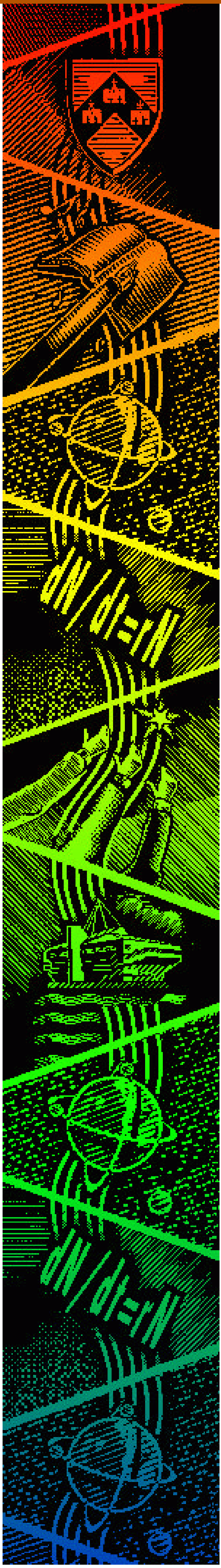

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Abstract

Recent models of monetary policy can have indeterminacy of equilibria. The indeterminacy property is often viewed as a difficulty of these models. We consider its significance using the learning approach to expectations formation by employing expectational stability as a robustness criterion for different equilibria. We derive the expectational stability and instability conditions for forward-looking multivariate models, both with and without lags, that cover a wide range of monetary policies proposed in the literature.

Key words: Adaptive learning, stability, sunspots, monetary policy.

JEL classification: E52, E31, D84.

1 Introduction

There has been a large amount of recent literature studying the performance of alternative monetary policies in dynamic macroeconomic settings, see for example the survey (Clarida, Gali, and Gertler 1999) and the papers in the 1999 Special Issue of the *Journal of Monetary Economics* and in the volume

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(Taylor 1999). An important feature of the recent setups is the forward-looking aspect of economic behavior, so that expectations about the future evolution of economic variables influence the current state of the economy.

The standard way of closing a model with expectations is to postulate rational expectations (RE). Attention is often directed at a specific rational expectations equilibrium (REE) that is thought to be the most natural one. In the general macroeconomics literature terms like fundamental equilibrium or minimal state variable (MSV) solution are used to describe this particular REE. It has been, however, pointed out that several of the recent models of monetary policy are plagued by the problem of indeterminacy, i.e. there are multiple, even continua of equilibria which include bubbles or sunspots, see e.g. the discussions in (Kerr and King 1996), (Bernanke and Woodford 1997), (Woodford 1999), (Clarida, Gali, and Gertler 1999), (Bullard and Mitra 2000b), and (Carlstrom and Fuerst 2000). In this paper we will use the terms *fundamental and non-fundamental equilibrium* when making the distinction between these different REE.

Our perspective on the problem of multiplicity of REE is to impose additional criteria that the REE should satisfy if it is to be reasonable or robust. Such criteria can be applied to both fundamental and non-fundamental REE. While different criteria have been suggested, the learning approach to expectation formation has recently gained some popularity.[1] In general terms this approach suggests that expectations might not always be fully rational, and the REE of interest should satisfy a natural stability criterion in expectations formation. In other words, if economic agents make forecast errors and adjust their forecast functions over time, the economy will reach the REE asymptotically. Perhaps the most widely used criteria are stability under adaptive learning and the closely related notion of expectational stability (E-stability). Equilibria which are not stable in this sense will not be reached after a perturbation from REE has occurred.

The notion that the REE should be robust to expectational errors and the consequent correction mechanisms is important from the applied viewpoint, since such errors can naturally arise in practice. For example, the economy might be subject to changes in its basic structure or in the practices and rules of policy makers, and the assumption that agents somehow have RE immediately after such changes is clearly strong and indeed may not be correct

[1](Evans and Honkapohja 2001c) provides a comprehensive treatment of the learning approach. See also the surveys (Evans and Honkapohja 1999) and (Marimon 1997).

empirically.

Most recently, monetary policy making has been analyzed from this learning viewpoint. The importance of this approach is argued in (Bullard and Mitra 2000b) who consider the determinacy and learnability (i.e. stability under adaptive learning) of the fundamental REE arising in a benchmark forward-looking model with different classes of monetary policy rules. Also taking the learning viewpoint (Evans and Honkapohja 2000) show that, if the policy maker conducts optimal monetary policy under discretion using an implied interest rate rule, the fundamental REE is not learnable in this kind of model. They also propose an alternative interest rate rule that is always E-stable.

These papers limit their attention to the fundamental REE even if other equilibria exist under the indeterminacy arising with specific policy rules. In this paper we rectify this limitation and consider learnability of the other types of non-fundamental REE in the standard model under different types of monetary policies.

The above motivation is made from a theoretical perspective, but there is an important practical motivation for conducting the study. Pursuit of optimal monetary policy on the part of the central bank or, flexible inflation targeting in the sense used by (Svensson 1999), implies that the instrument of monetary policy (the short-term nominal interest rate) should respond to inflation forecasts, see (Clarida, Gali, and Gertler 1999). There is evidence that monetary policy in a number of industrialized countries (like Germany, Japan, and the U.S.) has been forward looking since 1979, see (Clarida, Gali, and Gertler 1998). There is also some recent evidence to suggest that the European Central Bank (ECB) may have been forward looking since its inception in 1999, see (Alesina, Blanchard, Gali, Giavazzi, and Uhlig 2001). Moreover, in practice, a number of inflation-targeting central banks like those in England, Canada, and New Zealand are forward looking.

Though there is obviously a lot of evidence in favor of forward looking policy rules, a number of theoretical studies (mentioned above) have found an enormous indeterminacy problem with these rules. Different views have been taken on this problem. (Bernanke and Woodford 1997) have argued against inflation forecast targeting owing to this problem - these rules may lead to too much volatility in inflation and output which any central bank ought to avoid.

At the other end of the spectrum, indeterminacy is sometimes viewed as an unimportant curiosum. This position has been taken, for instance, in (McCallum 2001a) and (McCallum 2001b). One argument for regarding non-fundamental equilibria as "empirically irrelevant" is that the fundamental equilibria which result when the bank targets inflation forecasts can be learnable adaptively, as shown in (Bullard and Mitra 2000b). However, in order to complete this argument, one needs to also show that the non-fundamental equilibria associated with these policies are *unstable* under plausible learning schemes. (Woodford 1990) was first to show that adaptive learning rules can converge to stationary sunspot equilibrium in simple models with overlapping generations. If the non-fundamental equilibria possible under inflation forecast targeting happen to be stable under learning, then indeterminacies again have the potential to become empirically relevant. On the contrary, if these equilibria are unlearnable, then this would provide some support for the policies of inflation-targeting central banks.

In this regard, an interesting recent paper which does take indeterminacy as an empirically relevant possibility is (Clarida, Gali, and Gertler 2000). They estimate a forward looking policy reaction function for the postwar U.S. economy, both before and after the appointment of Paul Volcker as Fed Chairman in 1979. They conclude that monetary policy in the pre-Volcker era was compatible with the possibility of bursts of inflation and output that resulted from self-fulfilling changes in expectations of the private sector. In other words, the monetary policy of the Federal Reserve may have contributed to the high and volatile inflation of the 1960s and 1970s. In contrast, monetary policy in the Volcker-Greenspan era is compatible with the existence of a unique fundamental equilibrium and this may have contributed to low and stable inflation during this era. We use stability under adaptive learning to assess the plausibility of these explanations in Section 4.3.

The plan of the paper is as follows. We present the (by now) standard model of monetary policy in Section 2 and several examples of monetary policies considered in the literature. Section 3 characterizes and carries out a general analysis of E-stability for all of the known forms of non-fundamental equilibria in purely forward looking multivariate linear models. These general results are then applied to the model of monetary policy presented in Section 2. Section 4 extends the model to include lagged endogenous variables. This model covers several important examples of monetary policies proposed in the literature, in particular interest rules which react to past values of inflation (output) and/or interest rates. While the expectational stability conditions can be extended, the results are not theoretically clear-cut. We present the main methodology for the analysis of this case. Section 4.3 analyzes in some detail the model presented in Section 4 of (Clarida, Gali, and Gertler 2000). Finally, we discuss the implications of our results in Section 5.

2 The Basic Model of Monetary Policy

We conduct the analysis using the framework in Section 2 of (Clarida, Gali, and Gertler 1999). The structural model consists of two equations:

$$z_t = -\varphi(i_t - \hat{E}_t\pi_{t+1}) + \hat{E}_t z_{t+1} + g_t, \quad (1)$$
$$\pi_t = \lambda z_t + \beta\hat{E}_t\pi_{t+1} + u_t \quad (2)$$

where z_t is the "output gap" i.e. the difference between actual and potential output, π_t is the inflation rate, i.e. the proportional rate of change in the price level from $t-1$ to t and i_t is the nominal interest rate. $\hat{E}_t\pi_{t+1}$ and $\hat{E}_t z_{t+1}$ denote private sector expectations of inflation and output gap next period. We will use the same notation without the "^" to denote RE. All the parameters in (1) and (2) are positive. $0 < \beta < 1$ is the discount rate of the representative firm.

(1) is a dynamic "IS" curve that can be derived from the Euler equation associated with the household's savings decision. (2) is a "new Phillips curve" that can be derived from optimal pricing decisions of monopolistically competitive firms facing constraints on the frequency of future price changes.

g_t and u_t denote observable shocks following first order autoregressive processes

$$g_t = \mu g_{t-1} + \tilde{g}_t, \quad (3)$$
$$u_t = \rho u_{t-1} + \tilde{u}_t. \quad (4)$$

where $0 < |\mu| < 1, 0 < |\rho| < 1$ and $\tilde{g}_t \sim iid(0, \sigma_g^2), \tilde{u}_t \sim iid(0, \sigma_u^2)$. g_t represents shocks to government purchases as well as shocks to potential GDP. u_t represents cost push shocks to marginal costs.

Monetary policy is conducted by means of control of the nominal interest rate i_t. A number of different types of control have been analyzed in the literature and we present below several well-known examples.

Example 1. (Taylor rules based on contemporaneous data) Suppose that nominal interest rate is adjusted in accordance with contemporaneous data on inflation and output gap, so that

$$i_t = \chi_\pi \pi_t + \chi_z z_t.$$

The structural model becomes

$$\begin{pmatrix} z_t \\ \pi_t \end{pmatrix} = \frac{1}{\varphi^{-1}+\chi_z+\lambda\chi_\pi}\begin{pmatrix} \varphi^{-1} & 1-\beta\chi_\pi \\ \lambda\varphi^{-1} & \lambda+\beta(\varphi^{-1}+\chi_z) \end{pmatrix}\begin{pmatrix} \hat{E}_t z_{t+1} \\ \hat{E}_t \pi_{t+1} \end{pmatrix} + \frac{1}{\varphi^{-1}+\chi_z+\lambda\chi_\pi}\begin{pmatrix} \varphi^{-1} & -\chi_\pi \\ \lambda\varphi^{-1} & \varphi^{-1}+\chi_z \end{pmatrix}\begin{pmatrix} g_t \\ u_t \end{pmatrix}.$$

This model may be determinate or indeterminate. (Bullard and Mitra 2000b) show that the condition for determinacy is $\lambda(\chi_\pi - 1) + (1-\beta)\chi_z > 0$ and that there necessarily exists a positive eigenvalue more than 1 when the model is indeterminate. Thus there may exist other REE besides the fundamental equilibrium.

Example 2. (Taylor rules based on forward expectations) The nominal interest rate is now adjusted in accordance with expectations of output gap and inflation next period. For simplicity (and as a first approach) we assume that the expectations of private agents and policy makers are identical. Then

$$i_t = \chi_\pi \hat{E}_t \pi_{t+1} + \chi_z \hat{E}_t z_{t+1}$$

and the structural model becomes

$$\begin{pmatrix} z_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1-\varphi\chi_z & \varphi(1-\chi_\pi) \\ \lambda(1-\varphi\chi_z) & \beta+\lambda\varphi(1-\chi_\pi) \end{pmatrix}\begin{pmatrix} \hat{E}_t z_{t+1} \\ \hat{E}_t \pi_{t+1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} g_t \\ u_t \end{pmatrix}.$$

This model may be determinate or indeterminate. (Bullard and Mitra 2000b) show that the conditions for determinacy are $\chi_z < \varphi^{-1}(1+\beta^{-1})$, $\lambda(\chi_\pi - 1) + (1+\beta)\chi_z < 2\varphi^{-1}(1+\beta)$ and $\lambda(\chi_\pi - 1) + (1-\beta)\chi_z > 0$. It is also possible to show that, depending on the structural parameters and policy coefficients, there may exist eigenvalues which are more than 1 or less than -1 in the indeterminate case.

Optimal monetary policies can also lead to purely forward-looking structures. For example, postulating a standard quadratic objective function[2]

$$\min \frac{1}{2} E_t \left\{ \sum_{i=0}^{\infty} \beta^i \left[\alpha z_{t+i}^2 + \pi_{t+i}^2 \right] \right\}, \tag{5}$$

we consider optimal monetary policy under discretion. Carrying out the optimization and assuming that the economy is in the fundamental REE yields the following interest rate rule in terms of the observable exogenous shocks:[3]

$$i_t = \chi_u u_t + \chi_g g_t, \tag{6}$$

where policy optimization yields the values

$$\chi_u = \frac{(1-\rho)\lambda + \alpha\rho\varphi}{\varphi[\alpha(1-\rho\beta) + \lambda^2]}, \chi_g = \varphi^{-1}.$$

Example 3. (Interest rate rules based on observable shocks and a pure interest rate peg) More generally, one can consider the class of interest rate rules that depend on the observable fundamental shocks. These take the form (6), but with χ_u and χ_g not specified as the optimal values just given. If we think of (6) as a fixed monetary rule we obtain the purely forward-looking structural model

$$\begin{pmatrix} z_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1 & \varphi \\ \lambda & \beta + \lambda\varphi \end{pmatrix} \begin{pmatrix} \hat{E}_t z_{t+1} \\ \hat{E}_t \pi_{t+1} \end{pmatrix} + \begin{pmatrix} 1 - \varphi\chi_g & -\varphi\chi_u \\ \lambda(1-\varphi\chi_g) & 1 - \lambda\varphi\chi_u \end{pmatrix} \begin{pmatrix} g_t \\ u_t \end{pmatrix}. \tag{7}$$

Monetary policy pegging the interest rate at a certain target level would lead to the same coefficient matrix for the expectation variables since then $\chi_u = \chi_g = 0$. It is easy to check that the coefficient matrix of the vector of expectations has one eigenvalue greater than one (the other being between

[2] α is the relative weight for output deviations, and β is the discount rate. The policy maker is assumed to discount future at the same rate as the private sector. If desired, one could allow for a possible deviation of socially optimal output from potential output and a non-zero target value for the inflation rate.

[3] (Evans and Honkapohja 2000) call (6) the fundamentals form of the RE-optimal policy rule. We avoid this terminology, since we make the distinction between the fundamental and non-fundamental REE.

0 and 1), so that the model necessarily exhibits indeterminacy. Moreover, (Evans and Honkapohja 2000) show that the fundamental equilibrium for this model is not learnable (E-stable).

Example 4. (Expectational form of RE-optimal policy) In the fundamental equilibrium optimal monetary policy without commitment can be characterized in other ways besides (6) with the specified values for χ_u and χ_g. Under RE the optimal interest rate can also be written as

$$i_t = (1 + \frac{(1-\rho)\lambda}{\rho\alpha\varphi})E_t\pi_{t+1} + \varphi^{-1}g_t, \tag{8}$$

as pointed out in (Clarida, Gali, and Gertler 1999). We can alternatively think of (8) as a specified interest rate rule and consider the resulting structural model. It is again purely forward-looking and takes the form

$$\begin{pmatrix} z_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1 & -(1-\rho)\lambda/\rho\alpha \\ \lambda & \beta-(1-\rho)\lambda^2/\rho\alpha \end{pmatrix} \begin{pmatrix} \hat{E}_t z_{t+1} \\ \hat{E}_t \pi_{t+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_t.$$

(Evans and Honkapohja 2000) point out that in this case the model is either determinate or indeterminate, depending on the values of structural parameters and ρ. If $-1 < \rho < 0$, then equilibrium is necessarily indeterminate since in this case there are two positive eigenvalues with one exceeding 1. On the other hand, when $0 < \rho < 1$, indeterminacy obtains if $\rho < \lambda^2[\lambda^2+2\alpha(1+\beta)]^{-1}$ since in this case the characteristic polynomial has a root less than -1 (the other being between -1 and 0).

All of these examples lead to purely forward-looking frameworks. As was pointed out in the introduction, it is important to consider whether an equilibrium is robust to small expectational errors and mechanisms to correct them. This view has particular significance when the model exhibits indeterminacy, since stability under learning (or E-stability) can then provide a selection criterion between the fundamental and non-fundamental REE. In the next section we will provide a general analysis of E-stability and learning of the non-fundamental REE, as the stability of the fundamental REE has already been studied in the papers cited above.

3 Purely Forward-Looking Models

Consider a general bivariate linear model

$$\begin{aligned} x_t &= \Omega \hat{E}_t x_{t+1} + \Phi w_t & (9) \\ w_t &= \Psi w_{t-1} + v_t, & (10) \end{aligned}$$

where $x_t, w_t \in \mathbb{R}^2$ are, respectively, the vectors of endogenous and exogenous variables and all constants have been eliminated by centering the variables. The exogenous variables follow a stationary vector autoregressive process, so that the eigenvalues of Ψ are inside the unit circle. v_t is *iid*. Examples 1-4 all fit the framework (9)-(10). The limitation to a bivariate model, however, is not crucial, as many results extend to general multivariate frameworks. These will be noted below. For the main part we assume that the 2×2 matrix Ω is invertible (which is true for all of the Examples 1-4), but we will take note of the necessary modifications when Ω is singular.

3.1 Characterization of Non-Fundamental Solutions

In this subsection we impose RE, so that $\hat{E}_t x_{t+1} = E_t x_{t+1}$, the mathematical conditional expectation. The most common way to obtain non-fundamental solutions is to represent classes of solutions in terms of arbitrary (unanticipated) innovations to the expectations. Thus let $\eta_{t+1} = x_{t+1} - E_t x_{t+1}$ be any innovation process, so that it satisfies $E_t \eta_{t+1} = 0$ i.e. it is a (vector) martingale difference sequence.

The general class of solutions of (9)-(10) can be written in the form[4]

$$\begin{aligned} x_t &= \Omega^{-1} x_{t-1} - \Omega^{-1} \Phi w_{t-1} + \eta_t & (11) \\ w_t &= \Psi w_{t-1} + v_t \end{aligned}$$

or, introducing the notation $y_t = (x_t', w_t')'$, $u_t = (\eta_t', v_t')'$ in the VAR form

$$y_t = \mathcal{B} y_{t-1} + u_t, \text{ where } \mathcal{B} = \begin{pmatrix} \Omega^{-1} & -\Omega^{-1}\Phi \\ 0 & \Psi \end{pmatrix}. \quad (12)$$

Since there are many ways of specifying the innovation process η_t it is evident that in general there are indeterminacies of REE. The only restriction we

[4]There is a large literature on representing solutions to linear RE models, see e.g. (Broze and Szafarz 1991) or Part III in (Evans and Honkapohja 2001c).

have on η_t is that it must be a martingale difference sequence. However, a common further restriction is stationarity of the process (12) and we consider this next.

We first diagonalize the coefficient matrix of (12), so that $\mathcal{B} = Q\Lambda Q^{-1}$ and introduce the notation $Q^{-1} = (Q^{ij})$. We note that Λ is a diagonal matrix with the eigenvalues of $\mathcal{B}$ along its diagonal, i.e. $\Lambda = \lceil \lambda_1, ..., \lambda_4 \rfloor$. Since $\mathcal{B}$ is block-triangular, the last two eigenvalues are those of Ψ and are inside the unit circle. The remaining two eigenvalues of $\mathcal{B}$ (λ_1 and λ_2) are then given by those of Ω^{-1}. If both λ_1 and λ_2 are inside the unit circle, then (11) forms a stationary class of solutions. However, it may also be the case that one or both roots of Ω^{-1} are outside the unit circle. If both roots of Ω^{-1} are outside the unit circle, we have the so-called regular case and only the fundamental solution (which does not involve any lags x_{t-1}) is stationary.[5]

If just one of the roots is outside the unit circle, there exist stationary non-fundamental solutions that can be derived by using an extension of the diagonalization technique originally developed in (Blanchard and Kahn 1980). This procedure is normally applied to the original structural model (9)-(10). Since invertibility of Ω has been assumed, the same procedure can equally well be applied to the form (12), as we now show. The following proposition represents the class of non-fundamental stationary solutions to (9)-(10) in this case.

Proposition 1 *Assume, without loss of generality (w.l.o.g) that* $|\lambda_1| < 1$, $|\lambda_2| > 1$ *for the two eigenvalues of* Ω^{-1}. *The unique stationary solution takes*

[5] It takes the form $y_t = bw_t$, where matrix b satisfies the equation $b = \Omega b \Psi + \Phi$. Vectorizing this last equation we get the linear system of equations $(I - \Psi' \otimes \Omega)(\text{vec}\, b) = \text{vec}\, \Phi$ which normally has a unique solution.

the form

$$\begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} = \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 Q^{11} & \lambda_1 Q^{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{1,t-1} \\ x_{2,t-1} \end{pmatrix} \tag{13}$$
$$- \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}^{-1} \begin{pmatrix} Q^{13} & Q^{14} \\ Q^{23} & Q^{24} \end{pmatrix} \begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix}$$
$$+ \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 Q^{13} & \lambda_1 Q^{14} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{1,t-1} \\ w_{2,t-1} \end{pmatrix}$$
$$+ \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}^{-1} \begin{pmatrix} Q^{11} & Q^{12} & Q^{13} & Q^{14} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \\ v_{1,t} \\ v_{2,t} \end{pmatrix}.$$

Proof. See Appendix A.1. ■

At this point we make two observations.

Remark 1: If Ω is singular, then the representation (12) does not exist. Using the analogous diagonalization procedure on the coefficient matrix of the system

$$\begin{pmatrix} x_t \\ w_t \end{pmatrix} = \begin{pmatrix} \Omega & \Phi\Psi^{-1} \\ 0 & \Psi^{-1} \end{pmatrix} \begin{pmatrix} x_{t+1} \\ w_{t+1} \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & \Psi^{-1} \end{pmatrix} \begin{pmatrix} \eta_{t+1} \\ v_{t+1} \end{pmatrix},$$

where $\eta_{t+1} = x_{t+1} - E_t x_{t+1}$, the stationary RE solutions can be obtained, see Chapter 10, Appendix 2 of (Evans and Honkapohja 2001c).

Remark 2: If the system (9)-(10) is higher-dimensional, the same techniques can be used. However, different classes of stationary solutions may emerge when the above procedure is applied to the general solution class (12).

The above methodology does not readily yield all stationary solutions to (9)-(10). A further class of non-fundamental equilibria can be directly constructed as follows.[6] Suppose that economic agents condition their expectations on a sunspot process s_t which is a stationary Markov chain taking

[6] These kinds of solutions are a generalization of the class of sunspot equilibria introduced in (Chiappori, Geoffard, and Guesnerie 1992) for a linear framework with $AR(1)$ exogenous variables and shocks. They can be derived from the general form by suitably defining the innovation process η_t, as discussed in (Evans and Honkapohja 2001b) for scalar models without exogenous variables.

values in a finite set $\{1, ..., K\}$, $K \geq 2$. We denote its transition matrix by $\Pi = (\pi_{ij})$, where π_{ij} is the probability that the sunspot will be in state j next period if it is in state i in the current period.

At time t and with sunspot in state s at that time we consider a solution of the form

$$y_{t,s} = a_s + bw_t, \tag{14}$$

where $y_{t,s}$ denotes the vector of endogenous variables at time t and state s. The intercept vector is thus made dependent on the state of the sunspot process s_t. The following result shows that this kind of equilibria exist:

Proposition 2 *There exist sunspot equilibria of the form (14) if at least one eigenvalue of* Π *is equal to the inverse of an eigenvalue of* Ω*, i.e.* $|I_4 - \Pi \otimes \Omega| = 0$*, and where matrix* b *solves the equation* $b = \Omega b \Psi + \Phi$.[7]

Proof. Consider solutions of the form (14). Computing the conditional expectation

$$E_{t,s}y_{t+1} = b\Psi w_t + \sum_{i=1}^{K} \pi_{si} a_i,$$

where $E_{t,s}$ denotes the conditional expectation at time t and state s. The structural model (9) with conditioning on the state of the sunspots can be written as

$$y_{t,s} = \Omega E_{t,s} y_{t+1} + \Phi w_t.$$

Substituting in the expectations we get

$$y_{t,s} = (\Omega b \Psi + \Phi) w_t + \Omega \sum_{i=1}^{K} \pi_{si} a_i,$$

so that in the REE the equations

$$\begin{aligned} b &= \Omega b \Psi + \Phi & (15) \\ a_s &= \sum_{i=1}^{K} \pi_{si} \Omega a_i, \; s = 1, ..., K & (16) \end{aligned}$$

[7]Note that the linear equation for b is just the system for computing the coefficient matrix of the fundamental solution.

must hold. Letting $a = (a_1', ..., a_K')' \in \mathbb{R}^{2K}$, (16) can be re-written in matrix form as

$$(I - \Pi \otimes \Omega)a = 0$$

which must have a non-trivial solution. ■

We note that these *resonant frequency sunspot equilibria*[8] form a continuum of solutions, since the equilibrium value of $a = (a_1', ..., a_K')'$ is not unique. This is because, by Proposition 2, at least one eigenvalue of the coefficient matrix $\Pi \otimes \Omega$ is equal to one.

3.2 Learnability of Non-Fundamental REE

We now consider how the learnability of these non-fundamental equilibria can be analyzed for general forward looking models (9)-(10) and in the next section apply these results to the monetary policies of Examples 1-4. We employ the methodology exposited in (Evans and Honkapohja 2001c) as it is by now fairly standard. In this approach the conditions for learnability of REE are given by E-stability conditions.

From the literature it is known that in most cases E-stability provides precisely the conditions of the stability under least-squares (and related) learning schemes.[9] However, this theoretical connection sometimes fails for technical reasons. The main case of failure are the continua of RE solutions in linear models and we are indeed facing this situation here. Simulation studies for univariate linear models suggest that the connection between E-stability and convergence of real time learning does hold for solutions continua.[10] Though multivariate models have not been numerically studied for this question, there appears to be no reason why the situation would be different for them.

With these remarks in mind we employ the E-stability criterion in our analysis of learnability of the REE. The analysis of E-stability of the different

[8] This terminology is suggested in (Evans and Honkapohja 2001b).

[9] (Marcet and Sargent 1989) established the connection in models with a unique equilibrium This result was extended to local stability of multiple REE in (Evans and Honkapohja 1994a). (Woodford 1990) used the same general methodology to establish a global result about the possibility of convergence of learning to a sunspot equilibrium.

[10] See (Evans and Honkapohja 1994a) and Part III of (Evans and Honkapohja 2001c) for a discussion of these questions and for further references.

types of REE discussed above in the structural model (9)-(10) can generally be developed as follows.

We begin with the classes of REE taking the form (11) or (13). The analysis of E-stability begins with the (in general non-rational) perceptions of the agents. We thus introduce the perceived law of motion (PLM)

$$x_t = a + bx_{t-1} + cw_{t-1} + d\eta_t + ev_t,$$

where a, b, c, d and e are parameter matrices or vectors of appropriate dimensions. Note that this form of the PLM is the same as (11) and (13), but with parameter values that are in general different from any REE. Note also that we have allowed for a possible intercept in the PLM.

At any moment of time agents make forecasts using this PLM with given values of the parameters, so that the forecasts are given by

$$\begin{aligned}\hat{E}_t x_{t+1} &= a + b\hat{E}_t x_t + c\Psi w_{t-1} + cv_t \\ &= a + ba + b^2 x_{t-1} + (bc + c\Psi)w_{t-1} + bd\eta_t + (be + c)v_t.\end{aligned}$$

In this formulation we have made the assumption that, when making forecasts at time t, agent can observe the values of the exogenous variables and shocks but not of the endogenous variables at time t.[11] Substituting these forecasts into (9) leads to the actual law of motion (ALM) taking the form

$$x_t = \Omega[a + ba + b^2 x_{t-1} + (bc + c\Psi)w_{t-1} + bd\eta_t + (be + c)v_t] + \Phi\Psi w_{t-1} + \Phi v_t.$$

The ALM describes the temporary equilibrium of the economy when agents use the PLM with the specified parameter values when forming expectations.

We have obtained a mapping

$$(a, b, c, d, e) \to T(a, b, c, d, e)$$

from the PLM to ALM, where

$$T(a, b, c, d, e) = (\Omega(I + b)a, \Omega b^2, \Omega(bc + c\Psi) + \Phi\Psi, \Omega bd, \Omega(be + c) + \Phi).$$

[11] It can be shown that nonfundamental equilibria cannot be E-stable if the period t values of the endogenous variables are included in the information set, see Chapter 10 of (Evans and Honkapohja 2001c).

The different REE are fixed points of the T mapping. They must thus satisfy the matrix equations

$$a = \Omega(I+b)a \tag{17}$$

$$b = \Omega b^2 \tag{18}$$

$$c = \Omega(bc + c\Psi) + \Phi\Psi \tag{19}$$

$$d = \Omega bd \tag{20}$$

$$e = \Omega(be + c) + \Phi. \tag{21}$$

It can be seen that the equation for matrix b is a quadratic matrix equation. Clearly, $b = \Omega^{-1}$ solves this equation, but in general it has other solutions. Some of the solutions can be singular matrices and this possibility will be illustrated below.

Given a solution $\bar{b}$, equations (17) and (19) generically uniquely determine a and c ($\bar{a}$ and $\bar{c}$). Given $\bar{b}, \bar{c}$, (21) solves e uniquely. For sunspot equilibria, equation (20) has non-trivial solutions for d when, given $\bar{b}$, the matrix $I - \Omega\bar{b}$ is singular. This happens e.g. when $b = \Omega^{-1}$.

E-stability of a fixed point is defined using the ordinary differential equation

$$\frac{d}{d\tau}(a, b, c, d, e) = T(a, b, c, d, e) - (a, b, c, d, e). \tag{22}$$

Thus a fixed point $(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e})$ is said to be E-stable if it is locally asymptotically stable under (22). Since we will analyze continua of RE solutions, E-stability of a class of equilibria must also be defined. Following the literature, we say that a class of REE is E-stable if the dynamics under (22) converge to some member of the class.

Formally, this differential equation describes partial adjustment in continuous (artificial) time τ between the PLM that the agents use in forecasting and the actual outcome of the economy under these forecast functions. In view of this formal interpretation, E-stability is also sometimes used as highly stylized learning process.

To derive the E-stability and E-instability conditions we linearize (22). Since the system is matrix-valued, it must be vectorized. We use standard results from matrix algebra and analysis of multivariate linear models, see Chapter 10 of (Evans and Honkapohja 2001c), to obtain the coefficient matrices of the linearized and vectorized form of (22). This yields the necessary

E-stability condition that the real parts of the eigenvalues of the following matrices

$$\begin{aligned} DT_a(\bar{a}, \bar{b}) &= \Omega(I + \bar{b}) \\ DT_b(\bar{b}) &= \bar{b}' \otimes \Omega + I \otimes \Omega \bar{b} \\ DT_c(\bar{b}, \bar{c}) &= \Psi' \otimes \Omega + I \otimes \Omega \bar{b} \\ DT_d(\bar{b}) &= \Omega \bar{b} \end{aligned} \tag{23}$$

must have real parts less than one. The sufficient condition for E-instability is that at least one eigenvalues of these matrices has a real part greater than one.

This general analysis yields the result that the class of non-fundamental REE (11) for the structural model (9)-(10) is not E-stable:[12]

Proposition 3 *The solution class (11) is not E-stable.*

Proof. Using (23), if we evaluate $DT_b(b)$ at $b = \Omega^{-1}$, we get $(\Omega^{-1})' \otimes \Omega + I_4$, where I_4 denotes the 4×4 identity matrix. Using the properties of the eigenvalues of the Kronecker product of two matrices, we observe that the eigenvalues of $DT_b(\Omega^{-1})$ include an unstable root (with value 2) which proves the result. ■

Remark: This proposition also holds in higher than two-dimensional purely forward-looking models.

This proposition has, however, a limitation. In the analysis of E-stability we have not imposed the requirement that the RE solutions and the possible PLM be stationary. Of course, if both eigenvalues of Ω are outside the unit circle, then (11) forms a class of stationary sunspot solutions. If only one eigenvalue of Ω is outside the unit circle, then indeterminacy is, in a sense, lower-dimensional and (11) does not form a class of stationary sunspot solutions. We, therefore, turn to the analysis of E-stability for this class of stationary sunspot equilibria (SSE).

With attention being restricted to stationary solutions, we must focus on the class of REE given in (13). First, we note that, since E-stability is local concept, all non-rational PLM's sufficiently near these REE must be stationary. We have the following result.

[12](Evans 1989) analyzed E-instability of the solution class (12) for univariate forward-looking models.

Proposition 4 *The class of stationary REE (13) is not E-stable.*

Proof. We will show that instability arises from the eigenvalues of the matrix $DT_b(b)$ evaluated at this REE. For this we first need to compute the solution $\bar{b}$ which is given by

$$\bar{b} = \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 Q^{11} & \lambda_1 Q^{12} \\ 0 & 0 \end{pmatrix}. \tag{24}$$

The eigenvalues of $\bar{b}$ are 0 and λ_1 so that this solution is stationary. A necessary condition for $\bar{b}$ to be E-stable is that the eigenvalues of the matrix

$$DT_b(\bar{b}) = \bar{b}' \otimes \Omega + I \otimes \Omega \bar{b}$$

have real parts less than 1. However, one of its eigenvalues is always 2 which proves the result. See Appendix A.2 for a Mathematica routine calculating the eigenvalues of $DT_b(\bar{b})$. ■

Propositions 3 and 4 together show that for solutions classes (11) and (13) non-fundamental REE of 2-dimensional purely forward-looking models are E-unstable.

Remark: This proposition is currently limited to bivariate models, but we conjecture that it also holds generally.

Finally, we need to consider E-stability of the resonant frequency sunspot REE of the form (14).[13] Thus assume that agents have PLM of that form but a_s and b do not take the REE values given by equations (15) and (16). The right-hand sides of (15) and (16) define the $T-$mapping used in the analysis of E-stability in the standard way. Thus denote

$$\begin{aligned} T_a(a) &= (\Pi \otimes \Omega)a \\ T_b(b) &= \Omega b \Psi + \Phi \end{aligned}$$

in matrix form. Introducing the notation $\xi = (a, b)$, $T(\xi) = (T_a(a), T_b(b))$, E-stability is defined as usual by the differential equation

$$\frac{d\xi}{d\tau} = T(\xi) - \xi.$$

[13] E-stability of these kinds of sunspot equilibria for univariate models without exogenous shocks were considered in (Evans and Honkapohja 1994b), (Evans and Honkapohja 2001b) and (Evans and Honkapohja 2001a).

For these non-fundamental equilibria, a sufficient condition for E-instability is as follows:[14]

Proposition 5 *The class of sunspot equilibria of the form (14) are not E-stable if* Ω *has an eigenvalue with real part* > 1.

Proof. Consider the component $T_a(a) = (\Pi \otimes \Omega)a$ of the $T-$mapping constructed in the proof of Proposition 2. Its eigenvalues are the products of the eigenvalues of Π and Ω. Since 1 is an eigenvalue of the probability matrix Π, the matrix $\Pi \otimes \Omega$ has an eigenvalue with real part greater than one. ■

What about the possibility of E-stable resonant frequency sunspots? For the non-stochastic *scalar* model, where (in our notation) $\Omega < -1$, and a two-state sunspot process (Evans and Honkapohja 2001b) recently discovered that resonant frequency sunspot solutions are E-stable. Here we provide an extension of that result for the multivariate stochastic setup (9)-(10):

Proposition 6 *Assume that (i) all eigenvalues of* $\Psi' \otimes \Omega$ *have real parts* < 1 *and that (ii) with the exception of a single eigenvalue equal to* 1 *(which exists by Proposition 2) the other eigenvalues of* $\Pi \otimes \Omega$ *have real parts* < 1. *Then the class of resonant frequency sunspot equilibria are E-stable.*[15]

Proof. We first vectorize the matrix-valued differential equation

$$\frac{db}{d\tau} = \Omega b \Psi + \Phi - b.$$

This yields

$$\frac{d(vecb)}{d\tau} = (\Psi' \otimes \Omega - I)vecb + vec\Phi$$

which is stable by assumption (i).

[14] This result was first obtained in (Evans and Honkapohja 1994b) for scalar models without shocks.

[15] The different E-stability properties of the resonant frequency sunspot solutions and of the "$AR(1)$ form" (13) may seem surprising, since an appropriate specification of η_t in the latter gives the same RE solution as the former. However, this is reconciled by observing that the parametric form of the PLM can matter for the E-stability properties, see (Evans and Honkapohja 2001b) for a further discussion and references.

Next consider the (linear) differential equation for a. Its coefficient matrix $\Pi \otimes \Omega - I$ has a single eigenvalue equal to zero while the others are, by hypothesis, stable. The mathematical lemma in Appendix A.3 shows that for such systems we have convergence to the set of equilibrium points. ■

Propositions 2, 5 and 6 show the following corollary:

Corollary 7 *There exist E-stable SSEs when the parameter matrix Ω has a real eigenvalue < -1.*

This is accomplished by selecting the transition matrix Π so that (i) and (ii) of Proposition 6 can be met.[16]

3.3 Learnability in Examples 1-4.

We now discuss the implication of these results for the monetary policies in Examples 1-4. As pointed out, the non-fundamental equilibria of the form (11) and (13) are never E-stable, so that these equilibria can be ruled out by the criterion of learnability. In addition, Proposition 5 implies that non-fundamental equilibria of the form (14) are not E-stable also for the policy rules in Examples 1, 3, and for Example 4 when $-1 < \rho < 0$. This follows, because the "irregular" eigenvalues are greater than 1 in those models.[17]

These results seem to suggest that indeterminacies are indeed an unimportant curiosum once one takes into account the criterion of learnability. However, an important caveat here is that Proposition 6 and Corollary 7 show theoretically the existence of E-stable SSEs in certain cases. However, the interesting question is the possibility of learnable SSEs for *plausible* values of structural and policy parameters. As it turns out, this is indeed possible for Examples 2 and 4.

In Example 2, it is easy to show that a set of sufficient conditions for Ω to have one eigenvalue less than -1 (with the other in the interval $(-1, 1)$) are $\chi_\pi > 1$ and $\chi_z \geq 2\varphi^{-1}$, so that there may exist E-stable sunspots for plausible values of parameters if one uses, for instance, $\varphi^{-1} = .157$, as in

[16] If Ω has more than one real eigenvalue < -1, one selects Π so that the inverse of just one of these eigenvalues is an eigenvalue of Π. Note that if Ω has a pair of complex eigenvalues with modulus > 1 the analytical construction does not work. This is because the inverses of this pair must both be eigenvalues of Π if one of them is. Then the E-stability differential equation will have two zero eigenvalues and it is not stable.

[17] Proposition 5 also applies to Example 2 in some situations.

(Woodford 1999). This strengthens the worries concerning the indeterminacy problems with forward looking interest rules pointed out in (Bernanke and Woodford 1997), since some of the SSE's are learnable.

Proposition 6 is also applicable to Example 4 when $0 < \rho < 1$, which is the empirically plausible case. Note that, when α is close to 0, one is almost certain to have indeterminacies with the policy (8). This would correspond to a policy of (almost) *strict* inflation targeting in the sense used by (Svensson 1999). In addition, as mentioned in Example 4, Ω will have one root less than -1 and the other in the interval $(-1, 0)$ in this case so that Proposition 6 immediately suggests that E-stable sunspots are possible. Quite apart from the well known problem of large volatility in output, a policy of strict inflation targeting may also result in indeterminate E-stable equilibria. A large(r) value of α, on the other hand, reduces the possibility of indeterminacy. This perspective, therefore, supports a policy of flexible inflation targeting.

A theme that we will elaborate further in Section 5 is the connection between E-stability and the "Taylor principle", see (Woodford 2000) for a definition. Intuitively, the Taylor principle means that nominal interest rates rise by more than the increase in the inflation rate in the long-run. (Bullard and Mitra 2000b) showed earlier this connection for the fundamental REE /MSV solution: rules fulfilling the Taylor principle are learnable and rules violating the principle are unlearnable.

In this paper we find that the connection (in a sense) extends to the set of non-fundamental equilibria. Policies violating the Taylor principle lead to indeterminacy in Examples 1 and 2. Combining the results in (Bullard and Mitra 2000b) with ours, we find that both the MSV as well as all of the non-fundamental equilibria are unlearnable for these examples.[18]

On the other hand, while the Taylor principle suffices for determinacy in Example 1, it does not do so for Example 2. As pointed out, in Example 2 we may have learnable indeterminate equilibria. In other words, if a policy conforming with the Taylor principle is associated with indeterminacy, then all forms of solutions, fundamental and non-fundamental, are potentially learnable.

[18] Ω has an eigenvalue more than 1 in Examples 1 and 2 when the Taylor principle is violated since the characteristic polynomial of Ω, evaluated at 1, is negative under this condition.

4 Forward Looking Models with Lags

In this section we consider the stability of stationary sunspot equilibria in models with lags. Some recent models of monetary policy lead to such formulations.

Example 5. Suppose that the central bank sets the nominal interest rate based on lagged values of inflation and output. This rule is argued by (McCallum 1997) to be particularly realistic of actual central bank behavior since the bank does not usually have information about contemporaneous output and/or inflation when formulating policy. This rule is given by

$$i_t = \chi_\pi \pi_{t-1} + \chi_z z_{t-1}$$

and plugging this into the structural model (1) and (2), we get

$$\begin{pmatrix} z_t \\ \pi_t \end{pmatrix} = \begin{pmatrix} 1 & \varphi \\ \lambda & \beta + \lambda\varphi \end{pmatrix} \begin{pmatrix} \hat{E}_t z_{t+1} \\ \hat{E}_t \pi_{t+1} \end{pmatrix} + \begin{pmatrix} -\varphi\chi_z & -\varphi\chi_\pi \\ -\lambda\varphi\chi_z & -\lambda\varphi\chi_\pi \end{pmatrix} \begin{pmatrix} z_{t-1} \\ \pi_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} g_t \\ u_t \end{pmatrix}. \tag{25}$$

Other specifications of interest rate rules commonly used in applied work also lead to models with lagged endogenous variables. Typically, the class of *generalized* (inertial) Taylor type rules which (also) reacts to the lagged interest rate will lead to such formulations.

Consequently, we now consider the general class of bivariate models:[19]

$$x_t = \Omega \hat{E}_t x_{t+1} + \delta x_{t-1} + \Phi w_t \tag{26}$$
$$w_t = \Psi w_{t-1} + v_t, \tag{27}$$

where x_t, w_t are, respectively, the vectors of endogenous and exogenous variables. The exogenous variables follow a stationary VAR, so that the eigenvalues of Ψ are inside the unit circle. v_t is *iid*.

4.1 Characterization of Non-Fundamental Solutions

In this subsection we impose RE, so that $\hat{E}_t x_{t+1} = E_t x_{t+1}$. Let $\eta_{t+1} = x_{t+1} - E_t x_{t+1}$ be any innovation process, so that it satisfies $E_t \eta_{t+1} = 0$. For

[19] The modifications are obvious when x_t, w_t are of arbitrary (finite) dimension.

the main part we assume that the matrix Ω is invertible, but we will also encounter the case where Ω is singular and will show the modifications to the technique.

The general solution of (26)-(27) can be written in the form

$$\begin{aligned} x_t &= \Omega^{-1}x_{t-1} - \Omega^{-1}\delta x_{t-2} - \Omega^{-1}\Phi w_{t-1} + \eta_t \\ w_t &= \Psi w_{t-1} + v_t \end{aligned} \tag{28}$$

or, introducing the notation $y_t = (x_t', x_{t-1}', w_t')'$, $u_t = (\eta_t', v_t')'$, in the form

$$y_t = \mathcal{B}_1 y_{t-1} + \mathcal{L}u_t, \text{ where } \mathcal{B}_1 = \begin{pmatrix} \Omega^{-1} & -\Omega^{-1}\delta & -\Omega^{-1}\Phi \\ I & 0 & 0 \\ 0 & 0 & \Psi \end{pmatrix}, \mathcal{L} = \begin{pmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{pmatrix}. \tag{29}$$

Note that this formulation does not assume invertibility of δ. Since there are many ways for specifying the innovation process η_t, there can be indeterminacies of REE as before. A very common further restriction on (29) is stationarity of the process. The general characterization (29) will form a stationary class of solutions if all eigenvalues of $\mathcal{B}_1$ are inside the unit circle. However, it may also be the case that one or more roots of $\mathcal{B}_1$ are outside the unit circle and we consider this next.

The main methodology we use is similar to the one used in the previous section, so we illustrate it concretely with Example 5. The technique will work even though δ is non-invertible here, see (25). First, the coefficient matrix of (29) is diagonalized, so that $\mathcal{B}_1 = Q\Lambda Q^{-1}$, where Λ is a diagonal matrix with the eigenvalues of $\mathcal{B}_1$ along its diagonal, i.e. $\Lambda = \lceil \lambda_1, ..., \lambda_6 \rfloor$. Since $\mathcal{B}_1$ is block-triangular, the last two eigenvalues (denoted λ_5, λ_6) are just those of Ψ and, hence, inside the unit circle. The remaining eigenvalues of $\mathcal{B}_1$ are given by those of the top left corner block 2×2 matrix that has a zero determinant since δ is singular in Example 5. So (at least) one eigenvalue is 0 (denoted λ_4) and if 0 or 1 of the remaining three eigenvalues (λ_1, λ_2, or λ_3) are outside the unit circle, equilibrium is indeterminate. Equilibrium is unique if exactly 2 of these eigenvalues are outside the unit circle and it is explosive in the remaining scenario. In fact, all of these situations are possible for Example 5, as illustrated in Figure 2 of (Bullard and Mitra 2000b).

(Bullard and Mitra 2000b) studied the learnability of the fundamental equilibrium in Example 5. Here our aim is to study the learnability of non-fundamental equilibria which are possible for values of χ_π less than 1. We develop the technique for obtaining the stationary sunspot solutions when exactly one eigenvalue of $\mathcal{B}_1$ is outside the unit circle since this usually obtains for plausible values of structural parameters. Hence, assume, w.l.o.g., that $|\lambda_1| < 1$, $|\lambda_2| > 1$ and $|\lambda_3| < 1$. We use the notation $Q^{-1} = (Q^{ij})$ and $x_t = (x_{1,t}, x_{2,t})'$ below. Appendix A.4 shows that the unique stationary solution can be written in the form

$$\begin{aligned}
&\begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} \\
= &\begin{pmatrix} \lambda_1 Q^{11} - Q^{13} & \lambda_1 Q^{12} - Q^{14} \\ -Q^{23} & -Q^{24} \end{pmatrix} \begin{pmatrix} x_{1,t-1} \\ x_{2,t-1} \end{pmatrix} \\
&+ \begin{pmatrix} \lambda_1 Q^{13} & \lambda_1 Q^{14} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{1,t-2} \\ x_{2,t-2} \end{pmatrix} - \begin{pmatrix} Q^{15} & Q^{16} \\ Q^{25} & Q^{26} \end{pmatrix} \begin{pmatrix} w_{1,t} \\ w_{2,t} \end{pmatrix} \\
&+ \begin{pmatrix} \lambda_1 Q^{15} & \lambda_1 Q^{16} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_{1,t-1} \\ w_{2,t-1} \end{pmatrix} + \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_{1,t} \\ \eta_{2,t} \\ v_{1,t} \\ v_{2,t} \end{pmatrix}.
\end{aligned} \tag{30}$$

which generally gives a vector ARMA(2,1) process in x_t. To shorten notation, rewrite the previous equation as

$$\begin{aligned}
x_t &= \bar{b}_1 x_{t-1} + \bar{b}_2 x_{t-2} + \text{ other terms involving shocks,} \\
\bar{b}_1 &= \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 Q^{11} - Q^{13} & \lambda_1 Q^{12} - Q^{14} \\ -Q^{23} & -Q^{24} \end{pmatrix}, \\
\bar{b}_2 &= \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_1 Q^{13} & \lambda_1 Q^{14} \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{31}$$

Introducing the notation $y_t^1 = (x_t', x_{t-1}')'$, we can rewrite (31) in the form a vector auto-regression of order one, VAR(1), as

$$y_t^1 = \begin{pmatrix} \bar{b}_1 & \bar{b}_2 \\ I & 0 \end{pmatrix} y_{t-1}^1 + \text{ other terms involving shocks.}$$

Stationarity requires that the four eigenvalues of the above coefficient matrix be less than one in modulus.

4.2 Learnability of Non-Fundamental REE

Using the concept of E-stability, learnability of these non-fundamental equilibria can be analyzed as before using the standard methodology. We proceed from the structural model (26)-(27) and begin with the PLM of the agents

$$x_t = a + b_1 x_{t-1} + b_2 x_{t-2} + c w_{t-1} + d\eta_t + e v_t,$$

where a, b_1, b_2, c, d and e are parameter matrices or vectors of appropriate dimensions. The form of the PLM is the same as (28). Using this PLM, the forecasts are given by

$$\begin{aligned} \hat{E}_t x_{t+1} &= (I + b_1)a + (b_1^2 + b_2)x_{t-1} + b_1 b_2 x_{t-2} + (b_1 c + c\Psi)w_{t-1} + b_1 d\eta_t \\ &\quad + (b_1 e + c)v_t, \end{aligned}$$

where we have again assumed that agents observe only the time t values of the exogenous variables and shocks but not the time t values of endogenous variables in making their forecasts.[20]

Substituting these forecasts into (26) leads to the ALM

$$\begin{aligned} x_t &= \Omega(I + b_1)a + \{\Omega(b_1^2 + b_2) + \delta\}x_{t-1} + \Omega b_1 b_2 x_{t-2} + \\ &\quad \{\Omega(b_1 c + c\Psi) + \Phi\Psi\}w_{t-1} + \Omega b_1 d\eta_t + \{\Omega(b_1 e + c) + \Phi\}v_t. \end{aligned}$$

The mapping from the PLM to ALM, $(a, b_1, b_2, c, d, e) \to T(a, b_1, b_2, c, d, e)$, is given by

$$\begin{aligned} T(a, b_1, b_2, c, d, e) &= (\Omega(I + b_1)a, \Omega(b_1^2 + b_2) + \delta, \\ &\quad \Omega b_1 b_2, \Omega(b_1 c + c\Psi) + \Phi\Psi, \Omega b_1 d, \Omega(b_1 e + c) + \Phi). \end{aligned}$$

The different REE are fixed points of the T mapping that, therefore, satisfy the matrix equations

$$a = \Omega(I + b_1)a \quad (32)$$

$$b_1 = \Omega(b_1^2 + b_2) + \delta \quad (33)$$

$$b_2 = \Omega b_1 b_2 \quad (34)$$

$$c = \Omega(b_1 c + c\Psi) + \Phi\Psi \quad (35)$$

$$d = \Omega b_1 d \quad (36)$$

$$e = \Omega(b_1 e + c) + \Phi. \quad (37)$$

[20] As in the case of no lags, the equilibria will be E-unstable if the information set includes period t values of the endogenous variables.

It can be seen that the equation for matrices b_1 and b_2 form an independent sub-system and involve a (matrix) quadratic equation. Clearly, $b_1 = \Omega^{-1}$, $b_2 = -\Omega^{-1}\delta$ solve this sub-system, but in general there are other solutions. Some of the solutions can again be singular matrices and this will be illustrated below.

Given a solution $\bar{b}_1$, $\bar{b}_2$ equations (32) and (35) generically uniquely determine a and c ($\bar{a}$ and $\bar{c}$). Given $\bar{b}_1$ and $\bar{c}$, (37) uniquely determines e. For sunspot equilibria, given $\bar{b}_1$, the matrix $I - \Omega\bar{b}_1$ must be singular (e.g. when $\bar{b}_1 = \Omega^{-1}$) in which case the equation for d has nontrivial solutions and sunspot equilibria exist.

E-stability of a fixed point is defined by means of the ordinary differential equation

$$\frac{d}{d\tau}(a, b_1, b_2, c, d, e) = T(a, b_1, b_2, c, d, e) - (a, b_1, b_2, c, d, e). \tag{38}$$

Thus a fixed point $(\bar{a}, \bar{b}_1, \bar{b}_2, \bar{c}, \bar{d}, \bar{e})$ is said to be E-stable if it is locally asymptotically stable under (38). To derive the E-stability and instability conditions we linearize (38). The necessary E-stability conditions are that the real parts of all the eigenvalues of the following matrices

$$\begin{aligned} DT_a(\bar{a}, \bar{b}_1, \bar{b}_2) &= \Omega(I + \bar{b}_1) \\ DT_c(\bar{b}_1, \bar{b}_2, \bar{c}) &= \Psi' \otimes \Omega + I \otimes \Omega\bar{b}_1 \\ DT_d(\bar{b}_1, \bar{b}_2) &= \Omega\bar{b}_1 \end{aligned}$$

as well as the matrix

$$\begin{pmatrix} \bar{b}_1' \otimes \Omega + I \otimes \Omega\bar{b}_1 & I \\ \bar{b}_2' \otimes \Omega & I \otimes \Omega\bar{b}_1 \end{pmatrix} \tag{39}$$

have real parts less than one. On the other hand, the solution is E-unstable if any of the eigenvalues of these matrices has a real part exceeding one.

In this case, clear-cut theoretical results for E-instability are generally not available since the eigenvalues of (39), when evaluated at $b_1 = \Omega^{-1}$, $b_2 = -\Omega^{-1}\delta$, depend on δ. Nevertheless, these conditions can readily be applied to specific models, as we now illustrate.

Continuing with Example 5, we consider the model (25) numerically. Adopt the calibrated values of the structural parameters $\varphi = (.157)^{-1} \approx 6.37$, $\lambda = .024$, $\beta = .99$, given in (Woodford 1999). For illustrative purposes, also consider $\chi_\pi = .8$, $\chi_z = .1$, $\mu = \rho = .9$, which leads to indeterminate equilibria, see Figure 2 of (Bullard and Mitra 2000b). For these values the solution given by $b_1 = \Omega^{-1}$, $b_2 = -\Omega^{-1}\delta$ turns out to be non-stationary. Consequently, we use the sunspot solution given by (31) with

$$\bar{b}_1 = \begin{pmatrix} -0.29 & 1.03 \\ .02 & .79 \end{pmatrix}, \bar{b}_2 = \begin{pmatrix} 0.08 & 0.67 \\ 0.02 & 0.13 \end{pmatrix}$$

which results in a stationary sunspot solution. However, one eigenvalue of (39) is 2.28 which makes the solution E-unstable.[21]

(Bullard and Mitra 2000b) found two stationary MSV solutions in the indeterminate region of Example 5 and both of them were always unlearnable. We, therefore, again observe the failure of Taylor principle leading to E-instability of all types of solutions, fundamental and non-fundamental, as in Section 3.3.

4.3 The Model of Clarida, Gali, and Gertler (2000)

We now look at a model analyzed in Section 4 of (Clarida, Gali, and Gertler 2000) which is similar to the one considered in the previous section with some slight modifications noted below. The structural model continues to consist of (1) but (2) is replaced by a slightly modified equation, namely,

$$\pi_t = \lambda z_t + \beta \hat{E}_t \pi_{t+1} - \lambda u_t, \tag{40}$$

where the parameters are the same as in Section 2 and the shocks g_t and u_t continue to follow the processes (3) and (4). (Clarida, Gali, and Gertler 2000) use an interest rate rule of the form

$$i_t = \theta i_{t-1} + (1-\theta)\chi_\pi \hat{E}_t \pi_{t+1} + (1-\theta)\chi_z z_t. \tag{41}$$

which has an inertial component captured by θ and reacts to the contemporaneous output gap and future forecast of inflation.

Plugging this rule into (1) and (40) yields the reduced from

$$\begin{pmatrix} z_t \\ \pi_t \\ i_t \end{pmatrix} = \Omega \begin{pmatrix} \hat{E}_t z_{t+1} \\ \hat{E}_t \pi_{t+1} \\ \hat{E}_t i_{t+1} \end{pmatrix} + \delta \begin{pmatrix} z_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{pmatrix} + \kappa \begin{pmatrix} g_t \\ u_t \end{pmatrix}, \tag{42}$$

[21] A similar situation seems to prevail for other values of policy parameters in the indeterminate region.

where

$$
\begin{aligned}
\Omega &= \begin{pmatrix} k_0 & k_0\varphi\{1-(1-\theta)\chi_\pi\} & 0 \\ \lambda k_0 & \beta+\lambda k_0\varphi\{1-(1-\theta)\chi_\pi\} & 0 \\ k_0(1-\theta)\chi_z & k_0(1-\theta)(\varphi\chi_z+\chi_\pi) & 0 \end{pmatrix}, \\
\delta &= \begin{pmatrix} 0 & 0 & -k_0\varphi\theta \\ 0 & 0 & -\lambda k_0\varphi\theta \\ 0 & 0 & k_0\theta \end{pmatrix}, \kappa = \begin{pmatrix} k_0 & 0 \\ \lambda k_0 & -\lambda \\ k_0(1-\theta)\chi_z & 0 \end{pmatrix}, \\
k_0 &= \{1+(1-\theta)\varphi\chi_z\}^{-1}. \qquad (43)
\end{aligned}
$$

It is possible to show that indeterminacies can arise in this model. This model fits the general framework of (26)-(27), except that x_t is now three-dimensional. In this case neither δ nor Ω are invertible, so that for computing the indeterminate equilibria we need to apply the diagonalization technique directly on (42) rather than the autoregressive form used previously. Define the free variables as $x_t^1 = x_t = (z_t, \pi_t, i_t)'$ and the predetermined variables as $x_t^2 = (i_{t-1}, w_{1t}, w_{2t})'$ where $w_{1t} = g_t$ and $w_{2t} = u_t$.

The technique starts from the following general form ($e_t = (\tilde{g}_t, \tilde{u}_t)'$ below)

$$
\begin{aligned}
x_t^1 &= B_1 E_t x_{t+1}^1 + C x_t^2, \qquad (44) \\
x_t^2 &= R x_{t-1}^1 + S x_{t-1}^2 + \kappa_1 e_t. \qquad (45)
\end{aligned}
$$

where in our case we have $B_1 = \Omega$,

$$
\begin{aligned}
C &= \begin{pmatrix} -k_0\varphi\theta & k_0 & 0 \\ -k_0\lambda\varphi\theta & k_0\lambda & -\lambda \\ k_0\theta & k_0(1-\theta)\chi_z & 0 \end{pmatrix}, \\
R &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \rho \end{pmatrix}, \kappa_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
$$

and k_0 is defined in (43). Having put the model in this form we compute the matrix J given by (see Chapter 10, Appendix 2 of (Evans and Honkapohja 2001c) for the details)

$$
J = \begin{bmatrix} I & -C \\ R & S \end{bmatrix}^{-1} \begin{bmatrix} \Omega & 0 \\ 0 & I \end{bmatrix}. \qquad (46)
$$

Equilibrium will be unique if exactly 3 (of the 6) eigenvalues of J are inside the unit circle, while it will be indeterminate if fewer than 3 eigenvalues are inside the unit circle.

For the model (42), J takes the form[22]

$$J = \begin{pmatrix} 1 & \varphi & 0 & -\varphi & \mu^{-1} & 0 \\ \lambda & \beta + \lambda\varphi & 0 & -\lambda\varphi & \lambda\mu^{-1} & -\lambda\rho^{-1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{-(1-\theta)\chi_z}{\theta} & \frac{-(1-\theta)(\varphi\chi_z+\chi_\pi)}{\theta} & 0 & \frac{1+(1-\theta)\varphi\chi_z}{\theta} & \frac{-(1-\theta)\chi_z}{\mu\theta} & 0 \\ 0 & 0 & 0 & 0 & \mu^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho^{-1} \end{pmatrix}.$$

We now compute and examine the learnability of indeterminate equilibria in (Clarida, Gali, and Gertler 2000). They have suggested that monetary policy in the pre-Volcker era (i.e., $1960-1979$) led the economy to stationary sunspot equilibria. The calibrated parameter values they use are $\varphi = 1$, $\lambda = .3$, $\beta = .99$, $\theta = .68$, $\chi_z = .27$, $\mu = \rho = .9$. χ_π was consistently found to be less than one in this period and is the cause for indeterminacy. If we use the baseline estimate of $\chi_\pi = 0.83$ in Table 2 of their paper, we find that exactly 2 eigenvalues of J are inside the unit circle.

Appendix A.5 shows that the final solution for $x_t = (z_t, \pi_t, i_t)'$ is a vector ARMA process given by (69) with the corresponding solutions for $\bar{b}_1, \bar{b}_2$ given by (70) and (71). This (sunspot) solution will be stationary if all the eigenvalues of the matrix

$$\begin{pmatrix} \bar{b}_1 & \bar{b}_2 \\ I & 0 \end{pmatrix} \tag{47}$$

are inside the unit circle. For the period $1960-79$, the $\bar{b}_1$ and $\bar{b}_2$ matrices are

$$\bar{b}_1 = \begin{pmatrix} .41 & .50 & -1.5 \\ .44 & .54 & -.82 \\ .12 & .15 & .45 \end{pmatrix}, \bar{b}_2 = \begin{pmatrix} 0 & 0 & 1.02 \\ 0 & 0 & 1.10 \\ 0 & 0 & .30 \end{pmatrix}.$$

The maximum eigenvalue of (47) is .95 so that this solution is stationary. However, the eigenvalues of (39) have a pair of complex conjugates with real parts 2.1 so that the solution is not E-stable. This shows that even though there exist stationary sunspot equilibria in the pre-Volcker period, they are not learnable by private agents.

[22] Clearly, from its structure, it is apparent that 3 of the eigenvalues of J are always 0, ρ^{-1} and μ^{-1}.

It can also be shown that even the fundamental equilibria are not E-stable for these parameter configurations. The MSV (fundamental) solutions take the form

$$x_t = bx_{t-1} + cw_t$$

and solving the matrix quadratic, $\Omega b^2 - b + \delta = 0$, yields two stationary MSV solutions for b given by

$$\begin{pmatrix} 0 & 0 & -1.5 \\ 0 & 0 & -.82 \\ 0 & 0 & .45 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & .19 \\ 0 & 0 & 1.01 \\ 0 & 0 & .95 \end{pmatrix}. \tag{48}$$

If agents have a PLM of the form

$$x_t = a + bx_{t-1} + cw_t \tag{49}$$

then a necessary condition for E-stability is that the eigenvalues of the matrix $\Omega + \Omega\bar{b}$ have real parts less than one if agents use last period data on output, inflation, and interest rates to form their forecasts. However, it is easy to check that this condition is violated for both the solutions given in (48).[23] We note here that the estimated rule in the pre-Volcker era fails the Taylor principle and we (again) find all types of RE solutions, both MSV and SSEs, unstable under learning.[24]

These results offer a novel explanation for the high inflation in the pre-Volcker era. Since neither the fundamental nor the non-fundamental equilibria are E-stable, the high inflation of the 1960s and 1970s may have been due to the persistent learning dynamics of private sector agents. The forecasting errors made by agents did not disappear over time owing to the monetary policy being pursued by the Federal Reserve. On this interpretation, these errors were not due to the convergence of the economy to a sunspot equilibrium, as suggested in (Clarida, Gali, and Gertler 2000).

On the other hand, in the Volcker-Greenspan era the monetary policy followed was not compatible with the existence of a stationary sunspot equilibrium. Using the baseline estimates of $\chi_\pi = 2.15$, $\chi_z = .93$ and $\theta = .79$ in

[23] A similar conclusion follows if agents use contemporaneous information on output, inflation, and interest rates in forming their forecasts.

[24] Unlike the rules in Examples 1-5, this rule is inertial but it still fails the Taylor principle, see (Woodford 2000) for the definition of the Taylor principle in this case.

Table 2 of (Clarida, Gali, and Gertler 2000) for this period, one can check that there exists only one stationary MSV solution, namely

$$\begin{pmatrix} 0 & 0 & -1.22 \\ 0 & 0 & -0.64 \\ 0 & 0 & 0.43 \end{pmatrix},$$

which is E-stable if agents have a PLM of the form (49). This monetary policy satisfies the Taylor principle and was conducive to learnability of the unique MSV solution. This may in fact have contributed to the low inflation during this period.

5 Discussion and Concluding Remarks

We have carried out a general analysis of learnability for multivariate forward looking linear models (with and without lags) and all of the known forms of non-fundamental equilibria. These results apply to models of monetary policy which are being used to give advice to policy makers. We emphasize that learnability of fundamental equilibrium and unlearnability of non-fundamental equilibria are an important constraint that monetary policy makers should respect since otherwise, undesirable fluctuations may result. In addition, learnability puts restrictions that model parameters must satisfy if non-fundamental REE are going to be a useful framework for applications.

A theme that is apparent in our results is the connection between the Taylor principle and learnability. (Bullard and Mitra 2000b) earlier observed this for the fundamental REE and we find that this extends (in a certain sense) to the set of non-fundamental equilibria. Policies violating the Taylor principle usually result in indeterminacy, see (Bullard and Mitra 2000b), and this is put forward as a primary reason for avoiding them - this is true for the rules in Examples 1, 2, 3, and 5. In all of these cases we have found that all of the indeterminate equilibria are also unlearnable. This provides a novel explanation for the undesirability of rules violating the Taylor principle: they result in E-instability of both MSV and indeterminate solutions. Once we abstract from the assumption of RE on the part of private agents, these rules may lead to persistent learning dynamics as agents try to find (unsuccessfully) some equilibrium when in fact no learnable RE solution exists.

A concrete application of this idea is to the scenario in (Clarida, Gali, and Gertler 2000). They use estimated values for a forward looking policy rule, which violates the Taylor priciple, to suggest that the high and volatile inflation in the U.S. in the 1960s/70s may have been due to the indeterminate equilibria caused by the policy. Our analysis has shown that neither the fundamental nor the non-fundamental equilibria were learnable during this period so that the volatile period was perhaps a situation of agents trying unsuccessfully to find some equilibrium. A further analysis of this issue would certainly seem worth while.

At the other end of the spectrum, we have found that when an interest rule satisfying the Taylor principle is associated with indeterminacy, there may exist (some) learnable, non-fundamental equilibrium. In this regard, of particular importance, are the policies in Examples 2 and 4, i.e. when the bank uses inflation forecasts in its policy. Both fundamental and some non-fundamental RE solutions, are potentially learnable with these rules. Our analysis, therefore, provides in some cases support to the dangers pointed out in (Bernanke and Woodford 1997) from the learning viewpoint.

This result also underlies the importance of avoiding indeterminacies when forward looking policy rules conform to the Taylor principle. Indeterminacy can be avoided with moderate aggression to inflation and/or output forecasts and this (also) results in E-stability of the unique fundamental equilibrium, see (Bullard and Mitra 2000b).[25]

An additional way to reduce the possibility of indeterminacy is to make the interest rule react directly to its own past values. These *inertial* rules have been found to have desirable properties: they can lead to the existence of a unique learnable fundamental equilibrium and also have the potential to implement optimal policy of the central bank (see (Bullard and Mitra 2000a) and (Rotemberg and Woodford 1999)). It has been well documented that policymakers indeed show a clear tendency to smooth out changes in nominal interest rates, see (Rudebusch 1995). Note that this inertial component makes it easier to satisfy the Taylor principle, see (Woodford 2000) for the details. The interest rule estimated for the U.S. since the 1980s by (Clarida, Gali, and Gertler 2000) has this inertial component that in conjunction with its response to the inflation forecast and output gap fulfils the Taylor principle and leads to the existence of a unique learnable fundamental equilibrium.

[25]We note that relatively modest responses, particularly to the output gap, are also supported in the very different model of (Christiano and Gust 1999). Similarly, (Orphanides 2000) argues for *prudent* policies owing to the difficulties in measuring the output gap. Existence of parameter uncertainty also supports a less activist optimal policy for a central bank, see (Wieland 1998).

Our analysis, therefore, suggests that Taylor principle in rules with inertia may have contributed to the low and stable inflation to the present times.

In summary, we do not advocate policies that violate the Taylor principle. Policies satisfying the Taylor principle are recommended as long as they do not lead to indeterminacy. In addition, inflation-targeting central banks should adopt a policy of flexible inflation targeting instead of strict inflation targeting since the latter may easily lead to the existence of learnable, indeterminate equilibria. This is probably what most inflation-targeting central banks seem to do in practice.

Finally, on a broader perspective, the general results of this paper show that, in many cases, the non-fundamental equilibria are not learnable. This suggests that focus on the fundamental REE is often justifiable, provided this equilibrium is shown to be learnable. In this sense, the analysis lends some support to the arguments in (McCallum 2001a) and (McCallum 2001b).

A Appendices: Derivations

A.1 Proof of Proposition 1

Define the new variables $p_t = Q^{-1}y_t$.[26] This allows us to write the system (12) in the form

$$p_t = \Lambda p_{t-1} + Q^{-1}u_t. \tag{50}$$

The second equation of (50) can then be written as

$$p_{2,t} = \lambda_2 p_{2,t-1} + Q^{21}\eta_{1,t} + Q^{22}\eta_{2,t} + Q^{23}v_{1,t} + Q^{24}v_{2,t},$$

where the notation $Q^{-1} = (Q^{ij})$ has been used. Stationarity implies the restriction $p_{2,t} = 0$ or

$$Q^{21}x_{1,t} + Q^{22}x_{2,t} + Q^{23}w_{1,t} + Q^{24}w_{2,t} = 0. \tag{51}$$

The first equation is

$$p_{1,t} = \lambda_1 p_{1,t-1} + Q^{11}\eta_{1,t} + Q^{12}\eta_{2,t} + Q^{13}v_{1,t} + Q^{14}v_{2,t}. \tag{52}$$

[26] This is a modification of the well-known Blanchard-Kahn technique for obtaining stationary solutions to regular (i.e. "saddle-point stable") multivariate linear RE models. See, Appendix 2 of Chapter 10 in (Evans and Honkapohja 2001c) for the extension of the technique to irregular models.

These imply that one of components of the martingale difference sequence η_t is a linear combination of the other component and the *iid* shocks to the exogenous variables. Using the definition

$$p_{1,t} = Q^{11}x_{1,t} + Q^{12}x_{2,t} + Q^{13}w_{1,t} + Q^{14}w_{2,t}$$

we can write (52) as

$$\begin{aligned} Q^{11}x_{1,t} + Q^{12}x_{2,t} &= \lambda_1 Q^{11}x_{1,t-1} + \lambda_1 Q^{12}x_{2,t-1} - Q^{13}w_{1,t} - Q^{14}w_{2,t} \\ &\quad + \lambda_1 Q^{13}w_{1,t-1} + \lambda_1 Q^{14}w_{2,t-1} + \\ &\quad Q^{11}\eta_{1,t} + Q^{12}\eta_{2,t} + Q^{13}v_{1,t} + Q^{14}v_{2,t}. \end{aligned}$$

This equation and (51) make up the system (13) in the text.

A.2 Mathematica Routine used in Proposition 4

We give a brief description of the Mathematica routine used in computing the eigenvalues of $DT_b(\bar{b})$. For computing $\bar{b}$, we need only the top left 2×2 block of the diagonalization matrix for $\mathcal{B}$, namely Q. In addition, since $\mathcal{B}$ is block triangular, this matrix corresponds to the diagonalization of Ω^{-1}. Denote the 2×2 matrix $\Omega = (\Omega_{ij})$. The Jordan decomposition on Ω^{-1} yields the following diagonalization matrix

$$M = \begin{pmatrix} \frac{\Omega_{11} - \Omega_{22} + \sqrt{\Omega_{11}^2 + \Omega_{22}^2 - 2\Omega_{11}\Omega_{22} + 4\Omega_{12}\Omega_{21}}}{2\Omega_{21}} & \frac{\Omega_{11} - \Omega_{22} - \sqrt{\Omega_{11}^2 + \Omega_{22}^2 - 2\Omega_{11}\Omega_{22} + 4\Omega_{12}\Omega_{21}}}{2\Omega_{21}} \\ 1 & 1 \end{pmatrix}.$$

Note that, as mentioned above, M coincides with the top left 2×2 block of Q. The eigenvalues of Ω^{-1} are

$$\begin{aligned} \lambda_1 &= \frac{\Omega_{11} + \Omega_{22} - \sqrt{\Omega_{11}^2 + \Omega_{22}^2 - 2\Omega_{11}\Omega_{22} + 4\Omega_{12}\Omega_{21}}}{2\left(\Omega_{11}\Omega_{22} - \Omega_{12}\Omega_{21}\right)}, \\ \lambda_2 &= \frac{\Omega_{11} + \Omega_{22} + \sqrt{\Omega_{11}^2 + \Omega_{22}^2 - 2\Omega_{11}\Omega_{22} + 4\Omega_{12}\Omega_{21}}}{2\left(\Omega_{11}\Omega_{22} - \Omega_{12}\Omega_{21}\right)}. \end{aligned}$$

In general, we do not know whether λ_1 or λ_2 has the smaller modulus. Assume for now $|\lambda_1| < 1$, $|\lambda_2| > 1$. With this,

$$\bar{b} = M \begin{pmatrix} \lambda_1 M_{11}^{-1} & \lambda_1 M_{12}^{-1} \\ 0 & 0 \end{pmatrix}$$

where M^{-1}_{ij} denotes the (i, j) element of M^{-1} and $\bar{b}$ coincides with (24). It is then easy to check that one of the eigenvalues of $DT_b(\bar{b}) = \bar{b}' \otimes \Omega + I \otimes \Omega\bar{b}$ is 2.[27]

A.3 Mathematical Lemma used in Proposition 6

Consider the following linear system of differential equations

$$\dot{x} = Ax \tag{53}$$

with x an n dimensional vector. We assume that A can be written in the form $A = Q\Lambda Q^{-1}$, where the matrix of eigenvalues takes the form

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{54}$$

and all $n-1$ eigenvalues of Λ_1 have negative real parts. Λ_1 is thus invertible. Partition $Q^{-1} = (Q^{ij})$ as

$$Q^{-1} = \begin{pmatrix} Q^{11} & Q^{12} \\ Q^{21} & Q^{22} \end{pmatrix}$$

where Q^{11} is $(n-1) \times (n-1)$, Q^{12} is $(n-1) \times 1$, Q^{21} is $1 \times (n-1)$ and Q^{22} is a non-zero scalar. We assume that the matrix $Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}$ is invertible.

We prove the following auxiliary result:[28]

Lemma 8 *For any initial condition $x(0)$ the trajectory $x(t \mid x(0))$ of (53) converges to the set of equilibrium points $\{\hat{x} \mid A\hat{x} = 0\}$.*

Pre-multiplying (53) by Q^{-1}, we get

$$Q^{-1}\dot{x} = \Lambda Q^{-1}x \tag{55}$$

or defining the transformed variable, $y = Q^{-1}x$, we get $\dot{y} = \Lambda y$. So, using (54), we have $\dot{y}_1 = \Lambda_1 y_1$, and $\dot{y}_2 = 0$ where y_1 is $(n-1) \times 1$ and y_2 is

[27] We refer the reader to the web page www.valt.helsinki.fi/raka/seppo.htm for the full Mathematica routine used in this proof.

[28] We have not discovered this result in the mathematics literature, though we suspect that it is a known result.

scalar. Thus, $y_2(t) = y_2(0)$ for all t, i.e. given an initial condition $x(0)$, we can compute $y(0) = Q^{-1}x(0)$ and along the trajectory $x(t \mid x(0))$ the last component of $y(t) = Q^{-1}x(t \mid x(0))$ does not change over time.

Generally, we have

$$y_1(t) = Q^{11}x_1(t) + Q^{12}x_2(t), \tag{56}$$

$$y_2(t) = Q^{21}x_1(t) + Q^{22}x_2(t). \tag{57}$$

Using (55) we, therefore, get

$$Q^{11}\dot{x}_1 + Q^{12}\dot{x}_2 = \Lambda_1(Q^{11}x_1 + Q^{12}x_2), \tag{58}$$

$$Q^{21}\dot{x}_1 + Q^{22}\dot{x}_2 = 0. \tag{59}$$

where the time subscripts have been suppressed. Since Q^{22} is non-zero, we have $\dot{x}_2 = -(Q^{22})^{-1}Q^{21}\dot{x}_1$. Plugging this into (58) we get

$$[Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}]\dot{x}_1 = \Lambda_1(Q^{11}x_1 + Q^{12}x_2). \tag{60}$$

We know from (57) that $Q^{21}x_1(t) + Q^{22}x_2(t) = y_2(t) = y_2(0)$ which implies

$$x_2(t) = (Q^{22})^{-1}y_2(0) - (Q^{22})^{-1}Q^{21}x_1(t). \tag{61}$$

Substituting (61) into (60) yields

$$\begin{aligned}[Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}]\dot{x}_1 &= \Lambda_1[Q^{11}x_1 + Q^{12}(Q^{22})^{-1}y_2(0) - \\ &\quad Q^{12}(Q^{22})^{-1}Q^{21}x_1]\end{aligned}$$

or, after rearranging,

$$[Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}]\dot{x}_1 = \Lambda_1 Q^{12}(Q^{22})^{-1}y_2(0) + \Lambda_1[Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}]x_1 \tag{62}$$

Define $z = [Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}]x_1$. Then (62) yields

$$\dot{z} = \Lambda_1 Q^{12}(Q^{22})^{-1}y_2(0) + \Lambda_1 z$$

implying that

$$z \to \bar{z} = -Q^{12}(Q^{22})^{-1}y_2(0) \text{ as } t \to \infty. \tag{63}$$

Since $[Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}]$ is invertible,

$$x_1 \to \bar{x}_1 = [Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}]^{-1}\bar{z} \text{ as } t \to \infty.$$

Using (61), this in turn implies that

$$x_2 \to \bar{x}_2 = (Q^{22})^{-1}y_2(0) - (Q^{22})^{-1}Q^{21}\bar{x}_1 \text{ as } t \to \infty. \quad (64)$$

We now return to the original system (53). Obviously, any equilibrium point $\hat{x}$ satisfies $A\hat{x} = 0$ or $Q\Lambda Q^{-1}\hat{x} = 0$. It follows that $Q\Lambda\hat{y} = 0$ or, since Q is invertible, $\Lambda\hat{y} = 0$. Thus $\Lambda_1\hat{y}_1 = 0$, implying $\hat{y}_1 = 0$, since Λ_1 is invertible. In other words, $Q^{11}\hat{x}_1 + Q^{12}\hat{x}_2 = 0$.

The final step is to check whether $Q^{11}\bar{x}_1 + Q^{12}\bar{x}_2 = 0$ so that $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is indeed an equilibrium of the original system. Thus, calculate

$$\begin{aligned} Q^{11}\bar{x}_1 + Q^{12}\bar{x}_2 &= Q^{11}\bar{x}_1 + Q^{12}[(Q^{22})^{-1}y_2(0) - (Q^{22})^{-1}Q^{21}\bar{x}_1] \\ &= [Q^{11} - Q^{12}(Q^{22})^{-1}Q^{21}]\bar{x}_1 + Q^{12}(Q^{22})^{-1}y_2(0) \\ &= \bar{z} + Q^{12}(Q^{22})^{-1}y_2(0) = 0 \end{aligned}$$

as required, where we have used (63) and (64) above.

A.4 Details on Section 4.1

Define new variables $p_t = Q^{-1}y_t$. This allows us to write the system (29) in the form

$$p_t = \Lambda p_{t-1} + Q^{-1}\mathcal{L}u_t. \quad (65)$$

The second equation of (65) can then be written as

$$p_{2,t} = \lambda_2 p_{2,t-1} + l_{21}\eta_{1,t} + l_{22}\eta_{2,t} + l_{23}v_{1,t} + l_{24}v_{2,t},$$

where the notation $Q^{-1}\mathcal{L} = (l_{ij})$ has been used. Stationarity thus implies the restriction $p_{2,t} = 0$ or

$$Q^{21}x_{1,t} + Q^{22}x_{2,t} + Q^{23}x_{1,t-1} + Q^{24}x_{2,t-1} + Q^{25}w_{1,t} + Q^{26}w_{2,t} = 0. \quad (66)$$

where $Q^{-1} = (Q^{ij})$, and $x_t = (x_{1,t}, x_{2,t})'$. The first equation is

$$p_{1,t} = \lambda_1 p_{1,t-1} + l_{11}\eta_{1,t} + l_{12}\eta_{2,t} + l_{13}v_{1,t} + l_{14}v_{2,t} \quad (67)$$

Using the definition

$$p_{1,t} = Q^{11}x_{1,t} + Q^{12}x_{2,t} + Q^{13}x_{1,t-1} + Q^{14}x_{2,t-1} + Q^{15}w_{1,t} + Q^{16}w_{2,t}$$

we can write (67) as

$$\begin{aligned} Q^{11}x_{1,t} + Q^{12}x_{2,t} &= (\lambda_1 Q^{11} - Q^{13})x_{1,t-1} + (\lambda_1 Q^{12} - Q^{14})x_{2,t-1} + \lambda_1 Q^{13}x_{1,t-2} \\ &\quad + \lambda_1 Q^{14}x_{2,t-2} - Q^{15}w_{1,t} - Q^{16}w_{2,t} + \lambda_1 Q^{15}w_{1,t-1} \\ &\quad + \lambda_1 Q^{16}w_{2,t-1} + l_{11}\eta_{1,t} + l_{12}\eta_{2,t} + l_{13}v_{1,t} + l_{14}v_{2,t} \qquad (68) \end{aligned}$$

Equations (66) and (68) imply (30) in the text.

A.5 Details on Section 4.3

For the computation of irregular equilibria in the model of (Clarida, Gali, and Gertler 2000), we follow the technique illustrated in Chapter 10, Appendix 2 of (Evans and Honkapohja 2001c). We factor J as $\Lambda = Q^{-1}JQ$, where $Q^{-1} = \{q^{ij},\, i,j = 1,..,6\}$ and Λ are correspondingly partitioned as

$$\begin{aligned} Q^{-1} &= \begin{pmatrix} Q^{11}(1,1) & Q^{11}(1,2) & Q^{12}(1) \\ Q^{11}(2,1) & Q^{11}(2,2) & Q^{12}(2) \\ Q^{21}(1) & Q^{21}(2) & Q^{22} \end{pmatrix}, \\ \Lambda &= \begin{pmatrix} \Lambda_1^* & 0 & 0 \\ 0 & \Lambda_1^{\#} & 0 \\ 0 & 0 & \Lambda_2 \end{pmatrix}. \end{aligned}$$

Note that the diagonal matrix Λ_1^* above contains the eigenvalues of J with modulus less than one whereas $\Lambda_1^{\#}$ and Λ_2 are diagonal matrices containing the eigenvalues of J with modulus more than one. The free variables are also partitioned into the sets

$$x_t^1 = \begin{pmatrix} x_t^{1*} \\ x_t^{1\#} \end{pmatrix}.$$

If we use the baseline estimates in Table 2 of (Clarida, Gali, and Gertler 2000) for the period $1960 - 79$, the eigenvalues of J happen to be $\lambda_1 = 0$, $\lambda_2 = .63$, $\lambda_3 = 1.05$, $\lambda_4 = 2.21$, $\lambda_5 = \mu^{-1}$, $\lambda_6 = \rho^{-1}$, i.e., exactly 2 eigenvalues of J are inside the unit circle. Assume that $\Lambda_1^* = \{\lambda_1, \lambda_2\}, \Lambda_1^\# = \{\lambda_3\}$, and $\Lambda_2 = \{\lambda_4, \lambda_5, \lambda_6\}$. We have here $x_t^{1*} = \{z_t, \pi_t\}$, $x_t^{1\#} = \{i_t\}$, and

$$\begin{aligned}
Q^{11}(1,1) &= \begin{pmatrix} q^{11} & q^{12} \\ q^{21} & q^{22} \end{pmatrix}, Q^{11}(1,2) = \begin{pmatrix} q^{13} \\ q^{23} \end{pmatrix}, \\
Q^{11}(2,1) &= \begin{pmatrix} q^{31} & q^{32} \end{pmatrix}, Q^{11}(2,2) = \left(q^{33}\right), \\
Q^{12}(1) &= \begin{pmatrix} q^{14} & q^{15} & q^{16} \\ q^{24} & q^{25} & q^{26} \end{pmatrix}, Q^{12}(2) = \begin{pmatrix} q^{34} & q^{35} & q^{36} \end{pmatrix}, \\
Q^{21}(1) &= \begin{pmatrix} q^{41} & q^{42} \\ q^{51} & q^{52} \\ q^{61} & q^{62} \end{pmatrix}, Q^{21}(2) = \begin{pmatrix} q^{43} \\ q^{53} \\ q^{63} \end{pmatrix}, Q^{22} = \begin{pmatrix} q^{44} & q^{45} & q^{46} \\ q^{54} & q^{55} & q^{56} \\ q^{64} & q^{65} & q^{66} \end{pmatrix}.
\end{aligned}$$

$Q^{11} \equiv (Q^{11}(i,j))$ is then given by $(i, j = 1, 2)$

$$Q^{11} = \begin{pmatrix} q^{11} & q^{12} & q^{13} \\ q^{21} & q^{22} & q^{23} \\ q^{31} & q^{32} & q^{33} \end{pmatrix}.$$

Assume that Q^{11} is invertible and let $(Q^{11})^{-1} = \{q_{ij},\ i, j = 1, .., 3\}$. It can be checked that (see (Evans and Honkapohja 2001c) for the details) the final solution (with some abuse of notation) for $x_t = (z_t, \pi_t, i_t)'$ is

$$x_t = \bar{b}_1 x_{t-1} + \bar{b}_2 x_{t-2} + .. \tag{69}$$

which is a vector ARMA process (terms involving the shocks are omitted since they are not needed for E-stability). Here we have

$$\bar{b}_1 = \begin{pmatrix} \lambda_3^{-1} q_{13} q^{31} & \lambda_3^{-1} q_{13} q^{32} & q_{13}(\lambda_3^{-1} q^{33} - q^{34}) - q_{11} q^{14} - q_{12} q^{24} \\ \lambda_3^{-1} q_{23} q^{31} & \lambda_3^{-1} q_{23} q^{32} & q_{23}(\lambda_3^{-1} q^{33} - q^{34}) - q_{21} q^{14} - q_{22} q^{24} \\ \lambda_3^{-1} q_{33} q^{31} & \lambda_3^{-1} q_{33} q^{32} & q_{33}(\lambda_3^{-1} q^{33} - q^{34}) - q_{31} q^{14} - q_{32} q^{24} \end{pmatrix} \tag{70}$$

$$\bar{b}_2 = \begin{pmatrix} 0 & 0 & \lambda_3^{-1} q_{13} q^{34} \\ 0 & 0 & \lambda_3^{-1} q_{23} q^{34} \\ 0 & 0 & \lambda_3^{-1} q_{33} q^{34} \end{pmatrix}. \tag{71}$$

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