

2010/44



Multivariate concave and convex stochastic dominance

Michel Denuit, Louis Eeckhoudt,
Ilia Tsetlin and Robert L. Winkler



CORE

The logo for the Center for Operations Research and Econometrics (CORE) features the word "CORE" in a bold, black, sans-serif font. A thin, light blue arc curves over the letters, starting from the top left of the 'C' and ending at the top right of the 'E'.

DISCUSSION PAPER

Center for Operations Research
and Econometrics

Voie du Roman Pays, 34
B-1348 Louvain-la-Neuve
Belgium

<http://www.uclouvain.be/core>

CORE DISCUSSION PAPER
2010/44

Multivariate concave and convex stochastic dominance

Michel DENUIT¹, Louis EECKHOUDT²,
Ilia TSETLIN³ and Robert L. WINKLER⁴

July 2010

Abstract

Stochastic dominance permits a partial ordering of alternatives (probability distributions on consequences) based only on partial information about a decision maker's utility function. Univariate stochastic dominance has been widely studied and applied, with general agreement on classes of utility functions for dominance of different degrees. Extensions to the multivariate case have received less attention and have used different classes of utility functions, some of which require strong assumptions about utility. We investigate multivariate stochastic dominance using a class of utility functions that is consistent with a basic preference assumption, can be related to well-known characteristics of utility, and is a natural extension of the stochastic order typically used in the univariate case. These utility functions are multivariate risk averse, and reversing the preference assumption allows us to investigate stochastic dominance for utility functions that are multivariate risk seeking. We provide insight into these two contrasting forms of stochastic dominance, develop some criteria to compare probability distributions (hence alternatives) via multivariate stochastic dominance, and illustrate how this dominance could be used in practice to identify inferior alternatives. Connections between our approach and dominance using different stochastic orders are discussed.

Subject classifications: decision analysis: multiple criteria, risk; group decisions; utility/preference: multiattribute utility, stochastic dominance, stochastic orders

¹ Université catholique de Louvain, Institut des Sciences Actuarielles & Institut de Statistique, B-1348 Louvain-la-Neuve, Belgium. E-mail: michel.denuit@uclouvain.be

² IESEG School of Management, LEM, Université Catholique de Lille, France; Université catholique de Louvain, CORE, B-1348 Louvain-la-Neuve, Belgium. E-mail: eeckhoudt@fucam.ac.be

³ INSEAD, Singapore. E-mail: ilia.tsetlin@insead.edu

⁴ Fuqua School of Business, Duke University, Durham, USA. E-mail: rwinkler@duke.edu

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1. Introduction

One of the big challenges in decision analysis is the assessment of a decision maker's utility function. To the extent that the alternatives under consideration in a decision-making problem can be partially ordered based on less-than-full information about the utility function, the problem can be simplified somewhat by eliminating dominated alternatives. At the same time, partial orders can help in the creation of alternatives by providing an indication of the types of strategies that might be most promising. Stochastic dominance has been studied extensively in the univariate case, particularly in the finance and economics literature; early papers are Hadar & Russell (1969) and Hanoch & Levy (1969). For example, assuming that utility for money is increasing and concave can simplify many problems in finance and economics.

Moreover, stochastic dominance can be helpful not just in individual decision making, but perhaps even more so in group decision making, where the challenge of utility assessment is amplified by divergent preferences. Even though the group members can be expected to have different utility functions, these utility functions might share some common characteristics. Thus, if an alternative can be eliminated based on an individual's utility function being risk averse, it can be eliminated in group decision making if each member of the group is risk averse, even though the degree of risk aversion may vary considerably among the group.

Multiattribute consequences make the assessment of utility even more difficult, and extensions to multivariate stochastic dominance are tricky because there are many multivariate stochastic orders (Denuit et al. 1999, Müller & Stoyan 2002, Shaked & Shantikumar 2007, Denuit & Mesfioui 2010) on which the dominance can be based. Hazen (1986) investigates multivariate stochastic dominance when simple forms of utility independence (Keeney & Raiffa 1976) can be assumed. If utility independence cannot be assumed, the potential benefits of stochastic dominance are even greater. Studies of multivariate stochastic dominance include Levy & Paroush (1974), Levhari et al. (1975), Mosler (1984), Scarsini (1988), and Denuit & Eeckhoudt (2010). In this paper we use a stochastic order that is consistent with a basic preference assumption, can be related to characteristics such as risk aversion and correlation

aversion, and is a natural extension of the standard order typically used for univariate stochastic dominance. We also consider a stochastic order that is consistent with characteristics such as risk taking and correlation loving by reversing the basic preference assumption.

The objective of this paper is to study multivariate stochastic dominance for the above-mentioned stochastic orders. In §2, we define these stochastic orders, which form the basis for what we call n th-degree multivariate concave and convex stochastic dominance. We show a connection with a preference for combining good with bad in the concave case and with the opposite preference for combining good with good and bad with bad in the convex case. Then, we extend the concept of n th-degree risk to the multivariate case and consider infinite-degree concave and convex stochastic dominance, which can be related to utility functions that are mixtures of multiattribute exponential utilities. In §3, we develop some ways to facilitate the comparison of alternatives via multivariate stochastic dominance, focusing on the impact of background risk, on dominance results when the joint probability distribution for the attributes is multivariate normal, and on eliminating alternatives from consideration by comparing an alternative with a mixture of other alternatives. A simple hypothetical example is presented in §4, and in §5 we compare our multivariate stochastic dominance with dominance based on another family of stochastic orders possessing some interesting similarities and differences. A brief summary and concluding comments are given in §6.

2. Multivariate Stochastic Dominance and Preferences for Combining Good with Bad or Good with Good and Bad with Bad

2.1. Multivariate concave and convex stochastic dominance

We begin by defining some notation. A random vector is denoted by a tilde, $\tilde{\mathbf{x}}$, and $\mathbf{0}$ is a vector of zeroes. For two N -dimensional vectors \mathbf{x} and \mathbf{y} , $\mathbf{x} > \mathbf{y}$ if $x_j > y_j$ for $j = 1, \dots, N$, and $\mathbf{x} \geq \mathbf{y}$ if $x_j \geq y_j$ for all j and $\mathbf{x} \neq \mathbf{y}$. Also, $\mathbf{x} + \mathbf{y}$ denotes the component-wise sum, $(x_1 + y_1, \dots, x_N + y_N)$. We let $[\tilde{\mathbf{x}}, \tilde{\mathbf{y}}]$ denote a lottery with equal chances of getting $\tilde{\mathbf{x}}$ or $\tilde{\mathbf{y}}$.

Next, we formalize the notion of alternating signs for the partial derivatives of u , a multiattribute

utility function for a vector of N attributes.

DEFINITION 1. $\underline{\mathbb{U}}_n^N = \left\{ u \mid (-1)^{k-1} \partial^k u(\mathbf{x}) / \partial x_{i_1} \cdots \partial x_{i_k} \geq 0 \text{ for } k = 1, \dots, n \text{ and any } i_j \in \{1, \dots, N\}, j = 1, \dots, k \right\}$.

$\underline{\mathbb{U}}_n^N$ consists of all N -dimensional real-valued functions for which all partial derivatives of a given degree up to degree n have the same sign, and that sign alternates, being positive for odd degrees and negative for even degrees. Observe that if $u \in \underline{\mathbb{U}}_n^N$, then $u \in \underline{\mathbb{U}}_k^N$ for any $k < n$. Also, if $u \in \underline{\mathbb{U}}_n^N$, then for any $k < n$ and $i_j \in \{1, \dots, N\}, j = 1, \dots, k$, $(-1)^k \partial^k u(\mathbf{x}) / \partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k} \in \underline{\mathbb{U}}_{n-k}^N$. Now we use $\underline{\mathbb{U}}_n^N$ to define multivariate concave stochastic dominance.

DEFINITION 2. For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of n^{th} -degree concave stochastic dominance if $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$ for all $u \in \underline{\mathbb{U}}_n^N$, u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

THEOREM 1. Let $\tilde{\mathbf{x}}_m, \tilde{\mathbf{y}}_m, \tilde{\mathbf{x}}_n$, and $\tilde{\mathbf{y}}_n$ be mutually independent N -dimensional random vectors with $\tilde{\mathbf{x}}_i$ dominating $\tilde{\mathbf{y}}_i$ in the sense of i^{th} -degree concave stochastic dominance, $i = m, n$. Then $[\tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n]$ dominates $[\tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n]$ in the sense of $(n+m)^{\text{th}}$ -degree concave stochastic dominance.

Theorem 1 shows that concave stochastic dominance from Definition 2 is consistent with a preference for combining good with bad (up to degree n), where good and bad are understood in terms of lower-degree concave stochastic dominance. What if a decision maker prefers to combine good with good and bad with bad, as opposed to combining good with bad? Next we define convex stochastic dominance, which is consistent with such preferences.

DEFINITION 1*. $\bar{\mathbb{U}}_n^N = \left\{ u \mid \partial^k u(\mathbf{x}) / \partial x_{i_1} \cdots \partial x_{i_k} \geq 0 \text{ for } k = 1, \dots, n \text{ and any } i_j \in \{1, \dots, N\}, j = 1, \dots, k \right\}$.

$\bar{\mathbb{U}}_n^N$, consisting of all N -dimensional real-valued functions for which all partial derivatives of degree up to n are positive, is called $\mathbb{U}_{s\text{-dirconv}}$ by Denuit & Mesfioui (2010) and forms the basis for the s -increasing directionally convex order. Similar to $\underline{\mathbb{U}}_n^N$, if $u \in \bar{\mathbb{U}}_n^N$, then $u \in \bar{\mathbb{U}}_k^N$ for any $k < n$. Also, if $u \in \bar{\mathbb{U}}_n^N$, then for any $k < n$ and $i_j \in \{1, \dots, N\}, j = 1, \dots, k$, $\partial^k u(\mathbf{x}) / \partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k} \in \bar{\mathbb{U}}_{n-k}^N$.

DEFINITION 2*. For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of n^{th} -degree convex stochastic dominance if $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$ for all $u \in \bar{\mathbb{U}}_n^N$, u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

THEOREM 1*. Let $\tilde{\mathbf{x}}_m, \tilde{\mathbf{y}}_m, \tilde{\mathbf{x}}_n$, and $\tilde{\mathbf{y}}_n$ be N -dimensional independent random vectors with $\tilde{\mathbf{x}}_i$ dominating $\tilde{\mathbf{y}}_i$ in the sense of i^{th} -degree convex stochastic dominance, $i = m, n$. Then $[\tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n]$ dominates $[\tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n]$ in the sense of $(n + m)^{\text{th}}$ - degree convex stochastic dominance.

Definition 2 extends the standard definition of univariate stochastic dominance to the multivariate case. As Theorem 1 shows, it preserves a preference for combining good with bad (Eeckhoudt & Schlesinger 2006, Eeckhoudt et al. 2009). Definition 2* and Theorem 1* develop similar orderings based on the opposite preference for combining good with good and bad with bad, and show the connection between convex and concave stochastic dominance that follows from the fact that $u(\mathbf{x}) \in \underline{\mathbb{U}}_n^N$ if and only if $-u(\underline{\mathbf{x}} + \bar{\mathbf{x}} - \mathbf{x}) \in \bar{\mathbb{U}}_n^N$.

REMARK. The multivariate convex stochastic dominance in Definition 2* is different from what Fishburn (1974) calls convex stochastic dominance. Fishburn's usage of "convex" does not relate to the utility function. Instead, it refers to dominance results for convex combinations, or mixtures, of probability distributions in the univariate case, which we will extend to the multivariate case in §3.3 and use to eliminate alternatives in decision-making problems in §4. To clarify the distinction, we will use the term "mixture dominance" when referring to the type of stochastic dominance developed by Fishburn (1974, 1978). In contrast, our multivariate convex stochastic dominance can be thought of as "risk-taking stochastic dominance" because $u \in \bar{\mathbb{U}}_n^N$ for any $n > 1$ implies that u is multivariate risk seeking in the sense of Richard (1975). Not only do we have $\partial^2 u / \partial x_i^2 \geq 0$ for $i = 1, \dots, N$, but the underlying condition of preferring to combine good with good and bad with bad can be viewed as a form of risk taking. Similarly, our multivariate concave stochastic dominance from Definition 2 can be thought of as "risk-averse stochastic dominance" because $u \in \underline{\mathbb{U}}_n^N$ for any $n > 1$ means that u is multivariate risk averse

(Richard 1975). Also, the preference for combining good with bad associated with $u \in \underline{\mathbb{U}}_n^N$ implies that u is correlation averse (Epstein & Tanny 1980, Eeckhoudt et al. 2007, Denuit et al. 2010), which can be viewed as a form of risk aversion. Similarly, the preference for combining good with good and bad with bad associated with $u \in \overline{\mathbb{U}}_n^N$ implies that u is correlation loving, a form of risk taking.

2.2. The notion of n^{th} -degree risk in the multivariate case

By Definition 2 (2*), concave (convex) stochastic dominance of degree n implies stochastic dominance of any higher degree. To isolate a higher-degree effect in the univariate case, Ekern (1980) introduced the concept of n^{th} -degree risk. This subsection extends that concept to the multivariate case and relates it to concave and convex stochastic dominance.

DEFINITION 3. For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $-\infty < \underline{\mathbf{x}} < \bar{\mathbf{x}} < \infty$, $\tilde{\mathbf{y}}$ has more n^{th} -degree risk than $\tilde{\mathbf{x}}$ if $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$ for all u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$ such that

$$(-1)^{n-1} \partial^n u(\mathbf{x}) / \partial x_{i_1} \cdots \partial x_{i_n} \geq 0 \text{ for any } i_j \in \{1, \dots, N\}, j = 1, \dots, n.$$

THEOREM 2. *The random vector $\tilde{\mathbf{y}}$ has more n^{th} -degree risk than the random vector $\tilde{\mathbf{x}}$ if and only if*

- (i) $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of n^{th} -degree concave stochastic dominance, and
- (ii) the k^{th} moments of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are identical for $k = 1, \dots, n-1$: $\mathbb{E}[\tilde{x}_{i_1} \tilde{x}_{i_2} \cdots \tilde{x}_{i_k}] = \mathbb{E}[\tilde{y}_{i_1} \tilde{y}_{i_2} \cdots \tilde{y}_{i_k}]$ for any $i_j \in \{1, \dots, N\}, j = 1, \dots, k$.

REMARK. In the univariate case, Ekern (1980) defines a person as being “ n^{th} -degree risk averse” if the n^{th} derivative of her utility function is positive (negative) when n is odd (even). Our interpretation of multivariate concave stochastic dominance as risk-averse stochastic dominance is consistent with the extension of the notion of being n^{th} -degree risk averse to the multivariate case.

THEOREM 2*. *The random vector $\tilde{\mathbf{y}}$ has more n^{th} -degree risk than the random vector $\tilde{\mathbf{x}}$ if and only if*

- (i) $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ ($\tilde{\mathbf{y}}$ dominates $\tilde{\mathbf{x}}$) in the sense of n^{th} -degree convex stochastic dominance when n is odd (even), and
- (ii) the k^{th} moments of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are identical for $k = 1, \dots, n-1$.

The proof of Theorem 2* is similar to the proof of Theorem 2.

COROLLARY TO THEOREMS 2 AND 2*. *If n is odd (even) and the k^{th} moments of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are identical for $k=1, \dots, n-1$, then $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of n^{th} -degree concave stochastic dominance if and only if $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ ($\tilde{\mathbf{y}}$ dominates $\tilde{\mathbf{x}}$) in the sense of n^{th} -degree convex stochastic dominance.*

Thus, if all moments of degree less than n are identical, convex dominance goes along with higher n^{th} moments for both odd and even n . With concave dominance, this holds only for odd n . For even n , concave dominance goes along with lower n^{th} moments. These results relate stochastic dominance to ordering by moments, in the sense that convex dominance likes all moments to be higher, whereas concave dominance likes odd moments to be higher and even moments to be lower.

2.3. Infinite-degree dominance

Now we explore what emerges if a preference between combining good with bad, or combining good with good and bad with bad, holds for any n . In this case dominance relations are defined via $\underline{\mathbb{U}}_{\infty}^N$ and $\overline{\mathbb{U}}_{\infty}^N$, which extend $\underline{\mathbb{U}}_n^N$ and $\overline{\mathbb{U}}_n^N$.

DEFINITION 4. $\underline{\mathbb{U}}_{\infty}^N = \left\{ u \mid (-1)^{k-1} \partial^k u(\mathbf{x}) / \partial x_{i_1} \cdots \partial x_{i_k} \geq 0 \text{ for } k=1, 2, \dots \text{ and any } i_j \in \{1, \dots, N\}, j=1, \dots, k \right\}$,

and $\overline{\mathbb{U}}_{\infty}^N = \left\{ u \mid \partial^k u(\mathbf{x}) / \partial x_{i_1} \cdots \partial x_{i_k} \geq 0 \text{ for } k=1, 2, \dots \text{ and any } i_j \in \{1, \dots, N\}, j=1, \dots, k \right\}$.

DEFINITION 5. For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of infinite-degree concave (convex) stochastic dominance if $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$ for all $u \in \underline{\mathbb{U}}_{\infty}^N$ ($u \in \overline{\mathbb{U}}_{\infty}^N$), u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

Increasing the degree of dominance (n) restricts the set of utility functions with respect to which two random vectors are compared. Similarly, expanding the domain of definition of u (i.e., decreasing $\underline{\mathbf{x}}$ and/or increasing $\bar{\mathbf{x}}$) also restricts the set of utility functions, and thus increases the set of random vectors that can be ordered by stochastic dominance. As shown in Theorem 3, any $u \in \underline{\mathbb{U}}_{\infty}^N$, u defined on $[\underline{\mathbf{x}}, \infty)$, or $u \in \overline{\mathbb{U}}_{\infty}^N$, u defined on $(-\infty, \bar{\mathbf{x}}]$, is a mixture of multiattribute exponential utilities. Theorem 4 then shows that infinite-order dominance can be operationalized via multiattribute exponential utilities.

THEOREM 3. Consider a function $u(\mathbf{x})$, defined on $[\underline{\mathbf{x}}, \infty)$. Then $u \in \underline{\mathbb{U}}_\infty^N$ if and only if there exists a (not necessarily finite) measure F on $[\mathbf{0}, \infty)$ and constants b_1, \dots, b_N with $b_i \geq 0$, $i = 1, \dots, N$, such that

$$u(\mathbf{x}) = u(\underline{\mathbf{x}}) + \int_0^\infty \cdots \int_0^\infty \left(1 - \exp\left(-\left(r_1(x_1 - \underline{x}_1) + \dots + r_N(x_N - \underline{x}_N)\right)\right)\right) dF(r_1, \dots, r_N) + \sum_{i=1}^N b_i(x_i - \underline{x}_i). \quad (1)$$

Viewing the linear terms in (1) as limiting forms of exponential utilities (as $r_i \rightarrow 0$ with $r_j = 0$ for $j \neq i$)

and rescaling, we can express any $u \in \underline{\mathbb{U}}_\infty^N$, u defined on $[\underline{\mathbf{x}}, \infty)$, as a mixture of multiattribute exponential utilities,

$$u(\mathbf{x}) = -\int_0^\infty \cdots \int_0^\infty \exp(-r_1 x_1 - \dots - r_N x_N) dF(r_1, \dots, r_N). \quad (2)$$

Similarly, any $u \in \bar{\mathbb{U}}_\infty^N$, u defined on $(-\infty, \bar{\mathbf{x}}]$, can be expressed as

$$u(\mathbf{x}) = \int_0^\infty \cdots \int_0^\infty \exp(r_1 x_1 + \dots + r_N x_N) dF(r_1, \dots, r_N). \quad (3)$$

A proof for the concave case in Theorem 3 is given in Tsetlin & Winkler (2009), and the proof for the convex case is similar. From Theorem 3, we can state the following result without a proof.

THEOREM 4. The random vector $\tilde{\mathbf{x}}$ dominates the random vector $\tilde{\mathbf{y}}$ in the sense of infinite-degree concave stochastic dominance for u defined on $[\underline{\mathbf{x}}, \infty)$ if and only if $\mathbb{E}[\exp(-r_1 \tilde{y}_1 - \dots - r_N \tilde{y}_N)] \geq \mathbb{E}[\exp(-r_1 \tilde{x}_1 - \dots - r_N \tilde{x}_N)]$ for all $\mathbf{r} \in [\mathbf{0}, \infty)$, and $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of infinite-degree convex stochastic dominance for u defined on $(-\infty, \bar{\mathbf{x}}]$ if and only if $\mathbb{E}[\exp(r_1 \tilde{x}_1 + \dots + r_N \tilde{x}_N)] \geq \mathbb{E}[\exp(r_1 \tilde{y}_1 + \dots + r_N \tilde{y}_N)]$ for all $\mathbf{r} \in [\mathbf{0}, \infty)$.

Theorem 4 provides a convenient criterion for comparing multivariate probability distributions. Note that the expectations in Theorem 4 correspond to moment generating functions for distributions of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$. If we define $M_{\tilde{\mathbf{x}}}(\mathbf{r}) = \mathbb{E}[\exp(r_1 \tilde{x}_1 + \dots + r_N \tilde{x}_N)]$, then for concave stochastic dominance we need $M_{\tilde{\mathbf{x}}}(\mathbf{r}) \leq M_{\tilde{\mathbf{y}}}(\mathbf{r})$ for all $\mathbf{r} \in (-\infty, \mathbf{0}]$, and for convex stochastic dominance we need $M_{\tilde{\mathbf{x}}}(\mathbf{r}) \geq M_{\tilde{\mathbf{y}}}(\mathbf{r})$ for all $\mathbf{r} \in [\mathbf{0}, \infty)$.

REMARK. The domain of definition of u is crucial for the result stated in Theorem 4. For instance, if

$\tilde{\mathbf{x}} = (x_1, x_2) = (0.5, 0.5)$ and $\tilde{\mathbf{y}} = [(0,1), (1,0)]$, then by examining the expectations in Theorem 4 we can show that $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ by infinite-degree concave stochastic dominance for u defined on $[\underline{\mathbf{x}}, \infty)$ (e.g., on $[0, \infty)$). However, consider $u(\mathbf{x}) = x_1 + x_2 - x_1x_2$, u defined on $[0, 1]$. Theorem 4 does not apply here, and taking expectations with respect to u yields $\mathbb{E}[u(\tilde{\mathbf{x}})] = 0.75 < \mathbb{E}[u(\tilde{\mathbf{y}})] = 1$. Therefore, $\tilde{\mathbf{x}}$ does not dominate $\tilde{\mathbf{y}}$ by infinite-degree concave stochastic dominance. If we increase the upper limit of the domain of this u above $\mathbf{1}$, then $u \notin \underline{\mathbb{U}}_\infty^2$ because $\partial u(\mathbf{x}) / \partial x_i < 0$, $i = 1, 2$, when $\mathbf{x} > \mathbf{1}$. A similar situation can occur for any N , including the univariate case ($N = 1$). As noted previously, expanding the domain of definition of u restricts the set of utility functions with respect to which random vectors are compared. In the example, the set of utility functions $u \in \underline{\mathbb{U}}_\infty^2$ defined on $[0, 1]$ is larger than the set of utility functions $u \in \underline{\mathbb{U}}_\infty^2$ defined on $[0, \infty)$. The former set includes $u(\mathbf{x}) = x_1 + x_2 - x_1x_2$, whereas the latter does not.

3. Comparing Alternatives via Multivariate Stochastic Dominance

Here we present several results that are useful for comparing alternatives according to the stochastic dominance relations from §2. In §3.1 we show conditions under which dominance orderings remain unchanged in the presence of background risk, with independence playing an important role. In §3.2 we consider the case in which the distributions of the consequences are multivariate normal. In §3.3 we use mixture dominance to show that an alternative, even if not dominated by any single alternative, can be eliminated from consideration if it is dominated by a mixture of other alternatives.

3.1. Stochastic dominance with additive and multiplicative background risk

When one faces a choice between two (or more) risky alternatives, this decision is often not made in isolation, in the sense that there are other risks that affect the decision maker but are outside of the decision maker's control. Therefore, it is important to know whether a stochastic dominance ordering established in the absence of background risk will remain the same when background risk is present.

Consider a choice between two projects, with consequences characterized by random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$. In the presence of additive background risk, represented by the random vector $\tilde{\mathbf{a}}$, we are

interested in comparing $\tilde{\mathbf{a}} + \tilde{\mathbf{x}}$ and $\tilde{\mathbf{a}} + \tilde{\mathbf{y}}$. In the presence of multiplicative background risk, represented by the random vector $\tilde{\mathbf{m}}$, the appropriate comparison is between $\tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ and $\tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}}$, where $\mathbf{m} \otimes \mathbf{x}$ denotes the component-wise product, $(m_1 x_1, \dots, m_N x_N)$. If both additive and multiplicative background risks are present, $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ and $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}}$ are compared.

THEOREM 5. *Let $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{a}},$ and $\tilde{\mathbf{m}}, \tilde{\mathbf{m}} \succeq \mathbf{0}$, be N -dimensional random vectors such that for any fixed \mathbf{a} and \mathbf{m} , $\tilde{\mathbf{x}} | \mathbf{m}, \mathbf{a}$ dominates $\tilde{\mathbf{y}} | \mathbf{m}, \mathbf{a}$ in the sense of n^{th} -degree concave (convex) stochastic dominance. Then $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}}$ in the sense of n^{th} -degree concave (convex) stochastic dominance.*

The result of Theorem 5 is quite intuitive. If $\tilde{\mathbf{x}}$ is preferred to $\tilde{\mathbf{y}}$ for each possible value of \mathbf{a} and \mathbf{m} , then $\tilde{\mathbf{x}}$ is preferred to $\tilde{\mathbf{y}}$ even if we are uncertain about the exact values of \mathbf{a} and \mathbf{m} . If the project risk is independent of the background risk, the situation is further simplified.

COROLLARY 1 TO THEOREM 5. *Let $\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{a}},$ and $\tilde{\mathbf{m}}, \tilde{\mathbf{m}} \succeq \mathbf{0}$, be N -dimensional random vectors such that $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are independent of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{m}}$. If $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of n^{th} -degree concave (convex) stochastic dominance, then $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}}$ in the sense of n^{th} -degree concave (convex) stochastic dominance.*

Thus, independent background risk preserves stochastic dominance orderings. Note that no assumption is made about the relationship between the background risks $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{m}}$; they can be dependent. The assumption of independence of the project risk and the background risk is crucial, however. If background risk is not independent of project risk, preferences with and without background risk might be the opposite (Tsetlin & Winkler 2005). For example, suppose that a manager is considering adding a new project to an existing portfolio of projects. Let $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ represent the consequences of two potential new projects, and let $\tilde{\mathbf{a}}$ represent the consequences of the existing portfolio. Even if the manager is multivariate risk averse and $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in terms of multivariate concave stochastic dominance, she might prefer the new project associated with $\tilde{\mathbf{y}}$ (i.e., prefer $\tilde{\mathbf{a}} + \tilde{\mathbf{y}}$ to $\tilde{\mathbf{a}} + \tilde{\mathbf{x}}$) if the correlations between the components of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{y}}$ are smaller than those for $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{x}}$.

Theorem 5 and its Corollary 1 can also be used to compare random vectors that are functions of other random vectors, which can be ordered by stochastic dominance. For instance, if the consequences of a particular alternative can be represented as $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ and any of the mutually independent random vectors $\tilde{\mathbf{x}}$, $\tilde{\mathbf{a}}$, and $\tilde{\mathbf{m}}$ is improved in the sense of stochastic dominance, what can we say about the resulting changes to this alternative?

COROLLARY 2 TO THEOREM 5. *Let $\tilde{\mathbf{x}}_1, \tilde{\mathbf{y}}_1, \tilde{\mathbf{x}}_2,$ and $\tilde{\mathbf{y}}_2$ be mutually independent N -dimensional independent random vectors with $\tilde{\mathbf{x}}_i$ dominating $\tilde{\mathbf{y}}_i$ in the sense of n^{th} -degree concave (convex) stochastic dominance, $i = 1, 2$. Then $\tilde{\mathbf{x}}_1 + \tilde{\mathbf{x}}_2$ dominates $\tilde{\mathbf{y}}_1 + \tilde{\mathbf{y}}_2$ in the sense of n^{th} -degree concave (convex) stochastic dominance. If $\tilde{\mathbf{x}}_1 \succeq \mathbf{0}, \tilde{\mathbf{y}}_1 \succeq \mathbf{0}, \tilde{\mathbf{x}}_2 \succeq \mathbf{0},$ and $\tilde{\mathbf{y}}_2 \succeq \mathbf{0},$ then $\tilde{\mathbf{x}}_1 \otimes \tilde{\mathbf{x}}_2$ dominates $\tilde{\mathbf{y}}_1 \otimes \tilde{\mathbf{y}}_2$ in the sense of n^{th} -degree concave (convex) stochastic dominance.*

REMARK. It might be that, e.g., $\tilde{\mathbf{x}}_1 + \tilde{\mathbf{x}}_2$ dominates $\tilde{\mathbf{y}}_1 + \tilde{\mathbf{y}}_2$ in the sense of stochastic dominance of degree lower than n . For example, consider the univariate case (i.e., $N = 1$) with $\tilde{x}_1 = 1, \tilde{x}_2 = \tilde{y}_1 = 0,$ and $\tilde{y}_2 = [-c, c]$. Then \tilde{x}_i dominates \tilde{y}_i in the sense of second-degree concave stochastic dominance for $i = 1, 2,$ but also note that \tilde{x}_1 dominates \tilde{y}_1 in the sense of first-degree stochastic dominance. In this case $\tilde{x}_1 + \tilde{x}_2 = 1,$ and $\tilde{y}_1 + \tilde{y}_2 = [-c, c]$. For $c \leq 1,$ $\tilde{x}_1 + \tilde{x}_2$ dominates $\tilde{y}_1 + \tilde{y}_2$ in the sense of first-degree stochastic dominance, but for $c > 1,$ $\tilde{x}_1 + \tilde{x}_2$ dominates $\tilde{y}_1 + \tilde{y}_2$ only in the sense of second-degree concave stochastic dominance.

Theorem 5 and its corollaries show that, e.g., adding a non-negative random vector improves a multivariate distribution in the sense of first-degree concave and convex stochastic dominance. They also imply that if a set of N variables can be divided into two stochastically independent subgroups, and one of these groups is improved in the sense of n^{th} -degree concave (convex) stochastic dominance, then the joint distribution over all N variables is improved in the sense of n^{th} -degree concave (convex) stochastic dominance. In particular, if N random variables are independent, then their joint distribution is improved in the sense of n^{th} -degree concave (convex) stochastic dominance whenever the marginal distribution of

any of the variables is improved in the sense of n^{th} -degree concave (convex) stochastic dominance.

3.2. Comparison of multivariate normal distributions via infinite-degree dominance

The multivariate normal distribution is the most commonly encountered multivariate distribution, is very tractable, and is a reasonable representation of uncertainty in many situations. Müller (2001) provides several results on the stochastic ordering of multivariate normal distributions. The expectations appearing in Theorem 4 are especially tractable in this case, and thus the comparison of two multivariate normal distributions based on infinite-degree (concave and convex) stochastic dominance criterion is greatly simplified. If the random vector $\tilde{\mathbf{x}}$ is multivariate normal with mean vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_N)$ and covariance matrix $\boldsymbol{\Sigma} = (\rho_{ij}\sigma_i\sigma_j)$, then $\mathbb{E}[\exp(r_1\tilde{x}_1 - \dots + r_N\tilde{x}_N)] = \exp(\mathbf{r}\boldsymbol{\mu}^t + (\mathbf{r}\boldsymbol{\Sigma}\mathbf{r}^t/2))$, where a superscript

t denotes transposition and $\mathbf{r}\boldsymbol{\mu}^t + (\mathbf{r}\boldsymbol{\Sigma}\mathbf{r}^t/2) = \sum_{i=1}^N r_i\mu_i + \left(\sum_{i=1}^N \sum_{j=1}^N r_i r_j \rho_{ij} \sigma_i \sigma_j / 2 \right)$. Thus, we have the following

corollary to Theorem 4.

COROLLARY TO THEOREM 4. *Let $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ be multivariate normal vectors with mean vectors $\boldsymbol{\mu}_x$ and $\boldsymbol{\mu}_y$ and covariance matrices $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma}_y$. Then $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of infinite-degree concave stochastic dominance if and only if $-\mathbf{r}\boldsymbol{\mu}_y^t + (\mathbf{r}\boldsymbol{\Sigma}_y\mathbf{r}^t/2) \geq -\mathbf{r}\boldsymbol{\mu}_x^t + (\mathbf{r}\boldsymbol{\Sigma}_x\mathbf{r}^t/2)$ for all $\mathbf{r} \in [\mathbf{0}, \infty)$, and $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of infinite-degree convex stochastic dominance if and only if $\mathbf{r}\boldsymbol{\mu}_x^t + (\mathbf{r}\boldsymbol{\Sigma}_x\mathbf{r}^t/2) \geq \mathbf{r}\boldsymbol{\mu}_y^t + (\mathbf{r}\boldsymbol{\Sigma}_y\mathbf{r}^t/2)$ for all $\mathbf{r} \in [\mathbf{0}, \infty)$.*

Thus, increasing any mean μ_i leads to stochastic dominance improvement (both concave and convex). Decreasing any correlation ρ_{ij} leads to concave (convex) stochastic dominance improvement (deterioration). Decreasing any standard deviation σ_i leads to concave (convex) stochastic dominance improvement (deterioration) if $\rho_{ij} \geq 0$ for all j . However, if $\rho_{ij} < 0$ for some j , things are more complicated. Overall, adding independent noise to attribute i leads to the increase of σ_i and to the decrease of the absolute value of correlations ρ_{ij} . Thus, increasing σ_i without changing correlations is

equivalent to adding independent noise to attribute i and then to adjusting the correlations ρ_{ij} up (if ρ_{ij} is positive) or down (if ρ_{ij} is negative). For concave (convex) stochastic dominance, adding independent noise is bad (good), and adjusting correlations up (down) is bad (good). If all correlations are positive, increasing any standard deviation leads to convex (concave) stochastic dominance improvement (deterioration). If some correlations are negative, the effect might go either way. Tsetlin & Winkler (2007) established similar confounding effects of increasing standard deviations in target-oriented situations.

3.3. Elimination by mixtures

If an alternative (represented by a random vector) is dominated by some other alternative when the decision maker's utility falls in a particular class (e.g., $u \in \underline{\mathbb{U}}_n^N$ or $\underline{\mathbb{U}}_\infty^N$ for concave stochastic dominance and $u \in \bar{\mathbb{U}}_n^N$ or $\bar{\mathbb{U}}_\infty^N$ for convex stochastic dominance), then the dominated alternative can be eliminated from further consideration, thereby simplifying the decision-making problem. Mixture dominance, developed by Fishburn (1974) as "convex stochastic dominance" for the univariate case, allows us to eliminate an alternative even if it is not dominated by any other single alternative, as long as it is dominated by a mixture of other alternatives, which is a weaker condition (Fishburn 1978). We define mixture dominance for the multivariate case and then extend Fishburn's (1978) result regarding elimination by mixtures.

DEFINITION 6. For the random vectors $\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_k$ and utility class \mathbb{U}^* , $\tilde{\mathbf{x}}_{-k} = (\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_{k-1})$ dominates

$\tilde{\mathbf{x}}_k$ in the sense of mixture dominance with respect to \mathbb{U}^* if there exists $\mathbf{p} = (p_1, \dots, p_{k-1}) \geq \mathbf{0}$,

$$\sum_{i=1}^{k-1} p_i = 1, \text{ such that } \sum_{i=1}^{k-1} p_i \mathbb{E}[u(\tilde{\mathbf{x}}_i)] \geq \mathbb{E}[u(\tilde{\mathbf{x}}_k)] \text{ for all } u \in \mathbb{U}^*.$$

From Definition 6, the mixture can be thought of as a two-step process. In the first step, an alternative (a random vector $\tilde{\mathbf{x}}_i$) is chosen from $\tilde{\mathbf{x}}_{-k}$, where p_i represents the probability of choosing $\tilde{\mathbf{x}}_i$. Then at the second step, the uncertainty about $\tilde{\mathbf{x}}_i$ is resolved. Mixture dominance means that this mixture has a higher expected utility than $\tilde{\mathbf{x}}_k$ for all $u \in \mathbb{U}^*$.

THEOREM 6. *If $\tilde{\mathbf{x}}_{-k}$ dominates $\tilde{\mathbf{x}}_k$ in the sense of mixture dominance with respect to \mathbb{U}^* , then for every $u \in \mathbb{U}^*$, there is an $i \in \{1, \dots, k-1\}$ such that $\mathbb{E}[u(\tilde{\mathbf{x}}_i)] \geq \mathbb{E}[u(\tilde{\mathbf{x}}_k)]$.*

Note that the $\tilde{\mathbf{x}}_i$ in Theorem 6 can be different for different $u \in \mathbb{U}^*$. The importance of Theorem 6 is that if $u \in \mathbb{U}^*$ and $\tilde{\mathbf{x}}_{-k}$ dominates $\tilde{\mathbf{x}}_k$ in the sense of mixture dominance with respect to the utility class of interest, then we can eliminate $\tilde{\mathbf{x}}_k$ from consideration even if none of $\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_{k-1}$ dominates $\tilde{\mathbf{x}}_k$ individually. Reducing the set of alternatives that need to be considered seriously is always helpful. Since some of the mixing probabilities can be zero, note that we can eliminate an alternative if it is dominated in the sense of mixture dominance by any subset of the other alternatives. Of course, mixture dominance with respect to $\underline{\mathbb{U}}_n^N$, $\underline{\mathbb{U}}_\infty^N$, $\bar{\mathbb{U}}_n^N$, or $\bar{\mathbb{U}}_\infty^N$ is of particular interest because it invokes concave or convex stochastic dominance and relates to a preference for combining good with bad or the opposite preference for combining good with good and bad with bad.

4. Examples

An important aspect of decision analysis is the assessment of a decision maker's (DM's) utility function, and this is especially challenging in a multiattribute context. The problem is somewhat simplified if some potential alternatives can be eliminated from consideration without having to assess the full utility function, and that is where multivariate stochastic dominance can be helpful. In this section, we present a simple hypothetical example to illustrate the concepts from §§2-3 without getting distracted by complicating details.

Suppose that a telecom company is entering a new market and deciding among different entry strategies. For simplicity, assume that the DM focuses on two attributes, x_1 (the net present value (NPV) of profits for the first five years, in millions of dollars) and x_2 (the market share in percentage terms at the end of the five-year period). To begin, it is not surprising to find that the DM prefers more of each of these attributes to less. For example, she prefers $(x_1, x_2) = (300, 40)$ to $(200, 30)$. This is simple first-degree multivariate stochastic dominance.

Next, if the DM concludes that she is risk averse with respect to NPV, then $(250, 30)$ would be preferred to $[(300, 30), (200, 30)]$, a risky alternative that yields $(300, 30)$ or $(200, 30)$ with equal probabilities. Similarly, if she is risk averse with respect to market share, then $(250, 35)$ would be preferred to $[(250, 30), (250, 40)]$. These two choices are consistent with second-degree concave stochastic dominance but not sufficient to indicate that she would always want to behave in accordance with second-degree concave stochastic dominance. For example, the risk aversion with respect to NPV and market share is not sufficient to dictate her choice between the two risky alternatives $[(300, 40), (200, 30)]$ and $[(300, 30), (200, 40)]$. She states a preference for the latter and decides after some thought that she is, in general, correlation averse. Thus, her preferences are consistent with second-degree concave stochastic dominance.

In practice, most comparisons between competing alternatives are not as clear-cut as the above examples. In other words, once obviously inferior alternatives have been eliminated, it may be hard to find many cases where one alternative dominates another. However, by looking at three or more alternatives, we may still be able to eliminate alternatives via mixture dominance, as discussed in §3.3.

For a simple example, consider the choice among three alternatives: $(300, 30)$, $(200, 40)$, and $[(300, 40), (200, 30)]$. The first alternative gives a higher NPV, the second alternative gives a higher market share, and the third alternative is risky, with equal chances of either the high NPV and the high market share or the low NPV and the low market share. Note that a 50-50 mixture of the first two alternatives, $[(300, 30), (200, 40)]$, dominates the third alternative by second-degree concave stochastic dominance, consistent with the DM's preference for combining good with bad. By Theorem 6, then, we can eliminate the third alternative.

Of course, if the DM has the opposite preference for combining good with good and bad with bad, then convex stochastic dominance is relevant and the second-degree dominance orderings in the above examples will be reversed. For example, $[(300, 30), (200, 30)]$ dominates $(250, 30)$ by second-degree convex stochastic dominance. Similarly, $[(300, 40), (200, 30)]$, dominates $[(300, 30), (200, 40)]$ by

second-degree convex stochastic dominance, reflecting the fact that the DM is correlation loving.

The above comparisons among alternatives might have to be made in the presence of background risk. For example, the DM might be uncertain about the financial results of other ongoing projects of the telecom company, implying additive background risk with respect to the first attribute (NPV). She might also be uncertain about competitors' moves, which could translate into additive background risk with respect to the second attribute (market share). Finally, suppose that the company operates internationally and wants to express its NPV in another currency. In this case, the appropriate exchange rate, in the absence of hedging, would operate as multiplicative background risk with respect to the first attribute. As shown in §3.1, if the consequences of each alternative are independent of the background risk, then any stochastic dominance orderings are preserved and any resulting elimination of alternatives remains optimal under such background risk.

5. Comparisons with Other Multivariate Stochastic Orders

Many multivariate stochastic orders have been studied, and the appropriate order upon which to base multivariate stochastic dominance is not as obvious as it is in the univariate case. Once we move from $N = 1$ to $N > 1$, the relationship among the attributes complicates matters both in terms of the joint probability distribution and in terms of the utility function. We focus here on multivariate s -increasing orders, a family of stochastic orders for which some interesting connections and comparisons with our multivariate concave and convex stochastic dominance can be drawn. This helps to highlight potential advantages and disadvantages of our approach.

We begin by presenting the multivariate s -increasing concave order, where $s = (s_1, \dots, s_N)$ is a vector of positive integers, and defining stochastic dominance in terms of this order. This is a natural generalization of the bivariate (s_1, s_2) -increasing concave orders introduced by Denuit et al. (1999) and studied by Denuit & Eeckhoudt (2010) and Denuit et al. (2010).

DEFINITION 7.

$$\mathbb{U}_{s\text{-}icv}^N = \left\{ u \mid (-1)^{\sum_{i=1}^N k_i - 1} \partial^{\sum_{i=1}^N k_i} u(\mathbf{x}) / \partial x_1^{k_1} \dots \partial x_N^{k_N} \geq 0 \text{ for } k_i = 0, 1, \dots, s_i, i = 1, \dots, N, \sum_{i=1}^N k_i \geq 1 \right\}.$$

DEFINITION 8. For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate s -increasing concave order if $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$ for all $u \in \mathbb{U}_{s-icv}^N$, u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

If $s_1 = \dots = s_N = s$, we say that the order is an s -increasing concave order. Special cases of this are the lower orthant order when $s = 1$ and the lower orthant concave order when $s = 2$ (Mosler 1984). With the lower orthant order, for example, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ if $P(\tilde{\mathbf{x}} \leq \mathbf{c}) \leq P(\tilde{\mathbf{y}} \leq \mathbf{c})$ for all $\mathbf{c} \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

Our multivariate concave stochastic dominance, based on $\underline{\mathbb{U}}_n^N$, has a convex counterpart, based on $\bar{\mathbb{U}}_n^N$. Similarly, \mathbb{U}_{s-icv}^N and dominance in terms of the s -increasing concave order have convex counterparts (Denuit and Mesfioui 2010).

DEFINITION 7*. $\mathbb{U}_{s-icx}^N = \left\{ u \left| \partial^{\sum_{i=1}^N k_i} u(\mathbf{x}) / \partial x_1^{k_1} \dots \partial x_N^{k_N} \geq 0 \text{ for } k_i = 0, 1, \dots, s_i, i = 1, \dots, N, \sum_{i=1}^N k_i \geq 1 \right. \right\}$.

DEFINITION 8*. For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate s -increasing convex order if $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$ for all $u \in \mathbb{U}_{s-icx}^N$, u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

The s -increasing concave order and the s -increasing convex order are closely related, because $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the s -increasing concave order if and only if $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}$ dominates $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}$ in the s -increasing convex order. This follows from the fact that if $u \in \mathbb{U}_{s-icv}^N$, then $-u(\underline{\mathbf{x}} + \bar{\mathbf{x}} - \mathbf{x}) \in \mathbb{U}_{s-icx}^N$. An s -increasing convex order with $s_1 = \dots = s_N = s$ is an s -increasing convex order. Analogous to the concave case, the s -increasing convex order with $s = 1$ is the upper orthant order, under which $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ if $P(\tilde{\mathbf{x}} > \mathbf{c}) \geq P(\tilde{\mathbf{y}} > \mathbf{c})$ for all $\mathbf{c} \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

The lower and upper orthant orders, based on lower orthants $\{\mathbf{x} \mid \mathbf{x} \leq \mathbf{c}\}$ and upper orthants $\{\mathbf{x} \mid \mathbf{x} > \mathbf{c}\}$ for a given \mathbf{c} , highlight an important way in which moving from the univariate to the multivariate case makes stochastic orders and stochastic dominance more complex. In the univariate case, $P(\tilde{x} \leq c) + P(\tilde{x} > c) = 1$ for any c . When $N \geq 2$, $P(\tilde{\mathbf{x}} \leq \mathbf{c}) + P(\tilde{\mathbf{x}} > \mathbf{c}) \leq 1$ for any $\mathbf{c} \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, and this

becomes more of an issue as N increases because the lower and upper orthants for a given \mathbf{c} represent only 2 of the 2^N orthants associated with \mathbf{c} .

Theorem 7 provides conditions characterizing stochastic dominance in the sense of the multivariate s -increasing concave and convex orders via partial moments, without reference to utilities. The following remark indicates an alternative characterization in terms of integral conditions.

THEOREM 7. *Let $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ be random vectors with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $-\infty < \underline{\mathbf{x}} < \bar{\mathbf{x}} < \infty$, and denote $x_+ = \max\{x, 0\}$. Then*

(i) *$\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate s -increasing concave order if and only if*

$$\mathbb{E}\left[\prod_{i=1}^N (c_i - \tilde{x}_i)_+^{k_i-1}\right] \leq \mathbb{E}\left[\prod_{i=1}^N (c_i - \tilde{y}_i)_+^{k_i-1}\right] \text{ for all } c_i \in [\underline{x}_i, \bar{x}_i] \text{ if } k_i = s_i \text{ and } c_i = \bar{x}_i \text{ if } k_i = 1, \dots, s_i - 1, \\ i = 1, \dots, N.$$

(ii) *$\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate s -increasing convex order if and only if*

$$\mathbb{E}\left[\prod_{i=1}^N (\tilde{x}_i - c_i)_+^{k_i-1}\right] \geq \mathbb{E}\left[\prod_{i=1}^N (\tilde{y}_i - c_i)_+^{k_i-1}\right] \text{ for all } c_i \in [\underline{x}_i, \bar{x}_i] \text{ if } k_i = s_i \text{ and } c_i = \underline{x}_i \text{ if } k_i = 1, \dots, s_i - 1, \\ i = 1, \dots, N.$$

REMARK. Alternative necessary and sufficient conditions for dominance in the multivariate s -increasing concave and convex orders involve integral conditions. Let $F_{\tilde{\mathbf{x}}}$ be the cumulative distribution function $P(\tilde{\mathbf{x}} \leq \mathbf{x})$ for $\tilde{\mathbf{x}}$. Starting with $F_{\tilde{\mathbf{x}}}^{(1, \dots, 1)} = F_{\tilde{\mathbf{x}}}$, define recursively the integrated left tails of $\tilde{\mathbf{x}}$ as

$$F_{\tilde{\mathbf{x}}}^{(k_1, \dots, k_i+1, \dots, k_N)}(\mathbf{x}) = \int_{\underline{x}_i}^{x_i} F_{\tilde{\mathbf{x}}}^{(k_1, \dots, k_i, \dots, k_N)}(x_1, \dots, z_i, \dots, x_N) dz_i \quad (4)$$

for $k_1, \dots, k_N \geq 1$. The lower partial moments in Theorem 7(i) can be expressed via integrated left tails:

$\mathbb{E}\left[\prod_{i=1}^N (c_i - \tilde{x}_i)_+^{k_i-1}\right] = \left[\prod_{i=1}^N (k_i - 1)!\right] F_{\tilde{\mathbf{x}}}^{(k_1, \dots, k_N)}(\mathbf{c})$. Then $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate s -increasing concave order if and only if $F_{\tilde{\mathbf{x}}}^{(k_1, \dots, k_N)}(\mathbf{c}) \leq F_{\tilde{\mathbf{y}}}^{(k_1, \dots, k_N)}(\mathbf{c})$ for all $c_i \in [\underline{x}_i, \bar{x}_i]$ if $k_i = s_i$ and $c_i = \bar{x}_i$ if $k_i = 1, \dots, s_i - 1$, $i = 1, \dots, N$. When $N = 1$, (4) is the standard integral condition for univariate stochastic dominance.

An expression similar to (4), involving integrated right tails of $\tilde{\mathbf{x}}$, holds for the multivariate s -increasing convex order (Denuit & Mesfioui 2010). If $G_{\tilde{\mathbf{x}}}(\mathbf{x}) = P(\tilde{\mathbf{x}} > \mathbf{x})$ and $G_{\tilde{\mathbf{x}}}^{(1, \dots, 1)} = G_{\tilde{\mathbf{x}}}$, define recursively

$$G_{\tilde{\mathbf{x}}}^{(k_1, \dots, k_i+1, \dots, k_N)}(\mathbf{x}) = \int_{x_i}^{\bar{x}_i} G_{\tilde{\mathbf{x}}}^{(k_1, \dots, k_i, \dots, k_N)}(x_1, \dots, z_i, \dots, x_N) dz_i \quad (5)$$

for $k_1, \dots, k_N \geq 1$. Then $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate s -increasing convex order if and only if $G_{\tilde{\mathbf{x}}}^{(k_1, \dots, k_N)}(\mathbf{c}) \geq G_{\tilde{\mathbf{y}}}^{(k_1, \dots, k_N)}(\mathbf{c})$ for all $c_i \in [x_i, \bar{x}_i]$ if $k_i = s_i$ and $c_i = x_i$ if $k_i = 1, \dots, s_i - 1$, $i = 1, \dots, N$.

Mosler (1984) showed that stochastic dominance in terms of two special cases of the multivariate s -increasing concave order is related to multiplicative utilities. First, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in terms of the lower orthant order ($s = 1$) if and only if $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$ for all multiplicative utilities of the form $u(\mathbf{x}) = -\prod_{i=1}^N (-u_i(x_i))$, where $u_i(x_i) \leq 0$ and $du_i(x_i)/dx_i \geq 0$ for all x_i , $i = 1, \dots, N$. Second, this dominance extends to the lower orthant concave order ($s = 2$) if each $u_i(x_i)$ is also concave. Theorem 8 extends these results to the multivariate s -increasing concave order for any s , showing that this order corresponds to the preferences of decision makers having utility functions consistent with mutual utility independence (Keeney & Raiffa 1976).

THEOREM 8. *For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $-\infty < \underline{\mathbf{x}} < \bar{\mathbf{x}} < \infty$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate s -increasing concave order if and only if*

$$(-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{x}_i)\right] \leq (-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{y}_i)\right] \text{ for all } u_i \leq 0, u_i \in \underline{\mathbb{U}}_{s_i}^1, i = 1, \dots, N.$$

We now compare our multivariate dominance with dominance for the multivariate s -increasing orders. There are some close similarities between the two approaches as well as some important differences. In terms of infinite-degree stochastic dominance, the two approaches are equivalent, because

$$\lim_{\min\{s_i\} \rightarrow \infty} \underline{\mathbb{U}}_{s-icv}^N = \underline{\mathbb{U}}_{\infty}^N \text{ and } \lim_{\min\{s_i\} \rightarrow \infty} \underline{\mathbb{U}}_{s-icx}^N = \bar{\mathbb{U}}_{\infty}^N. \text{ However, this equivalence does not hold for finite } n \text{ and } s.$$

At a very basic level, our multivariate stochastic dominance is a natural extension of standard univariate stochastic dominance in that both are based on a preference between combining good with bad

and combining good with good and bad with bad. A preference for combining good with bad leads to multivariate concave dominance and the most common univariate dominance. The opposite preference leads to multivariate convex dominance and a risk-taking version of univariate dominance. The preference condition is easy for decision makers to understand and therefore easy to check. If the decision maker has a consistent preference one way or the other, this implies corresponding constraints on the utility function via \underline{U}_∞^N and \bar{U}_∞^N , but the discussion about preferences does not require direct consideration of utility.

Dominance in the sense of the s -increasing orders cannot be related to a simple preference assumption, but it can be characterized in terms of integral conditions that are extensions of the integral conditions for standard univariate dominance. In contrast, our multivariate dominance admits no such integral conditions. From a practical standpoint, however, the integral conditions in (4) and (5) might be difficult to verify as N increases or $\sum_{i=1}^N s_i$ increases.

Of course, not all decision makers share the same preferences. Thus, the preferences of different decision makers can be consistent with different classes of utility functions and therefore with different definitions of dominance. The approach to multivariate stochastic dominance developed here is intuitively appealing and should fit the preferences of some decision makers. As such, it is a useful addition to the stochastic dominance toolbox.

6. Summary and Conclusions

The concept of stochastic dominance has been widely studied in the univariate case, and there is widespread agreement on an underlying stochastic order for such dominance. This standard order is consistent with a basic preference condition, a preference for combining good with bad, as opposed to combining good with good and bad with bad. Many multivariate stochastic orders have been studied. However, most lack sufficient connections with the standard univariate stochastic dominance order and are not based on an intuitive preference condition that is easy to explain to decision makers. We fill this gap by defining multivariate n^{th} -degree concave stochastic dominance and n^{th} -degree risk in a way that

naturally extends the univariate case because it is consistent with the same basic preference assumption. As in the univariate case, multivariate infinite-degree stochastic dominance is equivalent to an exponential ordering. We also develop the notion of multivariate convex stochastic dominance, which is consistent with a preference for combining good with good and bad with bad, as opposed to combining good with bad.

After developing our notion of multivariate stochastic dominance, we present some results that are useful in applying our multivariate stochastic dominance relations to rank alternatives. We show that independent additive or multiplicative background risk does not change stochastic dominance orderings, discuss the ordering of multivariate normal distributions, and show how stochastic dominance can be applied to the choice among several alternatives using elimination by mixtures. We further illustrate our results with simple examples, and we discuss the connection of our approach with one based on a family of multivariate orders having some similarities to the order we use.

Many situations involve multiple decision makers, and somewhat divergent preferences can make decision making challenging. Even if each member of the group assesses a utility function (a challenging task itself, particularly in a multiattribute setting), it would be surprising for all members of the group to have identical utilities. However, the preferences of group members might be somewhat similar, especially when they are making a decision for their company and not a personal decision. They most likely will agree on a preference for more of each attribute to less or can define the attributes in such a way as to guarantee that preference, so that first-order stochastic dominance is applicable. They might also agree that the company's situation makes it prudent to be risk averse and that in general, a preference for combining good with bad is reasonable. This implies that they all should be willing to use a utility function $u \in \underline{\mathbb{U}}_n^N$ for any $n > 1$ and therefore to use multiattribute concave stochastic dominance to eliminate some alternatives from consideration.

Making a decision in a multiattribute situation is likely to be a multistage process. Some alternatives might be eliminated using stochastic dominance; choice among other alternatives might require more careful preference assessments, with emphasis on particular tradeoffs. That in turn might

lead to clarification of objectives and attributes, and generation of new promising alternatives (Keeney 1992). The results of our paper can be useful in that kind of decision process.

Acknowledgments

The financial support of the ‘‘Onderzoeksfonds K.U. Leuven’’ (GOA/07: Risk Modeling and Valuation of Insurance and Financial Cash Flows, with Applications to Pricing, Provisioning, and Solvency) is gratefully acknowledged by Michel Denuit. Ilia Tsetlin was supported in part by the INSEAD Alumni Fund.

Appendix

PROOF OF THEOREM 1. Consider any $u \in \underline{\mathbb{U}}_{n+m}^N$, and denote $v(\mathbf{z}) = \mathbb{E}[u(\tilde{\mathbf{y}}_m + \mathbf{z})] - \mathbb{E}[u(\tilde{\mathbf{x}}_m + \mathbf{z})]$. Now $0.5\mathbb{E}[u(\tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n)] + 0.5\mathbb{E}[u(\tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n)] \geq 0.5\mathbb{E}[u(\tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n)] + 0.5\mathbb{E}[u(\tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n)]$ is equivalent to $\mathbb{E}[u(\tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n)] - \mathbb{E}[u(\tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n)] \geq \mathbb{E}[u(\tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n)] - \mathbb{E}[u(\tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n)]$, or $\mathbb{E}[v(\tilde{\mathbf{x}}_n)] \geq \mathbb{E}[v(\tilde{\mathbf{y}}_n)]$. It remains to show that $v(\mathbf{z}) \in \underline{\mathbb{U}}_n^N$. For any $k = 1, \dots, n$ and any $i_j \in \{1, \dots, N\}, j = 1, \dots, k$,

$$(-1)^{k-1} \partial^k v(\mathbf{z}) / \partial z_{i_1} \cdots \partial z_{i_k} = (-1)^{k-1} \left(\mathbb{E}[\partial^k u(\tilde{\mathbf{y}}_m + \mathbf{z}) / \partial z_{i_1} \cdots \partial z_{i_k}] - \mathbb{E}[\partial^k u(\tilde{\mathbf{x}}_m + \mathbf{z}) / \partial z_{i_1} \cdots \partial z_{i_k}] \right), \text{ and}$$

$$(-1)^k \partial^k u(\mathbf{x}) / \partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k} \in \underline{\mathbb{U}}_{m+n-k}^N \subset \underline{\mathbb{U}}_m^N. \text{ Therefore, } (-1)^{k-1} \partial^k v(\mathbf{z}) / \partial z_{i_1} \cdots \partial z_{i_k} \geq 0, \text{ so } v(\mathbf{z}) \in \underline{\mathbb{U}}_n^N. \square$$

PROOF OF THEOREM 1*. This is, essentially, a corollary to Theorem 1. Observe that $u(\mathbf{x}) \in \underline{\mathbb{U}}_n^N$ if and only if $-u(\underline{\mathbf{x}} + \bar{\mathbf{x}} - \mathbf{x}) \in \bar{\mathbb{U}}_n^N$. Therefore, $\tilde{\mathbf{x}}_i$ dominates $\tilde{\mathbf{y}}_i$ in the sense of i^{th} -degree convex stochastic dominance if and only if $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_i$ dominates $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}_i$ in the sense of i^{th} -degree concave stochastic dominance. By Theorem 1, $[\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}_m + \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_n, \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_m + \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}_n]$ dominates $[\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}_m + \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}_n, \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_m + \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_n]$ in the sense of $(n+m)^{\text{th}}$ -degree concave stochastic dominance, and thus $[\tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n]$ dominates $[\tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n]$ in the sense of $(n+m)^{\text{th}}$ -degree convex stochastic dominance. \square

PROOF OF THEOREM 2. For the ‘‘only if’’ part, (i) holds by the definition of $\underline{\mathbb{U}}_n^N$. For (ii), consider

$$u(\mathbf{x}) = x_{i_1} x_{i_2} \cdots x_{i_k} \text{ for any } i_j \in \{1, \dots, N\} \text{ and } k < n. \text{ For this } u(\mathbf{x}), (-1)^{n-1} \partial^n u(\mathbf{x}) / \partial x_{i_1} \cdots \partial x_{i_n} = 0 \text{ for any}$$

$i_j \in \{1, \dots, N\}, j = 1, \dots, n$. Therefore, $\mathbb{E}[\tilde{x}_{i_1} \tilde{x}_{i_2} \cdots \tilde{x}_{i_k}] \geq \mathbb{E}[\tilde{y}_{i_1} \tilde{y}_{i_2} \cdots \tilde{y}_{i_k}]$. Similarly, for $u(\mathbf{x}) = -x_{i_1} x_{i_2} \cdots x_{i_k}$, $\mathbb{E}[\tilde{y}_{i_1} \tilde{y}_{i_2} \cdots \tilde{y}_{i_k}] \geq \mathbb{E}[\tilde{x}_{i_1} \tilde{x}_{i_2} \cdots \tilde{x}_{i_k}]$. Thus, $\mathbb{E}[\tilde{x}_{i_1} \tilde{x}_{i_2} \cdots x_{i_k}] = \mathbb{E}[\tilde{y}_{i_1} \tilde{y}_{i_2} \cdots \tilde{y}_{i_k}]$. Now, suppose that (i) and (ii) hold. We need to prove that for any u such that $(-1)^{n-1} \partial^n u(\mathbf{x}) / \partial x_{i_1} \cdots \partial x_{i_n} \geq 0$ for any $i_j \in \{1, \dots, N\}, j = 1, \dots, n$, $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$. Since u is differentiable at least n times, all lower-degree derivatives exist and are bounded on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$. Therefore, there exist coefficients $c_{i_1 \dots i_k}$ for $k = 1, \dots, n-1$ and any $i_j \in \{1, \dots, N\}, j = 1, \dots, k$, such that $v(\mathbf{x}) = u(\mathbf{x}) + \sum c_{i_1 \dots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ and $v \in \underline{\mathbb{U}}_n^N$, where the summation is over all possible combinations of i_1, \dots, i_k . By (i), $\mathbb{E}[v(\tilde{\mathbf{x}})] \geq \mathbb{E}[v(\tilde{\mathbf{y}})]$, and by (ii), $\mathbb{E}[v(\tilde{\mathbf{x}})] - \mathbb{E}[v(\tilde{\mathbf{y}})] = \mathbb{E}[u(\tilde{\mathbf{x}})] - \mathbb{E}[u(\tilde{\mathbf{y}})]$. Therefore, $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$. \square

PROOF OF THEOREM 5. Consider any $u \in \underline{\mathbb{U}}_n^N$ ($u \in \bar{\mathbb{U}}_n^N$). For any fixed \mathbf{a} and \mathbf{m} , $v(\mathbf{x}/\mathbf{a}, \mathbf{m}) = u(\mathbf{a} + \mathbf{m} \otimes \mathbf{x})$, as a function of \mathbf{x} , belongs to $\underline{\mathbb{U}}_n^N$ ($\bar{\mathbb{U}}_n^N$). Therefore, $\mathbb{E}[v(\tilde{\mathbf{x}}/\mathbf{a}, \mathbf{m})] \geq \mathbb{E}[v(\tilde{\mathbf{y}}/\mathbf{a}, \mathbf{m})]$. Taking expectations with respect to $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{m}}$ yields $\mathbb{E}[u(\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}})]$. \square

PROOF OF THEOREM 6. For any $u \in \mathbb{U}^*$, there is a \mathbf{p} such that $\sum_{i=1}^{k-1} p_i \mathbb{E}[u(\tilde{\mathbf{x}}_i)] \geq \mathbb{E}[u(\tilde{\mathbf{x}}_k)]$. This is impossible unless $\mathbb{E}[u(\tilde{\mathbf{x}}_i)] \geq \mathbb{E}[u(\tilde{\mathbf{x}}_k)]$ for some $i \in \{1, \dots, k-1\}$. \square

PROOF OF THEOREM 7. Statement (ii) is proven in Denuit & Mesfioui (2010, Proposition 3.1). Statement (i) follows from (ii) and the duality between the concave and convex orders: $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate s -increasing concave order if and only if $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}$ dominates $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}$ in the multivariate s -increasing convex order. Therefore, from (ii), $\mathbb{E}\left[\prod_{i=1}^N (x_i + \bar{x}_i - \tilde{y}_i - c_i)_+^{k_i-1}\right] \geq \mathbb{E}\left[\prod_{i=1}^N (x_i + \bar{x}_i - \tilde{x}_i - c_i)_+^{k_i-1}\right]$ with $c_i = \underline{x}_i$ if $k_i < s_i$ and $c_i \in [\underline{x}_i, \bar{x}_i]$ if $k_i = s_i$, which is equivalent to (i). \square

PROOF OF THEOREM 8. Suppose that $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate s -increasing concave order, and let $v(\mathbf{x}) = -\prod_{i=1}^N (-u_i(x_i))$. If $u_i \leq 0$ and $u_i \in \underline{\mathbb{U}}_{s_i}^1, i = 1, \dots, N$, then $v \in \underline{\mathbb{U}}_{s\text{-icv}}^N$. Therefore, $\mathbb{E}[v(\tilde{\mathbf{x}})] \geq \mathbb{E}[v(\tilde{\mathbf{y}})]$, so that $(-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{x}_i)\right] \leq (-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{y}_i)\right]$. For the converse, suppose that

$(-1)^N \mathbb{E} \left[\prod_{i=1}^N u_i(\tilde{x}_i) \right] \leq (-1)^N \mathbb{E} \left[\prod_{i=1}^N u_i(\tilde{y}_i) \right]$ for all $u_i \leq 0$, $u_i \in \underline{\mathbb{U}}_{s_i}^1$, $i = 1, \dots, N$. For $i = 1, \dots, N$ and $k = 1, \dots, s_i - 1$, let $u_i(x_i) = -(c_i - x_i)_+^{k_i+1}$ with $c_i = \bar{x}_i$ if $k_i < s_i$ and $c_i \in [\underline{x}, \bar{x}]$ if $k_i = s_i$. Thus, $u_i \leq 0$ and $u_i \in \underline{\mathbb{U}}_{s_i}^1$ if $k_i < s_i$. For $k_i = s_i$, u_i belongs to the closure of $\underline{\mathbb{U}}_{s_i}^1$ (i.e., there exist a sequence of functions $v_j \in \underline{\mathbb{U}}_{s_i}^1$, $j = 1, \dots$, with $v_j \rightarrow u_i$). Thus, $\mathbb{E} \left[\prod_{i=1}^N (c_i - \tilde{x}_i)_+^{k_i-1} \right] \leq \mathbb{E} \left[\prod_{i=1}^N ((c_i - \tilde{y}_i)_+^{k_i-1}) \right]$, and by Theorem 7(i), \tilde{x} dominates \tilde{y} in the sense of the multivariate s -increasing concave order. \square

References

- Denuit, M., L. Eeckhoudt. 2010. Bivariate stochastic dominance and substitute risk (in)dependent utilities. *Decision Anal.* **7** in press.
- Denuit, M., L. Eeckhoudt, B. Rey. 2010. Some consequences of correlation aversion in decision science. *Ann. Oper. Res.* **176**(1) 259-269.
- Denuit, M., C. Lefèvre, M. Mesfioui. 1999. A class of bivariate stochastic orderings, with applications in actuarial sciences. *Insurance: Math. & Econom.* **24**(1) 31-50.
- Denuit, M., M. Mesfioui. 2010. Generalized increasing convex and directionally convex orders. *J. Applied Prob.* **47**(1) 264-276.
- Eeckhoudt, L., B. Rey, H. Schlesinger. 2007. A good sign for multivariate risk taking. *Management Sci.* **53**(1) 117-124.
- Eeckhoudt, L., H. Schlesinger. 2006. Putting risk in its proper place. *Amer. Econom. Rev.* **96**(1) 280-289.
- Eeckhoudt, L., H. Schlesinger, I. Tsetlin. 2009. Apportioning of risks via stochastic dominance. *J. Econom. Theory* **144**(3) 994-1003.
- Ekern, S. 1980. Increasing N th degree risk. *Econom. Letters* **6**(4) 329-333.
- Epstein, L.G., S.M. Tanny. 1980. Increasing general correlation: A definition and some economic consequences. *Canadian J. Econom.* **13**(1) 16-34.
- Fishburn, P.C. 1974. Convex stochastic dominance with continuous distribution functions. *J. Econom. Theory* **7**(2) 143-158.
- Fishburn, P.C. 1978. Convex stochastic dominance. G.A. Whitmore, M.C. Findlay, eds., *Stochastic Dominance: An Approach to Decision-Making under Risk*. Heath, Lexington, MA, 337-351.
- Hadar, J., W.R. Russell. 1969. Rules for ordering uncertain prospects. *Amer. Econom. Rev.* **59**(1) 25-34.
- Hanoch, G, H. Levy. 1969. The efficiency analysis of choices involving risk. *Rev. Econom. Studies.* **36**(3) 335-346.

- Hazen, G.B. 1986. Partial information, dominance, and potential optimality in multiattribute utility theory. *Oper. Res.* **34**(2) 296-310.
- Keeney, R.L. 1992. *Value-Focused Thinking*. Harvard University Press, Cambridge, MA.
- Keeney, R.L., H. Raiffa. 1976. *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. Wiley, New York.
- Levhari, D., J. Paroush, B. Peleg. 1975. Efficiency analysis of multivariate distributions. *Rev. Econom. Studies.* **42**(1) 87-91.
- Levy, H., J. Paroush. 1974. Toward multi-variate efficiency criteria. *J. Econom. Theory.* **7**(2) 129-142.
- Mosler, K.C. 1984. Stochastic dominance decision rules when the attributes are utility independent. *Management Sci.* **30**(11) 1311-1322.
- Müller, A. 2001. Stochastic ordering of multivariate normal distributions. *Ann. Inst. Statist. Math.* **53**(3) 567-575.
- Müller, A., D. Stoyan. 2002. *Comparison Methods for Stochastic Models and Risks*. Wiley, Chichester, UK.
- Richard, S.F. 1975. Multivariate risk aversion, utility independence and separable utility functions. *Management Sci.* **22**(1) 12-21.
- Scarsini, M. 1988. Dominance conditions for multivariate utility functions. *Management Sci.* **34**(4) 454-460.
- Shaked, M., J.G. Shantikumar. 2007. *Stochastic Orders*. Springer, New York.
- Tsetlin, I., R.L. Winkler. 2005. Risky choices and correlated background risk. *Management Sci.* **51**(9) 1336-1345.
- Tsetlin, I., R.L. Winkler. 2007. Decision making with multiattribute performance targets: The impact of changes in performance and target distributions. *Oper. Res.* **55**(2) 226-233.
- Tsetlin, I., R.L. Winkler. 2009. Multiattribute utility satisfying a preference for combining good with bad. *Management Sci.* **55**(12) 1942-1952.

Recent titles

CORE Discussion Papers

- 2010/1. Giorgia OGGIONI and Yves SMEERS. Degree of coordination in market-coupling and counter-trading.
- 2010/2. Yu. NESTEROV. Efficiency of coordinate descent methods on huge-scale optimization problems.
- 2010/3. Geert DHAENE and Koen JOCHMANS. Split-panel jackknife estimation of fixed-effect models.
- 2010/4. Parkash CHANDER. Cores of games with positive externalities.
- 2010/5. Gauthier DE MAERE D'AERTRYCKE and Yves SMEERS. Liquidity risks on power exchanges.
- 2010/6. Marc FLEURBAEY, Stéphane LUCHINI, Christophe MULLER and Erik SCHOKKAERT. Equivalent income and the economic evaluation of health care.
- 2010/7. Elena IÑARRA, Conchi LARREA and Elena MOLIS. The stability of the roommate problem revisited.
- 2010/8. Philippe CHEVALIER, Isabelle THOMAS and David GERAETS, Els GOETGHEBEUR, Olivier JANSSENS, Dominique PEETERS and Frank PLASTRIA. Locating fire-stations: an integrated approach for Belgium.
- 2010/9. Jean-Charles LANGE and Pierre SEMAL. Design of a network of reusable logistic containers.
- 2010/10. Hiroshi UNO. Nested potentials and robust equilibria.
- 2010/11. Elena MOLIS and Róbert F. VESZTEG. Experimental results on the roommate problem.
- 2010/12. Koen DECANCO. Copula-based orderings of multivariate dependence.
- 2010/13. Tom TRUYTS. Signaling and indirect taxation.
- 2010/14. Asel ISAKOVA. Currency substitution in the economies of Central Asia: How much does it cost?
- 2010/15. Emanuele FORLANI. Irish firms' productivity and imported inputs.
- 2010/16. Thierry BRECHET, Carmen CAMACHO and Vladimir M. VELIOV. Model predictive control, the economy, and the issue of global warming.
- 2010/17. Thierry BRECHET, Tsvetomir TSACHEV and Vladimir M. VELIOV. Markets for emission permits with free endowment: a vintage capital analysis.
- 2010/18. Pierre M. PICARD and Patrice PIERETTI. Bank secrecy, illicit money and offshore financial centers.
- 2010/19. Tanguy ISAAC. When frictions favour information revelation.
- 2010/20. Jeroen V.K. ROMBOUTS and Lars STENTOFT. Multivariate option pricing with time varying volatility and correlations.
- 2010/21. Yassine LEFOUILI and Catherine ROUX. Leniency programs for multimarket firms: The effect of Amnesty Plus on cartel formation.
- 2010/22. P. Jean-Jacques HERINGS, Ana MAULEON and Vincent VANNETELBOSCH. Coalition formation among farsighted agents.
- 2010/23. Pierre PESTIEAU and Grégory PONTIERE. Long term care insurance puzzle.
- 2010/24. Elena DEL REY and Miguel Angel LOPEZ-GARCIA. On welfare criteria and optimality in an endogenous growth model.
- 2010/25. Sébastien LAURENT, Jeroen V.K. ROMBOUTS and Francesco VIOLANTE. On the forecasting accuracy of multivariate GARCH models.
- 2010/26. Pierre DEHEZ. Cooperative provision of indivisible public goods.
- 2010/27. Olivier DURAND-LASSERVE, Axel PIERRU and Yves SMEERS. Uncertain long-run emissions targets, CO₂ price and global energy transition: a general equilibrium approach.
- 2010/28. Andreas EHRENMANN and Yves SMEERS. Stochastic equilibrium models for generation capacity expansion.
- 2010/29. Olivier DEVOLDER, François GLINEUR and Yu. NESTEROV. Solving infinite-dimensional optimization problems by polynomial approximation.
- 2010/30. Helmuth CREMER and Pierre PESTIEAU. The economics of wealth transfer tax.
- 2010/31. Thierry BRECHET and Sylvette LY. Technological greening, eco-efficiency, and no-regret strategy.

Recent titles

CORE Discussion Papers - continued

- 2010/32. Axel GAUTIER and Dimitri PAOLINI. Universal service financing in competitive postal markets: one size does not fit all.
- 2010/33. Daria ONORI. Competition and growth: reinterpreting their relationship.
- 2010/34. Olivier DEVOLDER, François GLINEUR and Yu. NESTEROV. Double smoothing technique for infinite-dimensional optimization problems with applications to optimal control.
- 2010/35. Jean-Jacques DETHIER, Pierre PESTIEAU and Rabia ALI. The impact of a minimum pension on old age poverty and its budgetary cost. Evidence from Latin America.
- 2010/36. Stéphane ZUBER. Justifying social discounting: the rank-discounting utilitarian approach.
- 2010/37. Marc FLEURBAEY, Thibault GAJDOS and Stéphane ZUBER. Social rationality, separability, and equity under uncertainty.
- 2010/38. Helmuth CREMER and Pierre PESTIEAU. Myopia, redistribution and pensions.
- 2010/39. Giacomo SBRANA and Andrea SILVESTRINI. Aggregation of exponential smoothing processes with an application to portfolio risk evaluation.
- 2010/40. Jean-François CARPANTIER. Commodities inventory effect.
- 2010/41. Pierre PESTIEAU and Maria RACIONERO. Tagging with leisure needs.
- 2010/42. Knud J. MUNK. The optimal commodity tax system as a compromise between two objectives.
- 2010/43. Marie-Louise LEROUX and Gregory PONTIERE. Utilitarianism and unequal longevity: A remedy?
- 2010/44. Michel DENUIT, Louis EECKHOUDT, Ilia TSETLIN and Robert L. WINKLER. Multivariate concave and convex stochastic dominance.

Books

- J. GABSZEWICZ (ed.) (2006), *La différenciation des produits*. Paris, La découverte.
- L. BAUWENS, W. POHLMEIER and D. VEREDAS (eds.) (2008), *High frequency financial econometrics: recent developments*. Heidelberg, Physica-Verlag.
- P. VAN HENTENRYCKE and L. WOLSEY (eds.) (2007), *Integration of AI and OR techniques in constraint programming for combinatorial optimization problems*. Berlin, Springer.
- P-P. COMBES, Th. MAYER and J-F. THISSE (eds.) (2008), *Economic geography: the integration of regions and nations*. Princeton, Princeton University Press.
- J. HINDRIKS (ed.) (2008), *Au-delà de Copernic: de la confusion au consensus ?* Brussels, Academic and Scientific Publishers.
- J-M. HURIOT and J-F. THISSE (eds) (2009), *Economics of cities*. Cambridge, Cambridge University Press.
- P. BELLEFLAMME and M. PEITZ (eds) (2010), *Industrial organization: markets and strategies*. Cambridge University Press.
- M. JUNGER, Th. LIEBLING, D. NADDEF, G. NEMHAUSER, W. PULLEYBLANK, G. REINELT, G. RINALDI and L. WOLSEY (eds) (2010), *50 years of integer programming, 1958-2008: from the early years to the state-of-the-art*. Berlin Springer.

CORE Lecture Series

- C. GOURIÉROUX and A. MONFORT (1995), *Simulation Based Econometric Methods*.
- A. RUBINSTEIN (1996), *Lectures on Modeling Bounded Rationality*.
- J. RENEGAR (1999), *A Mathematical View of Interior-Point Methods in Convex Optimization*.
- B.D. BERNHEIM and M.D. WHINSTON (1999), *Anticompetitive Exclusion and Foreclosure Through Vertical Agreements*.
- D. BIENSTOCK (2001), *Potential function methods for approximately solving linear programming problems: theory and practice*.
- R. AMIR (2002), *Supermodularity and complementarity in economics*.
- R. WEISMANTEL (2006), *Lectures on mixed nonlinear programming*.