2010/80

Voting over piece-wise linear tax methods

Juan D. Moreno-Ternero

CORE

DISCUSSION PAPER

Center for Operations Research
and Econometrics

Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium
http://www.uclouvain.be/core

CORE DISCUSSION PAPER
2010/80

Voting over piece-wise linear tax methods

Juan D. MORENO-TERNERO[1]

December 2010

Abstract

We analyze the problem of choosing the most appropriate method for apportioning taxes in a democracy. We consider a simple theoretical model of taxation and restrict our attention to piece-wise linear tax methods, which are almost ubiquitous in advanced democracies world- wide. We show that if we allow agents to vote for any method within a rich domain of piece-wise linear methods, then a majority voting equilibrium exists. Furthermore, if most voters have income below mean income then each method within the domain can be supported in equilibrium.

Keywords: voting, taxes, majority, single-crossing, Talmud.

JEL Classification: D72, H24

---

[1] Universidad de Malaga, Spain; Universidad Pablo de Olavide, Seville, Spain; Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium.

I am most grateful to Oriol Carbonell-Nicolau, Biung-Ghi Ju, François Maniquet, Bernardo Moreno, Ignacio Ortuno-Ortin, Eve Ramaekers, and an anonymous referee for helpful comments and suggestions. I also thank the audiences at the Institute of Food and Resource Economics (University of Copenhagen), the Society of Economic Design Meeting 2009 (Maastricht University), the CORE Weekly Workshop in Welfare Economics (Université catholique de Louvain) and the PAI SEJ426 Research Group (Universidad de Malaga) for many useful discussions. Financial support from the Spanish Ministry of Science and Innovation (ECO2008-03883) and from Junta de Andalucia (P08-SEJ-04154) is gratefully acknowledged.

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the author.

1 Introduction

The primary struggle among citizens in all advanced democracies is over the distribution of economic resources. Income taxation, besides being a major source of state funds, is one of the essential tools for solving such a struggle, which makes it a matter of concern for politicians and economists alike. The search for the perfect income tax structure is (and has been for a long time) a milestone and even though some consensus has been reached (e.g., almost all countries in the world use statutory tax schedules specified only in terms of the brackets and tax rates) the discussion is far from being over.[1] In this paper, we approach this issue from a political economy perspective, upon studying the political process in which tax methods are either chosen directly by voters, according to majority rule, or via elections in a perfectly representative democracy.

Academic interest in this area started to emerge after Foley (1967), who analyzed the problem of voting over taxes in an endowment economy. Foley focused on the case of flat taxes (with or without exemption; and allowing or excluding for the existence of negative taxes) and showed that, for such a class, there always exists a majority voting equilibrium, i.e., a (flat) tax method that cannot be overturned by any other member of the class through majority rule.[2]

In this paper, we plan to focus on the class of piece-wise linear tax methods (rather than flat taxes) which, as mentioned above, seems to be almost ubiquitous in advanced democracies worldwide. For such a class, however, Foley's result does not extend and a majority voting equilibrium fails to exist. In other words, any piece-wise linear tax method can be overturned by another piece-wise linear tax method through majority rule. This is actually not more than another instance of Condorcet's paradox of voting, which is perhaps best exemplified by the problem of determining the division of a cake by majority rule (or, equivalently, tax shares by majority

[1]In the 2008 US presidential election we had a recent instance of such a discussion. President (then, Senator) Obama proposed a tax plan that would make the tax system significantly more progressive by providing large tax breaks to those at the bottom of the income scale and raising taxes significantly on upper-income earners. Senator McCain instead advocated for a tax plan that would make the tax system more regressive, upon providing relatively little tax relief to those at the bottom of the income scale while providing huge tax cuts to households at the very top of the income distribution (e.g., Burman et al., 2008).

[2]Foley's work mostly relies on verbal discussion. A more formal treatment of his model (and some of his results) is provided by Gouveia and Oliver (1996).

rule from a given initial distribution of endowments).[3]

Such a result might lead one to despair of ever achieving a voting equilibrium for any democratic polity. Nevertheless, as Campbell (1975) puts it, majority voting is never allowed to operate by itself without restraints imposed by constitution and convention. We actually show that if we limit the class of admissible methods in a meaningful way, albeit not very restrictive, the existence of a majority voting equilibrium is guaranteed. As a matter of fact, under a mild assumption, we construct the precise equilibrium for any parameter configuration of the model and show an interesting feature of it: any tax method within the class can be a majority voting equilibrium, provided the predetermined level of aggregate fiscal revenue is properly chosen.

The class of admissible methods we consider emerges as a generalization of a method inspired by the Babylonian Talmud (e.g., O'Neill, 1982; Aumann and Maschler, 1985). The principle underlying behind these methods is to impose each taxpayer a burden of the same sort as that faced by the whole society. Namely, if the overall tax burden is below a certain fraction of the aggregate income, then no taxpayer can pay more than such a fraction of her gross income. Similarly, if the burden is above a certain fraction of the aggregate income, then no taxpayer can pay less than such a fraction of her gross income. The class encompasses a whole non-countable set of methods ranging from the "least" progressive (the *needs-blind* head tax) to the "most" progressive (the *incentives-blind* leveling tax) piece-wise linear tax methods. Thus, voters are confronted with a wide variety of choices to select the best tax method, even if we restrict their options to this class.[4]

As we shall see later in the text, our modeling choice for this work is somehow unconventional. More precisely, most of the contributions in this area assume the existence of a continuum (rather than a finite set) of taxpayers. The main reason for it is twofold. On the one hand, the aim of modeling large (rather than small) elections. On the other hand, to allow for the use of calculus and hence avoid some theoretical problems, such as those resulting from the non-convexity of the individual voting choice set, or from the fact that a change of a vote might make a discrete change in policy (e.g., Alesina and Rosenthal, 1996). Nevertheless, we find some of those problems interesting and hence believe that they should not be dismissed

[3]Hamada (1973) provides a general treatment regarding why cycling is ubiquitous for this problem.

[4]Restricting to a one-parameter family of tax methods in which the parameter reflects the degree of progressivity (or regressivity) of the method is a usual course of action in taxation models (e.g., Bénabou, 2002).

from the outset. That is the main reason why we opt in this paper for a discrete modeling assumption. Another important reason to do so is the intention to explore the existence of equilibrium in smaller elections, when the tax problem refers to collecting a given amount of revenue out of a small (and hence finite) community. This is also the spirit in part of the literature to which this paper relates too. A notable instance is Young (1988), which although not concerned with the political economy of income taxation, could be considered as the seminal paper to analyze the principle of equal sacrifice, and its connections with distributive justice (a recurrent theme of this paper), in taxation.

The rest of the paper is organized as follows. In Section 2, we introduce the model. In Section 3, we provide our main result regarding the existence of majority voting equilibrium for a large set of piece-wise linear tax methods. In Section 4, we explicitly construct the equilibrium under an additional condition. Finally, Section 5 concludes.

2 The model

We study taxation problems in a variable population model, first introduced by Young (1988).[5] The set of potential taxpayers, or *agents*, is identified by the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element N. For each $i \in N$, let $y_i \in \mathbb{R}_+$ be i's (taxable) *income* and $y \equiv (y_i)_{i \in N}$ the income profile. A (taxation) *problem* is a triple consisting of a population $N \in \mathcal{N}$, an income profile $y \in \mathbb{R}^N_+$, and a tax revenue $T \in \mathbb{R}_+$ such that $\sum_{i \in N} y_i \geq T$. Let $Y \equiv \sum_{i \in N} y_i$. To avoid unnecessary complication, we assume $Y = \sum_{i \in N} y_i > 0$. Let $\mathcal{D}^N$ be the set of taxation problems with population N and $\mathcal{D} \equiv \cup_{N \in \mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, y, T) \in \mathcal{D}$, a *tax profile* is a vector $x \in \mathbb{R}^N$ satisfying the following three conditions: (i) for each $i \in N$, $0 \leq x_i \leq y_i$, (ii) $\sum_{i \in N} x_i = T$ and (iii) for each $i, j \in N$, $y_i \geq y_j$ implies $x_i \geq x_j$ and $y_i - x_i \geq y_j - x_j$. We refer to (i) as *boundedness*, (ii) as *balancedness* and (iii) as *order preservation.* A (taxation) *method* on $\mathcal{D}$, $R \colon \mathcal{D} \to \cup_{N \in \mathcal{N}} \mathbb{R}^N$, associates with each problem $(N, y, T) \in \mathcal{D}$ a tax profile $R(N, y, T)$ for the problem.[6]

[5] O'Neill (1982) used earlier the same mathematical framework to analyze the problem of adjudicating conflicting claims. Readers are referred to Moulin (2002) or Thomson (2003) for extensive treatments of diverse problems (such as taxation, conflicting claims, bankruptcy, cost sharing, or surplus sharing) fitting this framework.

[6] In essence, the problem under consideration is a distribution problem, in which the total amount to be distributed is exogenous, and the issue is to determine methods providing an allocation for each admissible problem. There is another branch of the taxation

Instances of methods are the *head tax*, which distributes the tax burden equally, provided no agent ends up paying more than her income, the *leveling tax*, which equalizes post-tax income across agents, provided no agent is subsidized and the *flat tax*, which equalizes tax rates across agents.

All these methods are instances of *piece-wise* linear tax methods. Formally, a piece-wise linear tax method is a method associated to a vector of brackets, rates and lump-sum levies. For each bracket, a given tax rate is proposed and the corresponding lump-sum levies of the brackets are designed so that the schedule moves continuously from one bracket to another. More precisely, a method R is piece-wise linear if for each $(N, y, T) \in \mathcal{D}$ there exist sequences $\{\alpha_j, \beta_j, \lambda_j\}_{j=1}^{k}$ such that

(i) For each $j = 1, \ldots, k$, $\alpha_j, \lambda_j \in \mathbb{R}_+$ and $\beta_j \in \mathbb{R}$;

(ii) For each $j = 1, \ldots, k-1$, $\lambda_j \leq \lambda_{j+1}$;

(iii) For each $j = 1, \ldots, k$, $0 \leq \alpha_j \leq 1$.

(iv) For each $j = 1, \ldots, k-1$, $\alpha_j \lambda_j + \beta_j = \alpha_{j+1} \lambda_j + \beta_{j+1}$;

(v) For each $j = 2, \ldots, k$, $(1 - \alpha_j)\lambda_{j-1} \geq \beta_j \geq -\alpha_j \lambda_{j-1}$;

and, for each $i \in N$,

$$R_i (N, y, T) = \alpha_j y_i + \beta_j,$$

where j is such that $\lambda_{j-1} \leq y_i \leq \lambda_j$.

Note that item (iii) above guarantees that every tax schedule has slope less than one. Item (iv) guarantees that the path of taxes generated by the method is continuous. Finally, item (v) guarantees that the tax payed by each agent is neither negative nor higher than her pre-tax income.[7]

---

literature in which no reference to the amount of revenue to be raised is made (e.g., Mitra and Ok, 1997). In such a branch, the basic problem is to determine a tax function yielding the tax associated to each positive income level. An underlying assumption of the corresponding models is to assume the existence of a non-countable set of agents (a reasonable assumption only in the case of arbitrary large populations), which, as mentioned above, allows the use of calculus. A more general approach encompassing both possibilities is taken by Le Breton et al., (1996).

[7]Alternatively, if we do not impose item (v) in the parameter configuration of the method R, we shall impose that for each $i \in N$,

$$R_i (N, y, T) = \max\{0, \min\{\alpha_j y_i + \beta_j, y_i\} = \min\{y_i, \max\{\alpha_j y_i + \beta_j, 0\},$$

where j is such that $\lambda_{j-1} \leq y_i \leq \lambda_j$, and will also deem the resulting method to be a piece-wise linear method.

We will analyze the problem in which agents vote for tax methods according to majority rule. We assume that voters are self-interested: given a pair of alternatives, a taxpayer votes for the alternative that gives her the greatest post-tax income. We say that a method R is a majority voting equilibrium for a set of methods $\mathcal{S}$ if, for any $(N, y, T) \in \mathcal{D}$, there is no other method $R' \in \mathcal{S}$ such that, $y - R'(N, y, T) > y - R(N, y, T)$ for the majority of voters.

3 The existence of equilibrium

We start this section with a (non-surprising) negative result.

Theorem 1 *There is no majority voting equilibrium for the family of piece-wise linear tax methods.*

Even though the technical proof of this result might be cumbersome, its logic should be clear. It all amounts to realize that given a piece-wise linear tax method, one can construct another (piece-wise linear) method increasing taxes for a small group of taxpayers and reducing the burden for all the others, while keeping the tax revenue constant. The argument, which is even valid for two-piece linear methods, is similar to others used in related models (e.g., Hamada, 1973; Marhuenda and Ortuño-Ortín, 1998).

A caveat is worth mentioning. If more than half of the agents are paying zero taxes, we cannot reduce their burdens and thus the corresponding tax allocation could not be defeated through majority rule by any other allocation. Nevertheless, there is no method guaranteeing that more than half of the agents are paying zero taxes for any admissible problem (although there certainly exist methods doing so for specific problems). The most extreme case would be the leveling tax, which would always be the most preferred method by the agent with the lowest income. This method, however, can be defeated by other piece-wise linear methods in many problems (in which, needless to say, there is not a majority of the population facing a zero tax burden with the leveling tax).

Given the previous result, our aim now shifts to prove the existence of a majority voting equilibrium for a sufficiently large family of piece-wise linear tax methods. To do so, we start considering a (piece-wise linear) method inspired by the Babylonian Talmud, implementing an old principle of distributive justice by which each taxpayer should face a burden of the same sort as that faced by the whole society. More precisely, if the overall tax burden is below one half of the aggregate income (which could be considered as a psychological threshold), then no taxpayer can pay more than such a fraction of her gross income. Similarly, if the burden is above one half of the aggregate income, then no taxpayer can pay less than such a fraction of her gross income. Formally,

For all $(N, y, T) \in \mathcal{D}$, *and all* $i \in N$,

$$R_i(N, y, T) = \begin{cases} \min\left\{\frac{y_i}{2}, \lambda\right\} & \text{if } T \leq \frac{Y}{2} \\ \max\left\{\frac{y_i}{2}, y_i - \mu\right\} & \text{if } T \geq \frac{Y}{2} \end{cases}$$

where $\lambda > 0$ *and* $\mu > 0$ *are chosen so that* $\sum_{i \in N} R_i(N, y, T) = T$.

In the usual parlance of taxation, the "Talmud method" yields two possible types of tax schedules. If the aim is to collect a tax revenue below one half of the aggregate income, the tax rate is one half up to some income level (which is endogenously determined), and zero afterwards. If, on the contrary, the tax revenue is above one half of the aggregate income, the tax rate is one half first and then one. Thus, even though it is a well-justified method on normative grounds (e.g., Moulin, 2002; Thomson, 2003), it seems to be quite specific for real-life taxation purposes.

One way of generalizing the Talmud method would be by moving the threshold (and the tax rate) in the above definition from one half to any other possible fraction (of the aggregate and individual incomes). In doing so, we would obtain a non-countable set of piece-wise linear methods ranging from the leveling tax to the head tax (and having the Talmud method in the middle).[8] Those tax methods would also yield two possible types of tax schedules that could be described similarly to those originating from the Talmud method. More precisely, for tax revenues below a fraction θ of the aggregate income, the tax rate would be θ up to some income level, and zero afterwards. For tax revenues above such fraction, the tax rate would be θ first and then one.[9]

In order to accommodate less restrictive methods too, while preserving the principle behind the Talmud method, we allow for other minimum and maximum tax rates, instead of always imposing zero and one for those values. More precisely, we consider tax methods yielding two possible types of tax schedules; namely, for tax revenues below a fraction θ of the aggregate income, the tax rate would be θ up to some income level, and $\theta_{\min}$ after-

[8]The resulting family of methods was studied, in the dual framework of bankruptcy problems, by Moreno-Ternero and Villar (2006a).

[9]Note that the flat tax schedules would also be covered by those tax methods, although the flat tax itself could not be considered a method of the resulting family.

wards. For tax revenues above such fraction, the tax rate would be θ first and then $\theta_{\max}$. Formally, we have the next definition.

Definition 1 *The family of generalized talmudic tax methods* $\{R^\theta\}_{\theta \in [\theta_{\min}, \theta_{\max}]}$. *Let* $\theta_{\min}, \theta_{\max} \in [0,1]$ *be fixed and such that* $\theta_{\min} < \theta_{\max}$. *For each* $\theta \in [\theta_{\min}, \theta_{\max}]$, *we define the method* R^θ *as follows. For all* $(N, y, T) \in \mathcal{D}$, *and all* $i \in N$,

$$R_i^\theta(N, y, T) = \begin{cases} \min\{\theta y_i, \max\{\theta_{\min} y_i + \lambda, 0\}\} & \text{if } T \leq \theta Y \\ \max\{\theta y_i, \min\{y_i, \theta_{\max} y_i - \mu\}\} & \text{if } T \geq \theta Y \end{cases}$$

where λ *and* μ *are chosen so that* $\sum_{i \in N} R_i^\theta(N, y, T) = T$.[10]

In order to illustrate further the above definition, we describe the algorithm by which tax burdens are allocated according to the (generalized talmudic) method R^θ, as the revenue varies from zero to the aggregate income of a given group of taxpayers. More precisely, let y be a given (gross) tax profile such that $y_1 \leq y_2 \leq \cdots \leq y_n$ and imagine that the tax revenue T moves from 0 to the aggregate income $Y = \sum_{i=1}^n y_i$. For T sufficiently small, the revenue is only financed by n (the taxpayer with the highest income). As T increases, the remaining taxpayers are sequentially asked to pay taxes (once they are able to do so) at the tax rate $\theta_{\min}$. This continues until all taxpayers contribute a $\theta_{\min}$ fraction of their income. As T increases from that point, equal taxation (for the increment) prevails until 1 (the taxpayer with the lowest income) pays a fraction θ of her income. At that point, 1 stops contributing while equal contribution of each (revenue) increment prevails among the other taxpayers. This process continues (making the remaining taxpayers stop contributing, sequentially, once they contribute a θ fraction of their income) until $T = \theta Y$. The next increments of T are faced by n until $n-1$ (the taxpayer with the second highest income) can contribute at the rate $\theta_{\max}$, at which point she is invited to do so. As T increases from there, the remaining taxpayers are also asked sequentially (but now in the reverse ordering of incomes) to contribute at the rate $\theta_{\max}$. Once all agents are contributing at the rate $\theta_{\max}$ then equal taxation (for the increment) prevails until 1 contributes with her whole income. From there, equal taxation (for the increment) prevails for the remaining agents, with the proviso that taxpayers contributing their whole income (obviously) stop paying additional increments.

[10]For ease of exposition, we shall avoid to mention explicitly $\theta_{\min}$ and $\theta_{\max}$, while referring to each rule within the family, unless it is specifically needed.

It turns out, as the next result shows, that the family of generalized talmudic methods described above constitutes a rich domain of piece-wise linear tax methods for which majority voting equilibrium exists.

Theorem 2 *There is a majority voting equilibrium for the family of generalized talmudic tax methods.*

In order to prove Theorem 2, we need the following lemma, which is interesting on its own, and whose proof appears in the appendix.

Lemma 1 *Let* $0 \leq \theta_{\min} \leq \theta_1 \leq \theta_2 \leq \theta_{\max} \leq 1$ *and* $(N, y, T) \in \mathcal{D}$. *If* n *denotes the agent in* N *with the highest income then* $R_n^{\theta_1}(N, y, T) \geq R_n^{\theta_2}(N, y, T)$.

We also need to introduce the following concept:

A method R ***single-crosses*** R' *if for each* $(N, y, T) \in \mathcal{D}$, *there exists* $i \in N$ *such that one of the following statements holds:*

(i) $R_j(N, y, T) \leq R'_j(N, y, T)$ *for all* j *such that* $y_j \leq y_i$ *and* $R_j(N, y, T) \geq R'_j(N, y, T)$ *for all* j such that $y_j \geq y_i$.

(ii) $R_j(N, y, T) \geq R'_j(N, y, T)$ *for all* j *such that* $y_j \leq y_i$ *and* $R_j(N, y, T) \leq R'_j(N, y, T)$ *for all* j such that $y_j \geq y_i$.

The single-crossing property allows one to separate those agents who benefit from the application of one method or the other, depending on the rank of their incomes. It is well known that a sufficient condition for the existence of a majority voting equilibrium is that voters exhibit *intermediate* preferences over the set of alternatives (e.g., Gans and Smart, 1996). Thus, as we assume that voters are self-interested and therefore simply vote according to the post-tax incomes that methods offer to them, it suffices to show that, for any pair of values $\theta_1, \theta_2 \in [\theta_{\min}, \theta_{\max}]$, R^{θ_1} single-crosses R^{θ_2}. To do so, let $0 \leq \theta_{\min} \leq \theta_1 \leq \theta_2 \leq \theta_{\max} \leq 1$, with $\theta_{\min} < \theta_{\max}$, and $(N, y, T) \in \mathcal{D}$ be given. For ease of exposition, assume that $N = \{1, \ldots, n\}$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. Then, it is enough to show that there exists some $i^* \in N$ such that:

(i) $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., i^*$ and
(ii) $R_i^{\theta_1}(N, y, T) \geq R_i^{\theta_2}(N, y, T)$ for all $i = i^* + 1, ..., n$.

We distinguish five cases:

Case 1: $0 \leq T \leq \theta_{\min}(Y - ny_1)$.

In this case, the single-crossing property trivially follows as $R^{\theta_1}(N,y,T) \equiv R^{\theta_2}(N,y,T)$.

Case 2: $\theta_{\min}(Y - ny_1) < T \leq \theta_1 Y$.

In this case, by the definition of the family of generalized talmudic methods, $R_i^{\theta_j}(N,y,T) = \min\{\theta_j y_i, \theta_{\min} y_i + \lambda_j\}$, for all $i \in N$ and $j = 1,2$, where λ_1 and λ_2 are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0,...,n\}$ such that $T \leq \theta_1(\sum_{i=1}^{r_1} y_i) + (n - r_1)(\theta_1 - \theta_{\min})y_{r_1+1}$ and r_2 the smallest non-negative integer in $\{0,...,n\}$ such that $T \leq \theta_2(\sum_{i=1}^{r_2} y_i) + (n - r_2)(\theta_2 - \theta_{\min})y_{r_2+1}$. It is straightforward to show that $r_2 \leq r_1$. Thus,

$$\begin{aligned} R^{\theta_1}(N,y,T) &= (\theta_1 y_1, ..., \theta_1 y_{r_2}, ..., \theta_1 y_{r_1}, \theta_{\min} y_{r_1+1} + \lambda_1, ..., \theta_{\min} y_n + \lambda_1), \textit{ and} \\ R^{\theta_2}(N,y,T) &= (\theta_2 y_1, ..., \theta_2 y_{r_2}, \theta_{\min} y_{r_2+1} + \lambda_2, ..., \theta_{\min} y_{r_1+1} + \lambda_2, ..., \theta_{\min} y_n + \lambda_2), \end{aligned}$$

where $\lambda_1 = \frac{T - \theta_1(\sum_{i=1}^{r_1} y_i) - \theta_{\min}\left(\sum_{i=r_1+1}^{n} y_i\right)}{n - r_1}$ and $\lambda_2 = \frac{T - \theta_2(\sum_{i=1}^{r_2} y_i) - \theta_{\min}\left(\sum_{i=r_2+1}^{n} y_i\right)}{n - r_2}$. Consequently, $R_i^{\theta_1}(N,y,T) \leq R_i^{\theta_2}(N,y,T)$ for all $i = 1,...,r_2$ and, by Lemma 1, $R_i^{\theta_1}(N,y,T) \geq R_i^{\theta_2}(N,y,T)$ for all $i = r_1+1,...,n$. To conclude the proof of this case, we distinguish three subcases:

Subcase 2.1: $\lambda_2 + \theta_{\min} y_{r_2+1} < \theta_1 y_{r_2+1}$.

Then, $i^* = r_2 + 1$ and the single-crossing property holds.

Subcase 2.2: $\lambda_2 + \theta_{\min} y_{r_1} \geq \theta_1 y_{r_1}$.

Then, $i^* = r_1 + 1$ and the single-crossing property holds.

Subcase 2.3: $\lambda_2 \in [(\theta_1 - \theta_{\min})y_{r_2+1}, (\theta_1 - \theta_{\min})y_{r_1}]$.

Then, there exists some $k \in \{r_2+1, ..., r_1-1\}$ such that $(\theta_1 - \theta_{\min})y_{k+1} > \lambda_2 \geq (\theta_1 - \theta_{\min})y_k$. Thus, $i^* = k+1$ and the single-crossing property holds.

Case 3: $\theta_1 Y < T < \theta_2 Y$.

By the definition of the family of generalized talmudic methods, $R_i^{\theta_1}(N,y,T) = \max\{\theta_1 y_i, \theta_{\max} y_i - \mu\}$ and $R_i^{\theta_2}(N,y,T) = \min\{\theta_2 y_i, \theta_{\min} y_i + \lambda\}$ for all $i \in N$, where μ and λ are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0,...,n-1\}$ such that $T \geq \theta_1 Y + (\theta_{\max} - \theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n - r_1)y_{r_1+1})$. Furthermore, let r_2 be the smallest non-negative integer in $\{0,...,n-1\}$ such that $T \leq \theta_2(\sum_{i=1}^{r_2} y_i) + (n - r_2)(\theta_2 - \theta_{\min})y_{r_2+1}$. It is straightforward to show that $r_2 \leq r_1$. Thus,

$$\begin{aligned} R^{\theta_1}(N,y,T) &= (\theta_1 y_1, ..., \theta_1 y_{r_1}, \theta_{\max} y_{r_1+1} - \mu, ..., \theta_{\max} y_n - \mu), \textit{ and} \\ R^{\theta_2}(N,y,T) &= (\theta_2 y_1, ..., \theta_2 y_{r_2}, \theta_{\min} y_{r_2+1} + \lambda, ..., \theta_{\min} y_n + \lambda), \end{aligned}$$

where $\lambda = \frac{T - \theta_2(\sum_{i=1}^{r_2} y_i) - \theta_{\min}\left(\sum_{i=r_2+1}^{n} y_i\right)}{n - r_2}$ and $\mu = \frac{\theta_1(\sum_{i=1}^{r_1} y_i) + \theta_{\max}\left(\sum_{i=r_1+1}^{n} y_i\right) - T}{n - r_1}$. Consequently, $R_i^{\theta_1}(N,y,T) \leq R_i^{\theta_2}(N,y,T)$ for all $i = 1,...,\min\{r_1,r_2\}$. To conclude the proof of this case, we distinguish two subcases:

Subcase 3.1: $r_1 \geq r_2$.

Then, $R_i^{\theta_1}(N,y,T) \leq R_i^{\theta_2}(N,y,T)$ for all $i = 1,...,r_2$. By Lemma 1, $R_n^{\theta_1}(N,y,T) \geq R_n^{\theta_2}(N,y,T)$. Let k be the smallest non-negative integer in N such that $R_k^{\theta_1}(N,y,T) \geq R_k^{\theta_2}(N,y,T)$.[11] Two options are then open. If $k \geq r_1+1$, then $\theta_{\max}y_k - \mu = R_k^{\theta_1}(N,y,T) \geq R_k^{\theta_2}(N,y,T) = \lambda + \theta_{\min}y_k$. Thus, $(\theta_{\max}-\theta_{\min})y_{k'} \geq \mu+\lambda$ for all $k' = k,...,n$, or equivalently, $R_{k'}^{\theta_1}(N,y,T) \geq R_{k'}^{\theta_2}(N,y,T)$ for all $k' = k,...,n$ and the single-crossing property follows. If, on the other hand, $r_2+1 \leq k \leq r_1$, then $\theta_1 y_k = R_k^{\theta_1}(N,y,T) \geq R_k^{\theta_2}(N,y,T) = \lambda + \theta_{\min}y_k$. Thus, $(\theta_1 - \theta_{\min})y_{k'} \geq \lambda$ for all $k' = k,...,n$. In particular, $R_{k'}^{\theta_1}(N,y,T) \geq R_{k'}^{\theta_2}(N,y,T)$ for all $k' = k,...,r_1$. Now, as $R_{r_1+1}^{\theta_1}(N,y,T) = \theta_{\max}y_{r_1+1} - \mu \geq \theta_1 y_{r_1+1} \geq \lambda + \theta_{\min}y_{r_1+1}$ we obtain that $\mu+\lambda \leq (\theta_{\max}-\theta_{min})y_{r_1+1} \leq (\theta_{\max}-\theta_{min})y_{k'}$ for all $k' = r_1+1,...,n$. As a result, $R_{k'}^{\theta_1}(N,y,T) \geq R_{k'}^{\theta_2}(N,y,T)$ for all $k' = k,...,n$, and the single-crossing property follows.

Subcase 3.2: $r_1 < r_2$.

Then, $R_i^{\theta_1}(N,y,T) \leq R_i^{\theta_2}(N,y,T)$ for all $i = 1,...,r_1$. Furthermore, by Lemma 1, $R_n^{\theta_1}(N,y,T) \geq R_n^{\theta_2}(N,y,T)$. Let k be the smallest non-negative integer in N such that $R_k^{\theta_1}(N,y,T) \geq R_k^{\theta_2}(N,y,T)$. As before, we have two options. If $k \geq r_2+1$, then $\theta_{\max}y_k - \mu = R_k^{\theta_1}(N,y,T) \geq R_k^{\theta_2}(N,y,T) = \lambda + \theta_{\min}y_k$. Thus, $y_{k'}(\theta_{\max}-\theta_{\min}) \geq \mu+\lambda$ for all $k' = k,...,n$, or equivalently, $R_{k'}^{\theta_1}(N,y,T) \geq R_{k'}^{\theta_2}(N,y,T)$ for all $k' = k,...,n$, and the single-crossing property follows. If, on the other hand, $r_1+1 \leq k \leq r_2$, then, $\theta_{\max}y_k - \mu = R_k^{\theta_1}(N,y,T) \geq R_k^{\theta_2}(N,y,T) = \theta_2 y_k$. Thus, $(\theta_{\max}-\theta_2)y_{k'} \geq \mu$ for all $k' = k,...,n$. In particular, $R_{k'}^{\theta_1}(N,y,T) \geq R_{k'}^{\theta_2}(N,y,T)$ for all $k' = k,...,r_2$. Now, as $R_{r_2+1}^{\theta_2}(N,y,T) = \lambda + \theta_{\min}y_{r_2+1}$ we know that $\lambda \leq (\theta_2 - \theta_{\min})y_{r_2+1}$. As $\theta_2 y_{r_2+1} \leq \theta_{\max}y_{r_2+1} - \mu$, it follows that $\lambda + \theta_{\min}y_{r_2+1} \leq \theta_{\max}y_{r_2+1} - \mu$ or, equivalently, that $\lambda+\mu \leq (\theta_{\max}-\theta_{\min})y_{r_2+1} \leq (\theta_{\max}-\theta_{\min})y_{k'}$, for all $k' = r_2+1,...,n$. As a result, $R_{k'}^{\theta_1}(N,y,T) \geq R_{k'}^{\theta_2}(N,y,T)$ for all $k' = k,...,n$ and the single-crossing property follows.

Case 4: $\theta_2 Y \leq T < \theta_{\max}(Y - ny_1) + ny_1$.

In this case, by the definition of the family of generalized talmudic methods, $R_i^{\theta_j}(N,y,T) = \max\{\theta_j y_i, \theta_{\max}y_i - \mu_j\}$, for all $i \in N$ and $j = 1,2$, where μ_1 and μ_2 are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0,...,n\}$ such that $T \geq \theta_1 Y + (\theta_{\max}-\theta_1)((\sum_{i=r_1+1}^n y_i) - (n-r_1)y_{r_1+1})$. Furthermore, let r_2 be the smallest non-negative integer in $\{0,...,n\}$ such that $T \geq \theta_2 Y + (\theta_{\max}-\theta_2)((\sum_{i=r_2+1}^n y_i) - (n-r_2)y_{r_2+1})$. It

[11] Note that $k \geq r_2$.

is straightforward to show that $r_2 \leq r_1$. Thus,

$$\begin{aligned} R^{\theta_1}(N,y,T) &= (\theta_1 y_1, ..., \theta_1 y_{r_2}, ..., \theta_1 y_{r_1}, \theta_{\max} y_{r_1+1} - \mu_1, ..., \theta_{\max} y_n - \mu_1), \text{ and} \\ R^{\theta_2}(N,y,T) &= (\theta_2 y_1, ..., \theta_2 y_{r_2}, \theta_{\max} y_{r_2+1} - \mu_2, ..., \theta_{\max} y_n - \mu_2), \end{aligned}$$

where $\mu_1 = \frac{\theta_1(\sum_{i=1}^{r_1} y_i)+\theta_{\max}\left(\sum_{i=r_1+1}^{n} y_i\right)-T}{n-r_1}$ and $\mu_2 = \frac{\theta_2(\sum_{i=1}^{r_2} y_i)+\theta_{\max}\left(\sum_{i=r_2+1}^{n} y_i\right)-T}{n-r_2}$. By Lemma 1, $R_n^{\theta_1}(N,y,T) \geq R_n^{\theta_2}(N,y,T)$. Thus, $\mu_1 \leq \mu_2$. Consequently, $R_i^{\theta_1}(N,y,T) \leq R_i^{\theta_2}(N,y,T)$ for all $i = 1, ..., r_2$ and $R_i^{\theta_1}(N,y,T) \geq R_i^{\theta_2}(N,y,T)$ for all $i = r_1+1, ..., n$. Now, there are three subcases:

Subcase 4.1: $\mu_2 < (\theta_{max} - \theta_1) y_{r_2+1}$.

Then, $i^* = r_1 + 1$ and the single-crossing property holds.

Subcase 4.2: $\mu_2 \geq (\theta_{max} - \theta_1) y_{r_1}$.

Then, $i^* = r_2$ and the single-crossing property holds.

Subcase 4.3: $\mu_2 \in [(\theta_{max} - \theta_1) y_{r_2+1}, (\theta_{max} - \theta_1) y_{r_1}]$.

Then, there exists some $k \in \{r_2+1, ..., r_1-1\}$ such that $(\theta_{max}-\theta_1)y_{k+1} > \mu_2 \geq (\theta_{max} - \theta_1) y_k$. Thus, $i^* = k+1$ and the single-crossing property holds.

Case 5: $T \geq \theta_{\max}(Y - n y_1) + n y_1$.

In this case, the single-crossing property trivially follows as $R^{\theta_1}(N,y,T) \equiv R^{\theta_2}(N,y,T)$. ■

It is worth mentioning that the above proof of Theorem 2 does not extend to the whole domain of two-piece linear methods. To see this, take the Talmud method (T), and the method $R^{\frac{5}{8}}$, for $\theta_{min} = \frac{1}{4}$ and $\theta_{max} = 1$. Let $(N,y,T) = (\{1,2,3\}, (4,16,20), 15)$. It is straightforward to show that $T(N,y,T) = \left(2, \frac{13}{2}, \frac{13}{2}\right)$, whereas $R^{\frac{5}{8}}(N,y,T) = \left(\frac{5}{2}, \frac{23}{4}, \frac{27}{4}\right)$.

4 Further insights

The proof of Theorem 2 tells us that the majority voting equilibrium for the family of generalized talmudic tax methods is precisely the method preferred by the median voter, i.e., the median taxpayer. We now explore further the properties of the equilibrium whose existence has been shown in the previous section. In what follows, we make the following mild assumption, which reflects a well-established empirical fact in advanced democracies.

Assumption 0. In each taxation problem, the median income is below the mean income.

Our next result summarizes the main findings within this section. To ease the exposition of its statement, we assume, without loss of generality, that for each $(N, y, T) \in \mathcal{D}$, $N = \{1, \ldots, n\}$ with $n \geq 3$ odd, and $y_1 \leq y_2 \leq \cdots \leq y_n$. Let $m = \frac{n+1}{2}$ denote the median taxpayer of this problem. Furthermore, let $Y^m = \sum_{j=m}^{n} y_j - (n - m + 1)y_m$, and

$$\theta^* = \max\left\{\theta_{\min}, \frac{T - \theta_{\max}Y^m}{Y - Y^m}\right\}.$$

Theorem 3 *If Assumption 0 holds, and $\theta_{\min}Y \leq T \leq \theta_{\max}Y$, then R^{θ^*} is the majority voting equilibrium for the family of generalized talmudic tax methods $\{R^\theta\}_{\theta \in [\theta_{\min}, \theta_{\max}]}$.*

Proof. We start with a piece of notation. Let $(N, y, T) \in \mathcal{D}$ be given in the conditions described at the statement and let $k \in N$. Let us also consider the following thresholds:

$$\begin{aligned}
\theta_1^k &= \theta_{\max} + \frac{T - \theta_{\max}Y}{ny_1}, \\
\theta_2^k &= \frac{T - \theta_{\max}(\sum_{j=k}^{n} y_j - (n-k+1)y_k)}{\sum_{j=1}^{k-1} y_j + (n-k+1)y_k}, \\
\theta_3^k &= \frac{T - \theta_{\min}(\sum_{j=k}^{n} y_j - (n-k+1)y_k)}{\sum_{j=1}^{k-1} y_j + (n-k+1)y_k}, \\
\theta_4^k &= \theta_{\min} + \frac{T - \theta_{\min}Y}{ny_1}.
\end{aligned}$$

As $\theta_{\min}Y \leq T \leq \theta_{\max}Y$, it is straightforward to show that $\theta_1^k \leq \theta_2^k \leq \theta_3^k \leq \theta_4^k$, and that $\theta_2^k \leq \theta_{\max}$ and $\theta_3^k \geq \theta_{\min}$. It can actually be shown, after some algebraic computations, that

$$R_k^\theta(N, y, T) = \begin{cases}
\theta_{\max}y_k + \frac{T - \theta_{\max}Y}{n} & \textit{if } \theta_{\min} \leq \theta \leq \theta_1^k \\
f_k(\theta) & \textit{if } \theta_1^k \leq \theta \leq \theta_2^k \\
\theta y_k & \textit{if } \theta_2^k \leq \theta \leq \theta_3^k \\
g_k(\theta) & \textit{if } \theta_3^k \leq \theta \leq \theta_4^k \\
\theta_{\min}y_k + \frac{T - \theta_{\min}Y}{n} & \textit{if } \theta_4^k \leq \theta \leq \theta_{\max},
\end{cases} \tag{1}$$

where $f_k(\cdot)$ and $g_k(\cdot)$ are piece-wise linear decreasing functions.[12] A graphical illustration appears in Figure 1.

Let k now be the median agent, i.e., $k = m$. Then, by Assumption 0, it follows that $\theta_{\max}y_k + \frac{T - \theta_{\max}Y}{n} \leq \theta_{\min}y_k + \frac{T - \theta_{\min}Y}{n}$. As $\theta_3^k \geq \theta_{\min}$ and

[12] Note that $\theta_1^1 = \theta_2^1$ and $\theta_3^1 = \theta_4^1$, whereas $\theta_2^n = \theta_3^n$. Thus, the taxpayers with the lowest and highest incomes only have three pieces (two of them constant with respect to θ) in their preferences.

$\theta_2^k \leq \theta_{\max}$, there would be nine possible cases depending of the relative positions of the remaining θ^k-thresholds with respect to $\theta_{\min}$ and $\theta_{\max}$. For our purposes, and thanks to (1), they summarize in just two supra-cases. If $\theta_2^k < \theta_{\min}$ then the minimum of $R_k^\theta(N, y, T)$, and therefore the most preferred method by agent k, is achieved for $\theta = \theta_{\min}$. If, otherwise, $\theta_2^k \geq \theta_{\min}$, then the minimum of $R_k^\theta(N, y, T)$, and therefore the most preferred method by agent k, is achieved for $\theta = \theta_2^m$. This concludes the proof. ■

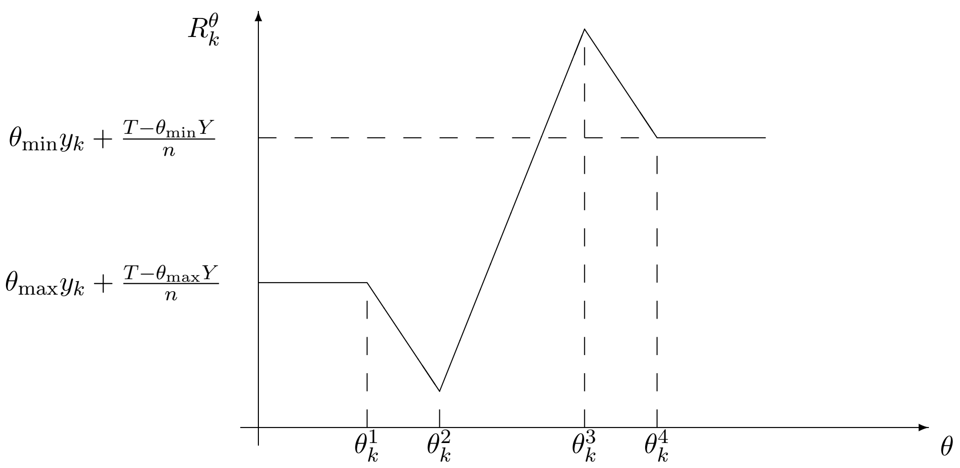

Figure 1: Individual preferences. This figure represents the tax burden proposed by the method R^θ, at the problem (N, y, T), for agent $k \in N$, as a function of the parameter θ.[13]

It is straightforward to show that if $T = \theta_{\max} Y$ then $\theta^* = \theta_{\max}$. Thus, the range of θ^* is the whole interval $[\theta_{\min}, \theta_{\max}]$ and, hence, we have the next corollary.

Corollary 1 *Under Assumption 0, any method within the family of generalized talmudic tax methods* $\{R^\theta\}_{\theta \in [\theta_{\min}, \theta_{\max}]}$ *can be the majority voting equilibrium for this family, for a given predetermined level of aggregate fiscal revenue* $\theta_{\min} Y \leq T \leq \theta_{\max} Y$.

Theorem 3 provides, under a mild assumption, an explicit expression for the majority voting equilibrium within the family of generalized talmudic tax

[13]For simplicity, we consider the second and fourth pieces as linear in the picture, although they are indeed piece-wise linear, as mentioned above.

methods, whose existence was shown in Theorem 2, as a function of the data of the tax problem (namely, the group of taxpayers and the predetermined level of aggregate fiscal revenue). Corollary 1 goes further and shows that, for a given group of taxpayers, and a given method within the family, there exists a predetermined level of aggregate fiscal revenue for which such a method is the equilibrium. Thus, if there is freedom to determine the level of aggregate fiscal revenue to be raised, a given method can be targeted to become the majority voting equilibrium. Another way of reading this corollary is as a neutrality condition for the family of generalized talmudic tax methods. In other words, the corollary is saying that there is no bias in favor, or against, any of the methods within the family as any of them can arise as an equilibrium.

5 Concluding remarks

We have dealt in this paper with the issue of designing the most appropriate income tax. There is a broad consensus worldwide about implementing piece-wise linear tax methods and therefore we have endorsed such a restriction in our (simple) modeling. A key aspect regarding the implementation of a piece-wise linear tax method is the choice of the corresponding brackets, rates and lump-sum levies. Here we have analyzed such aspect assuming that the tax parameters are chosen directly by voters according to majority rule. In spite of the impossibility result saying that if we allow agents to vote freely for any piece-wise linear tax method, no equilibrium can come out of it, we have obtained two positive results. First, we show that if we restrict the universe in a meaningful way an equilibrium does exist. Second, we show that, within such a restricted domain, basically any method can be the majority voting equilibrium, upon selecting precisely the level of aggregate fiscal revenue.

Our results also hold for the case of a perfectly representative democracy in which tax methods arise as a result of political competition.[14] More precisely, assume that there are two parties running in an election and that competition occurs only over tax policies. Given a pair of alternative policies, a taxpayer votes for the one she prefers (i.e., the one that gives her the greatest post-tax income). If she is indifferent, she votes for each policy with probability one-half. A *political equilibrium* would then be defined as a Nash equilibrium of the resulting game played by the two parties, where

[14] See, for instance, Roemer (1999) for a general analysis of the role of political competition in the design of income taxes.

they share a common policy space, and in which their payoff functions are their probabilities of victory (that obviously depend on their policy choices). Under these conventions, one could easily mimic the results of this paper replacing the concept of majority rule equilibrium by that of political equilibrium and obtaining that both parties cater to the median voter.

The restriction to piece-wise linear tax methods has not been our only assumption in the model. We have also assumed the existence of a finite set of taxpayers, in contrast to most of the related literature, where it seems customary to deal with taxes in a calculus framework. We have actually eschewed any reference to tax functions and presented our proofs for pre-tax and post-tax vectors. It turns out that simple inequalities, dispensing with any differentiability assumption, have shown to be powerful enough (and mathematically elegant) to prove our results. Our modeling choice has also allowed us to analyze interesting features that are normally bypassed in this area (such as the effect that a change of a vote might have over policies, or, more generally, the behavior of voters in small elections) because of dealing with a calculus framework.

We have also imposed a constraint on the tax structure indicating that there is a predetermined level of aggregate fiscal revenue that has to be raised. This is a standard feature of both optimal tax models and voting models (e.g., Romer, 1975). On the other hand, we have assumed that labor is perfectly inelastically supplied. Nevertheless, such assumption could be easily relaxed. In a more general model in which agents would have preferences over consumption and leisure, the preferred tax schedule of the median voter would also be the majority voting equilibrium, provided both preferences and tax schedules satisfy the single-crossing property (e.g., Gans and Smart, 1996). Minor restrictions (e.g., assuming both consumption and leisure are normal goods) would suffice to guarantee that preferences are single-crossing, and hence our results would still be relevant in this context.

To conclude, it is worth mentioning that our result regarding the existence of majority voting equilibrium offers as a byproduct an implication over the distributive power of the methods within the domain being considered. More precisely, and as a consequence of the single-crossing property they exhibit, it also holds that the methods within the domain are completely ranked according to the so-called Lorenz dominance criterion, the most fundamental criterion of income inequality.[15]

[15] Moreno-Ternero and Villar (2006b) prove this result directly for the case in which $\theta_{min} = 0$ and $\theta_{max} = 1$.

6 Appendix

Proof of Lemma 1

We distinguish five cases:

Case 1: $0 \leq T \leq \theta_{\min}(Y - ny_1)$.

In this case, the statement trivially follows as $R^{\theta_1}(N, y, T) \equiv R^{\theta_2}(N, y, T)$.

Case 2: $\theta_{\min}(Y - ny_1) < T \leq \theta_1 Y$.

In this case, by the definition of the family of generalized talmudic methods, $R_i^{\theta_j}(N, y, T) = \min\{\theta_j y_i, \theta_{\min} y_i + \lambda_j\}$, for all $i \in N$ and $j = 1, 2$, where λ_1 and λ_2 are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0, ..., n\}$ such that $T \leq \theta_1(\sum_{i=1}^{r_1} y_i) + (n - r_1)(\theta_1 - \theta_{\min})y_{r_1+1}$ and r_2 the smallest non-negative integer in $\{0, ..., n\}$ such that $T \leq \theta_2(\sum_{i=1}^{r_2} y_i) + (n - r_2)(\theta_2 - \theta_{\min})y_{r_2+1}$. It is straightforward to show that $r_2 \leq r_1$. Thus,

$$\begin{aligned} R^{\theta_1}(N, y, T) &= (\theta_1 y_1, ..., \theta_1 y_{r_2}, ..., \theta_1 y_{r_1}, \theta_{\min} y_{r_1+1} + \lambda_1, ..., \theta_{\min} y_n + \lambda_1), \textit{ and} \\ R^{\theta_2}(N, y, T) &= (\theta_2 y_1, ..., \theta_2 y_{r_2}, \theta_{\min} y_{r_2+1} + \lambda_2, ..., \theta_{\min} y_{r_1+1} + \lambda_2, ..., \theta_{\min} y_n + \lambda_2), \end{aligned}$$

where $\lambda_1 = \frac{T - \theta_1(\sum_{i=1}^{r_1} y_i) - \theta_{\min}\left(\sum_{i=r_1+1}^{n} y_i\right)}{n - r_1}$ and $\lambda_2 = \frac{T - \theta_2(\sum_{i=1}^{r_2} y_i) - \theta_{\min}\left(\sum_{i=r_2+1}^{n} y_i\right)}{n - r_2}$. Consequently, $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., r_2$. Assume, by contradiction, that $R_n^{\theta_1}(N, y, T) < R_n^{\theta_2}(N, y, T)$, i.e., $\lambda_1 < \lambda_2$. Then, $R_i^{\theta_1}(N, y, T) < R_i^{\theta_2}(N, y, T)$ for all $i = r_1 + 1, ..., n$. Finally, let $k \in \{r_2+1, ..., r_1-1\}$. Then, $R_k^{\theta_1}(N, y, T) = \theta_1 y_k \leq \theta_{min} y_k + \lambda_1 < \theta_{min} y_k + \lambda_2 = R_k^{\theta_2}(N, y, T)$. Thus,

$$T = \sum_{i=1}^{n} R_i^{\theta_1}(N, y, T) < \sum_{i=1}^{n} R_i^{\theta_2}(N, y, T) = T,$$

which represents a contradiction.

Case 3: $\theta_1 Y < T < \theta_2 Y$.

By the definition of the family of generalized talmudic methods, $R_i^{\theta_1}(N, y, T) = \max\{\theta_1 y_i, \theta_{\max} y_i - \mu\}$ and $R_i^{\theta_2}(N, y, T) = \min\{\theta_2 y_i, \theta_{\min} y_i + \lambda\}$ for all $i \in N$, where μ and λ are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0, ..., n-1\}$ such that $T \geq \theta_1 Y + (\theta_{\max} - \theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n - r_1)y_{r_1+1})$. Furthermore, let r_2 be the smallest non-negative integer in $\{0, ..., n-1\}$ such that $T \leq \theta_2(\sum_{i=1}^{r_2} y_i) + (n - r_2)(\theta_2 - \theta_{\min})y_{r_2+1}$. It is straightforward to show that $r_2 \leq r_1$. Thus,

$$\begin{aligned} R^{\theta_1}(N, y, T) &= (\theta_1 y_1, ..., \theta_1 y_{r_1}, \theta_{\max} y_{r_1+1} - \mu, ..., \theta_{\max} y_n - \mu), \textit{ and} \\ R^{\theta_2}(N, y, T) &= (\theta_2 y_1, ..., \theta_2 y_{r_2}, \theta_{\min} y_{r_2+1} + \lambda, ..., \theta_{\min} y_n + \lambda), \end{aligned}$$

where $\lambda = \frac{T-\theta_2\left(\sum_{i=1}^{r_2} y_i\right)-\theta_{\min}\left(\sum_{i=r_2+1}^{n} y_i\right)}{n-r_2}$ and $\mu = \frac{\theta_1\left(\sum_{i=1}^{r_1} y_i\right)+\theta_{\max}\left(\sum_{i=r_1+1}^{n} y_i\right)-T}{n-r_1}$. Consequently, $R_i^{\theta_1}(N,y,T) \leq R_i^{\theta_2}(N,y,T)$ for all $i = 1,...,\min\{r_1,r_2\}$. Assume, by contradiction, that $R_n^{\theta_1}(N,y,T) < R_n^{\theta_2}(N,y,T)$, i.e., $(\theta_{\max} - \theta_{\min})y_n < \mu + \lambda$. It follows from here that $(\theta_{\max} - \theta_{\min})y_k < \mu + \lambda$ for all $k \in N$. Thus, $R_i^{\theta_1}(N,y,T) < R_i^{\theta_2}(N,y,T)$ for all $i = \max\{r_1,r_2\},...,n$. Finally, let $k \in \{\min\{r_1,r_2\}+1,...,\max\{r_1,r_2\}-1\}$.

If $r_1 < r_2$ then $R_k^{\theta_1}(N,y,T) = \theta_{\max}y_k - \mu \geq \theta_1 y_k$ whereas $R_k^{\theta_2}(N,y,T) = \theta_2 y_k \leq \theta_{\min}y_k + \lambda$. Thus,

$$\begin{aligned} R_k^{\theta_1}(N,y,T) &= \theta_{\max}y_k - \mu \\ &< \theta_{\max}y_k - (\theta_{\max} - \theta_{\min})y_n + \lambda \\ &\leq \theta_{\max}y_k - \theta_{\max}y_n + \theta_2 y_n \\ &\leq \theta_2 y_k \\ &= R_k^{\theta_2}(N,y,T). \end{aligned}$$

If $r_1 > r_2$ then $R_k^{\theta_1}(N,y,T) = \theta_1 y_k \geq \theta_{\max}y_k - \mu$ whereas $R_k^{\theta_2}(N,y,T) = \lambda + \theta_{\min}y_k \leq \theta_2 y_k$. Thus,

$$\begin{aligned} R_k^{\theta_1}(N,y,T) &= \theta_1 y_k \\ &\leq \theta_1 y_n - \theta_{\min}(y_n - y_k) \\ &\leq (\theta_{\max} - \theta_{\min})y_n - \mu \\ &< \theta_{\min}y_k + \lambda \\ &= R_k^{\theta_2}(N,y,T). \end{aligned}$$

We have therefore shown that, in any case, $R_k^{\theta_1}(N,y,T) < R_k^{\theta_2}(N,y,T)$ for all $k \in \{\min\{r_1,r_2\}+1,...,\max\{r_1,r_2\}-1\}$. Thus,

$$T = \sum_{i=1}^{n} R_i^{\theta_1}(N,y,T) < \sum_{i=1}^{n} R_i^{\theta_2}(N,y,T) = T,$$

which represents a contradiction.

Case 4: $\theta_2 Y \leq T < \theta_{\max}(Y - ny_1) + ny_1$.

In this case, by the definition of the family of generalized talmudic methods, $R_i^{\theta_j}(N,y,T) = \max\{\theta_j y_i, \theta_{\max}y_i - \mu_j\}$, for all $i \in N$ and $j = 1,2$, where μ_1 and μ_2 are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0,...,n\}$ such that $T \geq \theta_1 Y + (\theta_{\max} - \theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n-r_1)y_{r_1+1})$. Furthermore, let r_2 be the smallest non-negative integer in

$\{0, ..., n\}$ such that $T \geq \theta_2 Y + (\theta_{\max} - \theta_2)((\sum_{i=r_2+1}^{n} y_i) - (n - r_2)y_{r_2+1})$. It is straightforward to show that $r_2 \leq r_1$. Thus,

$$R^{\theta_1}(N, y, T) = (\theta_1 y_1, ..., \theta_1 y_{r_2}, ..., \theta_1 y_{r_1}, \theta_{\max} y_{r_1+1} - \mu_1, ..., \theta_{\max} y_n - \mu_1),$$
$$R^{\theta_2}(N, y, T) = (\theta_2 y_1, ..., \theta_2 y_{r_2}, \theta_{\max} y_{r_2+1} - \mu_2, ..., \theta_{\max} y_n - \mu_2),$$

where $\mu_1 = \frac{\theta_1(\sum_{i=1}^{r_1} y_i) + \theta_{\max}\left(\sum_{i=r_1+1}^{n} y_i\right) - T}{n - r_1}$ and $\mu_2 = \frac{\theta_2(\sum_{i=1}^{r_2} y_i) + \theta_{\max}\left(\sum_{i=r_2+1}^{n} y_i\right) - T}{n - r_2}$. Consequently, $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., r_2$. Assume, by contradiction, that $R_n^{\theta_1}(N, y, T) < R_n^{\theta_2}(N, y, T)$, i.e., $\mu_1 > \mu_2$. Then, $R_i^{\theta_1}(N, y, T) < R_i^{\theta_2}(N, y, T)$ for all $i = r_1 + 1, ..., n$. Finally, let $k \in \{r_2 + 1, ..., r_1 - 1\}$. Then, $R_k^{\theta_1}(N, y, T) = \theta_1 y_k \leq \theta_2 y_k \leq \theta_{max} y_k - \mu_2 = R_k^{\theta_2}(N, y, T)$. Thus,

$$T = \sum_{i=1}^{n} R_i^{\theta_1}(N, y, T) < \sum_{i=1}^{n} R_i^{\theta_2}(N, y, T) = T,$$

which represents a contradiction.

Case 5: $T \geq \theta_{\max}(Y - ny_1) + ny_1$.

In this case, the statement trivially follows as $R^{\theta_1}(N, y, T) \equiv R^{\theta_2}(N, y, T)$.

■

References

[1] Alesina, A., Rosenthal, H., 1996, A Theory of Divided Government, Econometrica 64, 1311-1341.

[2] Aumann RJ, Maschler M, 1985, Game theoretic analysis of a bankruptcy problem from the Talmud, Journal of Economic Theory 36, 195-213.

[3] Bénabou, R., 2002, Tax and Education Policy in a Heterogeneous Agent Economy: What Levels of Redistribution Maximize Growth and Efficiency? Econometrica 70, 481-517.

[4] Burman, L., Khitatrakun, S., Leiserson. G., Rohaly, J., Toder, E., Williams, B., 2008, An Updated Analysis of the 2008 Presidential Candidates' Tax Plans. Tax Policy Center. Urban Institute and Brookings Institution.

[5] Campbell, D., 1975, Income distribution under majority rule and alternative taxation criteria, Public Choice 22, 23-35.

[6] Foley, D., 1967, Resource allocation and the public sector, Yale Economic Essays 7, 45-98.

[7] Gans, J. S., Smart, M., 1996, Majority voting with single-crossing preferences, Journal of Public Economics 59, 219-237.

[8] Gouveia, M., Oliver, D., 1996, Voting over flat taxes in an endowment economy, Economics Letters 50, 251-258

[9] Hamada, K., 1973, A Simple majority rule on the Distribution of Income, Journal of Economic Theory 6, 243-264.

[10] Le Breton, M., Moyes, P., Trannoy, A., 1996, Inequality Reducing Properties of Composite Taxation, Journal of Economic Theory 69, 71-103.

[11] Marhuenda F., Ortuño-Ortin, I., 1998, Income taxation, uncertainty and stability. Journal of Public Economics 67, 285-300.

[12] Mitra, T., Ok, E., 1997, On the equitability of progressive taxation, Journal of Economic Theory 73, 316-334.

[13] Moreno-Ternero J, Villar A., 2006a, The TAL-family of rules for bankruptcy problems. Social Choice and Welfare 27, 231-249.

[14] Moreno-Ternero J, Villar A., 2006b, On the relative equitability of a family of taxation rules. Journal of Public Economic Theory 8, 283-291.

[15] Moulin, H., 2002, Axiomatic cost and surplus-sharing, in: K. Arrow, A. Sen, K. Suzumura, (Eds.), The Handbook of Social Choice and Welfare, Vol.1, 289-357, North-Holland.

[16] O'Neill, B., 1982, A problem of rights arbitration from the Talmud, Mathematical Social Sciences 2, 345-371.

[17] Romer, T., 1975, Individual welfare, majority voting, and the properties of a linear income tax, Journal of Public Economics 4, 163-186.

[18] Roemer, J., 1999, The democratic political economy of progressive income taxation, Econometrica 67, 1-19.

[19] Thomson, W., 2003, Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey, Mathematical Social Sciences 45, 249-297.

[20] Young P., 1988, Distributive justice in taxation, Journal of Economic Theory 44, 321-335.

Recent titles

CORE Discussion Papers

2010/41. Pierre PESTIEAU and Maria RACIONERO. Tagging with leisure needs.
2010/42. Knud J. MUNK. The optimal commodity tax system as a compromise between two objectives.
2010/43. Marie-Louise LEROUX and Gregory PONTHIERE. Utilitarianism and unequal longevities: A remedy?
2010/44. Michel DENUIT, Louis EECKHOUDT, Ilia TSETLIN and Robert L. WINKLER. Multivariate concave and convex stochastic dominance.
2010/45. Rüdiger STEPHAN. An extension of disjunctive programming and its impact for compact tree formulations.
2010/46. Jorge MANZI, Ernesto SAN MARTIN and Sébastien VAN BELLEGEM. School system evaluation by value-added analysis under endogeneity.
2010/47. Nicolas GILLIS and François GLINEUR. A multilevel approach for nonnegative matrix factorization.
2010/48. Marie-Louise LEROUX and Pierre PESTIEAU. The political economy of derived pension rights.
2010/49. Jeroen V.K. ROMBOUTS and Lars STENTOFT. Option pricing with asymmetric heteroskedastic normal mixture models.
2010/50. Maik SCHWARZ, Sébastien VAN BELLEGEM and Jean-Pierre FLORENS. Nonparametric frontier estimation from noisy data.
2010/51. Nicolas GILLIS and François GLINEUR. On the geometric interpretation of the nonnegative rank.
2010/52. Yves SMEERS, Giorgia OGGIONI, Elisabetta ALLEVI and Siegfried SCHAIBLE. Generalized Nash Equilibrium and market coupling in the European power system.
2010/53. Giorgia OGGIONI and Yves SMEERS. Market coupling and the organization of counter-trading: separating energy and transmission again?
2010/54. Helmuth CREMER, Firouz GAHVARI and Pierre PESTIEAU. Fertility, human capital accumulation, and the pension system.
2010/55. Jan JOHANNES, Sébastien VAN BELLEGEM and Anne VANHEMS. Iterative regularization in nonparametric instrumental regression.
2010/56. Thierry BRECHET, Pierre-André JOUVET and Gilles ROTILLON. Tradable pollution permits in dynamic general equilibrium: can optimality and acceptability be reconciled?
2010/57. Thomas BAUDIN. The optimal trade-off between quality and quantity with uncertain child survival.
2010/58. Thomas BAUDIN. Family policies: what does the standard endogenous fertility model tell us?
2010/59. Nicolas GILLIS and François GLINEUR. Nonnegative factorization and the maximum edge biclique problem.
2010/60. Paul BELLEFLAMME and Martin PEITZ. Digital piracy: theory.
2010/61. Axel GAUTIER and Xavier WAUTHY. Competitively neutral universal service obligations.
2010/62. Thierry BRECHET, Julien THENIE, Thibaut ZEIMES and Stéphane ZUBER. The benefits of cooperation under uncertainty: the case of climate change.
2010/63. Marco DI SUMMA and Laurence A. WOLSEY. Mixing sets linked by bidirected paths.
2010/64. Kaz MIYAGIWA, Huasheng SONG and Hylke VANDENBUSSCHE. Innovation, antidumping and retaliation.
2010/65. Thierry BRECHET, Natali HRITONENKO and Yuri YATSENKO. Adaptation and mitigation in long-term climate policies.
2010/66. Marc FLEURBAEY, Marie-Louise LEROUX and Gregory PONTHIERE. Compensating the dead? Yes we can!
2010/67. Philippe CHEVALIER, Jean-Christophe VAN DEN SCHRIECK and Ying WEI. Measuring the variability in supply chains with the peakedness.
2010/68. Mathieu VAN VYVE. Fixed-charge transportation on a path: optimization, LP formulations and separation.
2010/69. Roland Iwan LUTTENS. Lower bounds rule!

Recent titles

CORE Discussion Papers - continued

2010/70. Fred SCHROYEN and Adekola OYENUGA. Optimal pricing and capacity choice for a public service under risk of interruption.
2010/71. Carlotta BALESTRA, Thierry BRECHET and Stéphane LAMBRECHT. Property rights with biological spillovers: when Hardin meets Meade.
2010/72. Olivier GERGAUD and Victor GINSBURGH. Success: talent, intelligence or beauty?
2010/73. Jean GABSZEWICZ, Victor GINSBURGH, Didier LAUSSEL and Shlomo WEBER. Foreign languages' acquisition: self learning and linguistic schools.
2010/74. Cédric CEULEMANS, Victor GINSBURGH and Patrick LEGROS. Rock and roll bands, (in)complete contracts and creativity.
2010/75. Nicolas GILLIS and François GLINEUR. Low-rank matrix approximation with weights or missing data is NP-hard.
2010/76. Ana MAULEON, Vincent VANNETELBOSCH and Cecilia VERGARI. Unions' relative concerns and strikes in wage bargaining.
2010/77. Ana MAULEON, Vincent VANNETELBOSCH and Cecilia VERGARI. Bargaining and delay in patent licensing.
2010/78. Jean J. GABSZEWICZ and Ornella TAROLA. Product innovation and market acquisition of firms.
2010/79. Michel LE BRETON, Juan D. MORENO-TERNERO, Alexei SAVVATEEV and Shlomo WEBER. Stability and fairness in models with a multiple membership.
2010/80. Juan D. MORENO-TERNERO. Voting over piece-wise linear tax methods.

Books

J. GABSZEWICZ (ed.) (2006), La différenciation des produits. Paris, La découverte.
L. BAUWENS, W. POHLMEIER and D. VEREDAS (eds.) (2008), *High frequency financial econometrics: recent developments*. Heidelberg, Physica-Verlag.
P. VAN HENTENRYCKE and L. WOLSEY (eds.) (2007), *Integration of AI and OR techniques in constraint programming for combinatorial optimization problems*. Berlin, Springer.
P-P. COMBES, Th. MAYER and J-F. THISSE (eds.) (2008), *Economic geography: the integration of regions and nations*. Princeton, Princeton University Press.
J. HINDRIKS (ed.) (2008), *Au-delà de Copernic: de la confusion au consensus* ? Brussels, Academic and Scientific Publishers.
J-M. HURIOT and J-F. THISSE (eds) (2009), *Economics of cities.* Cambridge, Cambridge University Press.
P. BELLEFLAMME and M. PEITZ (eds) (2010), *Industrial organization: markets and strategies*. Cambridge University Press.
M. JUNGER, Th. LIEBLING, D. NADDEF, G. NEMHAUSER, W. PULLEYBLANK, G. REINELT, G. RINALDI and L. WOLSEY (eds) (2010), *50 years of integer programming, 1958-2008: from the early years to the state-of-the-art*. Berlin Springer.

CORE Lecture Series

C. GOURIÉROUX and A. MONFORT (1995), Simulation Based Econometric Methods.
A. RUBINSTEIN (1996), Lectures on Modeling Bounded Rationality.
J. RENEGAR (1999), A Mathematical View of Interior-Point Methods in Convex Optimization.
B.D. BERNHEIM and M.D. WHINSTON (1999), Anticompetitive Exclusion and Foreclosure Through Vertical Agreements.
D. BIENSTOCK (2001), Potential function methods for approximately solving linear programming problems: theory and practice.
R. AMIR (2002), Supermodularity and complementarity in economics.
R. WEISMANTEL (2006), Lectures on mixed nonlinear programming.