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Abstract

In this paper, we extend the result of Dekel and Wollinsky [4] on rationalizable outcomes in first-price auctions. Dekel and Wollinsky [4] show that under certain conditions, each player chooses a unique bid conditional on her valuation. Their result however depends on the assumption that the number of players is sufficiently large (relative to the number of available bids). We first provide a different set of sufficient conditions for the uniqueness result. We then show that for the independent (possibly asymmetric) private value case, (i) the result holds if the distributions are such that the inverse hazard rate is sufficiently high for each valuation, implying that auctions need not necessarily be large, and (ii) if the distributions satisfy the conditions of Dekel and Wollinsky [4], they always satisfy ours.

Keywords: First-Price Auctions, Rationalizability, Dominance
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1 Introduction

It is standard to use the notion of (Bayesian) Nash equilibrium to analyze first-price auctions. Battigalli and Sciniscalchi [1] and Dekel and Wollinsky [4] are the first studies which departed from this tradition; they instead use the notion of *rationalizability.*[1] This alllows us to have a larger class of beliefs since they are required to be correct in equilibrium analyses. Moreover, Dekel and Wollinsky [4] (DW hereafter) show (i) that under certain conditions, each player has a unique rationalizable bid conditional on her valuation, and (ii) the conditions are satisfied for the case where the distributions are conditionally independent and identical, and the probability of each type is strictly positive.

Despite their surprising uniqueness result, they have one constraint which could be unrealistic for certain cases; *auctions need to be large.* More specifically, the (implicit) assumption in DW is that n, the number of players, is strictly higher than $m+1$, the number of grids for possible bids and valuations, i.e., $n > m+1$ $(n-1 \geq m+1)$.[2] Our goal in this paper is to provide a different set of conditions under which (i) the same uniqueness result is obtained, and (ii) auctions need not necessarily be large.

Our result does not require the assumption of $n > m+1$. In addition, for the independent case, we show that if the *inverse hazard rate* (the ratio of the probability density function over the cumulative distribution function, also known as the *inverse Mills ratio*) is sufficiently large for each valuation, the result holds.[3] This can be achieved even if the distributions are *asymmetric* (with the same support).[4] We provide an example under which the result still holds even if n is small *independent of the size of* m. Lastly, we show that for the independent case, if players' beliefs satisfy DW's conditions, they also satisfy ours.

The intuition behind DW's approach is as follows: Given your valuation v and conditional on the event that v is the highest valuation, if there are other players with valuation v, bidding the lowest bid among the ones which are rationalizable, $\underline{b}$, does not make sense. This is because the best scenario in which you win is every one else with v also bids $\underline{b}$ and the ones with lower valuations choose the bids lower than $\underline{b}$. In this case,

[1] Battigalli and Sciniscalchi [1] consider the continuous case while Dekel and Wollinsky [4] use the discrete case. See also Cho [2].

[2] If $n > m+1$ does not hold, $L_2 - U_2$ (DW p.181-182 in Proposition 1) is not well defined.

[3] Note that the assumption that the inverse hazard rate is high for each valuation excludes the case where there exists a valuation whose probability is zero.

[4] The issue of asymmetry has also been analyzed in the literature. See for example Maskin and Riley [5].

the winner is drawn with equal probability. The chance of winning is small if there are too many players with v.

Our approach provides a different intuition: Suppose, as in DW, that the valuation and bid space is a discrete grid and let d be the distance between adjacent points. Consider the case in which your valuation v is the highest possible valuation and the second highest valuation is $\tilde{v}$ ($v > \tilde{v}$). Each player with valuation $\hat{v} \leq \tilde{v}$ has a unique rationalizable bid $\hat{v} - d$. In this case, you do not want to make a bid $\tilde{v} - d$ since if it is likely that there are players with $\tilde{v}$, they will bid thier only surviving bid, $\tilde{v} - d$. Thus there is a chance that you lose. If instead you bid $\tilde{v}$ which is still less than v, you can guarantee the win.

Note that the intuition for our result does not necessarily require a large number of players. We need a player's concern that someone else would match her lower bid. If she believes that the event is very likely, she would choose a higher bid. This belief is reflected by the assumption that the inverse hazard rate is sufficiently high for the independent case. The case of large n is needed if players do not have such beliefs.

2 Preliminaries

The following is the set-up of DW which we adopt. Consider a first-price auction with the set of players $N = \{1, \ldots, n\}$. We assume $n \geq 3$.[5] Possible valuations are $V = \{0, d, 2d, \ldots, 1 - 2d, 1 - d, 1\}$. By denoting $d = \frac{1}{m}$, V contains $m + 1$ elements where $m \geq 2$.[6] Each player chooses her bid from V. Following DW, we assume that the various types of each player can have different beliefs regarding the other players' valuations and strategies. We now introduce the conditions imposed on the set of beliefs we consider.

2.1 Beliefs

Given player i's valuation v_i, let $p_i(\cdot \mid v_i) \in \Delta(V^{n-1})$ be player i's belief regarding other players' valuations. We assume that the following two conditions hold for the set of beliefs we consider:

Condition 1 *For each $i \in N$,*

$$p_i(v_j \leq 2d \text{ for all } j \neq i \mid 0) > 0, \text{ and}$$
$$p_i(v_j = 0 \text{ for all } j \neq i \mid d) > 0.$$

[5] Remember that the arguments in DW implies $n > m + 1 \geq 3$.

[6] If $m = 1$, Condition 1 suffices.

Condition 2 *For each $v_i \geq (\alpha+1)d$ where α is an positive integer ($\alpha \geq 1$),*

$$\sum_{l=1}^{n-1} p_i\left(\begin{array}{c} v_j \leq \alpha d \textit{ for all } j \neq i \\ \#\{k \neq i \textit{ s.t. } v_k = \alpha d\} = l \end{array} \middle| v_i \right)\left(\frac{l-1}{l+1}\right) > p_i(v_j \leq (\alpha-1)d \textit{ for all } j \neq i \mid v_i).$$

For the comparison purpose, we provide the conditions in DW below:

DW-1 For any $p_i(\cdot \mid v_i = v)$,

- $p_i(v_j < v \text{ for all } j \neq i \mid v_i = v) > 0$ for every $v > 0$, and
- $p_i(v_j = 0 \text{ for all } j \neq i \mid v_i = 0) > 0$.

DW-2 There exists $\bar{n}$ such that for any $n > \bar{n}$, i, and v, and any $p_i(\cdot \mid v_i = v)$,

$$p_i(\#\{j \neq i \text{ s.t. } v_j = v\} \leq m \mid v_j \leq v \text{ for all } j, v_i = v) < \frac{1}{n(m-1)+1}.$$

Note that

$$\begin{aligned} & \sum_{l=1}^{n-1} p_i\left(\begin{array}{c} v_j \leq \alpha d \text{ for all } j \neq i \\ \#\{k \neq i \text{ s.t. } v_k = \alpha d\} = l \end{array} \middle| v_i \right)\left(\frac{l-1}{l+1}\right) \\ \leq & \sum_{l=1}^{n-1} p_i\left(\begin{array}{c} v_j \leq \alpha d \text{ for all } j \neq i \\ \#\{k \neq i \text{ s.t. } v_k = \alpha d\} = l \end{array} \middle| v_i \right) \\ = & \; p_i\left(\begin{array}{c} v_j \leq \alpha d \text{ for all } j \neq i \\ \exists k \neq i \text{ s.t. } v_k = \alpha d \end{array} \middle| v_i \right). \end{aligned}$$

Given Condition 2 and

$$\begin{aligned} & p_i(v_j \leq \alpha d \text{ for all } j \neq i \mid v_i) \\ = & \; p_i(v_j \leq (\alpha-1)d \text{ for all } j \neq i \mid v_i) + p_i\left(\begin{array}{c} v_j \leq \alpha d \text{ for all } j \neq i \\ \exists k \neq i \text{ s.t. } v_k = \alpha d \end{array} \middle| v_i \right), \end{aligned} \tag{1}$$

for any $v_i \geq 2d$, we have

$$p_i(v_j < v_i \text{ for all } j \neq i \mid v_i) > 0. \tag{2}$$

This almost replaces DW-1. Since this does not include the cases for $v_i = 0, d$, we have Condition 1.

Condition 2 assures that a player with valuation v_i bids exactly $v_i - d$. The idea is, at each iteration, to use the bid αd to dominate the bid $(\alpha-1)d$ for players with valuations $v_i > \alpha d$. Since $v_i - \alpha d < v_i - (\alpha-1)d$, the bid αd must have a sufficiently larger probability of winning. Note that Condition 2 and (1) imply

$$p_i\left(\begin{array}{c} v_j \leq \alpha d \text{ for all } j \neq i \\ \exists k \neq i \text{ s.t. } v_k = \alpha d \end{array} \middle| v_i \right) > p_i(v_j \leq (\alpha-1)d \text{ for all } j \neq i \mid v_i). \tag{3}$$

In the independent case, this relationship is reflected in the requirement that the inverse hazard rate is sufficiently high.

Example 1. This example demonstrates that allowing correlations could lead to the violation of Condition 2. Suppose that there exists player i who believes that, conditional on her valuation being v, there are only two possible scenarios; (i) the other players have the same valuation v, and (ii) the other players' valuations are all (weakly) less than $v-d$ and there is exactly one player with valuation $v-d$. She assigns a high probability to the first so that DW-2 is not violated. The second scenario, however, implies that the left-hand side in Condition 2 is zero and hence it does not hold: First, for $l=1$, the associated probability is strictly positive while $(\frac{l-1}{l+1})=0$. Second, for each $l>1$, the corresponding probability is zero.

We later show that for the case of independent distributions, if the distributions satisfy the conditions in DW, they also satisfy ours. Hence, the violation of this statement (e.g., Example 1 above) can be observed only due to the possibility of correlations.

2.2 Dominance and Rationalizability

Given player i's valuation v_i, let $q_i \in \Delta(V^{n-1} \times V^{n-1})$ be player i's *forecast* over her opponents' valuations and bids.[7] Let $b_i \in V$ be player i's bid and $b_{-i} \in V^{n-1}$ be player i's opponents' bid profile.

Definition 1 *Given the set of bids $U_i^k(v_i) \subset V$ for each $i \in N$ and $v_i \in V$, a bid $b_i' \in U_i^k(v_i)$ strictly dominates $b_i \in U_i^k(v_i)$ for type v_i if*

$$\sum_{v_{-i},b_{-i}} u_i(b_i, b_{-i} \mid v) q_i(v_{-i}, b_{-i} \mid v_i) < \sum_{v_{-i},b_{-i}} u_i(b_i', b_{-i} \mid v) q_i(v_{-i}, b_{-i} \mid v_i)$$

for all $q_i \in \Delta(V^{n-1} \times V^{n-1})$ such that

- $q_i(v_{-i}, b_{-i} \mid v_i) > 0 \Rightarrow b_{-i} \in (U_j^k(v_j))_{j \neq i}$, *and*
- $\sum_{b_{-i}} q_i(v_{-i}, b_{-i} \mid v_i) = p_i(v_{-i} \mid v_i)$.

We say that $b_i \in U_i^k(v_i)$ is not strictly dominated if there does not exists $b_i' \in U_i^k(v_i)$ which strictly dominates b_i. Let $U_i^0(v_i) = V$ for each $i \in N$ and $v_i \in V$. For each $i \in N$ and $v_i \in V$, let

$$U_i^{k+1}(v_i) = \{b_i \in U_i^k(v_i) | b_i \text{ is not strictly dominated}\}$$

[7] The word "forecast" is adopted from Dekel, Fudenberg and Morris [3].

where $k \in \{0, 1, 2, \ldots\}$. The set of iteratively undominated bids for player i with valuation v_i is hence $\cap_{k=0}^{\infty} U_i^k(v_i)$.

The notion of strict dominance above is the adaptation of the same notion introduced by Dekel, Fudenberg, and Morris [3]. Dekel, Fudenberg, and Morris [3] introduce the concept of *interim correlated rationalizability* (ICR), and show that equivalence of ICR and iterated strict dominance.[8] Although the original definition of Dekel, Fudenberg and Morris [3] uses mixed strategies, we only consider pure-strategy dominance above. In addition, we assume that players are risk neutral.

3 Result

As in DW, our proof consists of two parts; we first identify an upper bound on bids for each type, and we then identify a lower bound. We show that these two bounds are identical, implying a unique bid for each type. We follow DW for the determination of the upper bounds, which is reproduced in Appendix.

Observation 1 *For any v_i, no bid strictly higher than* $\max\{0, v_i - d\}$ *survives iterated dominance.*

Note that for $v_i \in \{0, d\}$, the only surviving bid is $b_i = 0$.

Given Observation 1, we show that, for each $v_i \geq 2d$, $b_i = 0$ is dominated by $b_i = d$. First, suppose that for each $j \neq i$, $v_j \leq d$, and hence they only choose $b_j = 0$. Then, $b_i = d$ gives a payoff of $v_i - d > 0$ while $b_i = 0$ gives a payoff of $\frac{v_i}{n}$. Since we assume that $n \geq 3$, if follows that $v_i - d > \frac{v_i}{n}$ for any $v_i \geq 2d$.[9] If instead there is $j \neq i$ such that $v_j > d$, then $b_i = d$ weakly dominates $b_i = 0$. Condition 2 implies that for any $v_i \geq 2d$,

$$p_i(v_j \leq d \text{ for all } j \neq i \mid v_i) > 0.$$

Thus, the former occurs with positive probability, and our claim holds.

Observation 2 *For each $v_i \geq 2d$, the set of bids which still survive iterated dominance is $\{d, \ldots, v_i - d\}$.*

[8] Dekel, Fudenberg and Morris [3] define (i) payoff relevant state space, Θ, and (ii) type space $\mathcal{T} = (T_i, \pi_i)_{i \in N}$ where π_i represents player i's belief and maps T_i to $\Delta(T_{-i} \times \Theta)$. By letting $\Theta = \prod_{j \in N} T_j = V^n$, we utilize the beliefs we discussed above and define strict dominance with respect to V^n. The set of iteratively undominated strategies we obtain is hence the set of interim correlated rationalizable strategies.

[9] $(v_i - d) - \frac{v_i}{n} = \frac{n-1}{n} v_i - d \geq \frac{2}{3} 2d - d = \frac{1}{3} d > 0.$

Note that for $v_i = 2d$, the only surviving bid is $b_i = d$.

We now turn to establishing the unique bid for the remaining types $v_i \geq 3d$. Suppose that there exists $\alpha \geq 3$ such that for each $v_i \leq \alpha d$, the only remaining bid is $\max\{0, v_i - d\}$. For each $v_i \geq (\alpha+1)d$, we use $b_i = \alpha d$ to dominate $\tilde{b}_i = (\alpha-1)d$. Obviously, $v_i - b_i < v_i - \tilde{b}_i$ while b_i gives a higher chance of winning than $\tilde{b}_i$. We show that the latter effect dominates the former with any belief satisfying Condition 2, implying that b_i strictly dominates $\tilde{b}_i$ for each $v_i \geq (\alpha+1)d$. We then apply Proposition 1 iteratively. This demonstrates that for $v_i \geq 3d$, any $b_i < v_i - d$ is iteratively dominated, leaving $b_i = v_i - d$ as the only undominated bid.

Proposition 1 *Let $\alpha \geq 2$. Suppose that for each $v_i \geq (\alpha+1)d$,*

$$U_i^k(v_i) = \{(\alpha-1)d, \ldots, v_i - d\}$$

while for $v_i \leq \alpha d$ is $U_i^k(v_i) = \{\max\{v_i - d, 0\}\}$. Then, for each $v_i \geq (\alpha+1)d$, $b_i = (\alpha-1)d$ is strictly dominated by $b_i = \alpha d$.

Proof. Given $v_i \geq (\alpha+1)d$, the chance of winning by choosing $b_i = (\alpha-1)d$ is

$$\begin{aligned} W[(\alpha-1)d|v_i] &= p_i(v_j \leq (\alpha-1)d \text{ for all } j \neq i \mid v_i) \\ &\quad + \sum_{l=1}^{n-1} q_i \left(\begin{array}{c} \exists j \neq i \text{ s.t. } v_j \geq \alpha d \\ (\alpha-1)d \geq b_k \text{ for all } k \neq i \\ \#\{k \neq i \text{ s.t. } b_k = (\alpha-1)d\} = l \end{array} \middle| \, v_i \right) \left(\frac{1}{l+1} \right). \end{aligned} \tag{4}$$

Likewise, the chance of winning by choosing $b_i = \alpha d$ is

$$\begin{aligned} W[\alpha d|v_i] &= p_i(v_j \leq \alpha d \text{ for all } j \neq i \mid v_i) + q_i \left(\begin{array}{c} \exists j \neq i \text{ s.t. } v_j \geq (\alpha+1)d \\ (\alpha-1)d \geq b_k \text{ for all } k \neq i \end{array} \middle| \, v_i \right) \\ &\quad + \sum_{l=1}^{n-1} q_i \left(\begin{array}{c} \exists j \neq i \text{ s.t. } v_j \geq (\alpha+1)d \\ \alpha d \geq b_k \text{ for all } k \neq i \\ \#\{k \neq i \text{ s.t. } b_k = \alpha d\} = l \end{array} \middle| \, v_i \right) \left(\frac{1}{l+1} \right). \end{aligned} \tag{5}$$

It is clear that

$$W[\alpha d|v_i] > W[(\alpha-1)d|v_i]. \tag{6}$$

We need to show that, independent of opponents' bids (which still survive iterative dominance),

$$(v_i - \alpha d) W[\alpha d|v_i] > (v_i - (\alpha-1)d) W[(\alpha-1)d|v_i].$$

Given (6), this is hardest to satisfy when $v_i = (\alpha+1)d$. Hence, it suffices to show that

$$W[\alpha d|v_i] - 2W[(\alpha-1)d|v_i] > 0. \tag{7}$$

Equation (5) can be rewritten as

$$\begin{aligned}
W[\alpha d|v_i] &= p_i(v_j \le (\alpha-1)d \text{ for all } j \ne i \mid v_i) + p_i\left(\begin{array}{c} v_j \le \alpha d \text{ for all } j \ne i \\ \exists k \ne i \text{ s.t. } v_k = \alpha d \end{array} \middle| v_i \right) \\
&+ q_i\left(\begin{array}{c} \exists j \ne i \text{ s.t. } v_j \ge (\alpha+1)d \\ (\alpha-1)d \ge b_k \text{ for all } k \ne i \end{array} \middle| v_i \right) \\
&+ \sum_{l=1}^{n-1} q_i\left(\begin{array}{c} \exists j \ne i \text{ s.t. } v_j \ge (\alpha+1)d \\ \alpha d \ge b_k \text{ for all } k \ne i \\ \#\{k \ne i \text{ s.t. } b_k = \alpha d\} = l \end{array} \middle| v_i \right)\left(\frac{1}{l+1}\right). \qquad (8)
\end{aligned}$$

Equation (4) can be rewritten as

$$\begin{aligned}
W[(\alpha-1)d|v_i] &= p_i(v_j \le (\alpha-1)d \text{ for all } j \ne i \mid v_i) \\
&+ \sum_{l=1}^{n-1} p_i\left(\begin{array}{c} v_j \le \alpha d \text{ for all } j \ne i \\ \#\{k \ne i \text{ s.t. } v_k = \alpha d\} = l \end{array} \middle| v_i \right)\left(\frac{1}{l+1}\right) \\
&+ \sum_{l=1}^{n-1} q_i\left(\begin{array}{c} \exists j \ne i \text{ s.t. } v_j \ge (\alpha+1)d \\ (\alpha-1)d \ge b_k \text{ for all } k \ne i \\ \#\{k \ne i \text{ s.t. } b_k = (\alpha-1)d\} = l \end{array} \middle| v_i \right)\left(\frac{1}{l+1}\right). \qquad (9)
\end{aligned}$$

It is easy to see that

$$p_i\left(\begin{array}{c} v_j \le \alpha d \text{ for all } j \ne i \\ \exists k \ne i \text{ s.t. } v_k = \alpha d \end{array} \middle| v_i \right) = \sum_{l=1}^{n-1} p_i\left(\begin{array}{c} v_j \le \alpha d \text{ for all } j \ne i \\ \#\{k \ne i \text{ s.t. } v_k = \alpha d\} = l \end{array} \middle| v_i \right)$$

and

$$q_i\left(\begin{array}{c} \exists j \ne i \text{ s.t. } v_j \ge (\alpha+1)d \\ (\alpha-1)d \ge b_k \text{ for all } k \ne i \end{array} \middle| v_i \right) = \sum_{l=1}^{n-1} q_i\left(\begin{array}{c} \exists j \ne i \text{ s.t. } v_j \ge (\alpha+1)d \\ (\alpha-1)d \ge b_k \text{ for all } k \ne i \\ \#\{k \ne i \text{ s.t. } b_k = (\alpha-1)d\} = l \end{array} \middle| v_i \right).$$

Then, we have

$$\begin{aligned}
& W[\alpha d|v_i] - 2W[(\alpha-1)d|v_i] \\
= \;& -p_i(v_j \le (\alpha-1)d \text{ for all } j \ne i \mid v_i) + \sum_{l=1}^{n-1} p_i\left(\begin{array}{c} v_j \le \alpha d \text{ for all } j \ne i \\ \#\{k \ne i \text{ s.t. } v_k = \alpha d\} = l \end{array} \middle| v_i \right)\left(\frac{l-1}{l+1}\right) \\
& + \sum_{l=1}^{n-1} q_i\left(\begin{array}{c} \exists j \ne i \text{ s.t. } v_j \ge (\alpha+1)d \\ (\alpha-1)d \ge b_k \text{ for all } k \ne i \\ \#\{k \ne i \text{ s.t. } b_k = (\alpha-1)d\} = l \end{array} \middle| v_i \right)\left(\frac{l-1}{l+1}\right) \\
& + \sum_{l=1}^{n-1} q_i\left(\begin{array}{c} \exists j \ne i \text{ s.t. } v_j \ge (\alpha+1)d \\ \alpha d \ge b_k \text{ for all } k \ne i \\ \#\{k \ne i \text{ s.t. } b_k = \alpha d\} = l \end{array} \middle| v_i \right)\left(\frac{1}{l+1}\right).
\end{aligned}$$

Note that only the last two terms depend on opponents' bids and both are non-negative. Hence, if Condition 2 holds, which ensures that the sum of the first two terms is strictly positive, this expression is indeed strictly positive. *Q.E.D.*

It is then clear that the application of iterative dominance leads to the desired conclusion.

Proposition 2 *Under Conditions 1 and 2, a unique bid surviving iterative strict dominance is* $b_i = \max\{v_i - d, 0\}$ *for each* $v_i \in \{0, d, \dots, 1\}$.

4 Independent Distributions

DW also analyze the standard i.i.d. case, and demonstrate that if (i) players are symmetric, (ii) types are conditionally independent, and (iii) probability of each type is non-zero, DW-1 and DW-2 hold for sufficiently large n (with respect to m). In this section, we first show that under the same assumptions, our conditions also hold. Then, we drop the assumption of symmetry. Lastly, we show that if the distributions satisfy DW-1 and DW-2, then they also satisfy our conditions as well.

4.1 Symmetric Case

Let $p(v = \alpha d)$ be the probability that a player's type is αd. Then, the difference of the terms in Condition 2 becomes

$$\begin{aligned}
&\sum_{l=1}^{n-1} \binom{n-1}{l} [p(v \le (\alpha-1)d)]^{n-1-l} [p(v = \alpha d)]^l \left(\frac{l-1}{l+1}\right) - [p(v \le (\alpha-1)d)]^{n-1} \\
= \quad &[p(v \le (\alpha-1)d)]^{n-1} \left\{ \sum_{l=1}^{n-1} \binom{n-1}{l} \left[\frac{p(v = \alpha d)}{p(v \le (\alpha-1)d)}\right]^l \left(\frac{l-1}{l+1}\right) - 1 \right\}.
\end{aligned}$$

By letting

$$\frac{p(v = \alpha d)}{p(v \le (\alpha-1)d)} = \gamma,$$

the expression in the bracket becomes

$$\sum_{l=1}^{n-1} \binom{n-1}{l} \gamma^l - 2\sum_{l=1}^{n-1} \binom{n-1}{l} \left(\frac{1}{l+1}\right) \gamma^l - 1. \tag{10}$$

The first term of (10) is

$$\sum_{l=1}^{n-1} \binom{n-1}{l} \gamma^l = \sum_{l=0}^{n-1} \binom{n-1}{l} \gamma^l - 1 = (1+\gamma)^{n-1} - 1.$$

For the second term of (10), we have

$$\sum_{l=1}^{n-1}\left(\begin{array}{c} n-1 \\ l \end{array}\right)\left(\frac{1}{l+1}\right)\gamma^l = \frac{1}{n\gamma}\sum_{l=2}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)\gamma^l$$

$$= \frac{1}{n\gamma}\left\{\sum_{l=0}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)\gamma^l - 1 - n\gamma\right\} = \frac{1}{n\gamma}\{(1+\gamma)^n - 1\} - 1$$

Then, (10) becomes

$$\frac{(1+\gamma)^n}{\gamma}\left[\frac{\gamma}{1+\gamma} - \frac{2}{n}\right] + \frac{2}{n\gamma}. \tag{11}$$

Let

$$n(\alpha) = 2\left\lceil\frac{1+\gamma}{\gamma}\right\rceil$$

where $\lceil\cdot\rceil$ is the ceiling function. Let $n^* = \max\{n(\alpha) \mid \alpha \in \{0,1,\ldots,m\}\}$. It is clear that for any $n \geq n^*$, (11) is strictly positive, and hence so is (10).[10] That is, for any $n \geq n^*$, Condition 2 holds.

The expression in the bracket of (11) has an interesting implication. Note

$$\begin{aligned} & \frac{\gamma}{1+\gamma} - \frac{2}{n} \\ = \; & \frac{p(v=\alpha d)}{p(v \leq (\alpha-1)d) + p(v=\alpha d)} - \frac{2}{n} \\ = \; & \frac{p(v=\alpha d)}{p(v \leq \alpha d)} - \frac{2}{n}. \end{aligned} \tag{12}$$

Note that the first term is the inverse hazard rate. From this, it is clear that if the inverse hazard rate is high for each α, n does not have to be large.

Example 2. Consider the following simple example in which types are independently drawn from a unique distribution:

$$p(v=0) = \frac{1}{3^m}$$

$$p(v=\alpha d) = \frac{2}{3^{m+1-\alpha}} = \left(\frac{2}{3^{m+1}}\right)3^\alpha \text{ for } \alpha = 1,\ldots,m.$$

Note that for $\alpha \geq 1$,

$$p(v \leq \alpha d) = \frac{1}{3^m} + \left(\frac{2}{3^{m+1}}\right)\sum_{l=1}^{\alpha} 3^l = 3^{\alpha-m},$$

[10] We only need weak inequality because of the second term in (11).

and hence

$$p(v \le 1) = 1.$$[11]

For any $\alpha \ge 1$, we have

$$\frac{p(v = \alpha d)}{p(v \le \alpha d)} = \frac{\frac{2}{3^{m+1-\alpha}}}{3^{\alpha-m}} = \frac{2}{3}.$$

In this case, independent of m, $n = 3$ suffices.

4.2 Asymmetric Case

For the left-hand side of Condition 2, the opponents are divided into two groups; (i) those whose valuation is exactly αd, and (ii) those whose valuations are strictly lower than αd. Let $N_{-i} = N \backslash \{i\}$, $2^{N_{-i}}$ be the power set of the opponents and $\mathcal{N} = 2^{N_{-i}} \backslash \{\emptyset\}$. Let $N_k \in \mathcal{N}$ be a typical element of $\mathcal{N}$ for $k \in \{1, \dots, |\mathcal{N}|\}$ where $|\mathcal{N}| = 2^{n-1} - 1$. Let $p^i(\cdot)$ correspond to the distribution determining player i's valuation, which is commonly known. Then, we can rewrite the difference of the terms in Condition 2 as follows:

$$\begin{aligned}
&\sum_{k=1}^{|\mathcal{N}|} \left(\prod_{j \in N_k} p^j(v = \alpha d) \prod_{j' \in N_{-i} \backslash N_k} p^{j'}(v \le (\alpha - 1)d) \right) \left(\frac{|N_k| - 1}{|N_k| + 1} \right) \\
&- \prod_{j \in N_{-i}} p^j(v \le (\alpha - 1)d) \\
= &\prod_{j \in N_{-i}} p^j(v \le (\alpha - 1)d) \left[\sum_{k=1}^{|\mathcal{N}|} \left(\prod_{j \in N_k} \frac{p^j(v = \alpha d)}{p^j(v \le (\alpha - 1)d)} \right) \left(\frac{|N_k| - 1}{|N_k| + 1} \right) - 1 \right]
\end{aligned}$$

By letting

$$\underline{\gamma} = \min_{i \in N} \left\{ \frac{p^i(v = \alpha d)}{p^i(v \le (\alpha - 1)d)} \right\},$$

we can apply the same arguments for the symmetric case to the current case. That is, if the inverse hazard rate is sufficiently high for each valuation of each player, there is a unique bid for each valuation even if n is small.

4.3 Comparison of Conditions

In this section, we show that given that the distributions are independent (and possibly asymmetric), if they satisfy DW-1 and DW-2, they also satisfy Conditions 1 and 2. First, given the assumption that probability of each type is non-zero, DW-1 and Condition 1 both hold.

[11] Remember that $m = \frac{1}{d}$.

Suppose now that DW-2 holds. That is, there exists $\bar{n}$ such that for all $n > \bar{n}$, i and α, we have

$$p_i(\#\{k \neq i \text{ s.t. } v_k = \alpha d\} \leq m | v_j \leq \alpha d \text{ for all } j \neq i) < \frac{1}{n(m-1)+1}.$$

Remember also that DW's condition implies $n > m+1$.

Proposition 3 *Suppose that valuations are independently distributed. If players' beliefs satisfy DW-2, they also satisfy Condition 2.*

Proof. Since $p_i(v_j \leq \alpha d \text{ for all } j \neq i) > 0$, the expression in Condition 2 can be rewritten with the form of conditional probabilities:

$$\begin{aligned} & \sum_{l=1}^{n-1} p_i \left(\#\{k \neq i \text{ s.t. } v_k = \alpha d\} = l \mid v_j \leq \alpha d \text{ for all } j \neq i\right) \left(\frac{l-1}{l+1}\right) \\ > \; & p_i(v_k \leq (\alpha - 1)d \text{ for all } k \neq i \mid v_j \leq \alpha d \text{ for all } j \neq i). \end{aligned} \tag{13}$$

We show below that if DW-2 holds, (13) holds as well.

From the left-hand side expression in (13), we have

$$\begin{aligned} & \sum_{l=1}^{n-1} p_i \left(\#\{k \neq i \text{ s.t. } v_k = \alpha d\} = l \mid v_j \leq \alpha d \text{ for all } j \neq i\right) \left(\frac{l-1}{l+1}\right) \\ > \; & \sum_{l=m+1}^{n-1} p_i \left(\#\{k \neq i \text{ s.t. } v_k = \alpha d\} = l \mid v_j \leq \alpha d \text{ for all } j \neq i\right) \left(\frac{l-1}{l+1}\right) \\ > \; & \left(\frac{m}{m+2}\right) \sum_{l=m+1}^{n-1} p_i \left(\#\{k \neq i \text{ s.t. } v_k = \alpha d\} = l \mid v_j \leq \alpha d \text{ for all } j \neq i\right) \\ = \; & \left(\frac{m}{m+2}\right) \left[1 - p_i \left(\#\{k \neq i \text{ s.t. } v_k = \alpha d\} \leq m \mid v_j \leq \alpha d \text{ for all } j \neq i\right)\right] \end{aligned}$$

where (i) the first inequality holds since $n > m+1$ and (ii) the second inequality holds since $\left(\frac{l-1}{l+1}\right)$ is strictly increasing. Given DW-2, we have

$$\begin{aligned} & \left(\frac{m}{m+2}\right) \left[1 - p_i \left(\#\{k \text{ s.t. } v_k = \alpha d\} \leq m \mid v_j \leq \alpha d \text{ for all } j \neq i\right)\right] \\ > \; & \left(\frac{m}{m+2}\right) \left[1 - \frac{1}{n(m-1)+1}\right] \\ = \; & \left(\frac{m}{m+2}\right) \left[\frac{n(m-1)}{n(m-1)+1}\right] \end{aligned}$$

where the last term corresponds to the lower bound for the left-hand side of (13).

Since $m \geq 2$, from the right-hand side of (13), we have

$$p_i(v_k \leq (\alpha - 1)d \text{ for all } k \neq i \mid v_j \leq \alpha d \text{ for all } j \neq i)$$

$$
\begin{aligned}
&< \quad p_i(\#\{k \text{ s.t. } v_k = \alpha d\} \leq m \mid v_j \leq \alpha d \text{ for all } j \neq i) \\
&< \quad \frac{1}{n(m-1)+1}
\end{aligned}
$$

where the last inequality comes directly from DW-2. The last term gives the upper bound for the right-hand side of (13).

Hence, we need to show

$$\left(\frac{m}{m+2}\right)\left[\frac{n(m-1)}{n(m-1)+1}\right] > \frac{1}{n(m-1)+1}.$$

Since $n > m+1$, it suffices to show

$$(m+1)m(m-1) > m+2 \quad \Rightarrow \quad m^3 - 2m - 2 > 0.$$

This holds for any $m \geq 2$. *Q.E.D.*

5 Conclusion

In this paper, we show a set of conditions under which each player with valuation v chooses $b = \max\{v-d, 0\}$. Condition 2 in our study is different from DW-2. While DW's result requires a sufficiently large n (relative to m), ours does not rely on this. For the independent case, we show that n need not be necessarily large if the inverse hazard rate is sufficiently high for each type. This also applies to the case in which distributions are asymmetric. Moreover, if the distributions satisfy DW-2, they also satisfy Condition 2.

A Appendix

The first part of the dominance arguments, which is identical to that of Dekel and Wollinsky [4], identifies the upper bound of bid for each type. The only difference is that we instead use Conditions 1 and 2 (and hence (2)). First, we show that $b_i \leq 1-d$ for any v_i.

- For any $v_i \in \{0, \ldots, 1-d\}$, $b_i = 1$ is strictly dominated by $b_i = 0$. This is because the former guarantees a negative expected payoff (with a positive chance of winning) while the latter guarantees at least a payoff of zero.

- For a player with $v_i = 1$, $b_i = 1$ is strictly dominated by $b_i = 1-d$. For any state in which $v_j \leq 1-d$ for all $j \neq i$ and hence the highest bids are $1-d$, $b_i = 1$ is strictly dominated by $b_i = 1-d$ since the former guarantees a payoff of zero while the latter gives a strictly positive payoff. For any state in which there is player $j \neq i$ with $v_j = 1$, $b_i = 1$ is rather weakly dominated by $b_i = 1 - d$ since the latter does not win if an opponent with $v_j = 1$ chooses $b_j = 1$. Since the former class of states happens with a positive probability (Equation (2)), the clam holds.

Take $\alpha \in \{2, \ldots, m\}$. Suppose that for any $v_i \leq \alpha d$, the remaining bids are $b_i \leq (\alpha - 1)d$ while for any $v_i > \alpha d$, it is $b_i \leq v_i - d$. Then the arguments below show that for any $v_i \leq (\alpha - 1)d$, $(\alpha - 1)d$ is strictly dominated. Note that for the case of $v_i = 0, d$ we need to use Condition 1.

- For a player with $v_i \leq (\alpha - 2)d$, $b_i = (\alpha - 1)d$ is strictly dominated by $b_i = 0$. For any state in which $v_j \leq \alpha d$ for all $j \neq i$, the highest possible bid is $(\alpha - 1)d$. Hence, $b_i = (\alpha - 1)d$ gives a chance of winning, meaning a negative expected payoff, while $b_i = 0$ provides a non-negative payoffs. For any state in which there exists $v_j > \alpha d$, dominance relationship is rather weak. Since Equation (2) implies that the former states happen with a positive probability, our claim holds.[12]

- For a player with $v_i = (\alpha - 1)d$, $b_i = (\alpha - 1)d$ is strictly dominated by $b_i = (\alpha - 2)d$. For any state in which $v_j \leq (\alpha - 2)d$ for $j \neq i$, the highest possible bid is $(\alpha - 2)d$. The former guarantees a payoff of zero while the latter gives a positive expected payoff. For any state in which there exists $v_j > \alpha d$, dominance relationship is rather weak. Since the former states happen with a positive probability (Equation (2)), our claim holds.[13]

The repetition of the arguments above leads to the conclusion that, for each type $v_i \in V$, any $b_i \geq v_i$ does not survive iterative strict dominance except for $v_i = 0$ whose only surviving bid is $b_i = 0$.

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[12] Note that for $\alpha = 2$, this step implies that a player $v_i = 0$ only has $b_i = 0$, which requires the second equation in Condition 1.

[13] If $\alpha = 2$, this step implies that a player with $v_i = 1$ only has $b_i = 0$, which requires the first equation in Condition 1.

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