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**Abstract**

Learning with bounded memory in stochastic frameworks is incomplete in the sense that the learning dynamics cannot converge to an rational expectations equilibrium (REE). The properties of the dynamics arising from such rules are studied for models with steady states. If in standard linear models the REE is in a certain sense expectationally stable (E-stable), then the dynamics are asymptotically stationary and forecasts are unbiased. We also provide similar local results for a class of nonlinear models with small noise and their approximations.

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# 1 Introduction

There exists by now a sizeable literature that studies the dynamics of adaptive learning in macroeconomic and market equilibrium models. Two fundamental issues addressed in this literature are (i) can economic agents, who rely on "on-line" estimation rules and forecasting with data on relevant variables, learn to have rational expectations in the long run and (ii) what are the stable outcomes of such learning processes. The literature has been recently surveyed in (Evans and Honkapohja 1999) and (Marimon 1997).

A common starting point in this research is to postulate that economic agents behave like econometricians, i.e. they use standard econometric techniques to estimate the parameters of the stochastic process of the relevant variables and forecast the future values using these estimated parameter values. The assumed form of the stochastic process, the *perceived law of motion* (PLM), is taken to be correctly specified in the sense that with right parameter values it coincides with the rational expectations equilibria (REE) of interest.[1] In the most commonly studied frameworks *learning is complete* in the sense that the economy settles in an REE if the learning dynamics converges. For most circumstances the condition for the convergence of learning dynamics has turned out to be the so-called expectational stability (E-stability) condition. We will define E-stability precisely below.

The possibility of nonconvergence of learning dynamics has also been considered in the literature. It may be the case that the economy has no stable REE for particular values of the model parameters.[2] Another possibility is that *learning dynamics is incomplete* in the sense that it has no chance of converging to an REE for any parameter configuration, see e.g. Section 5 of (Evans and Honkapohja 1999) for a discussion and references. The incompleteness of learning may arise for different reasons. First, the PLM may be incorrectly specified. Second, the procedure for estimating the PLM may not yield exact convergence. Nevertheless, dynamics of incomplete learning may give a good approximation to actual economic data.[3]

Several papers in the literature have considered learning with a finite memory, and such rules have been shown to result in complete learning in various deterministic models. Given suitable values of structural parameters, the learning economy can indeed find an REE, see e.g. (Guesnerie and Woodford 1991), (Grandmont 1985), (Grandmont and Laroque 1986), (Balasko and Royer 1996), (Grandmont 1998), (Evans and Honkapohja 2000a), and Section 2, Chapter 7 of (Evans and Honkapohja 2000b), though for other

[1] Note that the dynamics are econometrically misspecified during learning, but the misspecification will disappear in the limit if the learning dynamics converges to an REE.

[2] This is discussed e.g. by (Grandmont and Laroque 1991), (Bullard 1994) and (Grandmont 1998).

[3] (Marcet and Nicolini 1998) argue that dynamics with certain type of incomplete learning provides a good description of the inflation processes in Latin America. (Sargent 1999) suggests that a similar form of incomplete learning may be an essential ingredient in the rise and decline of inflation in post-war America.

parameter values the REE may be unstable. In contrast, learning with a finite memory is known to lead to incomplete learning in the same models when random shocks are present. For example, if agents try to learn a steady state by computing a sample mean from a finite data set of fixed length, the resulting dynamics cannot converge to a rational expectations solution for any parameter values when a random shock is present, see (Evans and Honkapohja 1995b).

A different motivation for learning with bounded (or finite) memory is the observation that it may be optimal for any single agent to use a particular finite memory length if all other agents in the economy are using the same memory length. In other words, learning with bounded memory may arise as a self-confirming equilibrium which is in the spirit of (Sargent 1999). (Mitra 2000) considered this possibility in settings without self-referential aspects. His results are applicable to our framework and illustrate this possibility. This gives a further reason to characterize the dynamics arising from learning with bounded memory.

These considerations invite a further study into the nature of incomplete learning with finite memory when the economy is subject to random shocks. In this paper we show that, despite incompleteness, dynamics of learning can have several attractive properties in standard frameworks. Most importantly, E-stability has a key role for stationarity of the learning dynamics. Generally speaking, under E-stability the state of economy has a unique invariant distribution in the long run. Learning is then asymptotically unbiased in the sense that the mean of the first moment of the forecast is correct. There is also approximate convergence of the higher moments with the approximation improving as the support of the shock becomes small. Finally, we obtain some results on the influence of the memory length on the residual variance of the forecasts.

These properties seem relatively intuitive, but their precise statements require considerable care. In this paper we derive these results for standard frameworks, where agents try to learn a steady state. Several well-known models fall into the categories of models analyzed in this paper, and we start by discussing two examples.

*Example 1.* (The Muth market model) Consider a competitive market with a production lag. Demand is assumed to be a downward-sloping function of the market price, while supply depends on the expected price in consequence of a production lag. For simplicity, assume that suppliers are identical in their economic characteristics, including expectations and learning rules.

Postulate the demand function

$$q_t^d = C_1 - Bp_t$$

and the supply function

$$q_t^s = C_2 + DE_{t-1}^* p_t + v_t,$$

where $q_t^i, i = d, s$, denote quantities demanded and supplied, $p_t$ is the market price, $E_{t-1}^* p_t$ denotes the (in general non-rational) price expectation of the suppliers, and $v_t$ is an *iid* random shock with mean 0. $B, C_1, C_2$ and $D$ are positive parameters.

Using equality of supply and demand, the reduced form of this model takes the form

$$p_t = \alpha + \beta E^*_{t-1} p_t + u_t, \tag{1}$$

where $u_t = -B^{-1}v_t$, $\alpha = B^{-1}(C_1 - C_2)$ and $\beta = -B^{-1}D$.[4] A (stochastic) steady state equilibrium can be written in the form

$$\hat{p}_t = \frac{\alpha}{1-\beta} + u_t.$$

To model learning it is postulated that agents think that the economy is in a steady state but do not know the value of the constant. In other words, they have a PLM of the form $p_t = A + u_t$ and they form an estimate of the value of $A$ using past observations on prices. In this model the estimate is also the forecasted price $E^*_{t-1}p_t$. Computing the sample mean for a set of data is the standard statistical technique for estimating an unknown mean, so that a natural estimate of $A$ at time $t$ is given by

$$A_t = T^{-1} \sum_{i=1}^{T} p_{t-i}$$

if agents use past $T$ prices in computing the sample mean. Substituting the estimate into (1) yields

$$p_t = \alpha + \frac{\beta}{T} \sum_{i=1}^{T} p_{t-i} + u_t \tag{2}$$

which is an autoregressive process of order $T$ (an $AR(T)$ process). The forecast for the equilibrium price is also $\frac{1}{T}\sum_{i=1}^{T} p_{t-i}$ which is a random variable with a nontrivial asymptotic variance if $p_t$ follows (2). This shows that forecasts from finite-memory rules cannot converge to rational expectations equilibria. In this paper we are interested in the properties of the dynamics (2).

*Example 2.* Several common economic models lead to the reduced form

$$y_t = \alpha + \beta E^*_t y_{t+1} + v_t \tag{3}$$

in which the current value of the endogenous variable depends on its expected value for next period. Again $v_t$ is an *iid* random shock. (Sometimes an exogenous non-iid variable is added to the reduced form. We omit it for simplicity.)

For example, the demand for money is assumed to be a linear function of expected inflation in the simple monetary inflation model. Assuming a constant nominal stock of money then yields (3) as the reduced form. Other examples leading to (3) are the model of a small open economy with purchasing power parity on prices and open interest rate

[4] Some other models, e.g. a version of the (Lucas 1973) island model, also lead to the same reduced form. In Lucas' model $\beta > 0$.

parity, and the model of risk-neutral asset pricing in which the current asset price is the present value of expected price next period plus dividends.[5]

Model (3) has a stochastic steady state solution of the form $y_t = \hat{A} + v_t$, where $\hat{A} = \frac{\alpha}{1-\beta}$, and a natural learning rule to estimate the constant, assumed unknown, is computation the sample mean from a set of past values of $y_t$, i.e. the estimate in period $t$ is given by $A_t = T^{-1} \sum_{i=1}^{T} y_{t-i}$. (This assumes that current value of $y_t$ is not used in the estimation. This avoids a simultaneity problem in the model.) Again the dynamics of learning can be described by an $AR(T)$ process.

These two examples have a convenient linearity property, and the learning dynamics can be analyzed by standard techniques from time series analysis. We will study the first and second moments of the learning dynamics described by the $AR(T)$ process in Section 2. In Subsection 2.5 we will illustrate the possibility that learning with bounded memory may be a self-confirming equilibrium.

Nonlinear models with stochastic steady states also appear in the literature. In Section 3 we take up a general class of nonlinear models which was analyzed for complete learning by (Evans and Honkapohja 1995b). It turns out that, for models with small shocks, E-stability implies useful asymptotic properties for learning dynamics locally around a steady state when agents try to learn a (stochastic) steady state with a natural finite-memory rule. We also linearize the process and obtain an approximation which is an $ARMA(T,T)$ process.

Stationarity of this $ARMA$ approximation is briefly analyzed in Section 4. There we also consider a generalization of model (3) in Example 2 to incorporate observation errors. It is shown that E-stability yields stationarity of both processes.

Section 5 concludes.

# 2 Linear AR Models

## 2.1 Preliminaries

We start with the class of models mentioned in Examples 1 and 2 of the Introduction. Recall that these are of the following general form

$$x_t = \alpha + \beta E^*_{t-1} x_t + v_t, \tag{4}$$

or

$$x_t = \alpha + \beta E^*_t x_{t+1} + v_t \tag{5}$$

depending on the dating of the expectations and time period they concern. Here $x_t$ is an endogenous variable, $E^*_t x_{t+1}$ is the subjective expectation of $x_{t+1}$ held by agents at time $t$ and $v_t$ is a sequence of white noise shocks.

[5] See Section 3.3.1 of (Evans and Honkapohja 1999), or Section 7, Chapter 9 of (Evans and Honkapohja 2000b), for more detailed discussions and references.

We focus on steady state solutions of models (4) and (5).[6] These rational expectations solutions may be written as $x_t = \hat{A} + v_t$, where $\hat{A} = \frac{\alpha}{1-\beta}$. As discussed above, in modeling learning we postulate that agents think that they are in a steady state but do not know the value of the constant $\hat{A}$. In other words, they have a PLM of the form $x_t = A + u_t$ and they form an estimate of the value of $A$ using past observations on $x_t$. In the class of models covered by Example 1 the estimate is also the forecast $E^*_{t-1}x_t$ (or $E^*_t x_{t+1}$ for the class covered by Example 2).

Before considering learning we define precisely the concept of E-stability which, as noted, will play a key role. With the PLM of the above form agents use the estimated value of the constant as their forecast. If $A$ is the value of the forecast the temporary equilibrium or *actual law of motion* (ALM) of the economy is given by

$$x_t = \alpha + \beta A + u_t.$$

This defines a mapping from the PLM to the ALM which takes the form $T(A) = \alpha + \beta A$. E-stability is defined by considering the ordinary differential equation

$$\frac{dA}{d\tau} = T(A) - A.$$

If this differential equation is locally asymptotically stable (l.a.s.) at the REE $\hat{A} = \frac{\alpha}{1-\beta}$, then the equilibrium is said to be *weakly E-stable*. The formal E-stability condition is $T'(\hat{A}) = \beta < 1$. This formulation of E-stability is closely connected to convergence of real-time learning schemes, see (Evans and Honkapohja 1999) for a recent survey and (Evans and Honkapohja 2000b) for a detailed discussion.

This notion has been strengthened in several ways in the literature. For the results of this paper the concept of *iterative E-stability* turns out to be central. We stay that the REE is iteratively E-stable if it is locally asymptotically stable in iterations of the $T - map$, i.e. if the difference equation

$$A_{n+1} = T(A_n)$$

is locally asymptotically stable at $\hat{A}$. The formal condition for iterative E-stability is $\left|T'(\hat{A})\right| = |\beta| < 1$ in this case.

The notion of iterative E-stability is related to concepts of rationalizability in game theory, and the connection between these concepts has been explored by (Guesnerie 1992) and (Evans and Guesnerie 1993) in the context of rational expectations. We also remark that another related concept is *strong E-stability* in which the E-stability is required to be robust to overparameterizations of the PLM of the agents. For the linear frameworks (4) and (5) of this section weak and strong E-stability happen to coincide, while for the nonlinear models in the next section the condition for iterative E-stability is identical to that of strong E-stability.[7]

[6] As is well-known, (5) can have other solutions besides steady states.

[7] Discussions of the E-stability concepts for different frameworks are given in (Evans and Honkapohja 1995a), (Evans and Honkapohja 1999) and (Evans and Honkapohja 2000b).

## 2.2 Stationarity and Unbiasedness

After these preliminaries we begin to analyze learning dynamics with bounded memory for models (4) and (5). As noted above, computing the sample mean for a set of data is the standard way for estimating an unknown constant, so that an estimate of $A$ at time $t$ is given by

$$T^{-1}\sum_{i=1}^{T} x_{t-i} \tag{6}$$

if agents use past $T$ prices in computing the sample mean.[8] We will call $T$ the *memory length.* For some results we can in fact consider forecasting by a weighted sample mean, i.e.

$$\sum_{i=1}^{T} \mu_i x_{t-i}, \text{ where } \forall i : \mu_i \geq 0 \text{ and } \sum_{i=1}^{T} \mu_i = 1. \tag{7}$$

(6) is obviously a special case of (7).

Substituting the weighted mean into (4) or (5) yields

$$x_t = \alpha + \beta \sum_{i=1}^{T} \mu_i x_{t-i} + u_t \tag{8}$$

which is an autoregressive process of order $T$ (an $AR(T)$ process). The first question one needs to ask about such a process is whether it is stationary or not. This question is answered in the following proposition.

**Proposition 1** *(i) If the steady state is iteratively E-stable, i.e.* $|\beta| < 1$, *then* $x_t$ *is (covariance) stationary for all* $T \geq 1$.
*(ii) If it is weakly E-unstable, i.e.* $\beta > 1$, *then the process is non-stationary.*[9]

**Proof.** Consider the following equation

$$1 - \beta \Sigma_{i=1}^{T} \mu_i z^i = 0 \tag{9}$$

We need the roots of (9) to be outside the unit circle for stationarity.

Suppose that $|\beta| < 1$. Then we have

$$1 < \left|\frac{1}{\beta}\right| = \left|\Sigma_{i=1}^{T} \mu_i z^i\right| \leq \Sigma_{i=1}^{T} \mu_i \left|z\right|^i, \tag{10}$$

where the final inequality follows from the triangle inequality. Suppose that for a root $\hat{z}$ we have $|\hat{z}| \leq 1$. Then for all $i$, $|\hat{z}|^i \leq 1$. It then follows from (10) that

$$1 < \Sigma_{i=1}^{T} \mu_i \left|z\right|^i \leq \Sigma_{i=1}^{T} \mu_i = 1$$

[8] We note that, unlike in infinite memory learning, it is not possible to compute the sample mean of the previous $T$ prices from the corresponding average in the previous period, the number of observations and the new observation. In other words, bounded memory learning cannot be written in recursive fashion.

[9] The process is also non-stationary for $\beta = 1$.

which is a contradiction. This proves (i).

To prove (ii) consider the characteristic polynomial

$$p(\lambda) = \lambda^T - \beta\mu_1\lambda^{T-1} - \beta\mu_2\lambda^{T-2} - ... - \beta\mu_{T-1}\lambda - \beta\mu_T \tag{11}$$

If $\beta > 1$ we have $p(1) < 0$, so that by continuity $p(\lambda)$ must have a root greater than one. This proves (ii). (Note that if $\beta = 1$, then $\lambda = 1$ is a root.) ■

Weak E-stability has also a further implication:

**Corollary 2** *If the steady state is weakly E-stable, i.e. $\beta < 1$, then in the case (6) of equal weights $\exists T^* : \forall T \geq T^*$ the process is stationary.*

**Proof.** We consider only the case $\beta < -1$ due to Proposition 1. We basically replicate the proof of (Giona 1991). Consider the characteristic polynomial (11) which may be rewritten in this case as

$$p(\lambda) = \lambda^T + \frac{|\beta|\,(1-\lambda^T)}{(1-\lambda)T}$$

Define $q(\lambda) := (1-\lambda)p(\lambda)$. Observing that $\lambda = 1$ is not an eigenvalue (since $p(1) > 0$), the roots of $q(\lambda)$ are the same as that of $p(\lambda)$. The roots of $q(\lambda)$, on the other hand, are given by solving the equation

$$\lambda^T\{-\lambda + (1 - \frac{|\beta|}{T})\} + \frac{|\beta|}{T} = 0. \tag{12}$$

From (12) we have (on re-arranging)

$$|\lambda| = \left|1 - \frac{|\beta|}{T} + \frac{|\beta|}{T\lambda^T}\right| \leq \left|1 - \frac{|\beta|}{T}\right| + \frac{|\beta|}{T\left|\lambda^T\right|}$$

The proof now proceeds by contradiction. Assume that there exists some eigenvalue $\lambda$ such that $|\lambda| > 1$. Then we can choose $T^*$ such that $\forall T \geq T^*$ we have $\frac{|\beta|}{T} < 1$ and $\left|\lambda^T\right| > 2$. Consequently $\forall T \geq T^*$ it is true that

$$|\lambda| < 1 - \frac{|\beta|}{T} + \frac{|\beta|}{2T} = 1 - \frac{|\beta|}{2T} < 1$$

which contradicts $|\lambda| > 1$. Now let us assume that $\forall T \geq T^*$ there exists at least one eigenvalue on the unit circle, that is, $\lambda = e^{i\theta}$. In this case we have

$$1 = \left|\frac{|\beta|}{T}e^{-iT\theta} + (1 - \frac{|\beta|}{T})\right|$$

This last equation implies $\theta = 0$, but we have already ruled out $\lambda = 1$ as an eigenvalue. Hence $\forall T \geq T^*$, all the eigenvalues are inside the unit circle. ■

The results demonstrate that iterative and weak E-stability are closely connected to stationarity properties for learning dynamics with natural finite-memory rules for learning a stochastic steady state. With such rules exact convergence cannot obtain, but stationarity prevails if the underlying model has an iteratively E-stable REE and it may prevail even with just weak E-stability.

If the dynamics is stationary, it makes sense to consider further properties of learning with these finite-memory rules. Inspecting (8) it is immediately seen that the unconditional mean of $x_t$ converges to the rational steady state $\frac{\alpha}{1-\beta}$ under the postulated learning rule, and therefore the (unconditional) mean of the forecast also converges to the same value. Thus we have:

**Proposition 3** *If the dynamics (8) is stationary, then learning is asymptotically unbiased for all $T \geq 1$, i.e. the mean of the forecast $\sum_{i=1}^{T} \mu_i x_{t-i}$ converges, as $t \to \infty$, to the steady state $\hat{A} = \frac{\alpha}{1-\beta}$.*

This result shows that at each memory length the forecasts provide, on average, the correct estimate of the steady state.

## 2.3 Second Moment Properties

Here we are interested in determining the asymptotic variance of estimation errors when the process is stationary. We thus impose weak E-stability, i.e. $\beta < 1$ (and strengthen it if necessary). In order to compute this variance we first need to calculate the second moments of the $x_t$ process. This is a standard problem in time series econometrics, and one makes use of the Yule Walker equations, see e.g. Chapter 3 of (Hamilton 1994). The Yule Walker equations for this $AR(T)$ process yield a system of $T$ simultaneous linear equations which can be solved for the first $T$ auto-correlations of the process.

First define the $i-th$ auto-correlation as

$$\rho_i := \frac{Cov(x_{t-i,}x_t)}{Var(x_t)},$$

where $Cov(x_{t-i,}x_t)$ denotes the covariance between $x_{t-i}$ and $x_t$ and $Var(x_t)$ denotes the variance of $x_t$. To economize on notation we also define $a := \frac{\beta}{T}$. The Yule Walker equations in our case are

$$\begin{aligned}
\rho_1 &= a + a(\rho_1 + \rho_2 + ... + \rho_{T-1}) \\
\rho_2 &= a\rho_1 + a + a(\rho_1 + \rho_2 + ... + \rho_{T-2}) \\
\rho_3 &= a(\rho_2 + \rho_1) + a + a(\rho_1 + \rho_2 + ... + \rho_{T-3}) \\
&\quad ....... \\
\rho_{T-1} &= a(\rho_{T-2} + \rho_{T-3} + ... + \rho_1) + a + a\rho_1 \\
\rho_T &= a(\rho_{T-1} + \rho_{T-2} + ... + \rho_1) + a
\end{aligned}$$

The solution to this system is given in the following proposition:

**Proposition 4** *If $|\beta| < 1$, then the above system of equations has a unique solution $\rho_i = \rho = \frac{\beta}{(1-\beta)T+\beta}$, for all $i$ such that $1 \leq i \leq T$.*

**Proof.** The $T$ linear simultaneous equations need to be solved for the $T$ unknowns $\rho_1, \rho_2, ..., \rho_T$. However, on careful observation one sees that the following is true $\rho_1 = \rho_T$; $\rho_2 = \rho_{T-1}$; $\rho_3 = \rho_{T-2}$ or, in general, $\rho_j = \rho_{T-j+1}$.

This means that we can reduce the dimensionality of the equations to be solved for. As mentioned above, one can match the auto-correlations pairwise, so that we have to distinguish between two cases: when $T$ is even and when $T$ is odd. We first consider the case when $T$ is even.

CASE 1: $T = 2M$; $M$ is a positive integer greater than or equal to 2. In this case we can reduce the above set of $T$ equations into $M$ equations to solve for the $M$ unknowns $\rho_1, \rho_2, ..., \rho_M$. The resulting $M$ equations are

$$\begin{aligned}
\rho_1 &= a + a(\rho_1 + 2\rho_2 + ... + 2\rho_M) \\
\rho_2 &= a\rho_1 + a + a(\rho_1 + \rho_2 + 2\rho_3 + ... + 2\rho_M) \\
& ....... \\
\rho_{M-1} &= a(\rho_{M-2} + \rho_{M-3} + ... + \rho_1) + a + a(\rho_1 + ... + \rho_{M-1} + 2\rho_M) \\
\rho_M &= a(\rho_{M-1} + \rho_{M-2} + ... + \rho_1) + a + a(\rho_1 + ... + \rho_{M-1} + \rho_M).
\end{aligned}$$

There is an easy way to solve the above set of equations. First, subtract the second equation from the first to get $\rho_1 - \rho_2 = a(\ \rho_2 - \rho_1)$ or $(\rho_1 - \rho_2)(1 - a) = 0$. Since $|\beta| < 1$ implies that $|a| < 1$ for all $T \geq 1$, we get $\rho_1 = \rho_2$. Analogously, in general, subtracting equation $j + 1$ from equation $j$ (where $1 \leq j \leq M - 1$) one gets $(\rho_j - \rho_{j+1})(1 - a) = 0$, so that $\rho_j = \rho_{j+1}$.

This proves that all the auto-correlations are the same so that we can get the common value, say $\rho$, from a single equation. This yields $(1 - a - 2a(M - 1))\rho = a$ or

$$\rho = \frac{a}{1 - a - 2a(M - 1)} = \frac{a}{1 - a(T - 1)}.$$

CASE 2: $T = 2M + 1$; $M$ is a positive integer greater than equal to 2. In this case we can reduce the above set of $T$ equations into $M + 1$ equations to solve for the $M + 1$ unknowns $\rho_1, \rho_2, ..., \rho_M, \rho_{M+1}$. The resulting $M + 1$ equations are

$$\begin{aligned}
\rho_1 &= a + a(\rho_1 + 2\rho_2 + ... + 2\rho_M + 2\rho_{M+1}) \\
\rho_2 &= a\rho_1 + a + a(\rho_1 + \rho_2 + 2\rho_3 + ... + 2\rho_M + \rho_{M+1}) \\
& ........ \\
\rho_M &= a(\rho_{M-1} + \rho_{M-2} + ... + \rho_1) + a + a(\rho_1 + ... + \rho_M + \rho_{M+1}) \\
\rho_{M+1} &= a(\rho_M + \rho_{M-1} + ... + \rho_1) + a + a(\rho_1 + ... + \rho_{M-1} + \rho_M).
\end{aligned}$$

Note that in this case we get an extra equation corresponding to the unmatched autocorrelation at lag $M+1$. Here, analogously as for the first case, subtracting equation $j$ from equation $j + 1$ for all $1 \leq j \leq M$ we get $\rho_j - \rho_{j+1} = -a\rho_j + a\rho_{j+1}$ which implies $(\rho_j - \rho_{j+1})(1 - a) = 0$, and since $a \neq 1$ we get $\rho_j = \rho_{j+1}$.

This proves that again we have $\rho_i = \rho$ for all $1 \leq i \leq M + 1$. Using this fact we can now easily get $\rho$ from the first equation. This again gives us $\rho = \frac{a}{1 - a(T-1)}$.

Note that this also shows that the solution is unique. So finally we get the common value of $\rho$ for all $T \geq 1$ as

$$\rho = \frac{a}{1-a(T-1)} = \frac{\frac{\beta}{T}}{1-\frac{\beta}{T}(T-1)} = \frac{\beta}{(1-\beta)T+\beta}.$$

This proves the proposition for all $T \geq 4$. One can also check easily that the same is true for $T = 1, 2, 3$. ■

We are now in a position to get the asymptotic variance of $x_t$. First define $\gamma_i$ to be the *ith* autocovariance, so that

$$\rho_i = \frac{\gamma_i}{\gamma_0},$$

where $\gamma_0$ is the asymptotic variance of $x_t$. From (Hamilton 1994), p. 59, we have $\gamma_0 = a\sum_{i=1}^{T}\gamma_i + \sigma^2 = a\gamma_0\sum_{i=1}^{T}\rho_i + \sigma^2 = \gamma_0 aT\rho + \sigma^2 = \gamma_0(\frac{\beta^2}{(1-\beta)T+\beta}) + \sigma^2$. So finally solving for $\gamma_0$ yields

$$\gamma_0 = \sigma^2(1-\frac{\beta^2}{(1-\beta)T+\beta})^{-1} = \sigma^2(\frac{(1-\beta)T+\beta}{(1-\beta)(T+\beta)}).$$

Clearly $\gamma_0$ is decreasing in $T$ and in $\sigma^2$ if the process is stationary.

We finally turn to the forecast error to see how it behaves with $T$. Denote the forecast error of the least squares estimate from the REE based on memory $T$ as $Y_t(T)$. We have, by definition,

$$Y_t(T) = \sum_{i=1}^{T}\frac{1}{T}x_{t-i} - \hat{A}.$$

**Proposition 5** *If $|\beta| < 1$, the asymptotic variance of forecast error, $Var(Y_t(T))$, decreases monotonically with $T$ and increases with $\sigma^2$. As $T \to \infty$, $Var(Y_t(T)) \to 0$.*

**Proof.** $Var(Y_t(T)) = Var(\sum_{i=1}^{T}\frac{1}{T}x_{t-i})$. Thus

$$\begin{aligned} Var(Y_t(T)) &= (\frac{1}{T})^2[\sum_{i=1}^{T}Var(x_{t-i}) + \sum_{i=1}^{T}\sum_{j=1,i\neq j}^{T}Cov(x_{t-i},x_{t-j})] \\ &= (\frac{1}{T})^2[\sum_{i=1}^{T}\gamma_0 + 2\sum_{i=1}^{T}\sum_{j=1,i<j}^{T}Cov(x_{t-i},x_{t-j})] \\ &= (\frac{1}{T})^2[T\gamma_0 + 2\gamma_0\rho\{(T-1)+(T-2)+...+2+1\}] \\ &= (\frac{1}{T})^2[T\gamma_0 + 2\gamma_0\rho\frac{T(T-1)}{2}] = \frac{\gamma_0[1+(T-1)\rho]}{T} = \frac{\sigma^2}{(1-\beta)(\beta+T)}. \end{aligned}$$

If $|\beta| < 1$, this is clearly decreasing monotonically in $T$ and increasing in $\sigma^2$. Also note that $\lim_{T\to\infty} Var(Y_t(T)) = 0$. ■

We note that it can similarly be shown that the asymptotic variance of the forecast error as calculated from the actual price, i.e. $\sum_{i=1}^{T}\frac{1}{T}x_{t-i} - x_t$ for model (4) and

$\sum_{i=1}^{T} \frac{1}{T} x_{t-i} - x_{t+1}$ for model (5), respectively, are decreasing in $T$ and increasing in $\sigma^2$. In the limit $T \to \infty$ they both go to $\sigma^2$.

We illustrate the dynamics (8) for equal weights with simulations. We set $\alpha = 5, \beta = -4$ and assume a uniformly distributed shock with support $[-0.1, 0.1]$. The dynamics were run for 5000 periods and the figures show the last 100 periods. Figure 1A displays the actual value for $x_t$ for two simulations when memory lengths 5 and 50 were assumed (star denotes the dynamics with the higher memory length in Figures 1A and 1B). We note the clear reduction in the volatility for the longer memory length. Figure 1B shows the same simulation with memory lengths 50 and 500. Interestingly, this further ten-fold increase in memory length did not reduce the volatility much further. Finally, Figure 1C compares the dynamics with memory length 50 to the RE solution which in this case is just constant plus noise (squares indicate the RE values and triangles the learning solution). These two processes are apparently rather close to each other.

FIGURES 1A, 1B AND 1C ABOUT HERE

## 2.4 Generalization to Higher Order Models

The preceding results can be easily generalized for the steady states of some higher order linear models. For example, consider the model

$$x_t = \alpha + \beta_0 E_{t-1}^* x_t + \beta_1 E_{t-1}^* x_{t+1} + v_t$$

first analyzed in (Evans 1985) for E-stability.

If the agents have a PLM of the stochastic steady-state form

$$x_t = a + v_t$$

the iterative E-stability condition is $|\beta_0 + \beta_1| < 1$.[10] Assume now that agents make forecasts of the unknown constant $a$ by computing the sample mean $a_t = \sum_{i=1}^{T} \frac{1}{T} x_{t-i}$ from past data $x_{t-1}, \ldots, x_{t-T}$ and using the estimate as the forecast. The actual law of motion is given by

$$x_t = \alpha + (\beta_0 + \beta_1) a_t + v_t = \alpha + (\beta_0 + \beta_1) \sum_{i=1}^{T} \frac{1}{T} x_{t-i} + v_t,$$

which is an $AR(T)$ process. This process is a very minor modification to (8), and the above results apply to this framework.[11]

[10] See (Evans 1985). Evans' definition of E-stability is iterative E-stability in our terminology. (The distinction between E-stability and iterative E-stability was made only more recently.) Note that one could make a distinction between weak and strong iterative E-stability, but this is not needed in this paper.

[11] The new feature brought by the generalization is that this framework has other equilibria besides steady states.

## 2.5 Bounded Memory Learning as Self-Confirming Equilibrium

The preceding results characterize the dynamics of $x_t$ when all agents use the same finite memory length. In earlier work (Mitra 2000) showed that use of finite memory may be optimal for a single agent in short term forecasting of an exogenous stationary $AR(1)$ process. Here we show that in a self-referential setting it may sometimes be *optimal* for any single agent to use bounded memory learning when all other agents also do so.

In model (4) suppose that all agents use memory length $T = 1$ in their forecasting. Does any single agent have an incentive to use $T > 1$, given that all other agents continue to use $T = 1$? In this analysis we assume that the objective function for the agent is the minimization of the asymptotic ($t \to \infty$) Mean Squared Error (MSE) of his forecast error. It turns out that the agent does not want to deviate from using memory length $T = 1$ if $1 > \beta > 0.5$.

When all agents use $T = 1$, the true law of motion of the price $p_t$ is an $AR(1)$ process by (2). A single agent computes the forecast of $p_t$ by $T^{-1}\sum_{i=1}^{T} p_{t-i}$. Formally, the question is to find the value of $T$ that minimizes

$$\lim_{t\to\infty} E[T^{-1}\sum_{i=1}^{T} p_{t-i} - p_t]^2.$$

It can be shown that this is *exactly* the problem studied in (Mitra 2000), and we can appeal to those results. $T = 1$ minimizes the asymptotic MSE of the forecast error of a single agent *given* that other agents use $T = 1$ in their forecasting if $\beta > 0.5$. In this sense $T = 1$ is a self-confirming equilibrium in memory lengths.

We note that the optimality of $T = 1$ has nothing to do with any computational costs of acquiring more data. If, as might be realistic, one assumes that it is costly to acquire more data this result can probably be strengthened. The result above is only partial, and we leave a systematic study of the optimality of bounded memory learning to a separate paper.

# 3 Nonlinear Models

## 3.1 Preliminaries

In this section we consider learning of a steady state for the class of nonlinear models

$$x_t = H(G(x_{t+1}, v_{t+1})^e, v_t), \qquad (13)$$

discussed in (Evans and Honkapohja 1995b). Here $H$ and $G$ are given twice differentiable functions, $x_t$ is the value of the (scalar) variable of interest at time $t$, and $v_t$ is a sequence of independently and identically distributed random shocks with mean 0 and variance $\sigma_v^2$. $G(x_{t+1}, v_{t+1})^e$ denotes the subjective expectations of $G(x_{t+1}, v_{t+1})$ formed in period $t$. We will introduce some further assumptions later.

A *rational steady state* is a function $x(v)$ such that

$$\forall v : x(v) = H(E_w G(x(w), w), v),$$

where the expectation $E_w$ is taken with respect to a random variable which has the same distribution as the *iid* shocks $v_t$. For later purposes $\hat{x}$ denotes the steady state of the corresponding nonstochastic model, i.e. $\hat{x} = H(G(\hat{x}, 0), 0)$. (Evans and Honkapohja 1995b) provide an existence theorem for this kind of steady state when the support of the shock $v_t$ in (13) is sufficiently small.[12]

*Example 3.* (The basic overlapping generations model with shocks.) In the basic overlapping generations (OG) model with production agents supply labor $n_t$ and produce (perishable) output when young and consume $c_{t+1}$ when old. The utility function of the representative agent of generation $t$ is $U(c_{t+1}) - V(n_t)$. Holding money is the only means of saving, and there is a fixed quantity of money $M_0$. Output is assumed to be equal to labor supply plus and an additive productivity shock, so that output $q_t$ is given by

$$q_t = n_t + \lambda_t,$$

where $\lambda_t$ is an *iid* positive productivity shock. The budget constraints are $p_{t+1}c_{t+1} = M_t$ and $p_t q_t = M_t$. The first-order condition plus the market clearing condition $q_{t+1} = c_{t+1}$ and $p_t/p_{t+1} = q_{t+1}/q_t$ yields

$$(n_t + \lambda_t)V'(n_t) = E_t^*((n_{t+1} + \lambda_{t+1})U'(n_{t+1} + \lambda_{t+1})).$$

Since $(n + \lambda)V'(n)$ is strictly increasing in $n$, and letting $v_t \equiv \lambda_t - E(\lambda_t)$, this equation can be solved for $n_t$ Letting $x_t \equiv n_t$ the model can be put in the standard form (13).

Returning to the general framework, suppose that agents are trying to learn the steady state. Agents have to forecast the quantity $G(x_{t+1}, v_{t+1})^e$ which is a constant $E_w G(x(w), w)$ in the steady state. The learning problem for the agents is to find this value. The data are given by the past observations $G(x_1, v_1), G(x_2, v_2), ....., G(x_{t-1}, v_{t-1})$, and the agents are assumed to use the sample mean of these observations to forecast $G(x_{t+1}, v_{t+1})^e$.

We continue to focus on learning with a finite memory length $T$ : At date $t$ agents use $T$ past observations to estimate $G(x_{t+1}, v_{t+1})^e$. The estimate and forecast at date $t$, $\delta_t$, is given by

$$\delta_t = \sum_{i=1}^{T} \mu_i G(x_{t-i}, v_{t-i}),$$

where $\mu_i$ is a weight such that $\sum_{i=1}^{T} \mu_i = 1, \mu_i \geq 0$. The general results in this section hold for general weighting schemes, but in Section 4 attention will be focused on the most important case of the sample mean, i.e. the weights are equal $\mu_i = T^{-1}$.

[12] To our knowledge, existence of the stochastic steady state equilibria has not been analyzed in full generality.

Given the forecast $\delta_t$, the actual law of motion of $x_t$ from (13) is given by

$$x_t = H(\delta_t, v_t).$$

Substituting for $\delta_t$ in the above equation, we finally get the dynamical system

$$x_t = H(\sum_{i=1}^{T} \mu_i G(x_{t-i}, v_{t-i}), v_t) \tag{14}$$

(14) is the law of motion we are concerned with. This form can cover a wide variety of overlapping generations models with shocks to either preferences or technology. A special case arises if $G$ is independent of its second argument.[13]

## 3.2 Markovian Formulation

We now start to analyze the process (14) for some general properties. First, observe that (14) can be written as a Markov process in the following manner. Define the state vector

$$X_{t-1} = (x_{t-1}, x_{t-2}, .., x_{t-T}, v_{t-1}, v_{t-2}, .., v_{t-T})'.$$

Introducing the notation $X_{j,t-1}$ for the $j-th$ component of $X_{t-1}$, we can write

$$\begin{bmatrix} x_t \\ x_{t-1} \\ \cdots \\ x_{t-T+1} \\ v_t \\ v_{t-1} \\ .. \\ v_{t-T+1} \end{bmatrix} = \begin{bmatrix} \hat{H}(X_{t-1}, v_t) \\ X_{1,t-1} \\ \cdots \\ X_{T-1,t-1} \\ v_t \\ X_{T+1,t-1} \\ .. \\ X_{2T-1,t-1} \end{bmatrix}, \tag{15}$$

where

$$\hat{H}(X_{t-1}, v_t) \equiv H(\sum_{i=1}^{T} \mu_i G(x_{t-i}, v_{t-i}), v_t). \tag{16}$$

This can be written compactly as

$$X_t = F(X_{t-1}, v_t), \tag{17}$$

where the right hand side of (15) defines $F$.

Since $X_{t-1}$ and $v_t$ are independent, $X_t$ is a Markov process with some state space $A \subset \mathbb{R}^{2T}$. The first question we study is whether there exists a unique invariant probability for (17) and whether any initial probability distribution converges to this invariant

[13] For example, the dynamics in an overlapping generations model with a multiplicative shock to the disutility of labor and with agents trying to learn a stochastic steady state would follow a process in which the current value of the state variable depends only on the current shock and past states.

probability asymptotically. It should be emphasized that this analysis will be local since we will assume that the underlying noise is small (in a sense to be made precise shortly), since the existence of equilibria is known only for this case and since the linearization of (14) can be justified only in models with small noise.

### 3.3 Asymptotic Properties

It turns out that the existence of the unique invariant distribution with small noise and starting points in a neighborhood of the steady state can be established, if the steady state of the corresponding nonstochastic model is iteratively E-stable.[14] To obtain this condition suppose that in the nonstochastic model agents have a PLM $\theta$ about the expectations $G(\hat{x},0)^e$, where $\hat{x}$ is the unknown steady state. Then the $T-map$ is given by

$$T(\theta) = G(H(\theta,0),0)$$

yielding the condition for iterative E-stability given in Condition 2 below.

We now proceed to the general analysis of (14) or (17). We make the following assumptions.

**Condition 1** $v_t \in [-\varepsilon_v, \varepsilon_v]$ for all $t$.

**Condition 2** (Iterative E-stability) $\mid D_1H(G(\hat{x},0),0)D_1G(\hat{x},0) \mid < 1$.

We first show that with Conditions 1 and 2 and with $\varepsilon_v$ small enough, the state space of the Markov process (17) may be assumed to be compact. This is the content of the following lemma.

**Lemma 6** *Assume Conditions 1 and 2 and $\varepsilon_v$ small enough. Then the state space of (17), call it $N(\varepsilon_v) \subset A$, may be assumed to be compact.*

**Proof.** We want to show that if $\parallel X_{t-1} \parallel \leq \varepsilon$ and $\mid v_t \mid \leq \varepsilon_v$, then we have $\parallel X_t \parallel \leq \varepsilon$, too. (The vector norm will be specified later.) Given the structure of (15) it is clear that it is basically the first component of the vector which is problematic (as the other components are merely definitions). The first component of $X_t$ describes the process (14) for $x_t$. We will now prove that for suitable $\varepsilon_x$ if $\mid x_{t-i} \mid \leq \varepsilon_x$ for all $i = 1, 2, .., T$ and $\mid v_{t-i} \mid \leq \varepsilon_v$ for all $i = 0, 1, 2, .., T$ then we have $\mid x_t \mid \leq \varepsilon_x$ too. By choosing some appropriate vector norm (like the max norm), this in turn shows that if $\parallel X_{t-1} \parallel \leq \varepsilon$ and $\mid v_t \mid \leq \varepsilon_v$, then we have $\parallel X_t \parallel \leq \varepsilon$, too.

Assume, without loss of generality, that the perfect foresight steady state is $\hat{x} = 0$. We linearize (14) around the vector $(0, 0, ...., 0) \in \mathbb{R}^{2T+1}$ and get a second order residual term in the Taylor series expansion in the following manner:

$$x_t = \sum_{i=1}^{T} \alpha_i x_{t-i} + \beta_0 v_t + \sum_{i=1}^{T} \beta_i v_{t-i} + r(x_{t-1}, .., x_{t-T}, v_t, .., v_{t-T}), \tag{18}$$

[14] For this class of models iterative E-stability is in fact equivalent to strong E-stability, compare (Evans and Honkapohja 1995b).

where $\alpha_i = \mu_i D_1 H D_1 G$, $\beta_i = \mu_i D_1 H D_2 G$, for $i = 1, 2, .., T$ and $\beta_0 = D_2 H$; $D_i H$ being the partial derivative of $H$ with respect to the $ith$ argument, etc. These derivatives are evaluated at values corresponding to the nonstochastic steady state. Note also that, with the normalization $\hat{x} = 0$, the constant term in the Taylor series is 0.

Using the mean value theorem, the residual $r(.)$ satisfies

$$|r(x_{t-1}, .., x_{t-T}, v_t, .., v_{t-T})| \leq K(|x_{t-1}| + .. + |x_{t-T}| + |v_t| + |v_{t-1}| + .. + |v_{t-T}|).$$

Taking absolute values in (18) we get

$$\begin{aligned} |x_t| &\leq \mid \alpha_1 \mid |x_{t-1}| + .. + \mid \alpha_T \mid |x_{t-T}| + \mid \beta_0 \mid |v_t| + \mid \beta_1 \mid |v_{t-1}| + .. + \\ &\mid \beta_T \mid |v_{t-T}| + |r(x_{t-1}, .., x_{t-T}, v_t, .., v_{t-T})| \\ &\leq \sum_{i=1}^{T} (K + |\alpha_i|) \, |x_{t-i}| + \sum_{i=1}^{T} (K + |\beta_i|) \, |v_{t-i}| + (K + |\beta_0|) \, |v_t| \\ &\leq (TK + |D_1 H D_1 G|)\varepsilon_x + (TK + |D_1 H D_2 G|)\varepsilon_v + (K + |D_2 H|)\varepsilon_v \end{aligned}$$

if $|x_{t-i}| \leq \varepsilon_x$ for all $i$.

Since $\frac{\partial r}{\partial x_{t-j}}$ and $\frac{\partial r}{\partial v_{t-j}}$ are zero when evaluated at the origin (which is the steady state here), the constant $K$ can be made as close to zero as desired by restricting the analysis to a small enough neighborhood of the origin. We choose the neighborhood so that $1 - TK - |D_1 H D_1 G| > 0$. Then we choose $\varepsilon_x$ such that

$$(TK + |D_1 H D_1 G|)\varepsilon_x + (TK + |D_1 H D_2 G|)\varepsilon_v + (K + |D_2 H|)\varepsilon_v \leq \varepsilon_x$$

or in other words

$$\varepsilon_x \geq \frac{(TK + |D_1 H D_2 G| + K + |D_2 H|)\varepsilon_v}{1 - TK - |D_1 H D_1 G|} \tag{19}$$

By condition 2 and the choice of $K$ this inequality is well defined. This proves the lemma.[15] ■

We now prove that under suitable assumptions there exists a unique invariant probability for (17) and that the $n-$step transition probability of this Markov process converges weakly to this invariant probability, as $n \to \infty$, for every point in the state space. We use the results of (Bhattacharyya and Lee 1988) to prove these assertions.

By Lemma 6 the state space $N(\varepsilon_v)$ of (17) may be assumed to be compact. Henceforth, we assume that $\varepsilon_x$ in (19) is set so that equality holds. Thus $N(\varepsilon_v)$ is a compact metric space. We begin by proving the following lemma.

**Lemma 7** *There exists $\varepsilon_v$ sufficiently small such that for all $v_t \in [-\varepsilon_v, \varepsilon_v] = N_v$ the process (17) is a strict contraction on $N(\varepsilon_v)$, i.e. $\|F(h, v) - F(0, v)\| < \|h\|$ for all $h \in N(\varepsilon_v), h \neq 0$.*

[15] We note here that in general the constant $K$ will depend on $T$. In particular, the larger is $T$ the smaller must $K$ be. There are, however, special cases where a uniform result is obtainable when $\mu_i = o(T^{-1})$. This happens when the function $\hat{H}(X_{t-1}, v_t)$ is either independent of $v_t$ or additive and linear in it.

**Proof.** We continue to assume, without any loss of generality, that the deterministic steady state is $(0,0,..,0)$. First, note that

$$F(h,v) - F(0,v) = DF(0,v)h + r(h,v)$$

Here $DF$ denotes the Jacobian of $F$ with respect to first (vector) argument. It then follows that

$$\begin{aligned} \|F(h,v) - F(0,v)\| &= \|DF(0,v)h + r(h,v)\| \\ &\leq \|DF(0,v)\| \, \|h\| + \|r(h,v)\| \end{aligned} \tag{20}$$

The $2T \times 2T$ Jacobian matrix $DF(0,0)$ at the steady state takes the form

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where the $T \times T$ submatrices are given by

$$F_{11} = \begin{pmatrix} \mu_1\vartheta & \mu_2\vartheta & \mu_3\vartheta & \cdots & \mu_{T-1}\vartheta & \mu_T\vartheta \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$F_{12} = \begin{pmatrix} \mu_1\kappa & \cdots & \mu_T\kappa \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}, F_{21} = 0, F_{22} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Here we have introduced the notation $\vartheta = D_1HD_1G$, $\kappa = D_1HD_2G$.

Consider now the eigenvalues of $DF(0,0)$. From the partitioned form it is seen that the matrix $DF(0,0) - \lambda I_{2T\times 2T}$ is block triangular, so that its determinant is equal to the product of the determinants of $F_{11} - \lambda I_{T\times T}$ and $F_{22} - \lambda I_{T\times T}$. Thus the eigenvalues of $DF(0,0)$ consist of the eigenvalues of $F_{11}$ and $F_{22}$. Zero is the only eigenvalue of $F_{22}$. The special form of $F_{11}$ implies that its eigenvalues must satisfy the polynomial equation

$$p(\lambda) = \lambda^T - \vartheta\mu_1\lambda^{T-1} - ... - \vartheta\mu_{T-1}\lambda - \vartheta\mu_T = 0,$$

see e.g. (Hamilton 1994), Proposition 1.1. If $\lambda \neq 0$, dividing through by $\lambda^T$ and setting $z = \lambda^{-1}$ we obtain equation (9) with $\beta = \vartheta$. The argument in Proposition 1 yields that the roots of this equation must have modulus less than one if $\mid \vartheta \mid < 1$.

Thus Condition 2 implies that the spectral radius of the matrix $DF(0,0)$ is strictly less than one. Then there exists some matrix norm such that $\|DF(0,0)\| < 1$, see (Horn and Johnson 1985), Lemma 5.6.10. Now, by uniform continuity of the matrix norm in its elements, it follows that there exists $\varepsilon_v$ sufficiently small such that for $\forall v_t \in [-\varepsilon_v, \varepsilon_v] = N_v : \|DF(0, v_t)\| < \delta < 1$. Moreover, $\|r(h,v)\| \leq \hat{\delta} \|h\|^2$ for some $\hat{\delta}$ since $r(h,v)$ is 2nd order in $h$.

It now follows from (20) that for all $\|h\|$ small enough we have

$$\|F(h,v) - F(0,v)\| \leq (\|DF(0,v)\| + \hat{\delta}\,\|h\|)\,\|h\| < \|h\|\,.$$

This proves that $F(.,.)$ is a strict contraction on the state space $N(\varepsilon_v)$. ■

**Proposition 8** *There exists a unique invariant probability for the Markov process $F$ on the state space $N(\varepsilon_v)$ and the $n-$step transition probability of $F$ converges weakly to this invariant probability, as $n \to \infty$, for every point in $N(\varepsilon_v)$.*

**Proof.** Lemmata 6 and 7 show that the conditions of Corollary 2.3 of (Bhattacharyya and Lee 1988) are satisfied on $N$. ■

This result demonstrates how, with small noise, iterative E-stability of the steady state of the nonstochastic model yields attractive limiting properties for the learning dynamics with small noise. Obviously, if $v_t \equiv 0$ for all $t$, iterative E-stability also guarantees local stability of the dynamical system (14) or (17).

## 3.4 Error Bounds

To obtain further information on model (14), it appears necessary to revert to linearization. This will be done in the next section. Here we briefly discuss errors bounds for the residual term $r(x_{t-1},..,x_{t-T},v_t,..,v_{t-T})$ in the Taylor series expansion (18).

Rewrite (14) as

$$x_t = R(x_{t-1},..,x_{t-T},v_t,..,v_{t-T}) \tag{21}$$

Consider the linearization of (21). The residual $r(x_{t-1},..,x_{t-T},v_t,..,v_{t-T})$ in the Taylor series (18) consists of terms of the form $\frac{\partial^2 R}{\partial x_{t-i}\partial x_{t-j}}(\mathbf{X})(x_{t-i}x_{t-j})$, $\frac{\partial^2 R}{\partial v_{t-i}\partial v_{t-j}}(\mathbf{X})(v_{t-i}v_{t-j})$ and $\frac{\partial^2 R}{\partial x_{t-i}\partial v_{t-j}}(\mathbf{X})(x_{t-i}v_{t-j})$ at some point $\mathbf{X}$.[16] Assuming that all the second order partial derivatives are bounded, it can be shown that the mean residual is bounded above by the expression $M\varepsilon_v^2$. For example,

$$\begin{aligned}\left|E[\frac{\partial^2 R}{\partial x_{t-i}\partial v_{t-j}}(\mathbf{X})(x_{t-i}v_{t-j})]\right| &\leq E\left|\frac{\partial^2 R}{\partial x_{t-i}\partial v_{t-j}}(\mathbf{X})(x_{t-i}v_{t-j})\right| \\ &\leq M_1 E\left|x_{t-i}v_{t-j}\right| \leq M_1\sqrt{Ex_{t-i}^2 Ev_{t-j}^2} = M_1\varepsilon_x\varepsilon_v \leq M_2\varepsilon_v^2,\end{aligned}$$

[16]Note that we have again assumed, w.l.o.g, that the nonstochastic steady state $\hat{x} = 0$.

where $M_i$ are constants. Likewise, the absolute value of the means of the other terms $\frac{\partial^2 R}{\partial v_{t-i}\partial v_{t-j}}(\mathbf{X})(v_{t-i}v_{t-j})$ and $\frac{\partial^2 R}{\partial x_{t-i}\partial x_{t-j}}(\mathbf{X})(x_{t-i}x_{t-j})$ are bounded by expressions of the same form.

Consider next the second order moments of $x_t$. Viewing the $x_t$ process as a linear $ARMA(T,T)$ process plus a residual, these moments can be computed from the Yule Walker equations with error terms. If these residuals can be shown to be small, then the second order moments derived from the linearized process will be close to the true second order moments of $x_t$.

Recall that the Yule Walker equations are derived by multiplying $x_t$ in (21) by $x_{t-u}$ $(u = 1, 2, .., T)$ and then taking expectations of both sides. Thus an individual representative term in the residual of these equations is of the form

$$\frac{\partial^2 R}{\partial v_{t-i}\partial x_{t-j}}(\mathbf{X})(v_{t-i}x_{t-j}x_{t-u})$$

with the absolute value of the mean bounded above by

$$\left| E[\frac{\partial^2 R}{\partial v_{t-i}\partial x_{t-j}}(\mathbf{X})(v_{t-i}x_{t-j}x_{t-u})] \right| \leq M_3\varepsilon_v \left[E(x^4_{t-j})E(x^4_{t-u})\right]^{\frac{1}{4}} \leq M_4\varepsilon_v^3$$

where we have applied Hölder's inequality twice. Here we have also used the fact that $x_t^2 \leq \varepsilon_x^2 \leq \tilde{M}\varepsilon_v^2$ for some $\tilde{M}$. Similar bounds can be found for the other terms involved in the residuals of the Yule Walker equations.

These considerations yield the following conclusion:

**Remark 1** *The mean absolute residuals in the Yule Walker equations are of third order in the support of the noise.*

To illustrate the goodness of the approximation we simulated the model in Example 3 with the utility functions $U(c) = c^{1-s}/(1-s)$, $V(n) = n^{1+e}/(1+e)$ with parameter values $s = 4.0$ and $e = 1$. The productivity shock $\lambda$ was assumed to be uniformly distributed with support $[0.3, 0.9]$. Figure 2 illustrates the dynamics from the original nonlinear model and its linear approximation (diamonds are the dynamics of the original model and squares its approximation).

FIGURE 2 ABOUT HERE

# 4 Linear ARMA Models

In the preceding section it was seen that the linearization of the general nonlinear framework yielded an $ARMA(T,T)$ process as an approximation of the original model. In this section we study further the dynamics of the linearized process (but with the restriction that learning is based on estimation by the sample mean).

The significance of the $ARMA$ setup is not limited to this case. A closely related framework can arise in finite-memory learning in some linear frameworks when observation errors prevail. We present an example before proceeding to the main result.

*Example 4.* (A model with observation errors.) Consider model (3) as in Example 2. Assume now that agents do not directly observe $y_t$, but only a variable $x_t = y_t + u_t$, where $u_t$ represents the observation error, assumed to be $iid$ with zero mean, constant variance $\sigma_u^2$, and independent of the shock $v_t$. As before, $v_t$ is assumed to be $iid$ with zero mean and constant variance $\sigma_v^2$.

Agents have the perceived law of motion

$$x_t = b + noise$$

and they estimate $b$ by the sample mean from $T$ past observations

$$b_t = T^{-1} \sum_{i=1}^{T} (y_{t-i} + u_{t-i}).$$

Substituting into (3) leads to a somewhat non-standard $ARMA(T,T)$-type process

$$y_t = \alpha + \frac{\beta}{T} \sum_{i=1}^{T} (y_{t-i} + u_{t-i}) + v_t$$

describing the dynamics of learning with memory length $T$. Note that this is not a standard $ARMA(T,T)$ process in that the current shock $v_t$ is permitted to have a different distribution from the lagged observation errors $u_{t-i}$. Such a process can nevertheless be tackled using standard time series techniques with minor modifications.

We now set up a framework that covers both this example and the linearization of the nonlinear model of Section 3. After centering both processes can be written as

$$z_t = a \sum_{i=1}^{T} z_{t-i} + cv_t + b \sum_{i=1}^{T} u_{t-i}, \tag{22}$$

where $z_t = y_t - Ey_t, a = \frac{\delta}{T}, b = \frac{\phi}{T}$. Formally in Example 4 we have $\delta = \phi = \beta$, and in the linearized model (18) $v_t = u_t$ and $\delta = D_1 H D_1 G, \phi = D_1 H D_2 G$.

Proceeding with the general analysis, we are interested in finding the condition for stationarity of this process. Again iterative E-stability yields this property:

**Proposition 9** *If* $\mid \delta \mid< 1$, *then the process (22) is (covariance) stationary.*

**Proof.** First, it is easily seen that the mean of $z_t$ is zero. We then consider the two cases $v_t \neq u_t$ and $v_t = u_t$ separately.

Case 1: $v_t \neq u_t$. Using the lag operator and substituting in the values for $a$ and $b$ we can write (22) in the form

$$[1 - \frac{\delta}{T}(L + L^2 + ... + L^T)]z_t = cv_t + \frac{\phi}{T}(L + L^2 + ... + L^T)u_t.$$

Using the method of proof in Proposition 1 it is seen that the polynomial in the lag operator $1 - \frac{\delta}{T}(L + L^2 + ... + L^T)$ in the left-hand side has all roots outside the unit circle if $| \delta |< 1$. Dividing both sides by this polynomial shows that $z_t$ is the sum of two independent covariance-stationary processes

$$\frac{cv_t}{1 - \frac{\delta}{T}(L + L^2 + ... + L^T)} \text{ and } \frac{\frac{\phi}{T}(L + L^2 + ... + L^T)u_t}{1 - \frac{\delta}{T}(L + L^2 + ... + L^T)}.$$

Therefore, it is itself covariance stationary.

Case 2: $v_t = u_t$. In this case, again using the method of proof in Proposition 1, it is easily seen that the process is stationary if $| \delta |< 1$. ■

The following further results are also evident.

**Remark 2** *If the condition for weak E-instability $\delta > 1$ holds, the process (22) is not stationary, and under weak E-stability $\delta < 1$ the process is stationary for $T$ sufficiently large.*

If the learning dynamics (22) is stationary, it is possible to proceed as in Section 2 and derive the asymptotic second moments of the process for $z_t$ using the technique to derive Yule Walker equations. Given knowledge of these moments one can also obtain the variance of the forecast error in terms of the memory length $T$ and the variance of the disturbances $u_t$ and $v_t$. Unfortunately, it appears that unambiguous analytic results on properties of the variance of the forecast error, which would be comparable to Proposition 5, are not available. We leave the lengthy details and further analysis to another paper.

# 5 Concluding Remarks

Frameworks in which adaptive learning is incomplete are beginning to receive attention. This paper has provided basic analytical results for dynamics of adaptive learning when the learning rule has a finite memory and the presence of random shocks precludes exact convergence to the REE.

We focused on the case of learning a stochastic steady state. Our central result is that the E-stability principle, which plays a central role in situations of complete learning, as discussed e.g. in (Evans and Honkapohja 1999) and (Evans and Honkapohja 2000b), retains its importance in the analysis of incomplete learning, though it takes a new form. In our setup E-stability guarantees the stationarity of the dynamics of the learning economy and the unbiasedness of the forecasts.

Several open issues merit a further study. Clearly, the nature of the linear approximation of the nonlinear framework in Section 3 is worthy of a further analysis. It would also seem useful to generalize our approach in the nonlinear setting for cycles and Markov chain sunspot equilibria which were studied for complete learning in (Evans and Honkapohja 1995b) and (Evans and Honkapohja 1994).

We note that our approach to incomplete learning is quite different from that of (Hommes and Sorger 1997) who introduce the notion of a consistent expectations equilibrium (CEE) in a similar nonlinear setup. In the CEE the perceived law of motion is linear and thus misspecified, but it is required that the sample mean and autocorrelations coincide with their theoretical counterparts. There can be different types of CEE, such as steady states, cycles or even chaotic solutions. The relationship between bounded memory learning and CEE are not clear-cut, but evidently processes such as (8) are approximately CEE when the memory length $T$ is sufficiently large by Propositions 3 and 4. This follows since the sample mean is unbiased and the covariances are small for large enough $T$.

Finally, we remark that with incomplete learning it is possible to imagine criteria for choosing among different learning rules, so that they would be equilibria within some specified class of rules. (Evans and Honkapohja 1993), (Brock and Hommes 1997) and (Sargent 1999) are examples of such view points. The partial result in Section 2.5 is in this spirit. We hope to consider these issues further in the future.

# References


Balasko, Y., and D. Royer (1996): "Stability of Competitive Equilibrium with Respect to Recursive and Learning Processes," *Journal of Economic Theory*, 68, 319–348.

Barnett, W., et al. (eds.) (1991): *Equilibrium Theory and Applications, Proceedings of the Sixth International Symposium in Economic Theory and Econometrics*. Cambridge University Press, Cambridge.

Bhattacharyya, R. N., and O. Lee (1988): "Ergocidity and Central Limit Theorems for a Class of Markov Processes," *Journal of Multivariate Analysis*, 27, 80–90.

Brock, W. A., and C. H. Hommes (1997): "A Rational Route to Randomness," *Econometrica*, 65, 1059–1095.

Bullard, J. (1994): "Learning Equilibria," *Journal of Economic Theory*, 64, 468–485.

Evans, G. W. (1985): "Expectational Stability and the Multiple Equilibria Problem in Linear Rational Expectations Models," *The Quarterly Journal of Economics*, 100, 1217–1233.

Evans, G. W., and R. Guesnerie (1993): "Rationalizability, Strong Rationality, and Expectational Stability," *Games and Economic Behaviour*, 5, 632–646.

Evans, G. W., and S. Honkapohja (1993): “Adaptive Forecasts, Hysteresis and Endogenous Fluctuations,” *Federal Reserve Bank of San Francisco Economic Review*, 1993(1), 3–13.

——— (1994): “On the Local Stability of Sunspot Equilibria under Adaptive Learning Rules,” *Journal of Economic Theory*, 64, 142–161.

——— (1995a): “Adaptive Learning and Expectational Stability: An Introduction,” in (Kirman and Salmon 1995), chap. 4, pp. 102–126.

——— (1995b): “Local Convergence of Recursive Learning to Steady States and Cycles in Stochastic Nonlinear Models,” *Econometrica*, 63, 195–206.

——— (1999): “Learning Dynamics,” in (Taylor and Woodford 1999), chap. 7.

——— (2000a): “Convergence for Difference Equations with Vanishing Time Dependence, with Applications to Adaptive Learning,” *Economic Theory*, 15, 717–725.

——— (2000b): *Learning and Expectations in Macroeconomics*. Princeton University Press, forthcoming, Princeton, New Jersey.

Giona, M. (1991): “Dynamics and Relaxation Properties of Complex Systems with Memory,” *Nonlinearity*, 4, 911–925.

Grandmont, J.-M. (1985): “On Endogenous Competitive Business Cycles,” *Econometrica*, 53, 995–1045.

——— (1998): “Expectations Formation and Stability of Large Socioeconomic Systems,” *Econometrica*, 66, 741–781.

Grandmont, J.-M., and G. Laroque (1986): “Stability of Cycles and Expectations,” *Journal of Economic Theory*, 40, 138–151.

Grandmont, J.-M., and G. Laroque (1991): “Economic Dynamics with Learning: Some Instability Examples,” in (Barnett et al. 1991), chap. 11, pp. 247–273.

Guesnerie, R. (1992): “An Exploration of the Eductive Justifications of the Rational-Expectations Hypothesis,” *American Economic Review*, 82, 1254–1278.

Guesnerie, R., and M. Woodford (1991): “Stability of Cycles with Adaptive Learning Rules,” in (Barnett et al. 1991), pp. 111–134.

Hamilton, J. D. (1994): *Time Series Analysis*. Princeton University Press, Princeton NJ.

Hommes, C. H., and G. Sorger (1997): “Consistent Expectations Equilibria,” *Macroeconomic Dynamics*, 2, 287–321.

Horn, R., and C. Johnson (1985): *Matrix Analysis*. Cambridge University Press.

Kirman, A., and P. Salmon (eds.) (1995): *Learning and Rationality in Economics.* Basil Blackwell, Oxford.

Kreps, D., and K. Wallis (eds.) (1997): *Advances in Economics and Econometrics: Theory and Applications, Volume I.* Cambridge University Press, Cambridge.

Lucas, R. E. J. (1973): "Some International Evidence on Output-Inflation Tradeoffs," *American Economic Review*, 63, 326–334.

Marcet, A., and J. P. Nicolini (1998): "Recurrent Hyperinflations and Learning," Working Paper 1875, CEPR.

Marimon, R. (1997): "Learning from Learning in Economics," in (Kreps and Wallis 1997), chap. 9, pp. 278–315.

Mitra, K. (2000): "Is More Data Better?," mimeo.

Sargent, T. J. (1999): *The Conquest of American Inflation.* Princeton University Press, Princeton NJ.

Taylor, J., and M. Woodford (eds.) (1999): *Handbook of Macroeconomics, Volume 1.* Elsevier, Amsterdam.

**Figure 1 A : Plot of the actual value of $x_t$ for a typical simulation with T = 5 and 50. Noise is uniform with support [-0.1, 0.1]. The simulations have been run for 5000 periods with the final 100 values plotted. $\alpha$ = 5 and $\beta$ = -4 for the simulation.**

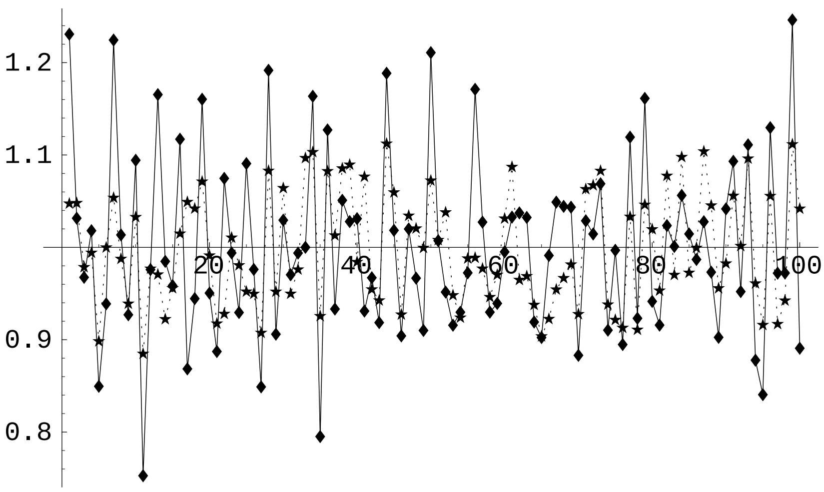


**Figure 1 B : Plot of the actual value of $x_t$ for a typical simulation with T = 50 and 500. Noise is uniform with support [-0.1, 0.1]. The simulations have been run for 5000 periods with the final 100 values plotted. $\alpha$ = 5 and $\beta$ = -4 for the simulation.**

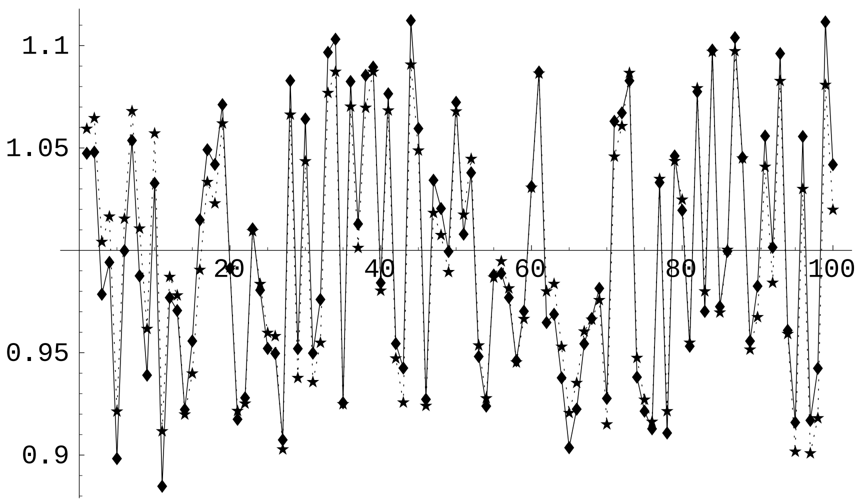

**Figure 1 C : Plot of the actual value of $x_t$ and the RE value for a typical simulation with T = 50. Noise is uniform with support [−0.1, 0.1]. The simulations have been run for 5000 periods with the final 100 values plotted. $\alpha$ = 5 and $\beta$ = −4 for the simulation.**

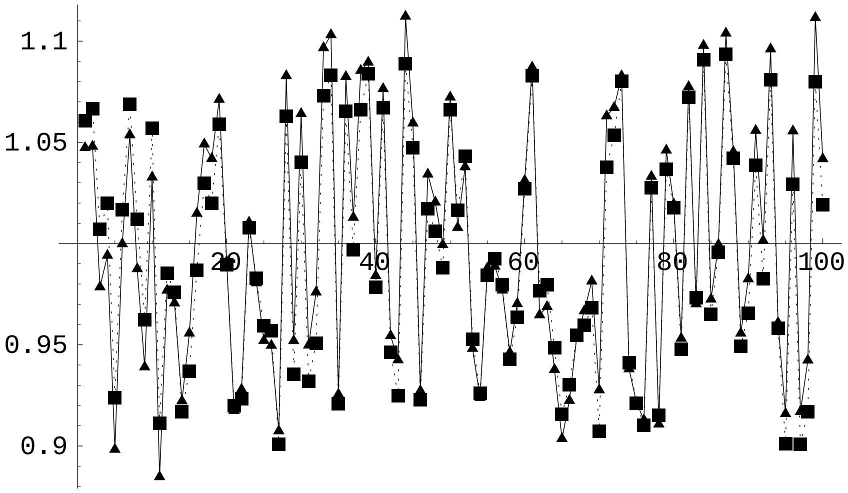


**Figure 2 : Plot of the actual value of $x_t$ and its linear approximation for a typical simulation with T = 50. The productivity shock $\lambda$ is assumed to be uniform with support [0.3, 0.9]. The simulations have been run for 5000 periods with the final 50 values plotted.**