b_{11}^1 & b_{12}^2 & b_{13}^1 \\ b_{21}^1 & b_{22}^2 & b_{23}^1 \\ b_{31}^1 & b_{32}^2 & b_{33}^1 \end{vmatrix} $$

When we multiply $\bar{B}_m$ by $\Re_{ml,n}$ we get

$$ {}_{(l_1\cdots l_m)_1}\bar{B}_m + \cdots + {}_{(l_1\cdots l_m)_f}\bar{B}_m $$

where $(l_1 \cdots l_m)_i, \ \ i = 1, \cdots, f$ denotes the set of all the f different combinations of m numbers which take values from 1 to n and sum to l. As an example, consider the case where $n = 3$, $m = 2$ and $l = 3$:

$$ \Re_{23,3}\,\bar{B}_2 = {}_{12}\bar{B}_2 + {}_{21}\bar{B}_2 = [\begin{vmatrix} b_{11}^1 & b_{12}^2 \\ b_{21}^1 & b_{22}^2 \end{vmatrix} + \begin{vmatrix} b_{11}^1 & b_{13}^2 \\ b_{31}^1 & b_{33}^2 \end{vmatrix} + \begin{vmatrix} b_{22}^1 & b_{23}^2 \\ b_{32}^1 & b_{33}^2 \end{vmatrix}]+ $$

$$ +[\begin{vmatrix} b_{11}^2 & b_{12}^1 \\ b_{21}^2 & b_{22}^1 \end{vmatrix} + \begin{vmatrix} b_{11}^2 & b_{13}^1 \\ b_{31}^2 & b_{33}^1 \end{vmatrix} + \begin{vmatrix} b_{22}^2 & b_{23}^1 \\ b_{32}^2 & b_{33}^1 \end{vmatrix}] $$

$$ {}^{i_1}\bar{A}_{lm} = \sum_{m=0}^{n-1} \Re'_{(m+1)l,n}\,{}^{i_1}\bar{A}_m, \ \ {}^{i_1}\bar{A}_m = \prod_{k=1}^{m}(\sum_{f_k=f_{k-1}+1}^{n-(m-k)})\prod_{k=1}^{m}({}^{i_1}\bar{A}_{1,f_k}^{1,f_k})(-1)^m, \ f_0 = 1 \tag{B.2b} $$

where ${}^{i}\tilde{A}$ denotes a $(n \times n)$ matrix. It is obtained from matrix $\tilde{B}$ by substituting its ith column with the column vector $\tilde{a}$. As an example, consider the case where $n = 3, \ \ i = 3$:

$$ {}^{3}\tilde{A} = \begin{bmatrix} b_{11} & b_{12} & a_1 \\ b_{21} & b_{22} & a_2 \\ b_{31} & b_{32} & a_3 \end{bmatrix} $$

${}^{i_1}\tilde{A}$ denotes a $(n \times n)$ matrix. It is obtained from matrix ${}^{i}\tilde{A}$ by moving the ith row (column) into the first row (column). As an example, consider the case where $n = 3, i = 3$:

$$ {}^{3_1}\tilde{A} = \begin{bmatrix} a_3 & b_{31} & b_{32} \\ a_1 & b_{11} & b_{12} \\ a_2 & b_{21} & b_{22} \end{bmatrix} $$

Of all the $(m + 1) \times (m + 1)$ submatrices of the $(n \times n)$ matrix ${}^{i_1}\tilde{A}$, ${}^{i_1}\tilde{A}_m$ denotes the sum of those which include elements of its first row and column. As an example, consider the case where $n = 3, \ \ i = 3$ and $m = 1$:

$$ {}^{3_1}\tilde{A}_1 = \begin{bmatrix} a_3 & b_{31} \\ a_1 & b_{11} \end{bmatrix} + \begin{bmatrix} a_3 & b_{32} \\ a_2 & b_{22} \end{bmatrix} $$

$$\Re'_{(m+1)l,n} = \begin{cases} \Re_{(m+1)l,n} & \text{if } l = m+1, \cdots, (m+1)n \\ 0 & \text{otherwise} \end{cases}$$

$$\Re_{(m+1)l,n} = \prod_{k^\star=1}^{m+1} \mho_{g_{k^\star}=max[1,l-[(m+1-k^\star)n+\sum_{t=1}^{k^\star-1} g_t]]}^{min[l-\sum_{t=1}^{k^\star-1} g_t-(m+1-k^\star),n]} g_{k^\star}$$

$\Re_{(m+1)l,n}$ denotes the set of all the combinations of $(m+1)$ numbers which take values from 1 to n and sum to l. As an example, consider the case where $n = 2,\ \ l = 5$ and $m = 2$:

$$\Re_{35,2} = 221, 212, 122$$

From (B.2) using (B.1a), (B.1b), (B.2a) and (B.2b) after some algebra we get

$$B(L)h_{it} = \omega_i^\star + A_i(L)v_t, \tag{B.3}$$

$$B(L) = \sum_{l=0}^{n^2} \bar{B}_{lm} L^l = \prod_{l=1}^{n^2}(1 - B_l^o L),\ \ A_i(L) = \sum_{l=1}^{n^2} {}^{i_1}\bar{A}_{lm}\, L^l = \sum_{l=1}^{n^2} A_{il} L^l \tag{B.3a}$$

where

$$\bar{B}_{lm} = \sum_{m=1}^{n^2} \Re'_{ml,n}\hat{\beta}_m,\ \ \hat{\beta}_m = \hat{\beta}_{1m} + \hat{\beta}_{2m},\ \hat{\beta}_{21} = -\sum_{i=1}^{n} w_i a_i$$

$$\hat{\beta}_{1m} = \prod_{k=1}^{m}(\sum_{f_k=f_{k-1}+1}^{n-(m-k)})\prod_{k=1}^{m}(\beta_{f_k})(-1)^m,\ \ \hat{\beta}_{2m} = \sum_{i=1}^{n} w_i a_i \prod_{k=1}^{m-1}(\sum_{\substack{f_k=f_{k-1}+1 \\ f_k \neq i}}^{n-(m-1-k)})\prod_{k=1}^{m-1}(\beta_{f_k})(-1)^m$$

where $\Re'_{ml,n}$ is defined as above.

${}_{k_1k_2\cdots k_m}\hat{\beta}_m$ denotes the $\hat{\beta}_m$ where now the ith terms in each of the products of m terms $(i = 1, \cdots, m)$ are taken from the $\tilde{B}_{k_i}$ matrix. As an example consider the case where $n = 4$ and $m = 3$.

$${}_{121}\hat{\beta}_3 = -[(\beta_1^1\beta_2^2\beta_3^1 + \beta_1^1\beta_2^2\beta_4^1 + \beta_1^1\beta_3^2\beta_4^1 + \beta_2^1\beta_3^2\beta_4^1) + w_1a_1^1(\beta_2^2\beta_3^1 + \beta_2^2\beta_4^1 + \beta_3^2\beta_4^1)$$

$$+ w_2a_2^1(\beta_1^2\beta_3^1 + \beta_1^2\beta_4^1 + \beta_3^2\beta_4^1) + w_3a_3^1(\beta_1^2\beta_2^1 + \beta_1^2\beta_4^1 + \beta_2^2\beta_4^1) + w_4a_4^1(\beta_1^2\beta_2^1 + \beta_1^2\beta_3^1 + \beta_2^2\beta_3^1)]$$

When we multiply $\hat{\beta}_m$ by $\Re_{ml,n}$ we get

$$_{(l_1\cdots l_m)_1}\hat{\beta}_m + \cdots + {}_{(l_1\cdots l_m)_f}\hat{\beta}_m$$

where $(l_1 \cdots l_m)_j,\ j = 1, \cdots, f$ denotes the set of all the f different combinations of m numbers which take values from 1 to n and sum to l. As an example, consider the case where $n = 3$, $m = 2$ and $l = 3$:

$$\Re_{23,3}\hat{\beta}_m = {}_{12}\hat{\beta}_2 + {}_{21}\hat{\beta}_2 = (\beta_1^1\beta_2^2 + \beta_1^1\beta_3^2 + \beta_2^1\beta_3^2) + (\beta_1^2\beta_2^1 + \beta_1^2\beta_3^1 + \beta_2^2\beta_3^1) +$$

$$+ [w_1a_1^1(\beta_2^2 + \beta_3^2) + w_2a_2^1(\beta_1^2 + \beta_3^2) + w_3a_3^1(\beta_1^2 + \beta_2^2)]$$

$$+ [w_1a_1^2(\beta_2^1 + \beta_3^1) + w_2a_2^2(\beta_1^1 + \beta_3^1) + w_3a_3^2(\beta_1^1 + \beta_2^1)]$$

$$^{i_1}\bar{A}_{lm} = \sum_{m=0}^{n^2-1} \Re'_{(m+1)l,n}\hat{a}_{im}, \ \ \hat{a}_{im} = a_i \prod_{k=1}^{m}\Big(\sum_{\substack{f_k=f_{k-1}+1 \\ f_k \neq i}}^{n-(m-k)}\Big)\prod_{k=1}^{m}(\beta_{f_k})(-1)^m, \ \ \hat{a}_{i0} = a_i, \ \ f_0 = 0. \qquad (B.3d)$$

where $\Re'_{(m+1)l,n}$ is defined as above.

$_{k_1k_2\cdots k_m}\hat{a}_{im}$ denotes the $\hat{a}_{im}$ where now the first term in each of the products of m terms $(j = 1, \cdots, m)$ are taken from the $\tilde{a}_{k_1}$ matrix and the next j-1 terms are taken from the $\tilde{B}_{k_j}$ matrix. As an example consider the case where $n = 4$ and $m = 2$.

$$_{121}\hat{a}_{13} = a_1^1\beta_2^2\beta_3^1 + a_1^1\beta_2^2\beta_4^1 + a_1^1\beta_3^2\beta_4^1$$

When we multiply $\hat{a}_{im}$ by $\Re'_{(m+1)l,n}$ we get

$$_{(l_1\cdots l_{m+1})_1}\hat{a}_{im} + \cdots + {}_{(l_1\cdots l_{m+1})_f}\hat{a}_{im}$$

where $(l_1 \cdots l_{m+1})_j,\ j = 1, \cdots, f$ denotes the set of all the f different combinations of $m+1$ numbers which take values from 1 to n and sum to l. As an example, consider the case where $n = 3$, $m = 2$ and $l = 4$:

$$\Re'_{24,3}\hat{a}_{1m} = {}_{112}\hat{a}_{1m} + {}_{121}\hat{a}_{1m} + {}_{211}\hat{a}_{1m} = a_1^1\beta_2^1\beta_3^2 + a_1^1\beta_2^2\beta_3^1 + a_1^2\beta_2^1\beta_3^1$$

■