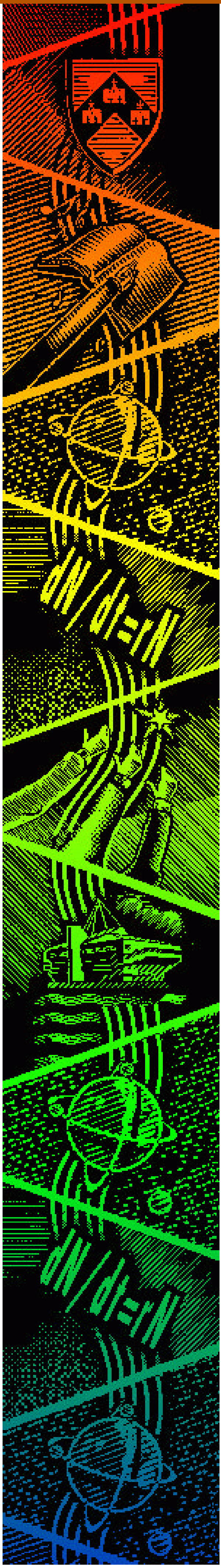

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A Simple Version and Extension of Arrow's Theorem in the Edgeworth Domain

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Abstract

This paper presents an Arrow-type result which can be simply demonstrated to hold within the standard domain of welfare economics: in the $(m \times n)$ Edgeworth Box, a best allocation must assign all goods to a single individual. Allowing the Social Welfare Function to take account of envy-freeness, or other related constructions, does not significantly resolve this problem.

Keywords Arrow, Social Welfare Function, Edgeworth Box, Envy-Freeness.

JEL codes D63, D71

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1 Introduction

Probably the most familiar, if not the simplest, problem in welfare economics is as follows. There are fixed amounts of several goods to be allocated between several individuals, each of whom has conventionally-structured preferences over his (her) own assignment of goods. The problem is to rank, i.e., to place in order of social preference, the various feasible allocations of these goods, and for this ranking to respect individual preferences in two specific ways.

Firstly, the *Pareto* condition requires that allocation x is strictly preferred to ("better than") allocation y, if each individual strictly prefers his assignment in x to his assignment in y. Figure 1 is the familiar representation of the 2×2 case. With the solid indifference curves, a is a Pareto-efficient allocation, as is any other allocation on the dotted contract curve. For any allocation not on the contract curve, there exists another which is better, because Pareto-superior.

The second condition, *Independence of Irrelevant Alternatives*, requires the ranking of any pair of allocations $\{x,y\}$ to depend on individuals' preferences only over (their own assignments in) x and y. The force of this condition emerges if it is additionally required that a social preference ordering be constructible for any admissible, albeit hypothetical, profile of individual preferences. Given this, IIA imposes on the social preference ordering a requirement of consistency across different individual preference profiles.

In a less structured context, i.e., where the available alternatives are abstract items over which any individual may have any coherent preference ordering, the implications of these conditions are well known. Arrow's theorem states that Pareto and IIA can together be satisfied only by a *dictatorial* Social Welfare Function. That is, there is some specific individual j such that, for any profile of individual preferences, the social preference ordering corresponds exactly to j's individual ordering.

It is only relatively recently (Bordes, Campbell and Le Breton, 1995) that the equivalent result has been confirmed to hold in the more structured economic environment described above, i.e., in the *Edgeworth Domain*.[1] In this paper we demonstrate an Arrow-like proposition which is logically weaker than that of Bordes *et al*. However it is still a strong result, and can not only be obtained with great simplicity, but can also be generalised in a novel direction. It concerns the possibility of a best allocation, that is, one which has no other allocation ranked above it. Since in the Edgeworth Domain there is an infinite number of possible allocations, then it cannot be assumed that a best allocation always exists. But it can be easily shown that if it does exist then it must assign all of the available goods to one individual.

The basic argument can be sketched immediately, with reference to Figure 1. Allocation *a* assigns positive amounts of each good to each individual and, given the solid indifference curves, is Pareto-efficient. Suppose that it is a best allocation, and is therefore at least as good as each of *b* and *c*, as shown. (The requirements on *b* and *c* will soon become apparent, as will the fact that such allocations can always be located.)

Individual 1 prefers *a* to *b*, but could have preferred *b* to *a*, as indicated by a dashed indifference curve. In that case, 2's preferences being unchanged, *b* would have been Pareto-superior to *a*. Each individual's preferences with respect to $\{a,c\}$ are unchanged so that, for this hypothetical profile, IIA implies that *a* is at least as good as *c*. It follows that, for this profile, *b* is better than *c*.

Now consider another hypothetical profile where, instead, individual 2 prefers *c* to *a*, again as indicated by a dashed indifference curve. For this profile *c* is Pareto-superior to *a*. Given that each individual's preferences with respect to $\{a,b\}$ are unchanged, IIA implies here that *a* is at least as good as *b*. It follows that, for this profile, *c* is better than *b*. But individual preferences with respect to $\{b,c\}$ are invariant throughout; 1 prefers *c* to *b* and 2 prefers *b* to *c*. Hence there is a violation of IIA.

So we cannot, without contradiction, assume that *a* is a best allocation. But there is nothing special about *a*. For any Pareto-efficient allocation in the interior of the Edgeworth Box a similar argument can be constructed, in that other allocations corresponding to *b* and *c* may always be found. This would be the case even were the contract curve to coincide (in part) with the edge of the box. The only points immune to this argument, i.e., for which allocations such as *b* and *c* cannot be found, are the endpoints of contract curve, where one individual is assigned all the available goods. No other allocation can be best.

2 The Edgeworth Dictator

In this section we formalise and generalise the argument just outlined. There are *m* goods $(g=1,..,m)$, each available in a given quantity $w_g > 0$, and each infinitely divisible between *n* individuals $(i=1,..,n)$. The available endowment is therefore $w = (w_1,..,w_m)$.

An *assignment* $x_i = (x_{i1},..,x_{im})$ is any non-negative m-tuple, where x_{ig} is the quantity of good *g* assigned to individual *i*. Given *w*, an *allocation* $x = (x_1,..,x_n)$ is any n-tuple of assignments such that $\sum_i x_{ig} \leq w_g$. Let $\chi(w)$ denote the set of all allocations, given *w*.

Each individual has a (complete, transitive) ordering R_i defined on assignments, and assumed to be continuous, strictly increasing and strictly convex. Call this a *standard* preference ordering. In the usual manner, $x_i \mathrm{R}_i y_i$ denotes that *i* weakly prefers x_i to y_i. Strict preference is denoted by $x_i \mathrm{P}_i y_i \Leftrightarrow \neg\, y_i \mathrm{R}_i x_i$, and indifference by $x_i \mathrm{I}_i y_i \Leftrightarrow [x_i \mathrm{R}_i y_i \;\&\; y_i \mathrm{R}_i x_i]$.

A *profile* $\rho = (\mathrm{R}_1,..,\mathrm{R}_n)$ is any n-tuple of standard preference orderings, one for each individual. For the 2×2 case, $\chi(w)$ and ρ can be depicted in the familiar Edgeworth Box. In Figure 1 the solid indifference curves represent a given profile $\rho^0 = (\mathrm{R}_1^0, \mathrm{R}_2^0)$.

A social preference ordering R, defined on $\chi(w)$, is required at any profile ρ. The function $\mathrm{R} = f(\rho)$, i.e., the Social Welfare Function (SWF), is to satisfy:[2]

Pareto If $x_i \mathrm{P}_i y_i$ for all *i*, then $x\mathrm{P}y$

IIA The social ordering of any pair of allocations $\{x,y\}$ depends on each individual *i*'s ordering only of $\{x_i, y_i\}$

At any given profile ρ, the social ordering $\mathrm{R} = f(\rho)$ may identify a *best* allocation, i.e., some *x* such that $x\mathrm{R}y$ for all $y \in \chi(w)$. If *x* is a best allocation, then the Pareto condition implies that there can be no $y \in \chi(w)$ such that $\forall i{:}\, y_i \mathrm{P}_i x_i$. In Figure 1 this restriction is represented by the (dotted) contract curve; given profile ρ^0, neither of allocations *b* and *c* can be best. As we now show, however, IIA additionally implies that even *a* cannot be best.

Proposition 1 In the $m \times n$ Edgeworth Domain $(m \geq 2)$, given Pareto and IIA, at any profile ρ a best allocation (if one exists) must assign all of the available goods to one individual, i.e., is characterised by $x_j = w$ for some individual j.

Proof Assume to the contrary that at ρ^0 there exists a best allocation a which does not assign all goods to one individual. Then (allowing for an appropriate re-labelling of individuals and goods) there must be a pair of distinct individuals {1,2} and a pair of distinct goods {1,2} such that:

$$a_{11} > 0 \qquad a_{22} > 0$$

Given any $\{\delta_i, \varepsilon_i, \zeta_i\}$ such that:

$$0 < (n-2)\zeta_i < \varepsilon_i < \delta_i < a_{ii} \qquad (i=1,2)$$

define allocations b and c as follows (where $j \neq 1,2$):

$$b_{11} = a_{11} - \delta_1 \qquad b_{21} = a_{21} + \delta_1 - (n-2)\zeta_1 \qquad b_{j1} = a_{j1} + \zeta_1$$

$$b_{22} = a_{22} - \varepsilon_2 \qquad b_{12} = a_{12} + \varepsilon_2 - (n-2)\zeta_2 \qquad b_{j2} = a_{j2} + \zeta_2$$

$$c_{11} = a_{11} - \varepsilon_1 \qquad c_{21} = a_{21} + \varepsilon_1 - (n-2)\zeta_1 \qquad c_{j1} = a_{j1} + \zeta_1$$

$$c_{22} = a_{22} - \delta_2 \qquad c_{12} = a_{12} + \delta_2 - (n-2)\zeta_2 \qquad c_{j2} = a_{j2} + \zeta_2$$

$$a_{ig} = b_{ig} = c_{ig} \qquad (i=1,..,n \ ; \ g \neq 1,2)$$

Figure 2 illustrates this construction. For any standard preferences R_i:

(1) $c_1 \mathrm{P}_1 b_1$ & $b_2 \mathrm{P}_2 c_2$ & $b_j \mathrm{I}_j c_j$ $(j \neq 1,2)$

(2) $b_j \mathrm{P}_j a_j$ & $c_j \mathrm{P}_j a_j$ $(j \neq 1,2)$

There exist standard preferences R_1^1 and R_2^2 such that:

(3) $\quad b_1 P_1^1 a_1 \quad \& \quad c_2 P_2^2 a_2$

Given $\{\delta_1, \delta_2\}$ and ρ^0, there exist $\hat{\varepsilon}_i < \delta_i$ $(i=1,2)$ such that if, additionally, $\varepsilon_i < \hat{\varepsilon}_i$ then:

(4) $\quad c_1 P_1^0 a_1 \quad \& \quad b_2 P_2^0 a_2$

Assume that $\varepsilon_i < \hat{\varepsilon}_i$, so that (4) is satisfied.[3]

At $\rho^0 = (R_1^0, R_2^0, .., R_n^0)$, by assumption, both aR^0b and aR^0c. Then from (1)-(4) there exists a profile $\rho^1 = (R_1^1, R_2^0, .., R_n^0)$ at which:

by Pareto:	bP^1a.
by IIA:	aR^1c
and thus:	bP^1c

Similarly, there exists a profile $\rho^2 = (R_1^0, R_2^2, .., R_n^0)$ at which:

by Pareto:	cP^2a
by IIA:	aR^2b
and thus:	cP^2b

But from (1) this is a violation of IIA. So the initial assumption is false.

QED

Proposition 1 applies to any given profile. But let x be an allocation such that $x_j = w$ for some individual j, and let y be any other allocation. Given increasingness, each individual i's preference ordering of $\{x_i, y_i\}$ is invariant across all profiles. So IIA implies that if xRy at any profile then xRy at every profile. From Proposition 1 it then follows that the set of best allocations must be the same at every profile. Thus:

Proposition 2 If $\mathrm{R}=f(\rho)$ identifies a best allocation at some ρ' then, given Pareto and IIA, there is a non-empty set of individuals D such that, at every ρ, *x* is a best allocation if and only if $x_j=w$ for some $j\in \mathrm{D}$.

Proposition 3 If $\mathrm{R}=f(\rho)$ identifies a uniquely best allocation at some ρ' then, given Pareto and IIA, there is an individual *j* such that, at every ρ, *x* is a best allocation if and only if $x_j=w$.

Propositions 1-3 are somewhat weaker than the Arrovian result of Bordes *et al*, i.e., that the SWF must be dictatorial. But 1-3 are nevertheless substantial in themselves. Furthermore, they can be extended in a direction which, in one respect, represents an advance on Bordes *et al*. This involves a weakening of IIA, to be described in the next section.

3 Envy-freeness in the Social Welfare Function

An allocation *x* is *envy-free* if $\forall i,j$: $x_i \mathrm{R}_i x_j$, that is, if no individual *i* strictly prefers the assignment of some other individual *j*. Figure 3 illustrates for the 2×2 case, where to any allocation $x=(x_1,x_2)$ there corresponds a *swap* allocation $x'=(x_1',x_2')$ such that $x_i'=x_j$ $(i\neq j)$. In the Edgeworth Box, x' is obtained from *x* by a $180°$ rotation about the equal-assignment allocation $(w/2, w/2)$. Then *x* is envy-free if neither individual *i* strictly prefers x_i' to x_i. Given standard preferences in the Edgeworth Domain, there always exist allocations which, like *a* in Figure 3, are both envy-free and efficient.[4]

Consider the following principle:

EF If *x* is envy-free and *y* is not, then $x\mathrm{P}y$

Whatever the appeal of this principle, it is not consistent with IIA, as may be confirmed by a 2×2 example. Define allocations $d=((6,4),(4,6))$ and $e=((1,9),(9,1))$, and profiles $\{\rho,\rho'\}$, as represented by the following utility functions.

ρ	$u_1 = x_{11}^2 x_{12}$	$u_2 = x_{21} x_{22}^2$	so that for $i \neq j$:	
	$u_i(d_i) = 144$	$u_i(d_j) = 96$	$u_i(e_i) = 9$	$u_i(e_j) = 81$
ρ′	$u_1 = x_{11} x_{12}^2$	$u_2 = x_{21}^2 x_{22}$	so that for $i \neq j$:	
	$u_i(d_i) = 96$	$u_i(d_j) = 144$	$u_i(e_i) = 81$	$u_i(e_j) = 9$

In profile ρ, *d* is envy-free but *e* is not, so EF here requires that $d\mathrm{P}e$. In profile ρ′ the reverse is true, so EF here requires that $e\mathrm{P}d$. But each individual's ordering of own-assignments in *d* and *e* is the same in ρ′ as in ρ, so this is a violation of IIA.

There is, however, a natural weakening of IIA which permits EF. It is:

IIA* The social ordering of any pair of allocations $\{x, y\}$ depends on each individual *i*'s ordering only of $\{x_1,..,x_n; y_1,..,y_n\}$

IIA* allows the social ordering of $\{d,e\}$ to differ between profiles ρ and ρ′, as EF requires. But EF itself remains problematic, since it is inconsistent not only with IIA but also with Pareto, as can be seen in the above example. According to the invariant individual preferences over own-assignments, *d* is Pareto-superior to *e*. But at profile ρ′ EF requires that $e\mathrm{P}d$. So even with the weaker IIA*, a SWF which satisfies Pareto cannot also accommodate EF.[5]

Although EF therefore cannot be part of it, the question arises as to whether there is any non-dictatorial SWF which satisfies Pareto together with IIA*. We will show that this is doubtful, in that results similar to Propositions 2 and 3 can be obtained in this more general setting.

The outline argument is almost as simple as before. In Figure 3, assume that *a* is a best allocation at $\rho^0 = (\mathrm{R}_1^0, \mathrm{R}_2^0)$, so that $a\mathrm{R}^0 b$ and $a\mathrm{R}^0 c$. As shown, *a* is not only efficient but also envy-free; given ρ^0, each individual strictly prefers *a* to a'. Thus *b* (and *c*) can be located sufficiently close to *a* such that individual 1 strictly prefers b_1 to any of $\{a_2, b_2, c_2\}$. It is evidently then possible to construct an alternative indifference curve, such as the dashed curve shown, which passes through a_1 and below b_1, but which leaves undisturbed 1's preferences over all other relevant assignments.

So there exists a profile $\rho^1 = (\mathrm{R}_1^1, \mathrm{R}_2^0)$ at which *b* is Pareto-superior to *a*, but in which each individual's ordering of $\{a_1,a_2;c_1,c_2\}$ and of $\{b_1,b_2;c_1,c_2\}$ is the same as in ρ^0. At ρ^1, IIA* thus implies that $a\text{R}^1c$, from which it follows that $b\text{P}^1c$.

Similarly, there exists a profile $\rho^2 = (\text{R}_1^0, \text{R}_2^2)$ at which c is Pareto-superior to a, but in which each individual's orderings of $\{a_1,a_2;b_1,b_2\}$ and of $\{b_1,b_2;c_1,c_2\}$ is the same as in ρ^0. At ρ^2, IIA* thus implies that $a\text{R}^2b$, from which it follows that $c\text{P}^2b$. But this is a violation of IIA*. So a cannot be a best allocation at ρ^0.

In section 5 we formalise and generalise this argument. To do this, however, we need to show that $\{\text{R}_1^1, \text{R}_2^2\}$ can always be found which relevantly resemble $\{\text{R}_1^0, \text{R}_2^0\}$. The next section provides the analytical framework for this.

4 A parametric preference transformation

Individual 1's standard preference ordering R_1^0 can be represented by the function:

$$x_{12} = \phi(u_1; x_{11}, x_{13}, .., x_{1m})$$

which is continuous, increasing in u_1, and for any given u_1 defines a decreasing and strictly convex indifference surface. Given a_1 and $b_{11} = a_{11} - \delta_1$, as in Proposition 1, calibrate u_1 and ϕ such that:

$$a_{12} = \phi(0; b_{11}, a_{13}, .., a_{1m}) = \phi(1; a_{11}, a_{13}, .., a_{1m})$$

Given any b_{12} such that:

$$\phi(0; b_{11}, a_{13}, a_{13}, .., a_{1m}) < b_{12} \leq \phi(1; b_{11}, a_{13}, a_{13}, .., a_{1m})$$

it is required to transform ϕ into ψ, representing R_1^1 such that $b_1\text{P}_1^1a_1$, and thus:

$$a_{12} = \psi(1; a_{11}, a_{13}, a_{13}, .., a_{1m}) \qquad \text{and} \qquad b_{12} > \psi(1; b_{11}, a_{13}, a_{13}, .., a_{1m})$$

but also to delimit the transformation in (standard) preferences over assignments other than $\{a_1, b_1\}$.

For notational conciseness we will assume that $m=3$, but the following applies straightforwardly to any $m \geq 2$. Define:

$$\theta(x_{11},x_{13}) \;=\; \max\{\, x_{12} \mid \alpha x_{12}+(1-\alpha)a_{12} \;\leq\; \phi(1;[\alpha x_{11}+(1-\alpha)b_{11}],[\alpha x_{13}+(1-\alpha)a_{13}]) \\ \text{either (i) } \forall \alpha \geq 1, \text{ or (ii) } \forall \alpha>0 \text{ such that } \alpha x_{11}+(1-\alpha)b_{11} \leq a_{11} \,\}$$

To illustrate, Figure 4 shows $\phi(1;x_{11},x_{13})$, delineated in each of the three cross-sections defined by $x_{1g}=a_{1g}$ $(g=1,2,3)$, intersecting at assignment a_1. At a distance δ_1 directly below this is located (b_{11},a_{12},a_{13}), emanating from which are five representative rays. These are shown as dashed where they lie below the indifference surface and solid where above it.

The two lower rays, up to their respective points of tangency with the indifference surface, are on θ by virtue of both (i) and (ii). The locus of such tangencies is shown by the dotted arc on the indifference surface at $x_{11} \leq a_{11}$. The other three rays break through the indifference surface in the plane $x_{11}=a_{11}$. They are on θ by virtue of (ii), unless and until they disappear again below the surface at some $x_{11}>a_{11}$, as do two of these. The locus of such points is shown as two dotted curves.

Within the dotted boundary, therefore, θ comprises rays such as those shown (dashed or solid). But outside the boundary θ coincides with ϕ by virtue of (i). It is everywhere continuous, weakly convex, and non-increasing.

For any given $\lambda \in [0,1]$, $k^0<0$ and $k^1>1$, let $\tau(u_1)$ be any continuous function such that:

$$\tau(u_1) \;=\; \lambda \qquad \text{for } u_1 \in [0,1]$$

$$\tau(u_1) \text{ is increasing for } u_1 \in [k^0,0], \text{ and decreasing for } u_1 \in [1,k^1]$$

$$\tau(k^0) \;=\; \tau(k^1) \;=\; 0$$

The transformed preference function may now be defined as:

$$\psi(u_1;x_{11},x_{13}) \;=\; \phi(u_1;x_{11},x_{13}) \qquad \text{for } u_1 \leq k^0$$

$$\;=\; [1-\tau(u_1)]\,\phi(u_1;x_{11},x_{13}) \;+\; \tau(u_1)\,\theta(x_{11},x_{13}) \qquad \text{for } u_1 \in [k^0,1]$$

$$= \max\{\ [1-\tau(u_1)]\,\phi(u_1;x_{11},x_{13}) + \tau(u_1)\,\theta(x_{11},x_{13})\ ,$$
$$[1-\lambda]\,\phi(u_1;x_{11},x_{13}..) + \lambda\,\theta(x_{11},x_{13})\ \} \qquad \text{for } u_1 \in [1,k^1]$$

$$= \max\{\ \phi(u_1;x_{11},x_{13})\ ,$$
$$[1-\lambda]\,\phi(u_1;x_{11},x_{13}..) + \lambda\,\theta(x_{11},x_{13})\ \} \qquad \text{for } u_1 \geq k^1$$

This can be described with the aid of Figure 4. Consider first the indifference surface $\phi(1;x_{11},x_{13})$. Beyond the dotted boundary it coincides with θ, so the transformation leaves it intact here. Within the boundary, it lies above θ for $x_{11}<a_{11}$, and below it for $x_{11}>a_{11}$. So here the transformation depresses the surface at all $x_{11}<a_{11}$, and raises it at all $x_{11}>a_{11}$. In effect, it pushes the surface down towards the vertex (b_{11},a_{12},a_{13}), while pivoting it around the indifference curve $\phi(1;a_{11},x_{13})$, which therefore remains intact. The magnitude of the distortion is given by the value of λ.

Indifference surfaces above and below this are similarly transformed, in a manner which preserves continuity and increasingness with respect to u_1. For each such surface the distortion is unconfined by any equivalent dotted boundaries. However, its magnitude diminishes to zero for $u_1 \leq k^0$ and $u_1 \geq k^1$ except that, for the latter, increasingness requires each indifference surface to be raised where it lies below θ, i.e., below rays such as the upper three in Figure 4.

Like ϕ, the transformed function ψ is continuous, increasing in u_1, and for any given u_1 defines a decreasing and strictly convex indifference surface.[6] It contains an indifference surface $(u_1=1)$ passing through a_1 and, for a sufficiently large value of λ, below b_1. But the transformation leaves intact the represented preferences over each of the following sets of assignments:

$$\iota \ \equiv\ \{\, x_1 \mid x_{12} = \phi(1;x_{11},x_{13}) = \theta(x_{11},x_{13})\,\}$$

$$\kappa^0 \ \equiv\ \{\, x_1 \mid x_{12} \leq \phi(k^0;x_{11},x_{13})\,\}$$

$$\kappa^1 \ \equiv\ \{\, x_1 \mid x_{12} \geq \phi(k^1;x_{11},x_{13}) \ \&\ x_{12} \geq \theta(x_{11},x_{13})\,\}$$

$$\iota \cup \kappa^0 \cup \kappa^1 \qquad \text{and} \qquad \{b_1\} \cup \kappa^0 \cup \kappa^1$$

So far we have assumed arbitrarily given values of k^0, k^1 and δ_1. But let $x_{12} = a_{12} + \beta(a_{13} - x_{13})$ be the hyperplane which supports $\phi(1;a_{11},x_{13})$ at a_1, so that:

$$\forall x_{13}:\ a_{12} + \beta(a_{13} - x_{13}) \ \leq\ \phi(1;a_{11},x_{13})$$

Then:

$$\lim_{\delta_1 \to 0+} \theta(x_{11},x_{13}) \ = \ \max\{\ \phi(1;x_{11},x_{13})\ ,\ a_{12} + \beta(a_{13} - x_{13})\ \}$$

So any assignment $x_1 \neq a_1$ such that $x_{11} \leq a_{11}$ and $x_{12} = \phi(1;x_{11},x_{13})$, can be included in ι by taking a sufficiently small value of $\delta_1 > 0$. In Figure 4, the surface enclosed within the lower dotted arc contracts towards a_1, diminishing to zero, as δ_1 approaches zero.

Any assignment x_1 such that $x_{12} < \phi(1;x_{11},x_{13})$ can be included in κ^0 by taking sufficiently small $\delta_1 > 0$ and sufficiently large $k^0 < 0$. Similarly, any assignment x_1 such that $x_{12} > \phi(1;x_{11},x_{13})$ and $x_{12} \geq \theta(x_{11},x_{13})$, can be included in κ^1 by taking sufficiently small $k^1 > 1$.

5 IIA* and *t*-exclusive allocations

We describe allocation x as *t-exclusive* if $x_i = w/t$ for each i of some t individuals. Thus, a t-exclusive allocation shares all goods equally among t individuals, assigning nothing to the remaining $n-t$ individuals. Polar cases are $t=1$, where one individual is assigned all of the available goods, and $t=n$, where all individuals receive an equal share of all goods.

According to Proposition 1, Pareto and IIA together imply that a best allocation must be 1-exclusive. Using the apparatus provided in section 4, we now demonstrate:

Proposition 4 In the $m \times n$ Edgeworth domain $(m \geq 2)$, given Pareto and IIA*, at any profile ρ a best allocation (if one exists) must be t-exclusive.

Proof Assume to the contrary that at ρ^0 there exists a best allocation a which is not t-exclusive. For each good g, define:

$$\tilde{a}_g = \max_i \{a_{ig}\} \qquad \text{and} \qquad \mathrm{A}_g = \{ i \mid a_{ig} = \tilde{a}_g \}$$

Suppose that A_g is not identical for each good *g*. Then (re-labelling as necessary) there exists some individual 1 and some pair of goods $\{1,2\}$ such that $1 \in \mathrm{A}_1$ and $1 \notin \mathrm{A}_2$. Select any individual $2 \in \mathrm{A}_2$.

Suppose instead that A_g is identical for each good *g*. By assumption *a* is not *t*-exclusive, so (again re-labelling as necessary) there exists some individual 1 and good 1 such that:

$$a_{11} = \max_i \{a_{i1} \mid a_{i1} < \tilde{a}_1 \} > 0$$

Select any other good 2, and any individual $2 \in \mathrm{A}_2$.

In either case, therefore, we can find individuals $\{1,2\}$ and goods $\{1,2\}$ such that:

$$\begin{array}{lll} a_1 \neq a_2 & & \\ a_{11} > 0 & \& & \forall j \neq 1 : \text{either } a_{j1} \leq a_{11} \text{ or } \forall g : x_{jg} > x_{1g} \\ a_{22} > 0 & \& & \forall j \neq 2 : a_{j2} \leq a_{22} \end{array}$$

Define allocations *b* and *c* in terms of $\{\delta_i, \varepsilon_i, \zeta_i\}$ as in Proposition 1 where, given any $\delta_1, \delta_2 > 0$, it is possible to satisfy (4) by taking sufficiently small $\varepsilon_1, \varepsilon_2 > 0$. Reference to Figure 2 confirms the following additional observations, which draw on Proposition 1 and on the analysis in section 4.

Consider individual 1's preferences regarding $\{a_2, b_2, c_2\}$. Suppose that $a_1 \mathrm{P}_1^0 a_2$. For sufficiently small $\delta_1 > 0$ we have $a_1 \mathrm{P}_1^0 b_2$ (for any $\zeta_1 > 0$). Then, for sufficiently small $\delta_1 > 0$, and sufficiently large $k^0 < 0$, we have $\{a_2, b_2, c_2\} \subset \kappa^0$.

Suppose alternatively that $a_2 \mathrm{P}_1^0 a_1$. Since either $a_{21} \leq a_{11}$ or $\forall g : a_{2g} > a_{1g}$, then it follows that $a_{22} \geq \theta(a_{21}, a_{23})$. So for sufficiently small $\delta_2 > 0$ we have both $c_2 \mathrm{P}_1^0 a_1$ and $c_{22} \geq \theta(c_{21}, c_{23})$. Then for sufficiently small $k^1 > 1$ we have $\{a_2, b_2, c_2\} \subset \kappa^1$.

Suppose lastly that $a_1 \mathrm{I}_1^0 a_2$ which implies that $a_{21} \leq a_{11}$. For sufficiently small $\delta_1 > 0$ we have $a_2 \in \iota$. Given also that $a_1 \neq a_2$, then for sufficiently small $\delta_1 > 0$ and $\varepsilon_2 > 0$ we have both $b_2 \mathrm{P}_1^0 a_1$ and $b_{22} \geq \theta(b_{21}, b_{23})$, and thus for sufficiently small $k^1 > 1$ we have $b_2 \in \kappa^1$. Given any $\delta_2 > 0$, for sufficiently small $\varepsilon_1 > 0$ we have $a_1 \mathrm{P}_1^0 c_2$ (for any $\zeta_1 > 0$); thus for sufficiently small $\delta_1 > 0$ and sufficiently large $k^0 < 0$ we have $c_2 \in \kappa^0$.

Now consider individual 1's preferences regarding $\{a_j, b_j, c_j\}$ for any $j \neq 1,2$, in which case $b_j = c_j > a_j$. For any a_j such that $a_1 \mathrm{P}_1^0 a_j$, sufficiently small $\zeta_i > 0$ may be found such that $a_1 \mathrm{P}_1^0 b_j$, and thus for sufficiently large $k^0 < 0$ we have $\{a_j, b_j, c_j\} \subset \kappa^0$. For any a_j such that $a_j \mathrm{P}_1^0 a_1$, since either $a_{j1} \leq a_{11}$ or $\forall g: a_{jg} > a_{1g}$, then it follows that $a_{j2} > \theta(a_{j1}, a_{j3})$, and so for sufficiently small $k^1 > 1$ we have $\{a_2, b_2, c_2\} \subset \kappa^1$. For any a_j such that $a_1 \mathrm{I}_1^0 a_j$, and thus $a_{j1} \leq a_{11}$, a sufficiently small $\delta_1 > 0$ may be found such that $a_j \in \iota$; then for sufficiently small $k^1 > 1$ we have $\{b_j, c_j\} \subset \kappa^1$.

From all of this it follows that $\{\delta_i, \varepsilon_i, \zeta_i\}$ may be found to allow a standard preference ordering R_1^1 such that $b_1 \mathrm{P}_1^1 a_1$, but also such that the orderings over $\{a_1, .., a_n; c_1, .., c_n\}$ and over $\{b_1, .., b_n; c_1, .., c_n\}$ are identical to those in R_1^0. So there exists a profile $\rho^1 = (\mathrm{R}_1^1, \mathrm{R}_2^0, .., \mathrm{R}_n^0)$ at which:

by Pareto:	$b\mathrm{P}^1 a$.
by IIA*:	$a\mathrm{R}^1 c$
and thus:	$b\mathrm{P}^1 c$

The equivalent reasoning can be applied in respect of individual 2's preferences, preserving orderings over $\{a_1, .., a_n; b_1, .., b_n\}$ and over $\{b_1, .., b_n; c_1, .., c_n\}$. In fact this is slightly less demanding, given that $\forall j \neq 2: a_{j2} \leq a_{22}$. So there exists a profile $\rho^2 = (\mathrm{R}_1^0, \mathrm{R}_2^2, .., \mathrm{R}_n^0)$ at which:

by Pareto:	$c\mathrm{P}^2 a$
by IIA*:	$a\mathrm{R}^2 b$
and thus:	$c\mathrm{P}^2 b$

But this is a violation of IIA*. So the initial assumption is false.

QED

Proposition 4 applies to any given profile. But for any pair $\{x,y\}$ of t-exclusive allocations, each individual's preference ordering of $\{x_1,..,x_n;y_1,..,y_n\}$ is invariant across profiles. So IIA* implies that if xRy at any profile then xRy at every profile. From Proposition 4 it then follows that the set of best allocations must be the same at every profile at which it is non-empty.

However, there exist profiles at which no t-exclusive allocation other than $t=1$ is Pareto-efficient. Figure 3 illustrates a 2×2 example.[7] Suppose that at such a profile there is a best allocation. Then it must be 1-exclusive; moreover, so too must be a best allocation at any other profile. Thus (cf. Propositions 2 and 3):

Proposition 5 If $\mathrm{R}=f(\rho)$ identifies a best allocation at every ρ then, given Pareto and IIA*, there is a non-empty set of individuals D such that, at every ρ, x is a best allocation if and only if $x_j=w$ for some $j\in\mathrm{D}$.

Proposition 6 If $\mathrm{R}=f(\rho)$ identifies a best allocation at every ρ, and a uniquely best allocation at some ρ', then, given Pareto and IIA*, there is an individual j such that, at every ρ, x is a best allocation if and only if $x_j=w$.

This appears to be as far as the present form of argument can lead us. As in section 2, the SWF has not been shown to be fully dictatorial. Nevertheless, Propositions 5 and 6 are substantial in themselves. They confirm that even in the restricted Edgeworth Domain, and with the weakened IIA*, it remains impossible to make complete, consistent and equitable social choices on the basis of interpersonally non-comparable individual preferences.

Notes

1. The related literature, however, dates back to Maskin (1976).

2. It would be more conventional to define Pareto and IIA in terms of individual preferences over the relevant allocations *x* and *y*, with a separate stipulation that these correspond to (standard) preferences over own-assignments. But the formulation used here is more appropriate to what follows in sections 3 and 5.

3. This may be elaborated with reference to Figure 2. Given δ_2, individual 1 strictly prefers c_1 to a_1 if and only if $\varepsilon_1 < \gamma_1[(n-2)\zeta_2]$, this function corresponding to 1's indifference curve through a_1. Similarly, given δ_1, individual 2 strictly prefers b_2 to a_2 if and only if $\varepsilon_2 < \gamma_2[(n-2)\zeta_1]$. Given also that $(n-2)\zeta_i < \varepsilon_i$, the required critical values are determined simultaneously as $\hat{\varepsilon}_1 = \gamma_1[\hat{\varepsilon}_2]$ and $\hat{\varepsilon}_2 = \gamma_2[\hat{\varepsilon}_1]$.

4. For a relatively recent review of the literature on envy-freeness, see Arnsperger (1994).

5. The inconsistency between EF and Pareto appears to have gone largely unremarked, although Sugden (1981, section 4.4) discusses a closely related idea. Indeed, EF as such does not explicitly feature in the envy-freeness literature, perhaps reflecting either a tacit awareness of this inconsistency or a disinterest in the SWF framework.

6. For any given u_1, all the steps in the construction of ψ involve taking either the average or the maximum of some pair of continuous, decreasing and convex functions. These properties are preserved under these operations. For decreasingness and convexity, strictness is preserved in the average if in either of the original functions, and in the maximum if in both.

7. An $m \times n$ example is represented by the Cobb-Douglas utility functions $u_i = x_{i1}^{i} x_{i2} .. x_{im}$.

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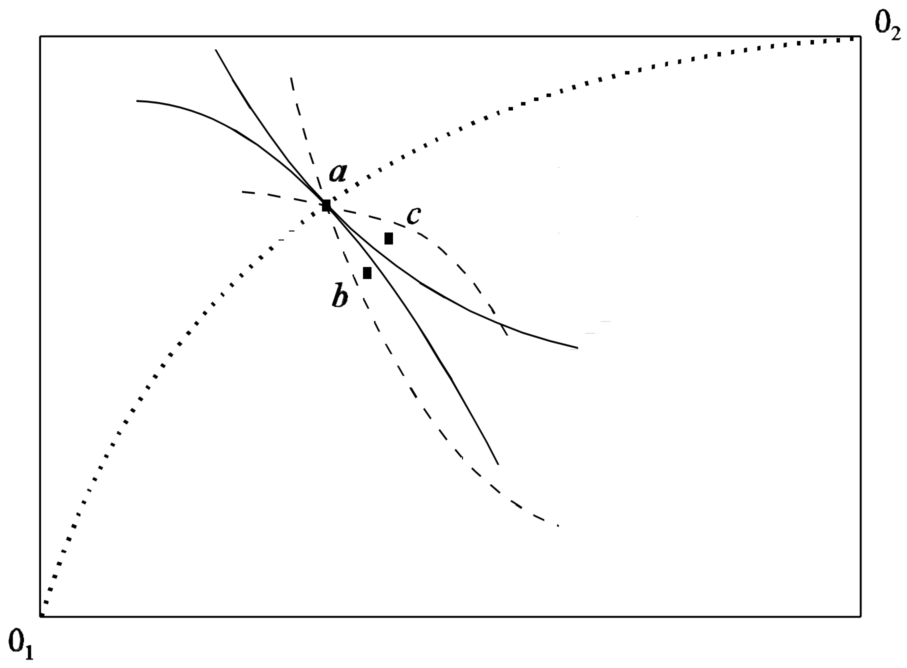

Figure 1: The 2×2 Edgeworth Domain

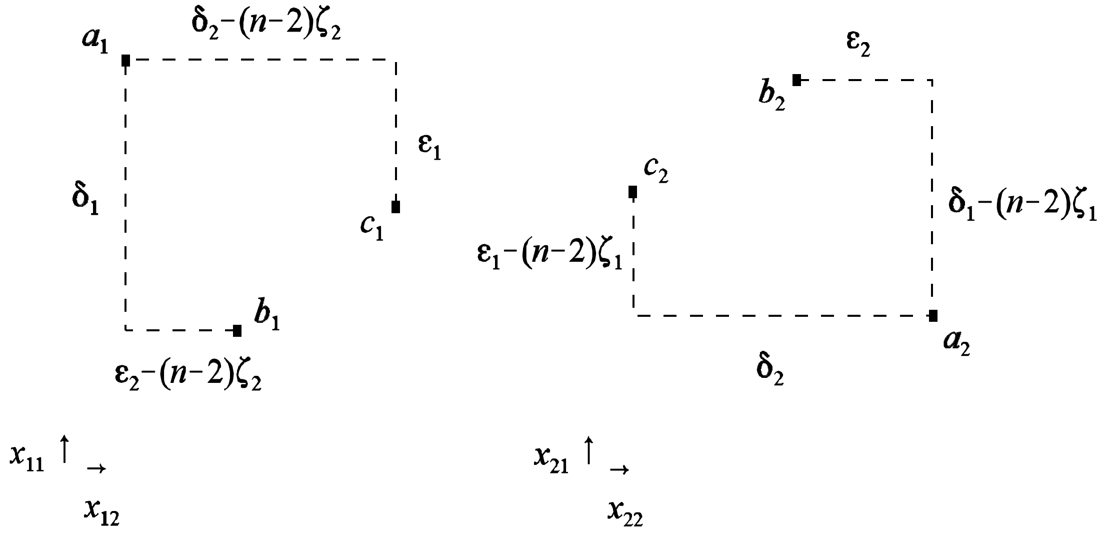

Figure 2: The construction of allocations *b* and *c*

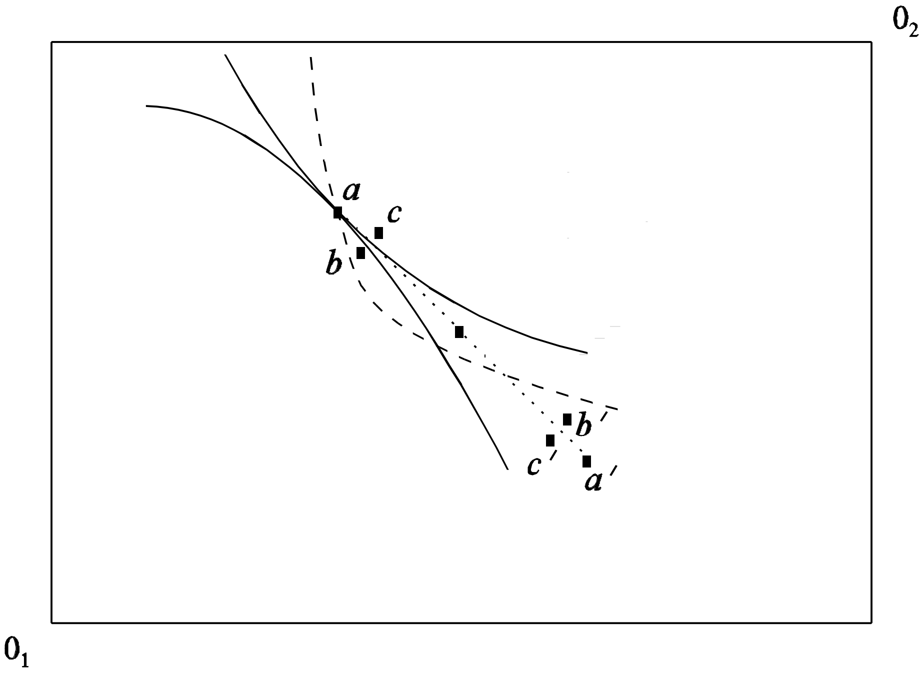

Figure 3: Swap allocations in the Edgeworth Domain

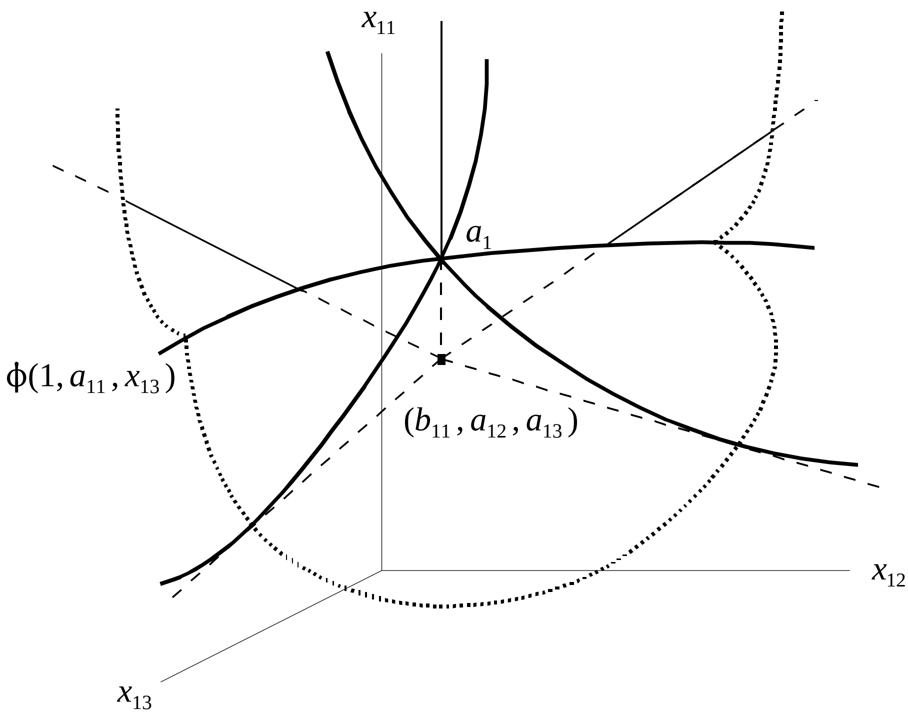

Figure 4: The construction of θ