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Lower bounds rule!

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DISCUSSION PAPER

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Abstract

We propose two axioms that introduce lower bounds into resource monotonicity requirements for rules for the problem of adjudicating conflicting claims. Suppose the amount to divide increases. The first axiom requires that two claimants whose lower bound changes equally experience an equal change in awards. The second axiom requires that extra resources are divided only among those claimants who experience a strictly positive change in their lower bound. We show that, in the two-claimant case, Concede-and-Divide is the only rule that satisfies both axioms when the axioms are defined over a large set of lower bounds that include the minimal rights lower bound and the secured lower bound. We also show that, in the n-claimant case where at least one claimant claims the total amount, the Minimal Overlap rule is the only rule that satisfies both axioms when the axioms are defined over the secured lower bound.

Keywords: claims problems, lower bounds, concede-and-divide, minimal overlap rule.

JEL Classification: D63, D74

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1 Introduction

The central question of the analysis of conflicting claims problems is how to divide a single infinitely divisible good among a group of individuals when the total amount to be divided is smaller than the total sum of individual claims. The objective is to design 'rules' that associate with each claims problem a division of the amount available over the claimants. The indispensable review article of the conflicting claims literature is Thomson (2003).

Depending on the specific claims problem under consideration, it might be advisable that some claimants are guaranteed to receive a 'lower bound', i.e. a minimal amount of the resource to be divided. The motivation to let some (or all) claimants benefit a lower bound could be prompted out of considerations of fairness, participation or incentive compatibility. A straightforward interpretation is the willingness to provide individuals with an insurance or the urge to reduce inequalities in outcomes. For a detailed analysis into the properties of lower bounds, we refer to Dominguez (2008). A rule guarantees a lower bound if, for each claims problem and for each claimant, the actual amount assigned to an individual is larger than or equal to her lower bound.

The two most prominent lower bounds in the literature are the minimal rights lower bound, proposed by Curiel et al. (1987), and the secured lower bound, proposed by Moreno-Ternero and Villar (2004). It is a well established result that all rules guarantee the minimal rights lower bound; see Thomson (2003). Most, though not all rules also guarantee the secured lower bound; see Moreno-Ternero and Villar (2004).

Lower bounds have appeared in characterizations of rules: either in axioms that explicitly impose the rule to guarantee the lower bound (see, among others, Herrero and Villar (2001), Moreno-Ternero and Villar (2004), Moreno-Ternero (2006), Yeh (2006)), either as an invariance property that requires the rule to assign the same awards vector (i) directly or (ii) indirectly, by first assigning the lower bound and then applying the rule to the appropriately revised problem (see, among others, Curiel et al. (1987), Dominguez and Thomson (2006), Dominguez (2008), Thomson and Yeh (2008)).

In this paper, we introduce lower bounds into resource monotonicity requirements for rules. We propose two axioms new to the literature. Both axioms express how changes in awards should depend on changes in lower bounds. The first axiom requires that two claimants whose lower bound changes equally should experience an equal change in their awards. The second axiom requires that an extra amount of resources should be divided only among those claimants who experienced a strictly positive change in their lower bounds. We show that, in the two-claimant case, Concede-and-Divide is the only rule that satisfies both axioms when the axioms are defined over a large set of lower bounds that include the minimal rights lower bound and the secured lower bound. We also show that, in problems with more than two claimants where at least one claimant claims the total amount to be divided, the Minimal Overlap rule is the only rule that satisfies both axioms when the axioms are defined over the secured lower bound.

The paper is organized as follows. In the next section, we start with an example to motivate our analysis. Preliminaries are presented in section 3. Our main axioms are introduced in section 4. Our analysis of the two-claimant case can be found in section 5, whereas section 6 is devoted to claims problems with more than two claimants. Concluding remarks are made in section 7.

2 An example to motivate our analysis

Consider the well-known three-agent estate-division problem originally formulated in the Talmud. A man bequeaths to his first, second and third wife an amount of 100, 200 and 300, respectively. Consider, for the moment, the minimal rights lower bound. The minimal right of an agent equals the amount that remains from the total estate when all other agents have received their claim (a precise formulation follows). Figure 1 depicts the minimal rights of the three wives as a function of the value of the estate.

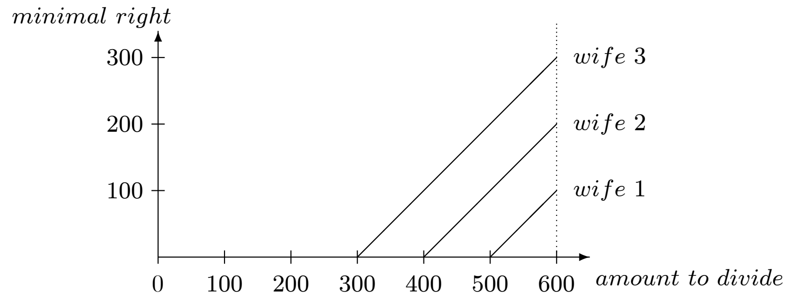

Figure 1: minimal rights.

The minimal rights lower bound embodies a specific principle of fairness. The minimal right of an agent could be given the interpretation of an uncontested part of the amount to divide, an amount of resources that the other claimants concede to a particular agent on the basis of their claims. From the example, it is clear that the whole estate is contested as long as its value does not exceed 300. But suppose that the estate is worth 400 instead of 300. Then, wife 1 and wife 2 together concede 100 to wife 3.

It is important to note that wife 1 and wife 2 only start making a concession when the value of the estate exceeds 300 (and not earlier). **The point we want to make is that a meaningful lower bound, such as the minimal rights lower bound, contains useful information about when exactly, depending on the amount to divide, a differential treatment of claimants can be justified, following a specific principle of fairness.** At the same time, a lower bound is not efficient in general (check, in figure 1, that the minimal rights of the three wives only add up to the value of the estate when the latter equals 600).

The goal of our analysis is exactly to combine the specific fairness principle implied by a meaningful lower bound with efficiency. More specifically, we aim at sorting out those rules that, depending on the amount to divide, discriminate between claimants in exactly the same way as meaningful lower bounds do. When the amount to divide increases, the two axioms that we introduce in this article require (1) that if the lower bound of one agent increases while the lower bound of another agent remains constant, the rule must be such that the award of the former increases while the award of the latter remains constant and (2) that if the lower bounds of two agents increase equally, the rule must be such that the awards of the two agents also increase equally.

3 Preliminaries

We present the fixed population version of the model. An amount $E \in \mathbb{R}_+$ of an infinitely divisible good has to be divided among a set N of claimants, with $|N| = n$. Each claimant $i \in N$ has a strictly positive claim $c_i \in \mathbb{R}_{++}$ over E; denote $c \equiv (c_1, \ldots, c_n)$ the vector of claims. A conflicting claims problem for N is a pair $(c, E) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, where the sum of claims $C = \sum_{i \in N} c_i$ is larger than or equal to E. Let $\mathcal{C}$ be the collection of conflicting claims problems that involve n claimants.

An awards vector $x \in \mathbb{R}^n_+$ of $(c, E) \in \mathcal{C}$ has the following properties: $0 \leqq x$ (non-negativity), $x \leqq c$ (claims boundedness) and $\sum_{i \in N} x_i = E$ (efficiency).[1] Denote $X(c, E)$ the set of awards vectors for the problem (c, E). A rule is a function that associates with each $(c, E) \in \mathcal{C}$ an awards vector in $X(c, E)$. Let R be our generic notation for a rule and $R(c, E)$ our generic notation for an awards vector. We only consider anonymous rules, i.e. rules for which the identity of the claimants does not matter. Accordingly, we limit attention to claims vectors with $c_1 \leq \ldots \leq c_n$. Let R^d denote the dual of a rule R. The rule R^d shares awards as the rule R shares losses, i.e. $R^d(c, E) = c - R(c, C - E)$ for all $(c, E) \in \mathcal{C}$. A rule is self-dual if $R = R^d$.

We can also apply the duality notion to properties of rules. We say that a property P has a dual property P^d if for every R it is true that R satisfies P iff R^d satisfies P^d. A property P is self-dual when both R and R^d satisfy P.

A graphical representation of a rule is by means of its *path of awards*. Given a rule R, for each claims vector c, the path of awards of R for c is the image of the

[1] Vector inequalities: $x \geqq y$, $x \geq y$ and $x > y$.

function $R(c, \cdot) : [0, C] \rightarrow X(c, \cdot)$, which maps each amount E with $0 \leq E \leq C$ into the awards vector assigned by the rule.

A lower bound vector $b \in \mathbb{R}^n_+$ of $(c, E) \in \mathcal{C}$ has the following properties: $0 \leqq b$, $b \leqq c$ and $\sum_{i \in N} b_i \leq E$. Denote $B(c, E)$ the set of lower bound vectors for the problem (c, E). A rule R satisfies a lower bound vector b if $R(c, E) \geqq b(c, E)$ for all $(c, E) \in \mathcal{C}$.

4 Two lower bounds and two axioms

Lower bounds

We introduce the two most prominent lower bounds in the literature.

The minimal rights vector $m(c, E)$, proposed by Curiel et al. (1987), assigns each claimant the difference between E and the sum of claims of all the other agents if this difference is strictly positive, or 0 otherwise. Formally, for all $(c, E) \in \mathcal{C}$ and all $i \in N$, $m_i(c, E) \equiv \max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\}$ and $m(c, E) \equiv (m_i(c, E))_{i \in N}$. The properties of efficiency, non-negativity and claims boundedness —properties that we incorporated in the definition of a rule— together guarantee that each claimant always at least receives her minimal right. In other words, all rules satisfy the minimal rights vector (Thomson, 2003).

The secured lower bounds vector $SLB(c, E)$, proposed by Moreno-Ternero and Villar (2004), assigns (i) one nth of c_i to each claimant holding a feasible claim ($c_i \leq E$) and (ii) one nth of E to each claimant holding an unfeasible claim ($c_i > E$). Formally, for all $(c, E) \in \mathcal{C}$, $SLB_i(c, E) \equiv \frac{1}{n} \min\{c_i, E\}$ for all $i \in N$ and $SLB(c, E) \equiv (SLB_i(c, E))_{i \in N}$. Most rules proposed in the literature satisfy the secured lower bounds vector. Notable exceptions are the proportional rule and the constrained equal losses rule; see Moreno-Ternero and Villar (2004) for definitions and proofs.

Axioms

We propose two axioms new to the literature. Both axioms express how the award vector should change in function of changes in the lower bound vector when the amount to divide increases. Denote ε an infinitesimally small increase in the amount to divide. One interpretation could be that, after a re-evaluation, the estate is estimated to be worth slightly more than originally thought ($E < E+\varepsilon \leq C$). In this new situation, the lower bound of none, some or all individuals could have changed, depending on c and E, i.e. $b(c, E+\varepsilon) \geqq b(c, E)$.

The first axiom, which we call "Equal treatment for equal changes in the lower bound b" (in short: Equal Treatment(b)) requires that two claimants whose lower bound changes equally should experience an equal change in their awards vector.

Equal Treatment(b): For all $i, j \in N$ and for all $(c, E) \in \mathcal{C}$, if $b_i(c, E+\varepsilon) - b_i(c, E) = b_j(c, E+\varepsilon) - b_j(c, E)$, then $R_i(c, E+\varepsilon) - R_i(c, E) = R_j(c, E+\varepsilon) - R_j(c, E)$.

What if claimants experience different changes in their lower bound? The second axiom, which we call "Priority for positive changes in the lower bound b" (in short: Priority(b)), is motivated by the idea that lower bounds contain useful information about which claimants deserve to gain from the re-evaluation. Suppose we can partition the set of claimants into two subsets: (i) those who do not experience a change in their lower bound after the re-evaluation and (ii) those who experience an increase in their lower bound after the re-evaluation. Suppose that the latter subset is non-empty. Then, the axiom requires that the extra amount of resources is divided among the claimants who experienced an increase in their lower bounds. In other words, the awards of the group of claimants whose lower bound stays constant should remain unchanged.

Denote $\tilde{N} = \{i \in N : b_i(c, E+\varepsilon) - b_i(c, E) = 0\} \subseteq N$ the subset of claimants whose lower bound has not changed after the re-evaluation.

Priority(b): For all $(c, E) \in \mathcal{C}$, if $N \backslash \tilde{N} \neq \varnothing$, then $R_i(c, E+\varepsilon) = R_i(c, E)$ for all $i \in \tilde{N}$.

5 The two-claimant case

Let $N \equiv \{1,2\}$. Consider the following alternative expressions for the minimal rights vector and the secured lower bound vector in the two-claimant case:

$$m(c,E) \equiv \begin{cases} (0,0) & \text{if } E \leq c_1 \\ (0, E-c_1) & \text{if } c_1 \leq E \leq c_2 \\ (E-c_2, E-c_1) & \text{if } c_2 \leq E \end{cases}$$

and

$$SLB(c,E) \equiv \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq c_1 \\ \left(\frac{c_1}{2}, \frac{E}{2}\right) & \text{if } c_1 \leq E \leq c_2 \\ \left(\frac{c_1}{2}, \frac{c_2}{2}\right) & \text{if } c_2 \leq E \end{cases}.$$

Now consider the following family of lower bounds.

$$b^{\gamma_1,\gamma_2,\gamma_3}(c,E) \equiv \begin{cases} (\gamma_1 E, \gamma_1 E) & \text{if } E \leq c_1 \\ (\gamma_1 c_1, \gamma_1 c_1 + \gamma_2 (E-c_1)) & \text{if } c_1 \leq E \leq c_2 \\ (\gamma_1 c_1 + \gamma_3 (E-c_2), \gamma_1 c_1 + \gamma_2 (c_2 - c_1) + \gamma_3 (E-c_2)) & \text{if } c_2 \leq E \end{cases}$$

$$\text{where } \begin{array}{l} \gamma_1 \in \left[0, \frac{1}{2}\right] \\ \gamma_2 \in (0,1] \\ \gamma_3 \in [0, 1-\gamma_1] \end{array}.$$

Denote $\hat{B}(c,E) \subset B(c,E)$ the set that comprises all possible members of this family for the problem (c,E). Both minimal rights and secured lower bound belong to this set; more specifically $b^{0,1,1}(c,E) \equiv m(c,E)$ and $b^{\frac{1}{2},\frac{1}{2},0}(c,E) \equiv SLB(c,E)$ for all $(c,E) \in \mathcal{C}$.

The two-claimant rule called Concede-and-Divide plays a central role in what follows. Under Concede-and-Divide, every claimant receives in a first step her minimal right (when strictly positive) conceded to her by the other claimant. In a second step, the amount of resources that remains after both concessions is divided equally. This equal division is preferable since, after being revised down by the minimal rights received in the first step, both claims become equal. Many other rules coincide in the case of two claimants with Concede-and-Divide; we refer to Thomson (2003) for an overview.

Concede-and-Divide, *CD*, selects for $|N| = 2$ and for all $(c, E) \in \mathcal{C}$, the awards vector

$$CD(c,E) \equiv \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } E \leq c_1 \\ \left(\frac{c_1}{2}, E - \frac{c_1}{2}\right) & \text{if } c_1 \leq E \leq c_2 \\ \left(\frac{E+c_1-c_2}{2}, \frac{E+c_2-c_1}{2}\right) & \text{if } c_2 \leq E \end{cases}.$$

Note that Concede-and-Divide coincides with the (efficient) lower bound $b^{\frac{1}{2},1,\frac{1}{2}}(c, E) \in \hat{B}(c, E)$.

Figure 2 depicts the path of awards of Concede-and-Divide.

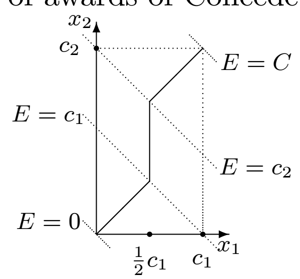

Figure 2: path of awards of Concede-and-Divide.

We now obtain the following characterization result.

Proposition 1 *For $|N| = 2$, for all $(c, E) \in \mathcal{C}$ and for all $b \in \hat{B}(c, E)$, a rule R satisfies Equal Treatment(b) and Priority(b) if and only if it is Concede-and-Divide.*

Proof. Let us first show that CD satisfies Equal Treatment(b) for all $b \in \hat{B}(c, E)$. Denote $\Delta b(c, E, E') = b(c, E') - b(c, E)$ with $E \leq E'$. First, let $E \leq c_1$, so $\Delta b(c, E, c_1) = (\gamma_1(c_1 - E), \gamma_1(c_1 - E))$ for all $b \in \hat{B}(c, E)$. Then $CD(c, c_1) - CD(c, E) = \left(\frac{c_1-E}{2}, \frac{c_1-E}{2}\right)$. Second, let $c_1 < E \leq c_2$ and remark that $\Delta b_1(c, c_1, E) = 0 \neq \Delta b_2(c, c_1, E) = \gamma_2(E - c_1)$ for all $b \in \hat{B}(c, E)$. Third, let $c_2 < E$, so $\Delta b(c, c_2, E) = (\gamma_3(E - c_2), \gamma_3(E - c_2))$. Then $CD(c, E) - CD(c, c_2) = \left(\frac{E-c_2}{2}, \frac{E-c_2}{2}\right)$.

Let us now show that CD satisfies Priority(b) for all $b \in \hat{B}(c,E)$. First, let $E \leq c_1$ and remark that $\Delta b(c,E,c_1) = (\gamma_1(c_1 - E), \gamma_1(c_1 - E))$ for all $b \in \hat{B}(c,E)$. So $\tilde{N} = \{1,2\}$ when $\gamma_1 = 0$ and $\tilde{N} = \varnothing$ when $\gamma_1 \neq 0$. In the latter case, Priority(b) is vacuously satisfied. Second, let $c_1 < E \leq c_2$, so $\Delta b(c,c_1,E) = (0, \gamma_2(E - c_1)) \geq (0,0)$ for all $b \in \hat{B}(c,E)$ since $\gamma_2 > 0$ and hence $\tilde{N} = \{1\}$. Then $CD_1(c,E) - CD_1(c,c_1) = 0$. Third, let $c_2 < E$, so $\Delta b(c,c_2,E) = (\gamma_3(E - c_2), \gamma_3(E - c_2))$ for all $b \in \hat{B}(c,E)$. So $\tilde{N} = \{1,2\}$ when $\gamma_3 = 0$ and $\tilde{N} = \varnothing$ when $\gamma_3 \neq 0$. In the latter case, Priority(b) is vacuously satisfied.

Conversely, let R be a two claimants rule satisfying Priority(b) and Equal Treatment(b) for all $b \in \hat{B}(c,E)$. Recall that R satisfies non-negativity and hence $R(c,0) = (0,0)$. First, let $E \leq c_1$, so $\Delta b(c,0,E) = (\gamma_1 E, \gamma_1 E)$ for all $b \in \hat{B}(c,E)$. By Equal Treatment(b), $R(c,E) = \left(0 + \frac{E}{2}, 0 + \frac{E}{2}\right) = CD(c,E)$. Also $R(c,c_1) = \left(\frac{c_1}{2}, \frac{c_1}{2}\right)$. Second, let $c_1 < E \leq c_2$, so $\Delta b(c,c_1,E) = (0, \gamma_2(E - c_1))$ for all $b \in \hat{B}(c,E)$ and $\tilde{N} = \{1\}$. By Priority, $R(c,E) = \left(\frac{c_1}{2}, \frac{c_1}{2} + E - c_1\right) = CD(c,E)$. Also $R(c,c_2) = \left(\frac{c_1}{2}, c_2 - \frac{c_1}{2}\right)$. Third, let $c_2 < E$, $\Delta b(c,c_2,E) = (\gamma_3(E - c_2), \gamma_3(E - c_2))$ for all $b \in \hat{B}(c,E)$. By Equal Treatment(b), $R(c,E) = \left(\frac{c_1}{2} + \frac{E - c_2}{2}, c_2 - \frac{c_1}{2} + \frac{E - c_2}{2}\right) = CD(c,E)$. ■

Figure 3 provides graphical intuition for Proposition 1.

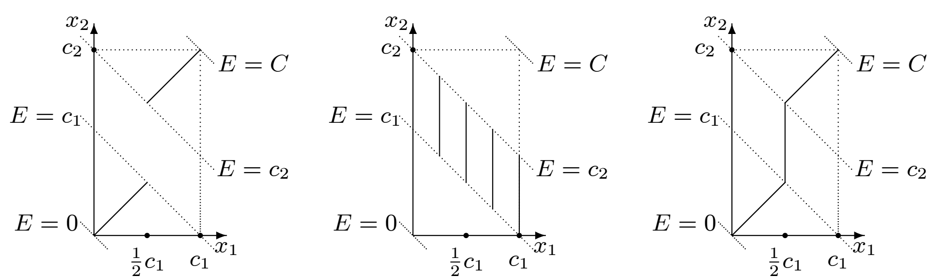

Figure 3: left panel: Equal Treatment(b) requires the path of awards to be a 45° line between the points $(0,0)$ and $\left(\frac{c_1}{2}, \frac{c_1}{2}\right)$ and again a 45° line between the points $\left(\frac{c_1}{2}, c_2 - \frac{c_1}{2}\right)$ and (c_1, c_2); middle panel: Priority(b) requires the path of awards to be a vertical line between $E = c_1$ and $E = c_2$. Different possible paths of awards are depicted; right panel: Equal Treatment(b) and Priority(b) together characterize Concede-and-Divide.

Note that (i) Equal Treatment(m) and Equal Treatment(SLB) and (ii) Priority(m) and Priority(SLB) are identical properties in the two-claimant case. Furthermore, remark that we can deduce from the left panel and the middle panel of Figure 3 that Equal Treatment(b) and Priority(b) are self-dual properties for all $b \in \hat{B}(c,E)$ in the two-claimant case.

6 The n-claimant case

Full domain of conflicting claims problems

For our analysis of the n-claimant case ($n > 2$), we focus in this section on the axioms Equal Treatment(b) and Priority(b) defined for $b = m(c,E)$ or $b = SLB(c,E)$. Other lower bounds are briefly discussed in the next section.

Whereas Equal Treatment(m) and Equal Treatment(SLB) are identical, self-dual properties in the case of two claimants, they turn out to be dual properties in general. A similar observation holds for Priority(m) and Priority(SLB). We omit the proof of this remark for reasons of parsimony.

Remark: *Equal Treatment(m) and Equal Treatment(SLB) are dual properties. Priority(m) and Priority(SLB) are dual properties.*

We believe that this remark reveals that the minimal rights lower bound and the secured lower bound, although very different in spirit and seemingly unrelated, are nevertheless connected with each other. Indeed, in the spirit of the analysis of Thomson and Yeh (2008), we can define two operators on the space of lower bounds, i.e. mappings that associate with each lower bound another lower bound. These operators are (i) $b'(c,E) \equiv \frac{1}{n}\min\{E, c - b(c, C-E)\}$ and (ii) $b''(c,E) \equiv \frac{1}{n}\min\{E, b(c,C) - b(c,C-E)\}$. The reader can check that in both cases it holds that if $b(c,E) = m(c,E)$, then $b'(c,E) = b''(c,E) = SLB(c,E)$ (but not vice versa, i.e. the above operators composed to themselves do not yield the identity).

We now obtain the following impossibility result on the full domain $\mathcal{C}$.

Proposition 2 *There does not exist a rule R that satisfies Equal Treatment(m) for all $(c,E) \in \mathcal{C}$. There does not exist a rule R that satisfies Equal Treatment(SLB) for all $(c,E) \in \mathcal{C}$.*

Proof. We give the proof for Equal Treatment(SLB). The proof for Equal Treatment(m) follows by duality. Suppose there exists a rule R that satisfies Equal Treatment(SLB) for all $(c,E) \in \mathcal{C}$. Let $N \equiv \{1,2,3\}$. Consider the following claims vector $c = (c_1, c_2, c_3)$ where $0 < c_1 < c_2 \leq c_3$ and the following claims problems $(c,0)$, (c,c_1), (c,c_3) and $(c, c_1+c_2+c_3)$. By non-negativity, $R(c,0) = (0,0,0)$ and by claims boundedness, $R(c, c_1+c_2+c_3) = (c_1,c_2,c_3)$. We have that $SLB(c,0) = (0,0,0)$, $SLB(c,c_1) = \left(\frac{c_1}{3}, \frac{c_1}{3}, \frac{c_1}{3}\right)$ and $SLB(c,c_3) = SLB(c, c_1+c_2+c_3) = \left(\frac{c_1}{3}, \frac{c_2}{3}, \frac{c_3}{3}\right)$. Hence, $\Delta SLB(c,0,c_1) = \left(\frac{c_1}{3}, \frac{c_1}{3}, \frac{c_1}{3}\right)$ and $\Delta SLB(c, c_3, c_1+c_2+c_3) = (0,0,0)$. Focussing on claimant 1, we must have that $R_1(c, c_1+c_2+c_3) = c_1 = \frac{c_1}{3} + \frac{c_1+c_2}{3}$ which implies that $c_1 = c_2$, a contradiction. ■

In contrast, many rules satisfy Priority(m) and Priority(SLB) on the full domain $\mathcal{C}$. For example, the Lexicographic rule (the rule distributing awards first to the highest claimant, then to the second highest claimant and so on, subject to nobody receiving more than her claim) is the most inegalitarian rule that satisfies Resource Monotonicity and Priority(SLB), i.e. the awards' distribution of the Lexicographic rule is Lorenz dominated by the awards' distribution of any other rule satisfying both axioms.

Restricted domains of conflicting claims problems

We restrict the analysis to two meaningful and dual subsets of conflicting claims problems. Denote $\tilde{\mathcal{C}} \subset \mathcal{C}$ the collection of all conflicting claims problems for which $E \leq c_n$, i.e. the collection of all conflicting claims problems for which there is at least one claimant who claims the total amount to be divided. Denote $\tilde{\mathcal{C}}^d \subset \mathcal{C}$ the dual collection of all conflicting claims problems for which $C - c_n \leq E$, i.e. the collection of all conflicting claims problems for which at least one claimant has a positive minimal right.

The well-known Minimal overlap rule, defined by O'Neill (1982), plays a central role in what follows. For the definition, we follow Thomson (2003). Instead of thinking of claims abstractly, imagine that the amount to divide is composed of individual and distinct units and that each claim is on specific units. Then, distribute claims over these units so as to maximize the fraction of the estate claimed by exactly one claimant, and subject to that, so as to maximize the fraction claimed by exactly two claimants, and so on. Finally, for each unit separately, apply equal division among all claimants claiming it.

Minimal overlap rule, MO. For all $(c, E) \in \mathcal{C}$, claims on units are arranged so that the number of units claimed by exactly $k+1$ claimants is maximized, given that the number of units claimed by k claimants is maximized, for $k = 1, \ldots, n-1$. Then, for each unit, equal division prevails among all claimants claiming it. Each claimant collects the partial compensations assigned to her for each of the units that she claims.

We obtain the following characterization result on the restricted domain $\tilde{\mathcal{C}}$.

Proposition 3 *For all $(c, E) \in \tilde{\mathcal{C}}$, a rule R satisfies Equal Treatment(SLB) and Priority(SLB) if and only if it is the Minimal Overlap rule.*

Proof. We follow Chun and Thomson (2005) for the definition of the Minimal Overlap rule for $(c, E) \in \tilde{\mathcal{C}}$.

Let $c_0 = 0$ and let $c_{k^*} < E \leq c_{k^*+1} \leq c_n$ with $k^* \in \{0, 1, \ldots, n-1\}$. Then $MO_i(c, E) = \frac{c_1}{n} + \frac{c_2 - c_1}{n-1} + \ldots + \frac{c_i - c_{i-1}}{n-i+1}$ for all $i = 1, \ldots, k^*$ and $MO_j(c, E) = MO_{k^*}(c, E) + \frac{E - c_{k^*}}{n-k^*}$ for all $j = k^* + 1, \ldots, n$.

It is straightforward to show that MO satisfies Equal Treatment(SLB) and Priority(SLB). The result follows when noting that

$$\Delta SLB(c, E, c_{k^*+1}) = \Delta MO(c, E, c_{k^*+1}) = \left(\underbrace{0, \ldots, 0}_{k^* \text{ times}}, \underbrace{\frac{c_{k^*+1} - E}{n - k^*}, \ldots, \frac{c_{k^*+1} - E}{n - k^*}}_{n-k^* \text{ times}} \right).$$

Conversely, let R be a rule satisfying Equal Treatment(SLB) and Priority(SLB). Since R satisfies non-negativity, $R(c, 0) = (0, \ldots, 0)$. First, let $E \leq c_1$, so

$\Delta SLB(c,0,E)=(\frac{E}{n},\ldots,\frac{E}{n})$. By Equal Treatment$(SLB)$, $R(c,E)=\left(0+\frac{E}{n},\ldots,0+\frac{E}{n}\right)=MO(c,E)$. Also $R(c,c_1)=\left(\frac{c_1}{2},\frac{c_1}{2}\right)$. Second, let $c_1<E\leq c_2$, so $\Delta SLB(c,c_1,E)=\left(0,\frac{E-c_1}{n-1},\ldots,\frac{E-c_1}{n-1}\right)$ and $\tilde{N}=\{1\}$. By Priority(SLB), $R_1(c,E)=R_1(c,c_1)=\frac{c_1}{2}=MO_1(c,E)$. By Equal Treatment$(SLB)$, $R_i(c,E)=\frac{c_1}{2}+\frac{E-c_1}{n-1}=MO_i(c,E)$ for all $i\in\{2,\ldots,n\}$. Also $R(c,c_2)=\left(\frac{c_1}{2},\frac{c_1}{2}+\frac{c_2-c_1}{n-1},\ldots,\frac{c_1}{2}+\frac{c_2-c_1}{n-1}\right)$. Third, let $c_2<E\leq c_3$, so $\Delta SLB(c,c_2,E)=\left(0,0,\frac{E-c_2}{n-2},\ldots,\frac{E-c_2}{n-2}\right)$ and $\tilde{N}=\{1,2\}$. By Priority(SLB), $R_1(c,E)=R_1(c,c_2)=\frac{c_1}{2}=MO_1(c,E)$ and $R_2(c,E)=R_2(c,c_2)=\frac{c_1}{2}+\frac{c_2-c_1}{n-1}=MO_2(c,E)$. By Equal Treatment$(SLB)$, $R_i(c,E)=\frac{c_1}{2}+\frac{c_2-c_1}{n-1}+\frac{E-c_2}{n-2}=MO_i(c,E)$ for all $i\in\{3,\ldots,n\}$. In general, let $c_{k^*}<E\leq c_{k^*+1}\leq c_n$ with $k^*\in\{0,1,\ldots,n-1\}$, so $\Delta SLB(c,c_{k^*},E)=\left(0,\ldots,0,\frac{E-c_{k^*}}{n-k^*},\ldots,\frac{E-c_{k^*}}{n-k^*}\right)$. By Priority$(SLB)$, $R_i(c,E)=R_i(c,c_{k^*})=\frac{c_1}{n}+\frac{c_2-c_1}{n-1}+\ldots+\frac{c_i-c_{i-1}}{n-i+1}=MO_i(c,E)$ for all $i=1,\ldots,k^*$. By Equal Treatment(SLB), $R_j(c,E)=MO_{k^*}(c,E)+\frac{E-c_{k^*}}{n-k^*}=MO_j(c,E)$ for all $j=k^*+1,\ldots,n$. ■

Note that for claims problems in $\tilde{\mathcal{C}}$, at least one claimant will experience a strictly positive change in her secured lower bound when the amount to divide increases. In contrast, for claims problems in $\mathcal{C}\backslash\tilde{\mathcal{C}}$, the secured lower bound of every claimant will remain unchanged (at $\frac{c_i}{n}$ for all $i\in N$) when the amount to divide increases. In other words, changes in the secured lower bound discriminate between claimants in a significant way only for claims problems in $\tilde{\mathcal{C}}$. Therefore, it is intelligible that Proposition 3 holds over the restricted domain $\tilde{\mathcal{C}}$.

The Minimal Overlap rule is not a self-dual rule. Therefore, let $MO^d(c,E)=c-MO(c,C-E)$ be the dual of the Minimal Overlap rule. We conclude this section with the dual characterization result on the dual restricted domain $\tilde{\mathcal{C}}^d$.

Corollary *For all $(c,E)\in\tilde{\mathcal{C}}^d$, a rule R satisfies Priority(m) and Equal Treatment(m) if and only if it is the dual of the Minimal Overlap rule.*

Again, note that for claims problems in $\tilde{\mathcal{C}}^d$, at least one claimant will experience a strictly positive change in her minimal rights lower bound when the amount to divide increases. In contrast, for claims problems in $\mathcal{C}\backslash\tilde{\mathcal{C}}^d$, the minimal rights lower bound of every claimant will remain unchanged (at 0 for all $i \in N$) when the amount to divide increases. In other words, changes in the minimal rights lower bound discriminate between claimants in a significant way only for claims problems in $\tilde{\mathcal{C}}^d$. Therefore, it is intelligible that the Corollary holds over the restricted domain $\tilde{\mathcal{C}}^d$.

7 Epilogue

From proposition 2, the reader might wonder whether there exist lower bounds for which the axioms Equal Treatment and Priority, defined over the lower bound under consideration, together characterize a unique rule R on the full domain of claims problems $\mathcal{C}$.

The answer to this question is clearly affirmative. To give an intuitive explanation for this result, consider the set of rules, denoted by $\mathcal{R}$, that have the following property: for any amount of resources E, an infinitesimally small increment ε of resources is (i) either shared equally among all claimants or (ii) some claimants are excluded (their awards do not change), while the other claimants share ε equally. Many well known rules belong to $\mathcal{R}$. Without giving formal definitions, we mention the Increasing-Constant-Increasing (ICI) and Constant-Increasing-Constant (CIC) families of rules of Thomson (2008), Piniles' rule and the Constrained Egalitarian rule of Chun, Schummer and Thomson (2001). As a notable exception, the Proportional rule does not belong to $\mathcal{R}$.

First, it is a straightforward result that, for any $R \in \mathcal{R}$, it holds that Equal Treatment(R) and Priority(R) together characterize R. Second, for any $R \in \mathcal{R}$, construct a lower bound b^R such that b^R increases equally among all claimants when (i) holds or b^R remains constant for those claimants whose awards do not change and increases equally for the other claimants when (ii) holds. For example, for all $(c, E) \in \mathcal{C}$, let $b^R(c, E) \equiv kR(c, E)$ for any $0 < k \leq 1$. Then it also holds that Equal Treatment(b^R) and Priority(b^R) together characterize R, explaining the result. However, a priori, it is not clear whether such a lower bound embodies a meaningful fairness condition. This observation reconfirms why we focus mainly on the minimal rights lower bound and the secured lower bound in this article.

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