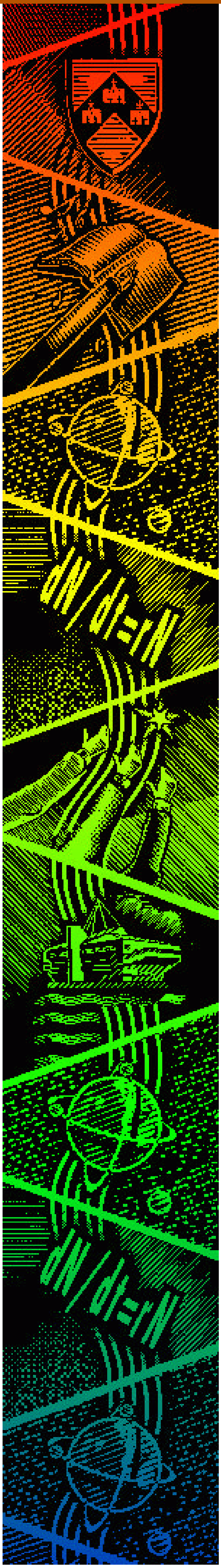

THE UNIVERSITY *of York*

Discussion Papers in Economics

No. 2000/43

Determinacy, Learnability, and Monetary Policy Inertia

by

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This version: January 18, 2001

Abstract. We evaluate Taylor-type monetary policy rules from the perspective of which classes of rules most reliably induce determinacy and learnability of a rational expectations equilibrium. The context is a simple, forward-looking model of the macroeconomy widely used in the rapidly expanding literature in this area. The policy rules we consider have an inertial component, whereby the central bank can respond cautiously to economic events. We document that policy inertia can help alleviate problems of indeterminacy and explosive instability of equilibrium in this model, and that learnability of equilibrium is not impaired by policymaker caution. We conclude that this might be an important reason why central banks in the industrialized economies display considerable inertia when adjusting monetary policy in response to changing economic conditions.

Keywords: Monetary policy rules, determinacy, learning, instrument instability.
JEL Classification E4, E5.

[*]Any views expressed are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis or the Federal Reserve System.

1. Monetary policy advice

1.1. Determinacy. A fundamental issue in the evaluation of alternative monetary policy rules, especially when the structural model has forward-looking elements, is the question of whether a proposed policy rule is associated with a determinate equilibrium or not. Starting with the work of Sargent and Wallace (1975), it has been shown that certain types of policy rules may be associated with very large sets of rational expectations equilibria, and this problem of indeterminacy has been seen as an important reason for excluding certain categories of rules.[1] Perhaps disconcertingly, this problem appears to be particularly acute for policy rules which may otherwise seem to be fairly realistic in terms of *actual* central bank behavior. For example, Clarida, Gali and Gertler (1998) have provided evidence which suggests that monetary policy for the major industrialized countries since 1979 has been *forward-looking:* Nominal interest rates are adjusted in response to *anticipated* inflation. This empirical finding is somewhat puzzling in light of the fact that such forward-looking rules are associated with equilibrium indeterminacy in many models (see, in particular, Bernanke and Woodford (1997)). Similarly, in many models policy rules which call for the monetary authority to respond aggressively to *past* values of endogenous variables (such as the previous quarter's deviations of inflation from a target level, or the output gap) can be associated with explosive instability of rational expectations equilibrium. Yet at the same time, such policy rules might also be thought of as fairly realistic in terms of actual central bank behavior in some contexts. Thus, at least two empirically relevant and seemingly ordinary-looking classes of policy rules seem to be associated with important theoretical problems, problems which might cause one to hesitate before recommending such rules to policymakers.

Christiano and Gust (1999), among others, have stressed the seriousness of these theoretical concerns for the design of stabilization policy. Even aside from broad modeling uncertainty, there is considerable sampling variability about the estimated parameters of a given model of the macroeconomy. When a candidate class of policy rules may or may not generate indeterminacy, or explosive instability, depending on the particular parameter values of the structural model and of the policy rule, it creates something of a minefield for policy design. One might, for instance, recommend a particular rule on the basis

[1] Some of the authors that discuss this issue most recently include Bernanke and Woodford (1997), Carlstrom and Fuerst (1999, 2000), Christiano and Gust (1999), Clarida, Gali and Gertler (2000), McCallum and Nelson (1999), Rotemberg and Woodford (1998, 1999), and Woodford (1999*a*).

that it would generate a determinate rational expectations equilibrium (REE), and that the targeted equilibrium would have desirable properties based on other criteria, such as utility of the representative household in the model. And yet, in reality, important parameters may lie (because of sampling variability alone) in a region associated with indeterminacy of equilibrium, or with explosive instability. Actually implementing the proposed rule could then lead to disastrous consequences. Thus, from the perspective of the design of stabilization policy, one would greatly prefer to recommend policy rules such that, even if the structural parameters actually take on values somewhat different from those that might be estimated, a determinate rational expectations equilibrium is produced.

1.2. Learnability. Even when a determinate equilibrium exists, however, the achievement of that equilibrium cannot be assured if agents do not possess rational expectations (RE) at every point in time. The notion that the REE should be robust to expectational errors is potentially important from the applied viewpoint, since such errors can naturally arise in practice. For example, the economy might be subject to changes in its basic structure or in the practices and rules of policymakers, and the assumption that agents somehow have RE immediately after such changes is clearly strong and indeed may not be correct empirically. Instead, it seems realistic to assume that agents must form expectations concerning economic events using the actual data produced by the economy. In general terms, the learning approach admits the possibility that expectations might not initially be fully rational, and that, if economic agents make forecast errors and try to correct them over time, the economy may or may not reach the REE asymptotically. Thus, beyond showing that a particular policy rule reliably induces a determinate REE, one needs to show the potential for agents to learn that equilibrium. In this paper, we assume the agents of the model do not initially have rational expectations, and that they instead form forecasts by using recursive learning algorithms—such as recursive least squares—based on the data produced by the economy itself. Our methodology is that of Evans and Honkapohja (1999, 2001). We ask whether the agents in such a world can learn the equilibria of the system induced by several classes of possible Taylor-type monetary policy feedback rules. We use the criterion of *expectational stability* (a.k.a. *E-stability*) to calculate whether rational expectations equilibria are stable under real time recursive learning dynamics or not. The research of Marcet and Sargent (1989*a*, 1989*b*) and Evans and Honkapohja (1999, 2001) has shown that the expectational stability of rational expectations equilibrium governs local convergence of real time recursive learning algorithms in a wide variety of macroeconomic models.[2]

We proposed learnability as a necessary additional criterion for evaluating alternative monetary policy feedback rules in Bullard and Mitra (2000). We suggested there that economists should only advocate policy rules which induce learnable REE. A monetary authority that adopts a particular monetary policy rule that is not associated with a learnable REE, simply assuming that private sector agents will coordinate on the equilibrium they are targeting, would encounter an unexpected surprise. Our analysis suggests that the private sector agents attempting to learn the equilibrium might in fact cause the macroeconomic system to diverge away from the targeted equilibrium. Learnable equilibria, on the other hand, do not have such problems, because in this case the learning dynamics tend toward, and eventually coincide with, the rational expectations dynamics. We therefore recommend learnable equilibria.

1.3. Monetary policy inertia. In this paper, we consider the effects of monetary policy inertia—the use of policy rules in which the authorities can move cautiously in response to unfolding events—on the determinacy and learnability of equilibrium. Inertia is one of the well-documented features of central bank behavior in industrialized countries: Policymakers show a clear tendency to smooth out changes in nominal interest rates in response to changes in economic conditions. Rudebusch (1995) has provided one statistical analysis of this fact. More casually, actual policy moves are discussed among central bankers and in the business press in industrialized countries as occurring as sequences of adjustments in nominal interest rates in the same direction. This is so much the case, in fact, that policy inertia has been the source of criticism of the efforts of central bankers, as suggestions are sometimes made that policymakers have been unwilling to move far enough or fast enough to respond effectively to incoming information about the economy.

Our purpose in this paper is to provide an analysis of the effects of monetary policy inertia on equilibrium determinacy and learnability in the context of a standard, small, forward-looking model which is currently the workhorse for the study of monetary policy rules.

[2] In this paper, we will use the terms "learnability," "expectational stability," "E-stability," and "stability in the learning dynamics" interchangably.

1.4. Model environment. We close our structural model using a variety of linear monetary policy feedback rules made famous by the seminal work of Taylor (1993, 1999a, 1999b). In each case, the central bank is viewed as adjusting a short-term nominal interest rate in response to deviations of inflation from some target level, to deviations of real output from some natural or long-run level, and, in order to capture interest rate smoothing, we also include a response to the deviation of the lagged interest rate from a long run value consistent with the steady state of the model. Importantly, we require that all of our rules are operational in the sense of McCallum (1999), who argues that central banks cannot react to *contemporaneous data* on output and inflation deviations because such information is not available to policymakers at the time decisions must be made. Accordingly, our classes of rules are as follows: (1) Policymakers react to lagged values of inflation deviations from target, the output gap, and interest rate deviations (the *lagged data* specification); (2) Policymakers respond to their expectations of *current quarter* values of inflation deviations and the output gap, in addition to lagged interest rate deviations (the *contemporaneous expectations* specification); and finally, (3) Policymakers react to future forecasts of inflation deviations, the output gap, as well as lagged interest rate deviations (the *forward-looking* specification).[3] We stress that all of these possibilities are operational in McCallum's sense.

1.5. Main results. We find that by placing a sufficiently large weight on lagged interest rate deviations in each of these classes of policy rules, the policy authorities can mitigate the threats of indeterminacy or explosive instability, and that this is one of the primary benefits of monetary policy inertia. We also argue that policy inertia does not hinder the learnability of rational expectations equilibrium. A key aspect of our argument is that we work in the context of forward-looking models motivated by microfoundations—the expectations of the private sector enter the model explicitly and do have an important impact on the results. Combining our results on determinacy and learnability with the Christiano-Gust caution leads us to recommend inertial policy rules as the most promising from the perspective of both generating determinacy and learnability of a rational expectations equilibrium.

[3] Bullard and Mitra (2000) studied the learnability of *simple* monetary policy rules, that is, of policy rules which only responded to inflation and output deviations, but not to lagged interest rate deviations. Thus, this paper complements our earlier analysis by evaluating the learnability of *generalized* policy rules, that is, those with inertia included.

As in Bullard and Mitra (2000), we find that *active,* Taylor-type rules (i.e., rules where the interest rate responds to inflation with a coefficient larger than one) can induce learnable equilibria in the systems we analyze. We also comment on superinertial policy rules—rules in which the reaction of the policy authorities to the lagged interest rate is described by a coefficient greater than one. Policy rules in this class have been explored by Rotemberg and Woodford (1999) and Woodford (1999*a*), among others, and have been found to induce equilibria which possess desirable qualities in terms of the long-run volatility of output and inflation. Rotemberg and Woodford (1999, p. 51) remark, "... our conclusion [is] that short-term interest rates should depend ... also upon their own past values—ideally, with a coefficient greater than one .... It is interesting to note that ... [such superinertial rules] ... do not lead to instrument instability." By *instrument instability*, Rotemberg and Woodford meant the non-existence of a stationary rational expectations equilibrium. We are able to show that operational, superinertial rules can induce *determinate* equilibria not only when these rules respond to lagged data (as examined by Rotemberg and Woodford (1999)) but also when these rules react to current or future forecasts of inflation and output deviations. We believe that this is an important finding. We also find that superinertial rules suffer no particular problem with respect to expectational stability. Thus there appears to be no "instrument instability" problem from the perspective of the learning dynamics with rules in this class.

1.6. Recent related literature. One could interpret our findings as a theory of why monetary policy inertia is observed in industrialized economies. In particular, our results suggest why other, non-inertial types of policies might leave the economy vulnerable to unexpected dynamics, and hence why central banks might willingly adopt inertial behavior. Recently, several very different theories have been proposed as to why policy inertia might be observed, for instance Woodford (1999a), Caplin and Leahy (1996), and Sack (1998). Our results are probably best viewed as complementary to these theories.

Evans and Honkapohja (2000) analyze learning in a model like the one analyzed in this paper, but where the monetary policy rule is derived from an optimal control exercise and is, consequently, non-inertial. They show that if the optimal control policy of the central bank assumes rational expectations (RE) on the part of private agents, then the rational expectations equilibrium (REE) is invariably rendered unstable when agents follow standard adaptive learning rules. On the other hand, if the optimal control policy is conditioned directly on the observed (subjective) private expectations, then the REE becomes stable under learning dynamics. In this paper, the forward-looking or the contemporaneous expectations Taylor-type rules are also based directly on the observed (subjective) future forecasts of inflation and output deviations of the private sector or their contemporaneous forecasts, respectively. We too find that the Taylor-type rules analyzed in this paper can induce learnability of equilibrium, and this is especially so when there is a sufficient degree of policy inertia. This shows the importance of appropriately conditioning monetary policy directly on the observed expectations of the private sector both in the context of optimal policies and Taylor-type rules.

With regard to recent empirical literature concerning policy rules, our results are comforting since actual interest rates are often modeled by a reaction rule where the *change* in the funds rate responds to deviations of inflation and output from their typical values (for an example in the U.S. case see Fuhrer and Moore (1995)). This means that the coefficient on the lagged interest rate in the policy rule is one. The same type of policy rules are also found to have desirable properties in terms of low output and inflation volatility across four different structural macroeconometric models of the U.S. economy in the study of Levin, Wieland, and Williams (1999). They report that (p. 264), "Our analysis provides strong support for rules in which the first difference of the federal funds rate responds to the current output gap and the deviation of the one-year inflation rate from a specified target."

1.7. Organization. In the next section we present the model we will analyze throughout the paper. We also discuss the types of linear policy feedback rules we will use to organize our analysis, and a calibrated case which we will employ. In the subsequent sections, we present conditions for determinacy of equilibrium for each of the different classes of policy rules. We then turn to the question of learnability of rational expectations equilibrium under our various specifications. We conclude with a summary of our findings.

2. Environment

2.1. A baseline model. We study a simple and small forward-looking macroeconomic model. This model has been developed by Woodford (1999*a*), based on some earlier work in a more elaborate, optimizing framework by Rotemberg and Woodford (1998, 1999). The model is intended to be a parsimonious description of the U.S. economy, with mechanisms that would remain prominent in nearly any model with complete microfoundations. We write Woodford's (1999*a*, p. 16) system as

$$x_t = \hat{E}_t x_{t+1} - \sigma \left(r_t - r_t^n - \hat{E}_t \pi_{t+1} \right) \quad (1)$$

$$\pi_t = \kappa x_t + \beta \hat{E}_t \pi_{t+1} \quad (2)$$

where x_t is the output gap, π_t is the period t inflation rate defined as the percentage change in the price level from $t-1$ to t, and r_t is the nominal interest rate; each variable is expressed as a deviation from its long run level. We normalize the targeted values of inflation, the output gap, and the interest rate to be zero since the values of these constants do not affect our analysis of determinacy and learnability. Since we will focus on learning we use the notation $\hat{E}_t \pi_{t+1}$ and $\hat{E}_t x_{t+1}$ to denote the possibly nonrational private sector expectations of inflation and output gap next period, respectively, whereas the same notation without the hat symbol will denote rational expectations (RE) values. In the nomenclature of the literature, equation (1) is sometimes called the intertemporal IS equation whereas equation (2) is sometimes called the aggregate supply equation or the new Phillips curve.[4] Equation (1) can be derived from log-linearizing the Euler equation associated with a representative household's saving decision. Equation (2) can be derived from optimal pricing decisions of monopolistically competitive firms facing constraints on the frequency of future price changes. The parameters σ, κ, and β are structural and are assumed to be positive on economic grounds. In particular, $\beta \in (0, 1)$ is the discount factor of the representative household and $\beta^{-1} - 1$ is the steady state real rate of interest for the economy. The interest elasticity of output, σ, corresponds to the intertemporal elasticity of substitution of consumption in the representative household's utility function. The parameter κ depends on the average frequency of price changes and the elasticity of demand faced by suppliers of goods. Prices are more nearly flexible the higher is κ. The "natural rate of interest" r_t^n is an exogenous stochastic term that follows the process

$$r_t^n = \rho r_{t-1}^n + \epsilon_t \quad (3)$$

where ϵ_t is *iid* noise with variance σ_ϵ^2, and $0 \leq \rho < 1$ is a serial correlation parameter.

[4] Woodford's model does not have any "cost push" shock as, for example, in Clarida, Gali and Gertler (1999). We could add such shocks to (2) as well as shocks to the monetary authorities' reaction function that are auto-regressive of order one without affecting the results in the paper.

2.2. Alternative specifications for setting interest rates. We close the system by supplementing equations (1), (2), and (3), which represent the behavior of the private sector, with a policy rule, which represents the behavior of the monetary authority. For our purposes, the policy instrument for the monetary authority is the nominal interest rate deviation. Our intention is to produce results across a variety of linear policy feedback rules that have been analyzed in the literature. Once we isolate the characteristics of rules that reliably produce both determinacy and learnability, then one could go about finding an optimal or best-performing rule from among the ones in this set.

Taylor (1993) popularized the use of interest rate rules which reacted to information on output and inflation observed at time t. Taylor's (1993) original motivation for considering such rules was in part that the policymaker respond in a simple and transparent way to available data. McCallum (1993, 1997, 1999) has often argued that such reaction functions are unrealistic, since actual policymakers do not have complete information on variables such as output and inflation in the quarter they must make a decision. We take McCallum's criticism seriously, and so we do not consider the contemporaneous data specification. Instead, we use several operational policy rules as described below.

One reaction to McCallum's criticism is to posit that the monetary authorities must react to last quarter's observations on inflation and the output gap, which could possibly be viewed as closer to the reality of central bank practice. This leads to our *lagged data* specification for our interest rate equation,

$$r_t = \varphi_\pi \pi_{t-1} + \varphi_x x_{t-1} + \varphi_r r_{t-1}. \tag{4}$$

Our complete system for the case of lagged data is, therefore, given by (1), (2), (4), and (3).

Another way of coping with McCallum's criticism is to assume that the authorities have to set their interest rate instrument in response to their current *expectations of* output gap and inflation, formed using information on output gap and inflation *last* period. This may also be viewed as close to the actual practice of central banks. Thus we consider versions of our systems where the policy feedback rule is

$$r_t = \varphi_\pi \hat{E}_t \pi_t + \varphi_x \hat{E}_t x_t + \varphi_r r_{t-1}. \tag{5}$$

We refer to this system as our *contemporaneous expectations* model and the complete system is given by (1), (2), (5), and (3). Note that in this formulation we may assume that $\hat{E}_t\pi_t$, $\hat{E}_t x_t$ denote the expectations of the central bank. Under learning, we will assume that both the private sector and the bank use identical learning algorithms to form their forecasts. This would be a reasonable first assumption and puts both the private sector and the central bank in a symmetric position. Alternatively, following Bernanke and Woodford (1997), it may be that the central bank simply targets the predictions of private sector forecasters so that $\hat{E}_t\pi_t$, $\hat{E}_t x_t$ will denote the expectations of the private sector. In the latter interpretation it will only be the private sector which is learning.

A final method of coping with McCallum's criticism is to assume that the authorities set their interest rate instrument in response to their *forecasts* of output gap and inflation, so that the policy rule itself is forward-looking. Some of the authors discussing forward-looking rules include Batini and Haldane (1999) and Bernanke and Woodford (1997). In fact, forward-looking rules have been found to describe the behavior of monetary policy, for instance, in Germany, Japan, and the US since 1979 as described in Clarida, Gali, and Gertler (1998). We consider simple versions of such forward-looking policy rules, ones in which the monetary authority looks just one quarter ahead when setting its interest rate instrument. This yields a specification, which we call the *forward expectations* model, in which the interest rate equation is

$$r_t = \varphi_\pi \hat{E}_t\pi_{t+1} + \varphi_x \hat{E}_t x_{t+1} + \varphi_r r_{t-1}. \tag{6}$$

We can again interpret the above equation in two ways. It may be that both policymakers and private agents have homogeneous expectations of the future, and in the analysis of learning we impute identical learning algorithms to both. Alternately, it may be that the central bank simply targets the predictions of private sector forecasters. The complete system for the case of *forward expectations* model is given by (1), (2), (6), and (3).

2.3. Methodology. Our determinacy analysis follows conventional practice. For the analysis of the model under learning, we assume the agents in the model do not have rational expectations at the outset, and instead, we replace rationally expected values with versions of least squares learning rules. Thus, the agents form expectations using the data actually generated by the economy. We think of the agents as using versions of recursive least squares learning rules. We use the results of Evans and Honkapohja (1999, 2001) and calculate the conditions for expectational stability (E-stability). Expectational stability is a notional time concept, but Evans and Honkapohja (1999, 2001) have shown that it

Table 1. Parameter configurations.

Parameter	Controls	Value or range
σ^{-1}	Intertemporal substitution	.157
κ	Price stickiness	.024
β	Household's discount factor	.99
φ_π	Coefficient on inflation	$0 \leq \varphi_\pi \leq 4$
φ_r	Coefficient on lagged interest rate	$0, .65, 5$
ϕ_x	Coefficient on output gap	$0 \leq \phi_x \leq 2.5$
ρ	Serial correlation of shock	.35

Table 1: Parameter configurations. We illustrate our analytical findings using these parameter values from Woodford (1999a).

is very closely associated with stability under real time adaptive learning. In particular, under quite general conditions, when E-stability holds recursive least squares learning is locally convergent to the rational expectations equilibrium. Evans and Honkapohja (1999, 2001) have also shown that, under the assumption that the fundamental disturbances have bounded support, if a rational expectations equilibrium is not E-stable, then the probability of convergence of the recursive least squares algorithm to the rational expectations equilibrium is zero. We define E-stability precisely later in the paper.

2.4. Parameters. Woodford (1999*a*) calibrated the parameters σ and κ of a similar model based on econometric estimates from Rotemberg and Woodford (1998, 1999). We use these values throughout the paper to illustrate our analytical findings. The value of β, which corresponds to the representative household's discount factor in the more general model, is set to .99 throughout, also following Woodford (1999*a*). Rotemberg and Woodford (1999) argue that the coefficients in the equations describing their economy are not dependent on parameters in the monetary authority's policy rule. This is their response to the Lucas critique. We follow their analysis and accordingly we study a number of possible policy rules. Calibrations of these rules correspond to values for the parameters φ_π, φ_r, and φ_x. Table 1 summarizes our calibration scenarios.

We organize our analysis as follows. We essentially consider three cases corresponding to the three different information structures (lagged data, contemporaneous expectations, and forward expectations) for the policy authority. For later reference, we call rules with $\varphi_\pi > 1$ *activist rules* and those with $\varphi_\pi \leq 1$ *passive rules*. We also call rules with $\varphi_r > 1$ *super-inertial rules*. In the next section, we consider the determinacy of rational expectations equilibrium, and then we follow that with a section analyzing the learnability of equilibrium. We maintain the following assumptions throughout the paper: $\varphi_\pi \geq 0$ and $\varphi_x \geq 0$, with at least one strictly positive, $\varphi_r > 0$, $\kappa > 0$, $\sigma > 0$, and $0 < \beta < 1$.

3. Inertia and determinacy

3.1. Lagged data in the policy rule. We start by considering the system with Taylor-type monetary policy feedback rules in which policymakers react to lagged values of inflation, output and interest rate deviations. *Simple* (no lagged interest rate) lagged data rules can easily lead to (locally) explosive situations—a phenomenon sometimes described as *instrument-induced instability.*[5] Indeed, Bullard and Mitra (2000) note that a sufficiently aggressive response to inflation and output deviations invariably leads to explosiveness, rendering quantitatively important portions of the parameter space dynamically unstable, as shown in their Figure 2, or similarly in Figure 2.15 of Rotemberg and Woodford (1999). We now show that this problem need not arise if the central bank displays sufficient inertia in setting its interest rate.

In this case, our policy rule is given by equation (4), so that the complete system becomes equations (1), (2), (4), and (3). We can iterate the equation (4) one time period forward and rewrite our system of three equations as

$$\begin{bmatrix} 1 & 0 & \sigma \\ -\kappa & 1 & 0 \\ \varphi_x & \varphi_\pi & \varphi_r \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ r_t \end{bmatrix} = \begin{bmatrix} 1 & \sigma & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ r_{t+1} \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix} r_t^n. \tag{7}$$

The matrix which is relevant for uniqueness is obtained by multiplying the inverse of the 3×3 left hand matrix with the right hand side matrix associated with the expectations variables. Since r_t is pre-determined, we need exactly two of the eigenvalues of this matrix to be inside the unit circle for determinacy. However, it is easier to work with the *inverse* of this relevant matrix given by

$$B_1 = \begin{bmatrix} 1 + \kappa\beta^{-1}\sigma & -\beta^{-1}\sigma & \sigma \\ -\kappa\beta^{-1} & \beta^{-1} & 0 \\ \varphi_x & \varphi_\pi & \varphi_r \end{bmatrix}. \tag{8}$$

[5] See, for example, McCallum (1999). We ask the reader to keep in mind that instability in this sense is distinct from the possibility of instability in the learning dynamics which we will discuss later in the paper.

We need exactly one eigenvalue of B_1 to be inside the unit circle for uniqueness. It is easy to show that with small values of φ_π and φ_x, a value of $\varphi_r > 1$ guarantees uniqueness of equilibria, whereas $\varphi_r < 1$ causes indeterminacy.[6] More generally, we can provide some intuition for the finding in Rotemberg and Woodford (1999) that large values of φ_r lead to a unique equilibrium with the following proposition.

Proposition 1. *A set of sufficient conditions for unique equilibria are*

$$\kappa(\varphi_\pi + \varphi_r - 1) + (1-\beta)\varphi_x > 0, \tag{9}$$

and

$$\varphi_r > Max\left\{1 - \kappa\sigma, \sigma(\varphi_x + \kappa\varphi_\pi)\right\}. \tag{10}$$

Proof. See Appendix *A*. ■

In particular, a sufficiently aggressive response to the lagged interest rate guarantees uniqueness. For example, values of $\varphi_r \geq 1$ combined with small values of κ, such as the one employed by Rotemberg and Woodford (1999), help to satisfy this condition, and create a relatively large region of determinate equilibria. We can also explain the finding in McCallum and Nelson (1999, pp. 34-35) that rules with large values of φ_π or φ_x still deliver dynamically stable results, so long as there is a sufficient level of policy inertia. Their first explanation for this surprising finding can be understood from our condition (10). One of the sufficient conditions for uniqueness is for $\varphi_r > \sigma(\varphi_x + \kappa\varphi_\pi)$, so consequently, this depends on the structural parameters σ and κ. Relatively small values of σ (that is, the slope of the intertemporal IS function with respect to the interest rate) and κ (the slope of the price adjustment equation) means that this condition is likely to be easily satisfied. The intuition of McCallum and Nelson (1999) is, therefore, verified here to this extent: Small values of these two parameters, which are crucial for the transmission of policy actions to inflation, reduce the possibility of explosive instrument instability. On the other hand, we also see (from conditions (9) and (10)) that for *any* given values of structural parameters, if policy is sufficiently inertial then the model *always* delivers dynamically stable results.[7] This is in striking contrast to the case when there

[6] If $\varphi_\pi = \varphi_x = 0$, then one of the eigenvalues of B_1 is φ_r, whereas of the other two positive eigenvalues, one is less than 1 and the other is greater than 1. By continuity, the same is true for values of φ_π and φ_x in a neighborhood of zero.

[7] We can also give some intuition for the phenomenon McCallum and Nelson describe in their footnote 32 (p. 36). If φ_π is made very large without correspondingly increasing φ_r or if σ is "increased sharply"

is no inertia ($\varphi_r = 0$), because, as we have stressed, it is relatively easy to get explosive situations in that case. Inertial behavior on the part of the central bank appears, from this perspective, to be quite desirable—instead of being the cause of instrument instability, it actually works toward defusing this tendency.

3.2. Contemporaneous expectations in the policy rule. Under contemporaneous expectations, the system is given by equations (1), (2), (5), and (3). We first define the vector of endogenous variables, $y_t = (x_t, \pi_t, r_t)$, and put our system in the following form

$$y_t = \beta_0 E_t y_t + \beta_1 E_t y_{t+1} + \delta y_{t-1} + \varkappa r_t^n, \tag{11}$$

where

$$\beta_0 = \begin{bmatrix} -\sigma\varphi_x & -\sigma\varphi_\pi & 0 \\ -\kappa\sigma\varphi_x & -\kappa\sigma\varphi_\pi & 0 \\ \varphi_x & \varphi_\pi & 0 \end{bmatrix}, \tag{12}$$

$$\beta_1 = \begin{bmatrix} 1 & \sigma & 0 \\ \kappa & \beta + \kappa\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{13}$$

and

$$\delta = \begin{bmatrix} 0 & 0 & -\sigma\varphi_r \\ 0 & 0 & -\kappa\sigma\varphi_r \\ 0 & 0 & \varphi_r \end{bmatrix}. \tag{14}$$

Note that it may perhaps be more natural to assume, as is done in Bullard and Mitra (2000), that expectations of both the private sector and the bank are formed at time $t-1$ in equation (11). However, the conditions for determinacy and (later on) learnability of equilibrium for the formulation given by equation (11) can be shown to be the same as for the case with $t-1$ dating of expectations. We maintain this since most of the related papers on monetary policy assume t dating of expectations.

It is easiest to obtain the conditions for determinacy in this case by using the Blanchard-Kahn technique, as exposited in Evans and Honkapohja (2001, ch. 10).[8] For the free variables we again have $x_t^1 = y_t$, and for the pre-determined variables, $x_t^2 = \{r_t^n, r_{t-1}\}$. We start with the formulation

$$x_t^1 = B_0 E_t x_t^1 + B_1 E_t x_{t+1}^1 + C x_t^2, \tag{15}$$

(even with moderate values of φ_π) then the sufficiency condition is likely to be violated. As an aside we note that with contemporaneous data in the policy rule, the sufficiency conditions do not involve the structural parameters, and hence one would not expect instability to arise in these cases.

[8]The slight difference is the changed dating of expectations which, as mentioned before, is not important for our purposes.

$$x_t^2 = Rx_{t-1}^1 + Sx_{t-1}^2 + \textit{white noise terms.} \tag{16}$$

It is easy to check that $B_0 = \beta_0$, $B_1 = \beta_1$,

$$C = \begin{bmatrix} \sigma & -\varphi_r\sigma \\ \kappa\sigma & -\kappa\varphi_r\sigma \\ 0 & \varphi_r \end{bmatrix}, \tag{17}$$

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

and

$$S = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix}. \tag{19}$$

The matrix J corresponding to Evans and Honkapohja (2001, ch. 10) is defined by

$$J = \begin{bmatrix} I - B_0 & -C \\ R & S \end{bmatrix}^{-1} \begin{bmatrix} B_1 & 0 \\ 0 & I \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} 1 & \sigma & 0 & \rho^{-1}\sigma & -\sigma \\ \kappa & \beta + \kappa\sigma & 0 & \kappa\rho^{-1}\sigma & -\kappa\sigma \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \rho^{-1} & 0 \\ -\frac{\varphi_x + \kappa\varphi_\pi}{\varphi_r} & -\frac{\varphi_x + (\kappa + \beta\sigma^{-1})\varphi_\pi}{\sigma^{-1}\varphi_r} & 0 & -\frac{\varphi_x + \kappa\varphi_\pi}{\rho\sigma^{-1}\varphi_r} & \frac{\sigma^{-1} + \varphi_x + \kappa\varphi_\pi}{\sigma^{-1}\varphi_r} \end{bmatrix}.$$

For a unique equilibrium we need three eigenvalues of J to be inside the unit circle. It is easy to see, from the structure of J, that two of its eigenvalues are 0 and ρ^{-1} and that the remaining three eigenvalues of J are those of the matrix

$$A = \begin{bmatrix} 1 & \sigma & -\sigma \\ \kappa & \beta + \kappa\sigma & -\kappa\sigma \\ \frac{-(\varphi_x + \kappa\varphi_\pi)}{\varphi_r} & \frac{-[\varphi_x + (\kappa + \beta\sigma^{-1})\varphi_\pi]\sigma}{\varphi_r} & \frac{[\sigma^{-1} + \varphi_x + \kappa\varphi_\pi]\sigma}{\varphi_r} \end{bmatrix}. \tag{21}$$

Consequently, for uniqueness under contemporaneous expectations, we need exactly two eigenvalues of A to be inside the unit circle. As it turns out, it is easier to prove our results by working with the *inverse* of this matrix which is given by

$$A^{-1} = \begin{bmatrix} 1 + \kappa\sigma\beta^{-1} + \sigma\varphi_x & \sigma(\varphi_\pi - \beta^{-1}) & \sigma\varphi_r \\ -\beta^{-1}\kappa & \beta^{-1} & 0 \\ \varphi_x & \varphi_\pi & \varphi_r \end{bmatrix}. \tag{22}$$

We need exactly one eigenvalue of A^{-1} to be inside the unit circle for determinacy.[9]

[9]We note here that the same condition would determine determinacy if the central bank used contemporaneous *data* in its policy rule as in Taylor (1993). In this sense, we can make the assumption of contemporaneous data in the policy rule operational by assuming that the central bank responds to contemporanous forecasts when setting the nominal interest rate, and that, from the point of view of determinacy, the conditions would be unchanged.

It is again easy to show that with small values of φ_π and φ_x, $\varphi_r > 1$ guarantees the uniqueness of equilibrium whereas $\varphi_r < 1$ causes indeterminacy. More generally, we can show that sufficiently inertial policy is enough to guarantee the existence of a unique stationary rational expectations equilibrium.

Proposition 2. *Under contemporaneous expectations policy rules, $\varphi_r \geq 1$ guarantees uniqueness of equilibrium.*

Proof. See Appendix *B*. ■

3.3. Forward expectations in the policy rule. The final information structure we consider is to assume that policymakers respond to forecasts of inflation and output deviations. This type of policy rule seems to describe well the monetary policy rules adopted by major industrialized countries. Clarida, Gali and Gertler (1998) present evidence which suggests that these central banks have in fact been forward looking since 1979: they respond to anticipated inflation instead of lagged inflation. However, as we have stressed, Bernanke and Woodford (1997) have argued that such rules can easily lead to problems of indeterminacy. Similarly, Bullard and Mitra (2000) verified that a sufficiently aggressive response to inflation and output deviations is associated with indeterminacy in this case. However, in this section we show that this problem can be circumvented by assuming a sufficiently aggressive response to the lagged interest rate on the part of the central bank. Consequently, our results suggest that central banks employing forward-looking policy rules need not encounter indeterminacy so long as their behavior displays sufficient interest rate smoothing.

With forward expectations the complete system is given by equations (1), (2), (6), and (3). We again have two free endogenous variables (x_t, π_t) and one predetermined variable, r_{t-1}, and our system is

$$\begin{bmatrix} 1 & 0 & 0 \\ -\kappa & 1 & 0 \\ 0 & 0 & \varphi_r \end{bmatrix} \begin{bmatrix} x_t \\ \pi_t \\ r_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & \sigma & -\sigma \\ 0 & \beta & 0 \\ -\varphi_x & -\varphi_\pi & 1 \end{bmatrix} \begin{bmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ r_t \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix} r_t^n. \tag{23}$$

This time the matrix that determines uniqueness is given by

$$B = \begin{bmatrix} 1 & \sigma & -\sigma \\ \kappa & \beta + \kappa\sigma & -\kappa\sigma \\ -\varphi_x \varphi_r^{-1} & -\varphi_\pi \varphi_r^{-1} & \varphi_r^{-1} \end{bmatrix} \tag{24}$$

and we need exactly two eigenvalues of B to be inside the unit circle for determinacy. We first note that with small values of φ_π and φ_x, $\varphi_r > 1$ guarantees uniqueness of equilibrium whereas $\varphi_r < 1$ causes indeterminacy. This is the same condition that governs uniqueness (or indeterminacy) of equilibrium in the case of contemporaneous expectations and lagged data policy rules. We are, however, able to give more general conditions for determinacy of equilibrium.

Proposition 3. *A set of sufficient conditions for unique equilibria are*

$$\varphi_r \geq 1, \tag{25}$$

$$[\kappa + 2\sigma^{-1}(1+\beta)](1+\varphi_r) > \kappa\varphi_\pi + (1+\beta)\varphi_x. \tag{26}$$

Proof. See Appendix *C.* ■

For given values of φ_π and φ_x, a large enough value of φ_r invariably leads to uniqueness. In particular, for all φ_π and φ_x, any value of φ_r such that

$$\varphi_r > Max\{1, [\kappa + 2\sigma^{-1}(1+\beta)]^{-1}[\kappa\varphi_\pi + (1+\beta)\varphi_x] - 1\}$$

always leads to a unique equilibrium. In particular, as long as the response to inflation and output deviations are not overly aggressive, any value of $\varphi_r \geq 1$ suffices for uniqueness. We can also have unique equilibria with passive rules as long as the central bank reacts aggressively to the lagged interest rate. As mentioned before, in earlier work it has been observed that forward-looking rules easily lead to indeterminacy. In particular, an aggressive response to inflation and/or output invariably causes this problem. For baseline values of the structural parameters such as those suggested by Woodford (1999a) this renders a quantitatively important part of the parameter space in (φ_π, φ_x) indeterminate (see Figure 3 in Bullard and Mitra (2000)). However, inertial behavior on the part of the monetary authority defuses this tendency.

3.4. Summary of the results on determinacy. We have argued that a sufficient degree of monetary policy inertia will always be associated with determinacy of rational expectations equilibrium in our systems. This is true across three types of operational, Taylor-type policy feedback rules—ones where the policy authorities respond to lagged data, to contemporaneous expectations, or to forward expectations of macroeconomic conditions. In each case, a sufficiently high degree of policy inertia renders equilibrium determinate, while the same cannot be said for the response to inflation or the response to output in the policy rule. For those parameters, a response which is too aggressive can, in some situations, lead to explosive instability or to indeterminacy.

We think the tendency of policy inertia to help generate determinacy may be an important reason why so much inertia is observed in the actual monetary policies of industrialized countries. However, too much policy inertia may cause another type of instability—that of the learning dynamics. We now turn to this topic.

4. Inertia and learnability

4.1. Lagged data in the policy rule. We now consider learning, beginning with the case in which the policy authority responds to lagged data. In this case, the complete system is given by equations (1), (2), (4), and (3). We analyze the expectational stability of the unique stationary minimum state variable (MSV) solution (see McCallum (1983)). For the analysis of learning, we need to compute the MSV solution and for this we need to obtain a relationship between the current endogenous variables (and their lags) and future expectations. This relationship is now obtained by first defining the vector of endogenous variables, $y_t = (x_t, \pi_t, r_t)$, and by putting our system in the form $y_t = \beta_1 \hat{E}_t y_{t+1} + \delta y_{t-1} + \varkappa r_t^n$ where β_1 and δ are given by

$$\beta_1 = \begin{bmatrix} 1 & \sigma & 0 \\ \kappa & \beta + \kappa\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

and

$$\delta = \begin{bmatrix} -\sigma\varphi_x & -\sigma\varphi_\pi & -\sigma\varphi_r \\ -\kappa\sigma\varphi_x & -\kappa\sigma\varphi_\pi & -\kappa\sigma\varphi_r \\ \varphi_x & \varphi_\pi & \varphi_r \end{bmatrix}. \tag{28}$$

The MSV solutions for this model take the form

$$y_t = \bar{a} + \bar{b} y_{t-1} + \bar{c} r_t^n \tag{29}$$

and the MSV solutions for $\bar{a}, \bar{b}$ and $\bar{c}$ are given by

$$\bar{a} = 0, \tag{30}$$

$$\bar{b} = (I - \beta_1 \bar{b})^{-1} \delta, \tag{31}$$

and

$$\bar{c} = (I - \beta_1 \bar{b})^{-1} (\varkappa + \rho \beta_1 \bar{c}). \tag{32}$$

Because equation (31) is a matrix quadratic, there are potentially multiple solutions for $\bar{b}$. The determinate case corresponds to the situation when there is a unique solution for $\bar{b}$ with all its eigenvalues inside the unit circle. The condition for this is that one eigenvalue of (8) be inside the unit circle as shown in Section 3.1. For the analysis of learning, we assume that agents have a *perceived law of motion* (PLM) of the form

$$y_t = a + by_{t-1} + cr_t^n \tag{33}$$

corresponding to the MSV solution. We then compute the following expectation (assuming that the time t information set does not include y_t)

$$\begin{aligned} \hat{E}_t y_{t+1} &= a + b\hat{E}_t y_t + c\rho r_t^n \\ &= a + b(a + by_{t-1} + cr_t^n) + c\rho r_t^n \\ &= (I+b)a + b^2 y_{t-1} + (bc + c\rho) r_t^n. \end{aligned}$$

Inserting the above computed expectations into the actual model one obtains the following actual law of motion (ALM) of y_t as

$$y_t = (\beta_1 + \beta_1 b)a + (\beta_1 b^2 + \delta)y_{t-1} + (\beta_1 bc + \beta_1 c\rho + \varkappa)r_t^n. \tag{34}$$

The mapping from the PLM to the ALM takes the form

$$T(a,b,c) = ((\beta_1 + \beta_1 b)a,\ \beta_1 b^2 + \delta,\ \beta_1 bc + \beta_1 c\rho + \varkappa). \tag{35}$$

Expectational stability is then determined by the matrix differential equation

$$\frac{d}{d\tau}(a,b,c) = T(a,b,c) - (a,b,c). \tag{36}$$

The fixed points of equation (36) give us the MSV solution $(\bar{a}, \bar{b}, \bar{c})$. We say that a particular MSV solution $(\bar{a}, \bar{b}, \bar{c})$ is expectationally stable if equation (36) is locally asymptotically stable at that point. Our system is in a form where we can apply the results of Evans and Honkapohja (2001, ch. 10). We assume that the private sector only has access to information on the previous period's values of output, inflation and interest rate in forming its forecasts. We believe this assumption to be realistic since contemporaneous values of these variables are rarely available in practice.[10] It can then be shown that for E-stability of any MSV solution with t-dating of expectations (and assuming that the time t

[10] We assume that the agents use information on the contemporaneous natural interest rate, r_t^n, in forming their forecasts; however, note that the results on E-stability are unaffected even if we assume that they only observe the last quarter's natural interest rate in forming their forecasts.

FIGURE 1. Lagged Data

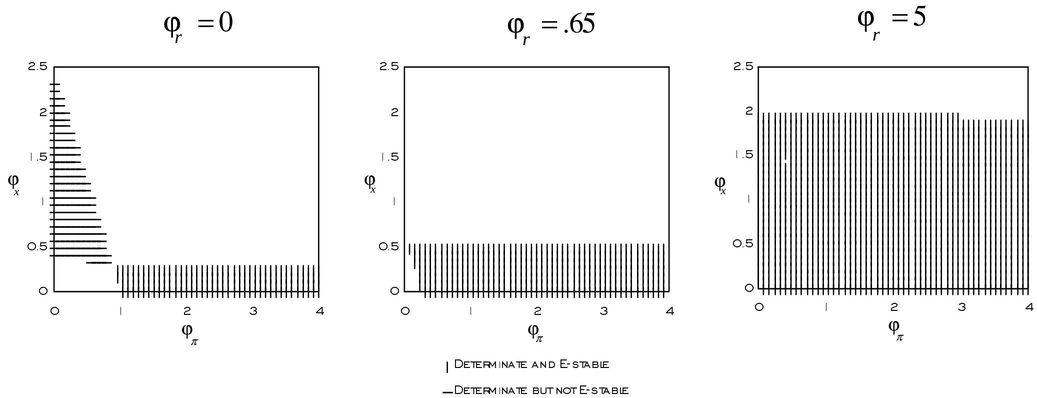

Figure 1: With $\varphi_r = 0$, the region of the parameter space associated with both determinate and learnable rational exepectations equilibria involves relatively small values for φ_x, and generally $\varphi_\pi > 1$. In the blank region, determinacy does not hold. When $\varphi_r = .65$, which is close to empirical estimates in the literature, the region of the parameter space associated with determinacy and learnability expands, relative to the no inertia case. For a large value of φ_r, such as $\varphi_r = 5$ as shown here, much of the pictured (φ_π, φ_x) space is associated with both determinacy and learnability.

information set is $(1, y'_{t-1}, r^n_t)')$ we require the eigenvalues of the following three matrices:

$$\bar{b}' \otimes \beta_1 + I \otimes (\beta_1 \bar{b}), \tag{37}$$

$$\rho\beta_1 + \beta_1 \bar{b}, \tag{38}$$

$$\beta_1 + \beta_1 \bar{b} \tag{39}$$

to have real parts less than one. If any eigenvalue of the above matrices has a real part larger than one, then the MSV solution is not E-stable, and hence it cannot be learned by boundedly rational agents using recursive least squares in their estimation exercise.

We illustrate regions of determinacy and E-stability for the case when the policy authorities react to lagged data in Figure 1. In this figure, we have employed the baseline parameter values described in Table 1. Figure 1 contains three panels, the first of which corresponds to the case where there is no policy inertia, so that $\varphi_r = 0$. This case was analyzed in Bullard and Mitra (2000). The figure is drawn in (φ_π, φ_x) space, holding all other parameters at their baseline values. Vertical lines in the figure denote parameter combinations that generate determinacy, and that also generate local stability in the learning dynamics. Horizontal lines, on the other hand, indicate parameter combinations that generate determinacy, but where the unique equilibrium is unstable in the learning dynamics. In this and all figures, the blank region is not associated with determinacy. The $\varphi_r = 0$ portion of this figure illustrates that determinacy does not always imply learnability. It also illustrates that active Taylor-type rules with little or no reaction to other variables (either the output gap or the lagged interest rate) tend to be associated with both determinacy and learnability. However, one judgement concerning this panel might be that of Christiano and Gust (1999), since parameter values within an empirically relevant range are sometimes associated with equilibria which are not determinate, or which are determinate but not learnable.

The second panel of Figure 1 illustrates how the situation in is improved when the degree of monetary policy inertia is increased from zero to $\varphi_r = .65$. This value is close to estimates of the degree of policy inertia based on U.S. postwar data. In this case, the region of the (φ_π, φ_x) space associated with both determinacy and learnability of equilibrium has been enlarged. The region associated with determinate, but unlearnable, rational expectations equilibria has been eliminated. This effect becomes even more pronounced in the third panel, where a very large value of φ_r is employed, specifically, $\varphi_r = 5$. In this case, a much larger portion of the space is determinate and learnable. Thus, we see that larger degrees of policy inertia enhance the prospects for determinacy considerably, relative to the case where there is no policy inertia at all. In addition, learnability does not appear to be jeopardized by large degrees of policy inertia, as the determinate equilibria are also learnable, even when φ_r is large.

These themes also carry over to other specifications for the policy rule, as we now show.

4.2. Contemporaneous expectations in the policy rule. When policymakers employ a rule in which they react to their expectations of current economic conditions, the complete model is given by equations (1), (2), (5), and (3) which can be rewritten in the form of equation (11) as shown in Section 3.2. The MSV solutions then take the form

$$y_t = \bar{a} + \bar{b}y_{t-1} + \bar{c}r_t^n$$

with $\bar{a} = 0$,

$$\bar{b} = (I - \beta_0 - \beta_1 \bar{b})^{-1} \delta, \tag{40}$$

and

$$\bar{c} = \beta_0 \bar{c} + \beta_1 (\bar{b}\bar{c} + \bar{c}\rho) + \kappa. \tag{41}$$

In general $\bar{b}$ will have multiple solutions because equation (40) is a matrix quadratic. In the case when there exists a unique stationary solution, however, only one solution for $\bar{b}$ will have all its eigenvalues less than one in absolute value. The condition for a unique stationary solution is that one eigenvalue of (22) be less than one in modulus, as shown in Section 3.2.

For the analysis of learning, we assume that agents have a PLM of the form

$$y_t = a + by_{t-1} + cr_t^n \tag{42}$$

We then compute the expectation

$$\begin{aligned} \hat{E}_t y_{t+1} &= a + b\hat{E}_t y_t + cr_t^n \\ &= a + b(a + by_{t-1} + cr_t^n) + cr_t^n \\ &= (I + b)a + b^2 y_{t-1} + (bc + c\rho) r_t^n. \end{aligned}$$

Note that if we interpret this model as one where the central bank is also learning, then we are effectively imputing identical learning algorithms to both the bank and the private sector. This seems to be reasonable first approximation. Inserting these computed expectations into the actual model one obtains the following actual law of motion (ALM) of y_t as

$$y_t = (\beta_0 + \beta_1 + \beta_1 b)a + (\beta_0 b + \beta_1 b^2 + \delta) y_{t-1} + (\beta_0 c + \beta_1 bc + \beta_1 c\rho + \varkappa) r_t^n. \tag{43}$$

Thus the mapping from the PLM to the ALM takes the form

$$T(a, b, c) = ((\beta_0 + \beta_1 + \beta_1 b)a,\ \beta_0 b + \beta_1 b^2 + \delta,\ \beta_0 c + \beta_1 bc + \beta_1 c\rho + \varkappa). \tag{44}$$

Finally, expectational stability is determined by the following matrix differential equation

$$\frac{d}{d\tau}(a, b, c) = T(a, b, c) - (a, b, c). \tag{45}$$

The fixed points of equation (45) give us the REE solution. We say that a particular REE $(\bar{a}, \bar{b}, \bar{c})$ is E-stable if equation (45) is locally asymptotically stable at the point. Our

FIGURE 2. Contemporaneous Expectations

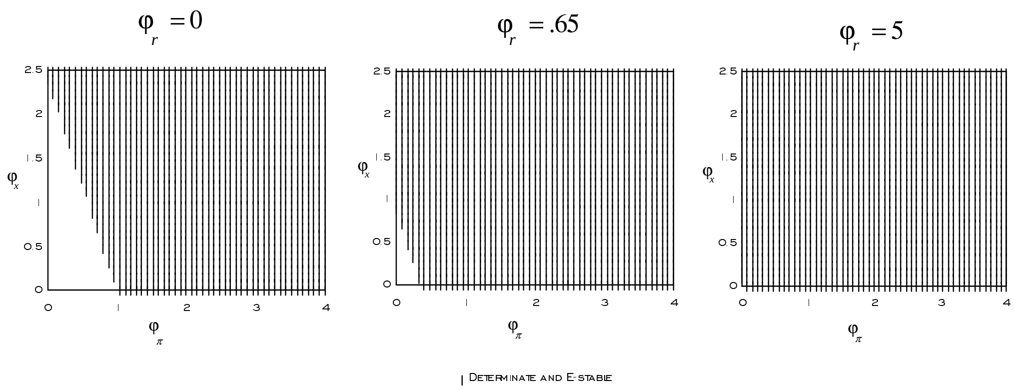

Figure 2: Under contemporaneous expectations Taylor-type rules, much of the pictured (φ_x, φ_π) space is associated with determinate and learnable rational expectations equilibria. With $\varphi_r = .65$, more of the pictured (φ_π, φ_x) space is determinate and learnable, relative to the case with $\varphi_r = 0$. With large values of φ_r, the entire pictured (φ_π, φ_x) space is associated with equilibria which are both determinate and learnable.

system is again in a form where we can apply the results of Evans and Honkapohja (2001, ch. 10). We again assume that the private sector only has access to information on the previous period's values of output, inflation and the interest rate in forming its forecasts. It can then be shown that for E-stability of any MSV solution (assuming that the time t information set for both the private sector and the central bank is $(1, y'_{t-1}, r^n_t)')$, we require the eigenvalues of the three matrices

$$\bar{b}' \otimes \beta_1 + I \otimes (\beta_0 + \beta_1 \bar{b}), \tag{46}$$

$$\rho\beta_1 + \beta_0 + \beta_1 \bar{b}, \tag{47}$$

and

$$\beta_0 + \beta_1 + \beta_1 \bar{b} \tag{48}$$

to have real parts less than 1. Otherwise, the solution is not E-stable.

Figure 2 illustrates determinacy and learnability in the contemporaneous expectations case using the baseline parameter values presented in Table 1. The first panel of the figure shows that a large portion of the (φ_π, φ_x) space is associated with both determinacy and learnability even when policy is noninertial, that is, with $\varphi_r = 0$ (again this case was analyzed in Bullard and Mitra (2000)). Nevertheless, the Christiano and Gust (1999) warning would still apply here.

The second and third panels show how the prospects for determinacy and learnability are enhanced by increased monetary policy inertia, first for an empirically relevant value of $\varphi_r = .65$, and then for a much larger, superinertial value of $\varphi_r = 5$. In the latter case, the entire pictured region becomes associated with both determinacy and learnability.

4.3. Forward expectations in the policy rule. With forward expectations the complete system is given by equations (1), (2), (6), and (3). We analyze the E-stability of the unique stationary MSV solution, under the assumption that both the private sector and the monetary policymakers have homogenous expectations, and that they are learning using identical versions of recursive least squares. For the analysis of learning, we need to find the MSV solution. Before doing so, we need to obtain a relationship between the current endogenous variables and their lags and future expectations. This relationship is now obtained by first defining the vector of endogenous variables, $y_t = (x_t, \pi_t, r_t)$, and by putting our system in the form

$$y_t = \beta_1 \hat{E}_t y_{t+1} + \delta y_{t-1} + \varkappa r_t^n, \tag{49}$$

where β_1 and δ are now given by

$$\beta_1 = \begin{bmatrix} \sigma(\sigma^{-1} - \varphi_x) & \sigma(1 - \varphi_\pi) & 0 \\ \kappa\sigma(\sigma^{-1} - \varphi_x) & \sigma(\kappa + \beta\sigma^{-1} - \kappa\varphi_\pi) & 0 \\ \varphi_x & \varphi_\pi & 0 \end{bmatrix}, \tag{50}$$

and

$$\delta = \begin{bmatrix} 0 & 0 & -\sigma\varphi_r \\ 0 & 0 & -\kappa\sigma\varphi_r \\ 0 & 0 & \varphi_r \end{bmatrix}. \tag{51}$$

As in the case of lagged data, the MSV solutions take the form (29) and the MSV solutions for $\bar{a}, \bar{b}$ and $\bar{c}$ are given by (30), (31) and (32). The determinate case corresponds to the situation when there is a unique solution for $\bar{b}$ with all its eigenvalues inside the unit circle. The condition for this is given in Section 3.3. For the analysis of learning, we assume that agents have a PLM of the form of equation (33) corresponding to the MSV solution. We then compute the required expectations as in the case of lagged data (assuming that the time t information set does not include y_t) and arrive at the actual law of motion (34). Note that again if we interpret this model as one where the central bank is also learning,

FIGURE 3. Forward Expectations

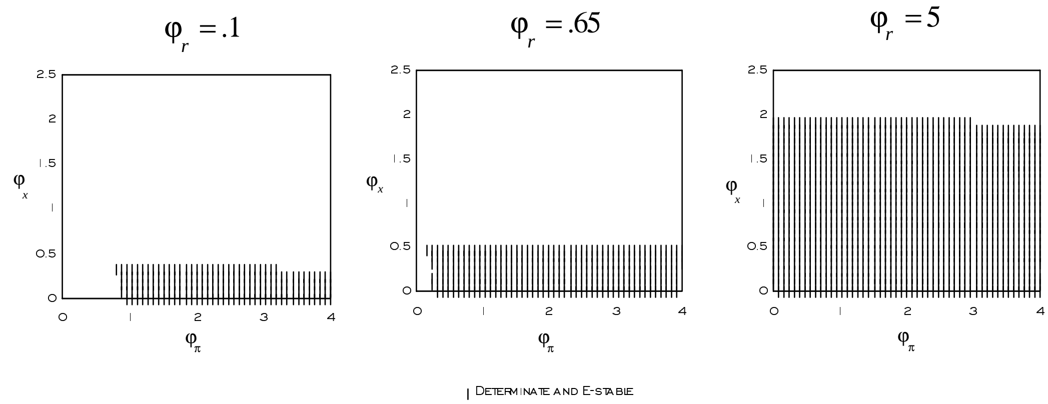

Figure 3: For small values of φ_r, forward-looking policy rules generate determinacy and learnability provided $\varphi_\pi > 1$ and φ_x is sufficiently small. For $\varphi_r = .65$, a larger region of the (φ_π, φ_x) space pictured is associated with both determinacy and learnability. Large values of φ_r generate relatively large regions of determinacy and learnability in (φ_π, φ_x) space.

then we effectively assume that both the policy authorities and the private sector use identical learning algorithms. The corresponding T map in this case is still given by equation (35). The fixed points of equation (36) give us the MSV solution $\left(\bar{a}, \bar{b}, \bar{c}\right)$. We say that a particular MSV solution $\left(\bar{a}, \bar{b}, \bar{c}\right)$ is expectationally stable if equation (36) is locally asymptotically stable at that point. For E-stability of any MSV solution with t dating of expectations (and assuming that the time t information set is $(1, y'_{t-1}, r^n_t)'$) we require the eigenvalues of the matrices (37), (38) and (39) have real parts less than one. If any eigenvalue of the above matrices has a real part larger than one, then the MSV solution is not E-stable.

Figure 3 illustrates how, even for this case where the policymakers are reacting to expectations of future inflation deviations and output gaps, policy inertia tends to enhance the prospects for determinacy and learnability of a rational expectations equilibrium. For very low values of φ_r, such as the value $\varphi_r = 0.1$ in the first panel, we again find that active Taylor-type rules with little or no reaction to other variables are associated with both determinacy and learnability of equilibrium (a similar figure obtains for $\varphi_r = 0$; see Figure 3 of Bullard and Mitra (2000)). However, the large region in the figure which is not associated with determinacy might be enough to limit recommendations of such rules via arguments such as those of Christiano and Gust (1999). However, the second and third panels of Figure 3 show that increased policy inertia can mitigate such concerns, creating a larger region of determinacy, and in addition, that in these cases determinate equilibria are also learnable.

4.4. Summary of the results under learning. In Figure 1, we illustrated a situation where a region of the parameter space that generated determinacy of rational expectations equilibrium failed to generate learnability. Significantly, that region was associated with a passive Taylor-type rule ($\varphi_\pi < 1$) as well as no inertial element of monetary policy. An active Taylor-type rule was found to often be associated with expectational stability in Bullard and Mitra (2000). Increasing the degree of monetary policy inertia appears to also be associated with learnability of rational expectations equilibrium in our setting. Even very large values of the policy inertia parameter do not disturb the stability of the learning dynamics according to these calculations.

We stress that the learnability criterion we employ is a minimal one in the following sense. We are giving the agents in the model the correct specification of the vector autoregression they need to estimate recursively in order to learn the equilibrium of the system in which they operate. We are also giving the agents initial conditions in the neighborhood of this equilibrium. Our view is that, if the agents cannot learn the equilibrium under such favorable circumstances, then the equilibrium seems to us to be a suspect for general instability in actual economies.

Evans and Honkapohja (2000) had found in the context of a similar model that the stability of optimal monetary policies is guaranteed if policy is based directly on the observed expectations of private agents. We too find that it is desirable to base the Taylor-type rules directly on these expectations. We note here that Hall and Mankiw (1994) had similarly emphasized the importance of targeting the predictions of private sector forecasters on the part of the central bank in the context of targeting a given growth rate of nominal GDP.[11]

Before concluding this section, we also remark that the sensitivity of aggregate de-

[11] There is a question of how to implement such a proposal. In fact, there are several ways of doing so. One may view the commercial forecasts published by various agencies as being the expectations of the private sector. In the U.S., for instance, there is a published consensus of respected private forecasters (Blue Chip Economic Indicators). See Romer and Romer (2000) for an extensive discussion of this issue.

mand to changes in interest rates, as measured by σ, is important for determinacy and learnability, at least for the class of rules based on lagged data and forward expectations. This is obvious from the conditions in Propositions 1 and 3 which show that relatively small values of σ promote determinacy of equilibrium. Typically, the values used for this parameter are much smaller than the one used in Woodford (1999a). For example, Clarida, Gali, and Gertler (2000) use a value of $\sigma = 1$ whereas McCallum and Nelson (1999) estimate a still smaller value of σ for the U.S. economy. If we plot the figures for rules based on lagged data and forward expectations with the parameter values used in Clarida, Gali, and Gertler (2000), for instance, then a much larger portion of the parameter space can be shown to be determinate and stable under learning dynamics.

5. Conclusions

One of the key issues when evaluating monetary policy rules since the work of Sargent and Wallace (1975) is whether they lead to a determinate outcome or not. *Simple* interest rate rules have been evaluated under this criterion in Bullard and Mitra (2000). The current paper can be taken to be a further step in evaluating *generalized* policy rules—rules in which the nominal interest rate instrument is also adjusted in response to the lagged interest rate. We provide analytical results which indicate how increased degrees of interest rate smoothing can contribute to a determinate equilibrium outcome across all of our specifications of monetary policy rules—a finding which we believe substantially alters their evaluation. For example, one of the main weaknesses of backward-looking policy rules (that is, rules which respond to lagged data) and of forward-looking rules is believed to be the fact that they easily lead to explosive instability or indeterminacy, respectively—a tendency which we found to be defused by interest rate smoothing. Consequently, both these types of policy rules—which are considered particularly realistic in terms of actual central bank behavior—should not be deemed undesirable on account of their determinacy properties, once policy inertia is taken into account.

Thus, a central message of this paper is that interest rate smoothing can substitute for the aggressive response of the interest rate to inflation in pushing the economy towards to an unique equilibrium. The intuition behind this phenomenon is provided in Rotemberg and Woodford (1999) and Woodford (1999a, 1999b). While it is true that for arbitrary paths of endogenous variables like inflation and output, "explosive" monetary policy rules would cause explosive paths of endogenous variables, in equilibrium the only paths that are possible are the ones which do not cause these variables to be non-stationary. A commitment to raise interest rates later, after inflation increases, is sufficient to cause an immediate contraction of aggregate demand in response to a shock that is expected to give rise to inflationary pressures. The reasons for this are two fold: firstly, aggregate demand is affected by expectations of *future* interest rates (or equivalently *long* rates) and not simply by current short rates and, secondly, the private sector is forward looking (has rational expectations). Persistent changes in the short term rates permit a large effect on long term rates and, therefore, a large effect of monetary policy on aggregate demand, pushing the economy back towards the equilibrium.

However, if we abstract from rational expectations on the part of private sector agents, then there could be a problem with such rules. For example, if agents try to learn the underlying model parameters through some version of adaptive learning, then they may not be able to coordinate on the unique equilibrium. Somewhat surprisingly, this type of instrument instability does not seem to plague even the superinertial class of policy rules we analyzed, regardless of the nature of the information set that policymakers employ. We think this is an interesting finding worthy of further analysis.

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A. Proof of Proposition 1

The characteristic polynomial of B_1, $p(\lambda)$, is given by

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \tag{52}$$

where

$$a_1 = -(1 + \beta^{-1} + \kappa\beta^{-1}\sigma + \varphi_r), \tag{53}$$

$$a_2 = \beta^{-1} + (1 + \beta^{-1} + \kappa\beta^{-1}\sigma)\varphi_r - \sigma\varphi_x, \tag{54}$$

$$a_3 = \frac{\kappa\varphi_\pi + \varphi_x - \sigma^{-1}\varphi_r}{\beta\sigma^{-1}}. \tag{55}$$

Note that $p(0) = a_3$ and

$$p(1) = \frac{\kappa(\varphi_\pi + \varphi_r - 1) + (1-\beta)\varphi_x}{\beta\sigma^{-1}} \tag{56}$$

$$p(-1) = \frac{\kappa(\varphi_\pi - \varphi_r - 1) + (1+\beta)\varphi_x - 2\sigma^{-1}(1+\beta)(1+\varphi_r)}{\beta\sigma^{-1}} \tag{57}$$

Condition (10) implies that $\varphi_r > \sigma\varphi_x$ which in turn implies that

$$\begin{aligned} a_2 &> \beta^{-1} + (1+\beta^{-1}+\kappa\beta^{-1}\sigma)\sigma\varphi_x - \sigma\varphi_x \\ &= \beta^{-1} + (\beta^{-1}+\kappa\beta^{-1}\sigma)\sigma\varphi_x > 0. \end{aligned} \tag{58}$$

Since $a_2 > 0$ and $a_3 < 0$ (by condition (10)), Descartes' Rule of signs implies that there are either three positive roots or one positive root and a pair of complex conjugates. Since $p(0) < 0$ and $p(1) > 0$ (by 9), there is a positive root, say λ_1, such that $0 < \lambda_1 < 1$. We will now use the theorem of *Fourier and Budan* to show that λ_1 is the only root between 0 and 1 when $\varphi_r > 1 - \kappa\sigma$ (see Barbeau (1989 p. 173)). First using (52) observe that

$$p'(\lambda) = 3\lambda^2 + 2a_1\lambda + a_2, \tag{59}$$

$$p''(\lambda) = 6\lambda + 2a_1, \tag{60}$$

and $p'''(\lambda) = 6$. Consequently, the signs of $p(0)$, $p'(0)$, $p''(0)$, $p'''(0)$ are respectively $-, +, -, +$ so that the number of sign changes is 3. The signs of $p(1)$ and $p'''(1)$ are both $+$. On the other hand, we have

$$\begin{aligned} p''(1) &= 2[3 - (1+\beta^{-1}+\kappa\beta^{-1}\sigma + \varphi_r)] \\ &= 2[2 - \beta^{-1} - \kappa\beta^{-1}\sigma - \varphi_r] \\ &< 2[2 - \beta^{-1}(1+\kappa\sigma) - (1-\kappa\sigma)] \\ &= 2[(1+\kappa\sigma)(1-\beta^{-1})] < 0. \end{aligned} \tag{61}$$

where we have used condition (10), namely $\varphi_r > 1 - \kappa\sigma$, in the above inequality. Since $p''(1) < 0$, irrespective of the sign of $p'(1)$, the number of sign changes in the sequence $p(1)$, $p'(1)$, $p''(1)$, $p'''(1)$ is always 2. By using the theorem of *Fourier and Budan,* the number of roots between 0 and 1 cannot be greater than 1. On the other hand, we already know that λ_1 is in this interval so that it is the only root in $(0, 1)$. If all the roots are real, then it immediately follows that the remaining roots must be greater than 1 ensuring uniqueness. If a pair of the roots are complex conjugates, we will use the fact that $\sum_{i=1}^{3} \lambda_i = Trace(B_1) = 1 + \beta^{-1} + \kappa\beta^{-1}\sigma + \varphi_r$ to prove our result. Let λ^r denote the real part of the complex eigenvalues. Since $0 < \lambda_1 < 1$ and $\varphi_r > 1 - \kappa\sigma$ we have

$$\begin{aligned} \lambda_2 + \lambda_3 &= 2\lambda^r > \beta^{-1} + \kappa\beta^{-1}\sigma + \varphi_r > 1 + \kappa\beta^{-1}\sigma + 1 - \kappa\sigma = \\ & 2 + \kappa\sigma(\beta^{-1} - 1) \end{aligned} \tag{62}$$

This means that $\lambda^r > 1$ so that the complex eigenvalues must be outside the unit circle ensuring uniqueness.

B. Proof of Proposition 2

The characteristic polynomial of A^{-1}, (22), is given by

$$p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \tag{63}$$

where

$$a_0 = -\beta^{-1}\varphi_r, \tag{64}$$

$$a_1 = \beta^{-1}\sigma[(\kappa + \sigma^{-1} + \beta\sigma^{-1})\varphi_r + \sigma^{-1} + \kappa\varphi_\pi + \varphi_x], \tag{65}$$

$$a_2 = -\beta^{-1}\sigma[\kappa + \sigma^{-1} + \beta\sigma^{-1} + \beta\sigma^{-1}\varphi_r + \beta\varphi_x], \tag{66}$$

By Descartes' Rule of signs there are either three positive roots or one positive root and a pair of complex conjugates (by observing the sign changes in the coefficients of $p(\lambda)$ and $p(-\lambda)$). Also $p(0) = a_0$ is negative while

$$p(1) = \frac{\kappa(\varphi_\pi + \varphi_r - 1) + (1-\beta)\varphi_x}{\beta\sigma^{-1}} \tag{67}$$

is positive when $\varphi_r \geq 1$. This means that there is always a positive root, say λ_1, such that $0 < \lambda_1 < 1$ (by using the continuity of $p(\lambda)$ in λ). The sum of the eigenvalues is given by

$$\sum_{i=1}^{3} \lambda_i = Trace(A^{-1}) = 1 + \beta^{-1} + \kappa\beta^{-1}\sigma + \varphi_r + \sigma\varphi_x. \tag{68}$$

On the other hand, the product of the eigenvalues is given by φ_r/β (the determinant of A^{-1}) which is more than 1 for all $\varphi_r > \beta$. Consequently, if all the roots are real, there exists at least one positive root, say λ_2, which exceeds 1 when $\varphi_r > \beta$. But this automatically means that the third positive root also exceeds 1 since $p(1) > 0$, $p(\lambda_2 + \varepsilon) < 0$ for small positive ε while $p(\infty) = \infty$.

On the other hand, if a pair of eigenvalues are complex conjugates (denoted by λ_2 and λ_3), then by (68), we have

$$\lambda_2 + \lambda_3 > \beta^{-1} + \kappa\beta^{-1}\sigma + \varphi_r + \sigma\varphi_x > 2 + \kappa\beta^{-1}\sigma + \sigma\varphi_x \tag{69}$$

where the first inequality uses the fact that $0 < \lambda_1 < 1$ and the second inequality uses the assumption that $\varphi_r \geq 1$. The real parts of the (complex) eigenvalues are, therefore, more than 1 so that they are outside the unit circle.

C. PROOF OF PROPOSITION 3

The characteristic polynomial of B, $p(\lambda)$, is given by

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \tag{70}$$

where

$$a_1 = -(1 + \beta + \kappa\sigma + \varphi_r^{-1}), \tag{71}$$

$$a_2 = \beta + (1+\beta)\varphi_r^{-1} - \sigma\varphi_r^{-1}\{\kappa(\varphi_\pi - 1) + \varphi_x\}, \tag{72}$$

$$a_3 = \beta(\varphi_x - \sigma^{-1})\sigma\varphi_r^{-1}. \tag{73}$$

We observe that $p(0) = a_3$ and

$$p(1) = -\sigma\varphi_r^{-1}[\kappa(\varphi_\pi + \varphi_r - 1) + (1-\beta)\varphi_x], \tag{74}$$

$$\begin{aligned} p(-1) &= \sigma\varphi_r^{-1}[\kappa(\varphi_\pi - \varphi_r - 1) + (1+\beta)\varphi_x - 2\sigma^{-1}(1+\beta)(1+\varphi_r)] \\ &= \sigma\varphi_r^{-1}[\kappa\varphi_\pi + (1+\beta)\varphi_x - \{\kappa + 2\sigma^{-1}(1+\beta)\}(1+\varphi_r)]. \end{aligned} \tag{75}$$

We consider three subcases in turn.

Case 1. $\varphi_x > \sigma^{-1}$. In this case, we have $p(0) > 0$. Conditions (25) and (26) ensure that $p(1) < 0$ and $p(-1) < 0$. Consequently, there is root each in the interval $(-1, 0)$ and $(0, 1)$. Also, $p(1) < 0$ implies that the third root is more than 1. So, we have uniqueness in this case.

Case 2. $\varphi_x < \sigma^{-1}$. We first note that $a_2 > 0$ when

$$\beta + (1+\beta)\varphi_r^{-1} > \sigma\varphi_r^{-1}\{\kappa(\varphi_\pi - 1) + \varphi_x\}, \tag{76}$$

that is, when

$$\beta\sigma^{-1}\varphi_r + (1+\beta)\sigma^{-1} > \kappa(\varphi_\pi - 1) + \varphi_x \tag{77}$$

which is ensured by condition (26). Moreover, since a_1 and a_3 are negative, we have by Descartes' Rule of signs that there are either three positive roots or there is one positive root and a pair of complex conjugates.

In this case, condition (25) again ensures that $p(1) < 0$ so that there is a root more than 1. The product of the eigenvalues, which is $\beta\varphi_r^{-1}(1-\sigma\varphi_x)$, is less than 1 by condition (25) so that there must be at least one eigenvalue with modulus less than 1. If the eigenvalues are complex, we immediately have uniqueness. On the other hand, if all the roots are real, then we have at least one root in the interval $(0,1)$. We can now use the theorem of *Fourier and Budan* to show that there are exactly two roots in $(0,1)$. The signs of $p(0)$, $p'(0)$, $p''(0)$, $p'''(0)$ are respectively $-,+,-,+$ so that the number of sign changes is 3. The signs of $p(1)$ and $p'''(1)$ are $-$ and $+$ respectively. Consequently, irrespective of the sign of $p'(1)$ and $p''(1)$, the number of sign changes in the sequence $p(1)$, $p'(1)$, $p''(1)$, $p'''(1)$ can be either 1 or 3. So the number of roots in $(0,1)$ is either 0 or 2 by the theorem of *Fourier and Budan.* But since we already know that there is 1 root in this interval, the number of roots between 0 and 1 is exactly 2. So, we again have uniqueness in this case.

Case 3. $\varphi_x = \sigma^{-1}$. In this case, B becomes singular so that one of the eigenvalues is 0. The remaining two eigenvalues are given by the following characteristic polynomial

$$q(\lambda) = \lambda^2 + a_1\lambda + a_2$$

where a_1 and a_2 are given by (71) and (72) after substituting in $\varphi_x = \sigma^{-1}$. We have $q(0) = a_2$ which was shown to be positive by condition (26) in *Case 2.* It can also be easily checked that $q(1)$ is identical to the expression for $p(1)$ given in (74) (with $\varphi_x = \sigma^{-1}$) so that it is negative by condition (25). It, therefore, follows that there must be one root in the interval $(0,1)$ and the other root must be greater than 1. So, B again has exactly 2 eigenvalues inside the unit circle.

Having considered all the subcases, the proposition is proved.