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Abstract

We examine patterns of acquiring non-native languages in a model with two linguistic communities with heterogeneous learning skills, where every individual faces the choice of self-learning the foreign language or acquiring it at a profit-maximizing linguistic school. We consider a one-school model with divisions in both communities and various two-school settings with a school in each community. We compare the number of learners and welfare implications under self- learning with those obtained under various schooling contexts. In particular, we show that for communities with similar size, introducing language schools always increases the number of learners with respect to the exclusive self-learning option.

Keywords: communicative benefits, linguistic equilibrium, learning costs.

JEL Classification: C72, D83, O52, Z13

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1 Introduction

In this paper, we compare the outcomes of two different modes of learning languages, *self-learning* and *learning at school*, in a simple model of language acquisition, based on previous work by Selten and Pool (1991), Church and King (1993), Shy (2001), and Gabszewicz, Ginsburgh and Weber (2010).

In their pioneering paper Selten and Pool (1991) formulate a general game-theoretical model of language acquisition. They introduce the notion of *communicative benefits*, that cover a wide range of economic, cultural and social advantages gained by learning languages. Church and King (1993) consider a special case of the Selten-Pool model with two linguistic communities, where every agent is initially proficient only in her own language, but can acquire the other one at a cost identical for all agents. Every agent is faced with a binary choice: either to learn the other language, or refrain from acquiring it. The communicative benefit of an individual increases with the number of those with whom she can communicate using a common language. Thus, the equilibrium outcome depends on a network externality since the strategic decision by an individual to learn the other language expands the communication links for others who speak that language. The larger the number of individuals in the other language group who learn the native tongue of an agent, the smaller the benefit from second language acquisition for that agent. Church and King (1993) show that only corner solutions exist in equilibrium: either no one learns any language in either community (if the cost of learning is sufficiently high), or everybody learns the foreign language in one community while nobody does in the other. The fact that only corner equilibria exist is due to the assumption that learning costs are homogeneous over the population: once learning is beneficial for one agent initially endowed with some language, it is also so for all those who speak the same language.

Ginsburgh, Ortuño-Ortín and Weber (2007) suggest that interior equilibria may exist, but focus on the empirical implications of the model, namely, the derivation of demand functions for languages. Gabszewicz, Ginsburgh and Weber (2010) introduce a heterogeneous variant of the Church-King model where individuals differ in their degree of learning aptitude and, therefore, face different individual learning costs. Agents opt to maximize their net communicative benefit defined as the difference between the communicative benefit discussed above, and their individual cost of acquiring a new language. Gabszewicz, Ginsburgh and Weber require that, in equilibrium, expectations of learners in one population about the number of learners in the other are fulfilled.

In this paper we offer an extensive menu of language learning mechanisms. In addition to self-learning examined in the aforementioned papers, we also introduce profit-maximizing language schools which charge tuition fees. We consider different scenarios, including the one that allows the school to open a division in each of two linguistic communities.[1] This introduces a multi-sided market interpretation of the model in which tuition fees are explicitly taken into account when defining learning costs of individuals. The multilingual school then appears as a two-sided platform embarking individuals from both communities. While two-sided markets[2] often involve *positive* cross network externalities, this is no longer the case in the bilingual context considered here. Agents are faced with *negative* externalities since the benefit obtained from learning the other language "at home" decreases with the number of agents who speak the home language in the other community.

Self-learning and learning at school are *a priori* very different mechanisms. The former introduces coordination among agents through the Nash equilibrium of a game with a number of players equal to the total number of citizens in the two communities. By contrast, the second mechanism requires the presence of an intermediary (the platform) to coordinate the decisions between the two sides of the market, and relies on profit maximization and prices (tuition fees). In spite of these discrepancies, we show that both mechanisms lead to similar features, but that welfare may become larger with the existence of schools. For instance, under both mechanisms, an increase of the population in the home community reduces the fraction of learners in the home community while the reverse holds when increasing the population in the other community. It is interesting to point out that while the raising learning costs reduce the fraction of learners in the smaller community, it is not necessarily the case with respect to the larger community. Moreover, under both mechanisms, self- and school-learning, the fraction of learners is larger in the smaller population. The presence of a negative externality in language learning generates some results that may seem counterintuitive. For instance it may happen that the introduction of language schools in both communities *reduces* the number of learners in one of them with respect to

[1] An example of such a school is the well-known *Berlitz School of Languages.*

[2] The notion was introduced by Rochet and Tirole (2003), and applied to several contexts such as credit cards (Wright, 2004) and the media industry (Gabszewicz, Laussel and Sonnac, 2001, Anderson and Coate, 2004).

the exclusive self-learning setting.

Note that the model with language schools can be viewed either as a two-sided market, or as a multiproduct monopoly. If it is considered as a market for a good or service the utility of which on one side (in our case, in one community) depends on the number of buyers on the other side (the other community), then the situation we describe is a two-sided market. However this dependence is not as significant as, say, in the credit cards' market. While, in the latter, the utility of a cardholder is zero if no shop accepts it, the utility of learning a foreign language is positive, even if nobody learns in the other community. If a two-sided market is viewed as a market for a good that has value only if the two sides are "on board," then the situation described above corresponds to a multiproduct monopoly: a language school may be operated in each community by different owners without qualitatively impairing the value of language schools for agents on both sides. Note that the conceptual problem raised above is rather a question of terminology: if the multiproduct monopolist acts as a profit maximizer taking into account the spillovers between the two markets, he would behave as the platform described above.

The alternative institutional arrangement with one school acting as a monopolist in each community, requires another type of implicit coordination between the two communities. In spite of being local monopolists, the demand for learning the foreign language at each local school depends not only on its own subscription fee, but also indirectly on the subscription fee set by the school in the other community. Indeed, when the latter sets a low fee, it attracts more learners, increasing thereby the number of those who learn the language of the former community. This in turn diminishes, in this community, the incentives to learn the foreign language and entails a decrease in demand for learning. It can be argued that local schools are unaware of this rather complicated demand interdependence, in which case they behave as myopic monopolists who do not take into account the spillovers described above. Alternatively, they can be sophisticated monopolists who are aware of these spillovers, in which case the interdependence between demands becomes explicit and makes the payoffs of each local monopolist depend not only on its own fee, but also on the fee set in the other community. We distinguish the effects of these two alternative assumptions on the fraction of language learners, and compare them to the corresponding fractions under self-learning and the multi-sided school operation.

The paper is organized as follows. The model is presented in Section 2, which also includes the main results obtained in the case of self-learning only. Section 3 describes the two-sided market solution and performs the comparative statics and welfare analysis with self-learning only. Section 4 contains the analysis of the setting where a local school acts as a monopolist in each separate community, both under the assumption of myopia and sophistication. In Section 5, the various modes of language learning are analyzed in a simplifying setting with equal populations in both communities. Section 6 concludes and suggests further research avenues. Proofs of the propositions are relegated to an appendix.

2 The model

We consider two communities (regions, countries) e and f, whose populations sizes are E and F, respectively. All individuals are assumed to be born unilingual and speak only their native language, e or f. However, they may consider acquiring the other language. This decision is based on two factors, *potential benefits* and *learning costs*.

Benefits. Following Church and King (1991), we assume that the gross communicative benefit of an individual is represented by the number of others (in her own and in the other community) with whom she shares a common language. Assume that a proportion l_f of citizens in f learns language e. A citizen in population e who refrains from learning language f can communicate with E fellow citizens and with the l_fF individuals of the other population who have acquired language e. That is, her communicative benefit is represented by $E + l_fF$. If that individual from community e learns language f, her communicative benefit will take into account all individuals from both populations $E + F$. However, to acquire the other language, she will have to face learning costs.

Learning costs. We assume that both communities consist of heterogeneous individuals uniformly ranked on the basis of a parameter $\theta \in [0, 1]$, representing their type, which is the inverse of their ability to learn a foreign language. Those with lower θ can acquire it with a relative ease, and, in particular, an individual with $\theta = 0$ can learn it in her sleep. A larger θ is the sign of more difficulty in learning the other language. We identify individuals by their type θ and introduce two options of acquiring the foreign language:

The first is *self-learning.* In this case we assume that every individual faces a personalized self-learning cost $C(\theta)$, which depends on her language ability. For simplicity, we assume that in both communities self-learning costs are given by $C(\theta) = c\theta$, where c is a positive parameter common to both populations.[3]

The second is *school-learning.* Indeed, the demand for foreign language learning creates an opening for language classes provided on a commercial basis and we assume now that individuals in both communities are offered an additional means of learning the other language by attending profit-maximizing *language schools,* that charge tuition fees p_e and p_f to populations e and f, respectively. While attending classes, individuals who learn still invest time and effort. We assume that an individual θ in community e (f), who enrolls in the school will face the total cost $p_e + r\theta$, $(p_f + r\theta$, respectively), where r, again for reasons of analytical simplicity, is a common parameter in both communities. The school allows reducing the personalized cost so that the school-generated parameter r is smaller than the self-learning cost factor c.

Assuming without loss of generality that the population size E is larger than F, we impose the following assumption on the parameters of the model:

Assumption

$$F \leq E < r < c, \tag{1}$$

where the inequality $E < r$ is a necessary and sufficient condition to generate positive demands for schooling (see below). Note also that this inequality guarantees that the school-driven learning cost r, and the self-learning cost c of the least able individual whose $\theta = 1$ in both populations, exceeds the maximum additional communicative benefit she can derive from learning the other language. This assumption ensures that not *all* individuals in e and f will learn the other language. It will be used in proving all the results.

Individuals in both populations have three options: to learn by themselves, to attend the division of the school located in their community or refrain from learning the other language. We now derive the communicative benefit of an agent of type θ in either of the three cases:

[3]This assumption is introduced for the purpose of analytical simplicity only. The commonality of the cost parameter can be also challenged on empirical grounds, see Ginsburgh and Weber (2010, chapter 3).

• Self-learning. Every agent of type θ who incurs the cost c of learning the other language on his own will be able to communicate with both populations:

$$E + F - c\theta. \tag{2}$$

• School-learning: Every agent of type θ who incurs the cost c of learning the other language at scool will be able to communicate with both populations:

$$E + F - p_e - r\theta. \tag{3}$$

• No learning: Without any investment, every agent will still be able to communicate with the members of her own community e and those in f who learn language e:

$$E + l_f F. \tag{4}$$

For a given tuition fee p_e for learning language f in community e, denote by $\theta^1(p_e)$ the individual (if she exists) who is indifferent between self-learning and learning at school, i.e., for whom the benefits, given by (2) and (3), are equal:

$$\theta^1(p_e) = \frac{p_e}{c - r}. \tag{5}$$

Note that individuals with $\theta < \theta^1(p_e)$ prefer self-learning over learning at school, whereas all those with $\theta > \theta^1(p_e)$ will make the opposite choice. Indeed, individuals with low θ face low cost of self-learning and would not find it beneficial to pay tuition. However, to estimate the demand addressed to the language school, we have to make sure that individuals with higher θ indeed enroll in the school and do not drop out of the language market alltogether. For this, denote by $\theta^2(p_e, l_f)$, the individual (if she exists) who is indifferent between learning at school and refraining from learning language f. She is the individual for whom the benefits, given by (3) and (4), are equal:

$$\theta^2(p_e, l_f) = \frac{F(1 - l_f) - p_e}{r}. \tag{6}$$

Individuals with $\theta < \theta^2(p_e, l_f)$ prefer learning at school rather than forego learning language f, whereas all those with $\theta > \theta^2(p_e, l_f)$ will refrain from learning. We restrict our attention to levels of tuition fees p_e and p_f that guarantee that the solutions are interior and that markets are not fully covered. We have the following result:

Proposition 1 *Given the tuition fee p_e and the fraction l_f of learners of language e in community f, there is an incomplete market coverage and a strictly positive apprenticeship in community e only if*

$$0 < \theta^1(p_e) < \theta^2(p_e, l_f) < 1. \tag{7}$$

In this case,

(i) individuals with $\theta \in [0, \theta^1(p_e))$ engage in self-learning;

(ii) individuals with $\theta \in [\theta^1(p_e), \theta^2(p_e, l_f))$ enroll in the language school;

(iii) individuals with $\theta \in [\theta^2(p_e, l_f), 1]$ refrain from learning.

A similar reasoning applies to community f. That is, given the tuition fee p_f and the fraction l_e of learners of language f in community e, there is an incomplete market coverage and a strictly positive apprenticeship in community f only if

$$0 < \theta^1(p_f) < \theta^2(p_f, l_e) < 1, \tag{8}$$

where $\theta^1(p_f) = \dfrac{p_f}{c - r}$, $\theta^2(p_f, l_e) = \dfrac{E(1 - l_e) - p_f}{r}$.

Then,

(iv) individuals with $\theta \in [0, \theta^1(p_f))$ engage in self-learning;

(v) individuals with $\theta \in [\theta^1(p_f), \theta^2(p_f, l_e))$ enroll in the language school;

(vi) individuals with $\theta \in [\theta^2(p_f, l_e), 1]$ refrain from learning.

Proposition 1 implies that the fraction of learners l_e in community e is given by[4]

$$l_e = \frac{F(1 - l_f) - p_e}{r}. \tag{9a}$$

Similarly, the fraction of learners l_f in community f is:

$$l_f = \frac{E(1 - l_e) - p_f}{r}. \tag{9b}$$

[4]In fact, the fraction l_e is determined by $l_e = \min\{(F(1 - l_f) - p_e)/r, 1\}$. However, since we restrict our examination to interior solution only, we proceed with the expressions in (9a) and (9b).

By solving the system (9a)-(9b), we derive the fraction of learners in both communities as functions of the tuition fees p_e and p_f:

$$l_e(p_e, p_f) = \frac{rp_e - rF + EF - p_f F}{EF - r^2}, \tag{10a}$$

$$l_f(p_e, p_f) = \frac{rp_f - rE + EF - p_e E}{EF - r^2}. \tag{10b}$$

Note that Proposition 1 allows us to derive the demand for apprenticeship in both populations, as only individuals from the intermediate range of θ will enroll in the language school (see *(ii)* and *(v)* in Proposition 1). The demands as functions of l_e and l_f are given by

$$E\left(\frac{F(1-l_f) - p_e}{r} - \frac{p_e}{c-r}\right) \text{ and } F\left(\frac{E(1-l_e) - p_f}{r} - \frac{p_f}{c-r}\right). \tag{11}$$

By substitution of (10a)-(10b) in (11), we obtain the demands $D_e(p_e, p_f)$ and $D_f(p_e, p_f)$ at the two divisions, namely,

$$D_e(p_e, p_f) = E\left(\frac{rF - EF - rp_e + p_f F}{r^2 - EF} - \frac{p_e}{c-r}\right), \tag{12a}$$

$$D_f(p_e, p_f) = F\left(\frac{rE - EF - rp_f + p_e E}{r^2 - EF} - \frac{p_f}{c-r}\right). \tag{12b}$$

Since we compare the various schooling options to self-learning, it is useful to describe here the main results obtained by Gabszewicz, Ginsburgh and Weber (2010) for that case. We identify the marginal types θ of individuals who are indifferent between self-learning and foregoing the study of the other language given by the cutoff values $\theta(l_e)$ and $\theta(l_f)$ in communities e and f, respectively. Assuming self-fulfilling expectations, we equate (2) and (4) and obtain the expressions for shares of learners in both communities:

$$\theta(l_f) = (1 - l_f)\frac{F}{c} \text{ and } \theta(l_e) = (1 - l_e)\frac{E}{c}.$$

Since $l_e = \theta(l_f)$ and $l_f = \theta(l_e)$, we solve this system to derive the equilibrium shares l_e^* and l_f^* of self-learners:

$$l_e^* = \frac{F(E-c)}{EF - c^2}; \;\; l_f^* = \frac{E(F-c)}{EF - c^2}.$$

Assumption (1) implies that the difference $l_f^* - l_e^*$ is none-negative (and positive if $F < E$) so that the number of learners in the large community cannot be larger than the one in the small community. The small community gains more from learning the other language than the large one, since its population has access to more speakers.

3 Language learning in a two-sided market

Now turn to the case of the language school that offers language learning in both communities. In this two-sided platform setting, the school chooses prices $\tilde{p}_e$ and $\tilde{p}_f$ which maximize total profits $\pi(p_e, p_f)$ over the two divisions:

$$\pi(p_e, p_f) = p_e D_e(p_e, p_f) + p_f D_f(p_e, p_f),$$

with $D_e(p_e, p_f)$ and $D_f(p_e, p_f)$ as defined in (12a) and (12b). By Lemma 1 in the Appendix, $\pi(p_e, p_f)$ is concave and optimal prices can be obtained from the first-order conditions:

$$\tilde{p}_e = \frac{F\,(c-r)\,(c-E)}{2(c^2 - EF)}, \tag{13a}$$

$$\tilde{p}_f = \frac{E\,(c-r)\,(c-F)}{2(c^2 - EF)}. \tag{13b}$$

By assumption (1), these prices are positive, and the positive demands for schooling are given by:

$$D_e(\tilde{p}_e, \tilde{p}_f) = \frac{(r-E)\,EF}{2(r^2 - EF)},$$

$$D_f(\tilde{p}_e, \tilde{p}_f) = \frac{(r-F)\,EF}{2(r^2 - EF)}.$$

Equilibrium shares of learners defined in (14a) and (14b) are finally obtained by substituting optimal prices (13a) and (13b) in (10a)-(10b):

$$\tilde{l}_e = \frac{F\,(cr^2 - rEF - cEF + c^2 r - c^2 E - r^2 E + 2E^2 F)}{2\,(c^2 - EF)\,(r^2 - EF)} \tag{14a}$$

$$\tilde{l}_f = \frac{E\,(cr^2 - rEF - cEF + c^2 r - c^2 F - r^2 F + 2F^2 E)}{2\,(c^2 - EF)\,(r^2 - EF)}. \tag{14b}$$

By Lemma 2, these equilibrium shares $\tilde{l}_e$ and $\tilde{l}_f$ lie in the interior of the unit-interval. Observe also that, since $F \leq E$, assumption (1) implies that the difference between (14b) and (14a) is non-negative:

$$\tilde{l}_f - \tilde{l}_e = \frac{(cr - EF)(c + r)(E - F)}{2(c^2 - EF)(r^2 - EF)} \geq 0. \tag{15}$$

This leads to the following proposition, that is similar to the one obtained in the case of self-learning only.

Proposition 2 *The fraction of learners under a two-sided platform cannot be smaller in the smaller community.*

We now turn to comparative statics results:

Proposition 3
(a) An increase in the home population reduces the fraction of learners at home: the values of $\frac{d\tilde{l}_e}{dE}$ and $\frac{d\tilde{l}_f}{dF}$ are both negative
(b) An increase in the population of the other population raises the fraction of learners at home: the values of $\frac{d\tilde{l}_e}{dF}$ and $\frac{d\tilde{l}_f}{dE}$ are both positive
(c) If $E > F$, an increase in the learning cost c reduces the fraction of learners in the smaller community F, while there is a threshold value $\hat{c}$, such that the fraction of learners in the larger community E increases for $c < \hat{c}$ and declines for $c > \hat{c}$. If $E = F$, an increase in the learning costs reduces the fraction of learners in both communities.

Proof. See Appendix.

The intuition for these results is as follows: (a) when there are more people to communicate with at home, the incentive to learn the language of the other community decreases; (b) an increase in the population of the other community makes it more compelling to learn their language; (c) when learning costs increase, there are less learners; the latter effect is however mitigated in the larger community under relatively low level of learning costs.

We now compare the shares of learners with and without schools. The difference $\tilde{l}_e - l_e^*$ obtains as

$$\tilde{l}_e - l_e^* = \frac{(-cE + FE + cr - rE)(c - r) F}{2(r^2 - FE)(c^2 - FE)}.$$

Since the denominator is positive, the sign of $\tilde{l}_e - l_e^*$ is determined by the sign of the numerator of

$$c - E\frac{r-F}{r-E}.$$

The sign of this expression could be negative if the population size E substantially exceeds F. However, it is always positive when $E = F$.

Similarly, $\tilde{l}_f - l_f^* > 0$ if, and only if

$$c > F\frac{r-E}{r-F},$$

which is always satisfied whenever $E \geq F$. Therefore,

Proposition 4

(a) In the smaller community, the number of learners is always larger than under self-learning only.
(b) However, in the larger community, there exists a threshold value of the self-learning cost such that the number of learners under a two-sided platform can be smaller than under self-learning only.
(c) If the two communities have the same size, the number of learners under a two-sided platform is always larger than under self-learning in both communities.

To conclude this section we compare welfare impact of the entry of a linguistic school. To do so, we compare the aggregate welfare of both communities with and without the language school.

With self-learning only, the welfare of population e is determined by its aggregate communicative benefit net of the learning costs:

$$W_e^* = (1 - l_e^*)E(E + l_f^* F) + l_e^* E(E + F) - cE\int_0^{l_e^*} \theta d\theta.$$

The first term represents the welfare of those $(1 - l_e^*)E$ citizens who do not learn language F; each of them gets a benefit equal to $E + l_e^* F$, since they can communicate with that number of e-speakers. The second term describes the net benefit of e-citizens who learn f; their mass is $l_e^* E$, and the benefit that each of them is $E + F$. The third term is the total cost of those who learn. The expression can be rewritten as

$$W_e^* = E^2 + EF[(1 - l_e^*)l_f^* + l_e^*] - \frac{1}{2}cE(l_e^*)^2.$$

The welfare of population f, W_f^* obtains by interchanging F (and f) and E (and e) in the last expression, and the aggregate welfare of the two communities is then

$$W^* = E^2 + F^2 + 2EF[l_e^* + l_f^* - l_e^* l_f^*] - \frac{1}{2}c[E(l_e^*)^2 + F(l_f^*)^2].$$

In the schooling option, the derivations are quite similar. One has to take into account two different types of fees in every community and the fact that the constant part of the schooling fees, $\tilde{p}_e$ and $\tilde{p}_f$, does not directly enter the calculations, since it represents the revenue for the school and the cost for customers. The welfare in community e is therefore given by:

$$\tilde{W}_e = E^2 + EF[\tilde{l}_e + \tilde{l}_f - \tilde{l}_e\tilde{l}_f] - \frac{1}{2}cE\tilde{l}_e^2 - rE\int_{\tilde{l}_e}^{\tilde{l}_e+\tilde{D}_e} \theta d\theta,$$

where $\tilde{D}_e = D_e(\tilde{p}_e, \tilde{p}_f)$ is the demand for learning in community E under profit-maximizing prices (13a) and (13b). Since $\tilde{W}_e$ can be simplified to

$$\tilde{W}_e = E^2 + EF[\tilde{l}_e + \tilde{l}_f - \tilde{l}_e\tilde{l}_f] - \frac{1}{2}cE\tilde{l}_e^2 - \frac{1}{2}rE\tilde{D}_e(\tilde{D}_e + 2\tilde{l}_e),$$

and the welfare of population f, W_f^* obtains by interchanging F (and f) and E (and e) in the last expression, the aggregate welfare of the two communities is:

$$\tilde{W} = E^2 + F^2 + 2EF[\tilde{l}_e + \tilde{l}_f - \tilde{l}_e\tilde{l}_f] - \frac{1}{2}c[E\tilde{l}_e^2 + F\tilde{l}_f^2]$$
$$-\frac{1}{2}r[E\tilde{D}_e(\tilde{D}_e + 2\tilde{l}_e) + F\tilde{D}_f(\tilde{D}_f + 2\tilde{l}_f)].$$

To simplify these expressions, we consider the case (as we will also do in other sections) where $E = F$. Then we have

$$\tilde{W} - W^* = \frac{N^2(c-r)}{4(c+N)^2(r+N)^2}\Delta,$$

where

$$\Delta = c^2N(N-1)^2 + (N-4)N^4 - 2Nr(2+N^2) + c(2N(2+N^2)(N-3)$$
$$-(16 - Nr(3-2N)).$$

It is easy to see that Δ is positive when N is sufficiently large (e.g, $N > 4$). Thus,

Proposition 5 *There exists a threshold N^* such that if assumption (1) holds with $E = F > N^*$, the introduction of a language school generates higher welfare than self-learning only, i.e. $\tilde{W} > W^*$.*

In that case, the introduction of language schools raises the welfare in both community. Indeed, since their sizes are equal, this is equivalent to an increase in joint welfare.

Obviously, by invoking a continuity argument, we may extend this result to the case with communities with unequal but similar sizes. Namely, we can state that there exists $\varepsilon > 0$ such that the Proposition 5 holds whenever $0 < E - F < \varepsilon$. However, this is not necessarily true if the size inequality between E and F is sufficiently large. Thus, introducing language schools does not necessarily yield improvements in terms of communicating benefits. This observation would call for a careful analysis of subsidizing learning costs, since network effects may end up reducing aggregate welfare benefits.

4 Language learning with two local schools

We now consider an alternative set-up which constitutes a natural reference point to which the two-sided market approach can be compared. We assume that there exists a language learning school in each community, behaving in its own community as a monopolist. As suggested in the introduction, the demand for learning the foreign language at each local school depends on its own subscription fee, but may depend also on the subscription fee set by the school in the other community. Indeed, when the school in say, f sets a low fee, it attracts more learners of e. This in turn reduces the incentives to learn f in community e and entails a decrease in demand at a given value for l_f.

Facing this situation, local monopolists can either be unaware of (or ignore) these spillovers and behave in a myopic way or, on the contrary, be sophisticated enough to take them into account when they set their own price. We examine the two situations.

4.1 Myopic monopolists

Here, the school in community e (resp.f) computes its demand at a fixed proportion of learners in the other community, so that their demands write as in (11), with corresponding profits

$$
\begin{aligned}
\pi_e(p_e, p_f) &= p_e E \left(\frac{F(1-l_f) - p_e}{r} - \frac{p_e}{c-r} \right) \\
\pi_f(p_1, p_2) &= p_f F \left(\frac{E(1-l_e) - p_f}{r} - \frac{p_f}{c-r} \right).
\end{aligned}
$$

Separate profit maximization by each school leads to the corresponding optimal local monopoly prices:

$$
\begin{aligned}
\overset{\circ}{p_e} &= \frac{F}{2c}(c-r)(1-l_f) \\
\overset{\circ}{p_f} &= \frac{E}{2c}(c-r)(1-l_e).
\end{aligned}
$$

Substituting these prices in the corresponding demand functions, and equating the resulting expressions to l_e and l_f, respectively, leads to:

$$
\begin{aligned}
l_e &= \frac{1}{r} \left(F(1-l_f) - \frac{1}{2c} F(c-r)(1-l_f) \right) \\
l_f &= \frac{1}{r} \left(E(1-l_e) - \frac{1}{2c} E(c-r)(1-l_e) \right).
\end{aligned}
$$

Solving this system in l_e and l_f, we finally obtain the equilibrium values for these variables, namely,

$$
\overset{\circ}{l_e} = \frac{(cE - 2cr + rE)(c+r)F}{(2crEF + c^2EF + r^2EF - 4c^2r^2)}, \tag{16a}
$$

and

$$
\overset{\circ}{l_f} = \frac{(cF - 2cr + rF)(c+r)E}{(2crEF + c^2EF + r^2EF - 4c^2r^2)}. \tag{16b}
$$

It is easy to check that, as in the two previous cases, the following result holds:

Proposition 6 *The fraction of learners is larger in the smaller community.*

4.2 Sophisticated monopolists

We finally consider the alternative set-up in which the local monopolists are aware that the number of foreign citizens who know the home language indirectly influences the home demand for the foreign language. This introduces a strategic interaction between schools, to the extent that an increase in price in the foreign community decreases the number of foreigners who are willing to learn the home language and, accordingly, increases the home demand for learning the foreign one. This strategic interaction can be formalized as a game with the two schools as players, prices as strategies and payoffs as profits. Using the expressions (12a) and (12b), these payoffs write as

$$\pi_e(p_e, p_f) = p_e E \left(\frac{rF - EF - rp_e + p_f F}{r^2 - EF} - \frac{p_e}{c - r} \right)$$

and

$$\pi_f(p_1, p_2) = p_f F \left(\frac{rE - EF - rp_f + p_e E}{r^2 - EF} - \frac{p_f}{c - r} \right).$$

It is easy to see that each payoff is concave in its own price. The first-order conditions are thus sufficient and necessary to obtain the equilibrium prices $\hat{p}_e$ and $\hat{p}_f$,:

$$\hat{p}_e = \frac{F\,(c-r)\,(2cr^2 - cEF - rEF - crE - r^2E + 2E^2F)}{4c^2r^2 - c^2EF - r^2EF - 6crEF + 4E^2F^2}$$

$$\hat{p}_f = \frac{E\,(c-r)\,(2cr^2 - cEF - rEF - crF - r^2F + 2EF^2)}{4c^2r^2 - c^2EF - r^2EF - 6crEF + 4E^2F^2}.$$

The equilibrium values of $\hat{l}_e$ and $\hat{l}_f$ can be obtained by substituting the above equilibrium values of prices in (10a) and (10b). This leads to

$$\hat{l}_e = \frac{(cFE + rFE + crE - 2cr^2 - 2FE^2 + r^2E)\,(cr - 2FE + r^2)\,F}{(6crFE + c^2FE + r^2FE - 4c^2r^2 - 4F^2E^2)\,(r^2 - FE)} \qquad (17a)$$

and

$$\hat{l}_f = \frac{(cFE + rFE + crF - 2cr^2 - 2F^2E + r^2F)\,(cr - 2FE + r^2)E}{(6crFE + c^2FE + r^2FE - 4c^2r^2 - 4F^2E^2)\,(r^2 - FE)}. \qquad (17b)$$

Subtracting (17b) from (17a), we get

$$\hat{l}_e - \hat{l}_f = \frac{-\left(cEF + rEF - 2cr^2\right)\left(cr - 2EF + r^2\right)(E - F)}{\left(4c^2r^2 - c^2EF - r^2EF - 6crEF + 4E^2F^2\right)\left(EF - r^2\right)}.$$

Lemma 3 of the appendix shows that the denominator of this expression is negative. If $E > F$, the numerator is positive, so that

Proposition 7 *The fraction of learners is larger in the community with the smaller population.*

5 Comparing numbers of language learners

Our analysis in previous sections has covered four institutional mechanisms of language learning: self-learning only, a two-sided platform, two local monopoly schools, either myopic or sophisticated. In Section 3, we compared in full generality the shares of learners in the two first situations – (i) self-learning only and (ii) school as a platform. However, the analytical challenges prevent us from performing the same comparison with the two other set-ups – (iii) myopic monopolists and (iv) sophisticated monopolists. We therefore assume throughout this section that the communities are of equal size and denote $N = E = F$. It is worth pointing out that using a continuity argument as we did ion Section3, our results can be extended to the case of two communities of "almost" equal sizes.

Let us now list the expressions of the equilibrium shares corresponding to the four situations, namely

$$l^* = \frac{N}{N + c} \qquad (i)$$

$$\tilde{l} = \frac{N\left(cr^2 - rN^2 - cN^2 + c^2r - c^2N - r^2N + 2N^3\right)}{2\left(c^2 - N^2\right)\left(r^2 - N^2\right)} \qquad (ii)$$

$$l^\circ = \frac{(cN - 2cr + rN)(c + r)N}{\left(2crN^2 + c^2N^2 + r^2N^2 - 4c^2r^2\right)} \qquad (iii)$$

$$\hat{l} = \frac{N\left(cr - 2N^2 + r^2\right)}{(N + r)\left(Nr - Nc + 2cr - 2N^2\right)}. \qquad (iv)$$

From direct comparisons between (i) and any of the other three cases (ii), (iii) and (iv), we obtain

Proposition 8 *If $E = F$, the introduction of language learning schools increases the fraction of learners with respect to self-learning only.*

Performing similar direct comparisons between the pairs $(\tilde{l}, l^{\circ})$, $(l^{\circ}, \hat{l})$ and $(\tilde{l}, \hat{l})$, respectively, it is easy to see that (i) $\tilde{l} < l^{\circ}$ and (ii) $l^{\circ} < \hat{l}$. It follows that $\tilde{l} < l^{\circ} < \hat{l}$.

The results of these comparisons are summarized in Proposition 9:

Proposition 9 *If $E = F$, the fractions of learners of the foreign language corresponding to (i) self-learning only, (ii) a two-sided platform, (iii) myopic local monopolies and (iv) sophisticated local monopolies satisfy the following inequalities*

$$l^* < \tilde{l} < l^{\circ} < \hat{l}.$$

The two propositions show that any type of school induces more agents to learn the language of the other community than if there is self-learning only. If the goal of a two-community state is to have more people learning the language of the other community, private schools always help. However, some forms of competition are better than others. The two-division platform *à la* Berlitz, which maximizes its joint profit over the two regions is the less satisfactory, since it is the institution that is closer to a monopoly. Myopic monopolies, one in each community, do better, because they both neglect the interaction between their markets and, thus, discard the competitive pressures generated by this interaction. By contrast, more competition is introduced when the two local monopolies behave in a sophisticated way by taking into account strategic choices in the other community.

6 Conclusion

The paper compares the outcomes of three different language learning procedures. In the first, learners proceed by using their own specific ability to learn the foreign language at a cost which varies within the population according to these abilities. In the two other settings, potential learners are assisted by language instructors. This reduces the individual learning cost, compared with self-learning. However the subscription fee adds up to this reduced individual learning cost, which makes *a priori* unclear whether it is better to learn by oneself or by using the school's services. According to the level of the tuition fees, some potential learners – those with low individual learning costs – prefer to learn by themselves, while those with higher learning costs may prefer to go to school.

When the language school consists of two branches, one in each community, and chooses tuition fees that maximize its joint profits, it can be considered as a two-sided platform, that turns out to be a rare example of *negative* cross-network externalities: the more people in a foreign community learn my own language, the smaller my own interest to learn theirs.

We also consider a setting in which there exists a school in each community that acts as a local monopolist who may or may not take into account the network effects of the other monopolist.

In spite of major differences between the three mechanisms, they turn out to be similar in their effects from several viewpoints. In fact, under all mechanisms, the fraction of learners is larger in the community with the smaller population. Furthermore, according to Propositions 1 and 3, both under self-learning and the two-sided platform, an increase in the "home" population reduces the fraction of learners in the home community, while an increase in the population of the "foreign" community raises the fraction of learners in the home one. We also compare the total fraction of learners in each community with and without schooling, and in the case of the equal-size communities we offer a complete ranking of the number of learners in all institutional settings as a function of the model parameters. We also point to some welfare implications in the case of the platform.

It is difficult to judge whether our conclusions are robust since they are obtained under a set of rather restrictive conditions, such as the linearity of the benefit function and Assumption (1), introduced to guarantee the existence of an interior solution. In spite of the difficulties inherent to this problem, our approach nevertheless offers some insights into language learning mechanisms.

7 References

Anderson, S. and S. Coate (2005), Market provision of broadcasting: A welfare analysis, *Review of Economic Studies* 72, 947-972.

Church, J. and I. King (1993), Bilingualism and network externalities, *Canadian Journal of Economics* 26, 337-345.

Gabszewicz, J., V. Ginsburgh and S. Weber (2010), Bilingualism and communicative benefits, *Annales d'Economie et de Statistique*, forthcoming.

Gabszewicz, J.,D. Laussel and N. Sonnac (2004), Programming and advertising competition in the broadcasting industry, *Journal of Economics and Management Strategy* 13, 657-669.

Ginsburgh, V., I. Ortuño-Ortín and S. Weber (2007), Learning foreign languages. Theoretical and empirical implications of the Selten and Pool model, *Journal of Economic Behavior and Organization* 64, 337-347.

Ginsburgh, V. and S. Weber (2010), *The Economics of Linguistic Diversity*, Princeton: Princeton University Press, in press.

Rochet, J-C and J. Tirole (2003), Platform competition in two-sided markets, *Journal of the European Economic Association* 1, 990-1029.

Selten, R. and J. Pool (1991), The distribution of foreign language skills as a game equilibrium, in R. Selten, ed., *Game Equilibrium Models*, vol. 4, Berlin: Springer-Verlag, 64-84.

Shy, O. (2001), *The Economics of Network Industries*, Cambridge: Cambridge University Press.

Wright, J. (2003), Optimal card payment systems, *European Economic Review* 47, 587-612.

8 Appendix

Lemma 1 *The function $\pi(p_e, p_f)$ is concave.*

Proof: The Hessian matrix of the profit function can be presented as:

$$\begin{pmatrix} -\frac{E}{c-r}\frac{2cr-2EF}{r^2-EF} & 2E\frac{F}{r^2-EF} \\ 2E\frac{F}{r^2-EF} & -\frac{F}{c-r}\frac{2cr-2EF}{r^2-EF} \end{pmatrix}$$

and the profit function is concave if:

$$D_1 = -\frac{E}{c-r}\frac{2cr-2EF}{r^2-EF} \leq 0,$$

$$D_2 = \frac{EF}{(c-r)^2}\left(\frac{2cr-2EF}{r^2-EF}\right)^2 - \left(\frac{2EF}{r^2-EF}\right)^2 \geq 0,$$

where D_1 and D_2 are the relevant determinants extracted from the Hessian matrix. Since, under (1), $EF > cr$ and $EF > r^2$, D_1 is obviously non-positive. After some simple algebraic transformations, the second determinant D_2 can be written as:

$$E^2F^2 + EF\left(c^2+r^2\right) + c^2r^2,$$

an expression that is strictly positive. Thus, the profit function is concave. QED.

Lemma 2 *The values of the equilibrium shares of learners in both communities, $\tilde{l}_e$ and $\tilde{l}$, lie strictly between* 0 *and* 1.

Proof: Consider community e. Since the denominator in (14a) is positive, the sign of $\tilde{l}_e$ is the sign of the numerator which is itself a second order polynomial $P(r) = A + Br + Cr^2$, where, by (1), $A = 2E^2F - c^2E - cEF < 0$, $B = c^2 - EF > 0$, and $C = c - E > 0$. It follows that $P(r)$ is increasing for $r > 0$. However, $P(E) = (c-E)(E-F)E > 0$ and since in our case, by (1), $r > E$, it follows that in the range of relevant r, the value of P is positive, that shows that $\tilde{l}_e > 0$. The same argument is used to prove that $\tilde{l}_f$ is positive as well.

To conclude the proof, observe that since equilibrium tuition fees are positive, equations (9a) and (9b) imply that both values, $\tilde{l}_e$ and $\tilde{l}_f$ are strictly smaller than one. QED.

Proof of Proposition 3

Proof of (a)

$$\frac{d\tilde{l}_e}{dE} = -\frac{1}{2}F\left(\frac{c(c-F)}{(EF-c^2)^2} + \frac{r(r-F)}{(EF-r^2)^2}\right),$$

which is negative by (1).

Proof of (b).

$$\frac{d\tilde{l}_e}{dF} = \frac{1}{2}\left(\frac{c^2(c-E)}{(EF-c^2)^2} + \frac{r^2(r-E)}{(EF-r^2)^2}\right),$$

which is positive by (1).

Proof of (c).

The derivative of $\tilde{l}_f$ and $\tilde{l}_e$ with respect to c, are given by

$$\frac{d\tilde{l}_f}{dc} = \frac{E(2cF - EF - c^2)}{(c^2 - EF)^2}$$

and

$$\frac{d\tilde{l}_e}{dc} = \frac{F(2cE - EF - c^2)}{(c^2 - EF)^2},$$

respectively. Consider first the case where $F < E$. The first derivative is always negative, as the second-order polynomial $2cE - EF - c^2$ has no real root. However, the second is positive for all values of $c \in \left[E - \sqrt{E^2 - EF}, E + \sqrt{E^2 - EF}\right]$ and negative elsewhere. Since, by (1), $c > E$, it follows that the above derivative is positive for $c \in (E, \hat{c})$, equal to 0 when $c = \hat{c}$ and negative for all $c > \hat{c}$, where $\hat{c} = E + \sqrt{E^2 - EF}$.

If $F = E < c$, both derivatives are negative as

$$\frac{d\tilde{l}_f}{dc} = \frac{d\tilde{l}_e}{dc} = -\frac{E}{(c+E)^2}$$

QED.

Lemma 3 *The sign of the denominator of* $\hat{l}_e - \hat{l}_f$ *is negative.*

Proof. Under assumption (1), $EF - r^2$ is negative. Accordingly, if we prove that the sign of the expression $4c^2r^2 - c^2EF - r^2EF - 6crEF + 4E^2F^2$ is positive, the proof of the lemma is complete. This expression rewrites as $4c^2r^2 + EF(4EF - 4cr - 2cr - c^2 - r^2)$. Again by assumption (1), the sign of the expression in the parenthesis is negative. Accordingly, if we take the highest admissible value for the product EF, namely, $EF = r^2$, and show that the expression is positive for this value, the same sign should hold for any smaller admissible value of the product EF. Substituting r^2 to EF in the above expression, we get

$$4c^2r^2 + r^2(4r^2 - 4cr - (c+r)^2) = r^2(3c^2 + 3r^2 - 6cr) = 3r^2(c-r)^2 > 0,$$

which completes the proof of the lemma. QED.

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