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Evolution and market behavior with endogenous investment rules

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Abstract

In a repeated market for short-lived assets, we investigate long run wealth-driven selection on the general class of investment rules that depend on endogenously determined current and past prices. We study the random dynamical system that describes the price and wealth dynamics and characterize local stability of long-run market equilibria. Instability, leading to asset mis-pricing and informational inefficiencies, turns out to be a common phenomenon generated by two different mechanisms. Firstly, conditioning investment decisions on asset prices implies that dominance of an investment rule on others, as measured by the relative entropy, can be different at different prevailing prices thus reducing the global selective capability of the market. Secondly, the feedback existing between past realized prices and current investment decisions can lead to a form of deterministic overshooting.

Keywords: Market Selection; Evolutionary Finance; Price Feedbacks; Asset Pricing; Informational Efficiency; Kelly rule.
JEL Classification: D50, D80, G11, G12

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1 Introduction

The Efficient Market Hypothesis (EMH), advanced as a general interpretative and normative framework nearly forty years ago (Fama, 1970), has grown to become a widely accepted working tool for the economic profession. Rooted in the evolutionary foundations of neoclassical economics (Alchian, 1950; Friedman, 1953), the EMH is broadly based on the "as if" argument that poorly informed investors are persistently loosing wealth in favor of the better informed. If this is true, those who are poorly informed are, in the long-run, driven out of the market so that the best available information about assets fundamentals is ultimately reflected in prevailing prices.

Despite its pervasive influence in economics, a general formal proof of the selective capability of markets and, consequently, of the ultimate convergence of asset prices toward fundamental values is still lacking. Only fairly recently scholarshiply work has started to investigate this issue. Several behavioral models based on evidence collected from laboratory experiments and real markets (see Barberis and Thaler, 2003, and references therein) contend both the positive and normative aspects of EMH. Rational behavior does not appear as a pervasive property of trading, nor does automatically guarantee, even if appropriately implemented, better performances and higher probability to "survive" the speculative struggle. The modeling effort of these studies has been, however, limited to partial equilibrium models with exogenous price dynamics. A general equilibrium model with an endogenous price dynamics has been firstly proposed in Blume and Easley (1992). They investigate wealth-driven market selection, and the information content of asset prices, on a class of investment rules that depend on the realization of exogenous variables, such as asset dividends. Two groups of contributions developed from their analysis.

A first group of works has focused on investment rules not necessarily coming from utility maximization and expressed as fraction of wealth to be invested in each asset (see Evstigneev et al., 2009, for a recent survey). Rules are allowed to depend on histories of exogenous market variables, in particular past dividends. Assuming that all agents consume the same fraction of their wealth, a robust finding is that investing proportionally to asset expected dividends, also named the Kelly rule after Kelly (1956), is the unique globally stable rule. The result holds for both short- and long-lived assets, as shown in Amir et al. (2005) and Evstigneev et al. (2008) respectively. When the Kelly rule is not present in the market, for instance when a complete knowledge of the underlying dividend process is lacking, rules with the lowest relative entropy with respect to the dividend process, or "nearest" to it, are gaining all the wealth in the long-run. As a result asset prices are brought as "close" as possible to their fundamental values and, in this sense, the market can be said informationally efficient.

A second group of works has instead focused on selection among investment decisions explicitly coming from utility maximization. In this case assets demand is not necessarily expressed as a fraction of wealth. The main objective of these works is to establish whether the market is able to select for agents whose beliefs, or information, are "closer" to the underlying dividend payment process. Assuming perfect foresight on realized prices and market completeness[1], Sandroni (2000) and Blume and Easley (2006) find that the "as if" statement is correct: no matter the functional form of the utility function they maximize, agents whose beliefs are "nearest" the correct ones are selected for in the long run, provided that they discount future consumption at the same rate.

[1]Notice that market completeness cannot be achieved unless agents coordinate on having rational expectations on prices. On the issue of market completeness see also footnote 5.

Both groups of contributions leave some relevant questions unanswered. In particular, it is not known how market selection works when agents do not coordinate to have perfect foresight on realized prices and, at the same time, prices enter as explicit parameters in the investment rules. The aim of the present paper is to investigate this issue. In particular, we extend the model in Blume and Easley (1992) to encompass investment rules that depend on current and past asset prices. In doing so we drop the assumption of perfect foresight, and, apart the technical requirement of certain regularity conditions, do not pose any further restriction on the functional forms of investment rules. Our aim is twofold. Firstly, we want to move closer to a formal general check of the "as if" statement, studying market selection and the ensuing asset prices behavior for a broader class of asset demands. Secondly, we want to better understand the functioning of markets when their role of information gatherers is directly acknowledged by traders. In fact, in a market where prices supposedly reflect fundamentals as close as possible, the use of the formers to infer about the latters is a rational behavior which does not imply the presence of informational asymmetries. When agents believe that market prices reflect the best available information and use them to guide their investment decision, is market efficiency increased or decreased? Relatedly, if agents rely on endogenous information, what is the long-run effect of those strategies that are designed to trade against asset mis-pricing?

The dependence of investment rules on current and, possibly, past prices implies the presence of a feedback effect in agents' demands which links past and present market performances. The same effect has already been investigated in several heterogeneous agents models. The main finding is that market instability and asset mis-pricing are in general possible (see Hommes, 2006; LeBaron, 2006, for a review). However, in these works market selection is postulated to operate according to ad-hoc fitness measures and not by looking at the natural measure of relative wealth (Levy et al., 2000; Farmer, 2002; Chiarella and He, 2001, are among the few exceptions). Moreover, results are often derived for specific investment behaviors and in a partial equilibrium framework. Both gaps have been partially filled by our previous works, see e.g. Anufriev and Bottazzi (2010), Anufriev et al. (2006) or Anufriev and Dindo (2010), which study wealth-driven market selection on the general class of price dependent investment rules. However, those works, being based on an essentially deterministic framework, do not discuss the information efficiency issue we are interested here.

Technically we investigate market selection and the informational role of prices by analyzing the random dynamical system that describes the price and wealth dynamics. The price dependence of investment rules commands a notion of economic equilibrium compatible with the way in which agents form their individual demand. In particular, a requirement of consistency between agents' expectation and realized market dynamics should be introduced. This requirement, which is not necessary when agents base their investment decisions on exogenous variables, leads to the notion of "procedural consistent equilibria" (see the discussion in Anufriev and Bottazzi, 2010) which are naturally identified with the deterministic fixed points of the random dynamical system describing the market evolution. We characterize such fixed points, or long-run market equilibria, and investigate their stochastic stability. We are able to derive general sufficient conditions ruling whether any given agent is locally dominating all others. Since our exercise can be accomplished when investment rules depend on current prices, a wide spectrum of behaviors can be modeled, including those derived from the maximization of any Constant Relative Risk Aversion (CRRA) expected utility.

Extending previous contributions, our analysis confirms the existence of a special rule, named S-rule, that turns out to be a price dependent generalization of the Kelly rule. When it is present in the market, it acts as the "local" champion meaning that it can destabilize any long-run informationally inefficient market equilibrium. At the same time, it determines a market equilibrium where risky assets are correctly priced proportionally to their expected revenues. This equilibrium is never unstable, no matter the number and type of other competing investment rules. However, when the S-rule is not used by any agent, the analysis of the informational efficiency of the market becomes much more complicated. In fact, the dependence of investment rules on prices brings instability and multiplicity of equilibria into the market, so that persistent asset mis-pricing can be observed, and prices do not need to reflect the best available information.

The presence of multiple equilibria and instability is essentially related to two causes. Firstly, inside the general class of investment rules we consider, the relative average wealth growth rate obtained using two different rules depends on prevailing prices. It may well happen that the first rule has a higher wealth growth rate at the prices determined by the second, while the second has a higher wealth growth rate at the prices determined by the first, so that none can prevail. A second source of instability is directly linked to the effect induced by price feedbacks. Even though a given rule has the highest relative average wealth growth at "its" prices than all other rules present in the market would have at "their" prices, it can happen that its price feedback is too strong and acts as a destabilizing force. In both cases, market prices do not converge to the level reflecting the "best" available information but instead keep displaying endogenous fluctuations.

The outline of this paper is as follows. In Section 2 we present our model. Section 3 proposes an example which, albeit its simplicity, will hopefully help in appreciating our findings and in understanding their causes. Section 4 contains our main results, that is, existence and local stability of long run market equilibria for any finite set of investment rules inside the general class considered here. In Section 5 we illustrate some implications of our results by discussing three specific issues. In Section 5.1 we define the S-rule, as the rule that never vanishes against any other rule, and show that it is a price dependent generalization of the Kelly rule. Section 5.2 explores the possibility of establishing an order relation on the space of investment rules by exploiting their relative market performance. The answer will be negative. Section 5.3 characterizes conditions under which a generic form of learning from prices does not vanish when trading with a S-rule investor. Section 6 concludes. All proofs are collected in the Appendix.

2 The model

Given the set $\Sigma = \{1, \ldots, s, \ldots, S\}$ of states of the world, define the set of sequences $\Omega := \prod_{-\infty}^{+\infty} \Sigma$ with elements $\omega = (\ldots, \omega_0 \ldots, \omega_t, \ldots)$, so that $\{\omega\}_t = \omega_t \in \Sigma$ for every t, and the complete σ-algebra $\mathbb{P} = 2^\Omega$. Let ρ be a measure on $\mathbb{P}$ so that $(\Omega, \mathbb{P}, \rho)$ is a well-defined probability space. We assume that the corresponding stochastic process with realizations in Ω is ergodic, that is, there exists a unique invariant measure $\pi = (\pi_1, \ldots, \pi_S)$ on Σ such that for every bounded statistics $g : \Sigma \to \mathbb{R}$ and almost all sequences ω it holds

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} g(\omega_t) = \sum_{s=1}^{S} g(s)\pi_s \,.$$

Given ρ, let θ be the Bernoulli shift operator on Ω, that is, for every component t it holds $\{\theta\omega\}_t = \{\omega\}_{t+1}$. Name θ^t the composition of t operators θ, so that $\{\theta^t\omega\}_{t'} = \{\omega\}_{t'+t}$.

We study a market where, in each period t, I agents are trading K short-lived risky assets. We denote agent i wealth in period t as w_t^i, and asset k price in period t as $p_{k,t}$, using the vector notations $w_t = (w_t^1, \ldots, w_t^I)$ and $\mathbf{p}_t = (p_{k,t}, \ldots, p_{K,t})$ when appropriate. Asset dividend payoffs are paid in terms of a consumption good, the numéraire of the economy, and are random variables defined over $(\Omega, \mathbb{P}, \rho)$. Payoffs are stationary and depend on the contemporaneous realization of the state of the world. Given the matrix D with non-negative elements $D_{s,k}$, the dividend of asset k at time t is defined as $d_{k,t}(\omega) = D_{\omega_{t+1},k}$. We assume that the matrix D is non-trivial, that is, for all k (for all s) there exists a s (a k) such that $D_{s,k} > 0$. The first assumption implies that there are no assets with zero payoff in every state. The second assumption rules out the possibility that the total wealth in a given period is zero. Since payoffs are stationary one can also define K random variables over Σ with $d_k(s) = D_{s,k}$ and then use $d_{k,t}(\omega) = d_k(\omega_t)$.

Asset demands are modeled as wealth fractions to be allocated to each asset. We denote with $\alpha^i_{k,t}$ the fraction of wealth that at period t agent i invests in asset k. Whereas previous contributions have assumed investment decisions to depend on partial histories of ω, we assume they are functions of endogenous market variables, in particular current and past asset prices.

Assumption 1. The fraction of wealth agent i invests on asset k at time t, $\alpha^i_{k,t}$, is given by

$$\alpha^i_{k,t} = \alpha^i_k(p_t) \quad k = 1, \ldots, K\,, \tag{2.1}$$

where the function $\alpha^i_k(\cdot)$ is agent i-asset k investment rule and p_t is the vector of current and past, up to lag L, asset prices, that is, $p_t = (\mathbf{p}_t, \mathbf{p}_t^1, \ldots, \mathbf{p}_t^L)$ with $\mathbf{p}_t^l = \mathbf{p}_{t-l}$.[2] The consumption of agent i at time t remains defined as $\alpha^i_0(p_t) = 1 - \sum_{k=1}^K \alpha^i_k(p_t)$. We further assume that for any price vector p investment rules satisfy the following properties

P1 Each agent i invests a positive fraction of her wealth and cannot borrow, that is, $0 < \sum_{k=1}^K \alpha^i_k(p) \leq 1$;

P2 Short positions are forbidden, that is, $\alpha^i_k(p) \geq 0$ for every asset k and agent i, and portfolios are sufficiently diversified, that is, $\sum_{k=1}^K \alpha^i_k(p) D_{s,k} > 0$ for every agent i and state of the world s;

P3 Demand is strictly positive for zero contemporaneous prices, that is, for every asset k and agent i, $\alpha^i_k(p_t) > 0$ if $p_{k,t} = 0$.

We shall name α^i the vector valued investment rule adopted by agent i and introduce the shorter notation $\alpha^i_t = \alpha^i(p_t)$. We denote with $\mathcal{A}$ the set of vector valued rules complying to **P1-P3**.

The role of Assumption 1 deserves a brief discussion. As common in this literature, we model asset demand in terms of wealth shares allocated to each asset. We extend previous works by assuming that shares α may depend on current and past prices. **P1** implies that agents cannot borrow, may decide not to consume, but can never end up with zero wealth by consuming it all. **P2** guarantees that agents' portfolios are sufficiently diversified so as to avoid having zero wealth for some realizations of the stochastic dividend process. Notice that if the dividend matrix D is diagonal (as in the case of Arrow securities) **P2** reduces to the condition $\alpha^i_k(p) > 0$. Rules that violate the constraints in **P1-P2** will possess zero wealth with probability one and we can safely neglect them. **P3** is introduced to guarantee that prevailing prices are strictly positive. Since each asset possesses a strictly positive expected

[2] The compact notation for lagged prices allows l to be equal to 0, in which case trivially $\mathbf{p}_t^0 = \mathbf{p}_t$ and $p^0_{k,t} = p_{k,t}$.

payoff, any rule should consider it a valuable investment at a sufficiently low price. Notice however that the αs can be infinitesimally small so that we do not consider **P3** as an actual limit on investment behaviors.

Given a set of investment rules and an initial wealth distribution, market clearing and intertemporal budget constraints determine the dynamics of asset prices and agents' wealths for all subsequent periods, that is, the time evolution of the market state variable $x_t = (w_t, p_t)$. In fact, given individual wealths, asset prices, and investment decisions at time t, each agent wealth at time $t+1$ is given by the scalar product of her individual asset holding with the vector of asset returns corresponding to the state of the world just realized or

$$w^i_{t+1} = \mathcal{W}^i(x_t;\omega) := \sum_{k=1}^{K} \frac{\alpha^i_{k,t} w^i_t}{p_{k,t}} d_k(\omega_{t+1}) \,, \quad i = 1, \ldots, I \,. \tag{2.2}$$

Given wealths and investment decisions at time time $t+1$, asset prices are computed by aggregating asset demands and imposing market clearing which, upon normalizing asset supply to 1, leads to

$$p_{k,t+1} = \sum_{i=1}^{I} w^i_{t+1} \alpha^i_{k,t+1} = \sum_{i=1}^{I} \mathcal{W}^i(x_t;\omega) \alpha^i_{k,t+1} \,, \quad k = 1, \ldots, K \,. \tag{2.3}$$

As assets are short-lived, the total wealth in each period is given by the sum of asset dividends payed for the state of the world just realized. We can thus use total wealth to introduce a convenient normalization of dividends and individual wealths. This procedure does not change equations (2.2-2.3) upon remembering that all variables are now understood to be normalized, so that it holds

$$\sum_{i=1}^{I} w^i_t = \sum_{k=1}^{K} D_{s,k} = 1 \,, \quad s = 1, \ldots, S \,, \quad t \in \mathbb{N} \,, \tag{2.4}$$

$$\sum_{k=1}^{K} p_{k,t} = \sum_{i=1}^{I} (1 - \alpha^i_0(p_t)) w^i_t \leq 1 \,, \quad t \in \mathbb{N}. \tag{2.5}$$

The previous normalization is, in fact, equivalent to assuming from the beginning that total wealth given by the sum of asset dividend for each realization of ω_t is equal to one so that, in particular, there is no aggregate risk in the economy. This is consistent with the assumption that investment rules are not state dependent. Notice that when normalizing total wealth in each period, we are implicitly changing the shape of the investment rule by changing their dependence upon prices into a dependence upon normalized prices. In particular we assume that the components of the investment rules α are defined over the compact set $[0, 1]^{(L+1)K}$.

If investment decisions do not depend on current prices, (2.3) uniquely determines the vector of market clearing prices at time $t+1$. Conversely, when the dependence on contemporaneous prices is present in some of the αs, prices are fixed by (2.3) through a system of K implicitly defined functions. Continuity of investment rules is sufficient to guarantee the existence of at least one vector of positive market clearing prices.

Theorem 2.1. *If for every agent $i = 1, \ldots, I$ it holds $\alpha^i \in \mathcal{A}$ and $\alpha^i \in \mathcal{C}^0$, then there always exists a vector $\mathbf{p}^*$ of positive prices satisfying* (2.3)*, that is, clearing the asset market.*

The uniqueness of the solution is in general not guaranteed, but smooth investment rules with a mild dependence on present prices constitute a sufficient condition, see for instance Theorem A.1 in the Appendix.

Since in the following sections our analysis will be mostly local, we are not particularly bothered by the possibility that the market clearing price vector is not unique. However when discussing the global dynamics, we shall assume that there exist K explicit global maps

$$p_{k,t+1} = f(x_t;\omega)\,, \quad k = 1,\ldots,K\,. \tag{2.6}$$

Summarizing, the market evolution can be written as a system of $I + K(L+1)$ equations

$$\mathcal{F}(\omega)x_t := \left[\begin{array}{lcl} \mathcal{W}(x_t;\omega) & := & \left[\begin{array}{lcl} w^1_{t+1} & = & \mathcal{W}^1(x_t;\omega) \\ \vdots & \vdots & \vdots \\ w^I_{t+1} & = & \mathcal{W}^I(x_t;\omega) \end{array}\right. \\ \mathcal{P}(x_t;\omega) & := & \left[\begin{array}{lcl} \mathcal{P}_1(x_t;\omega) & := & \left[\begin{array}{lcl} p_{1,t+1} & = & f_1(x_t;\omega) \\ p^1_{1,t+1} & = & p_{1,t} \\ p^2_{1,t+1} & = & p^1_{1,t} \\ \vdots & \vdots & \vdots \\ p^L_{1,t+1} & = & p^{L-1}_{1,t} \end{array}\right. \\ \vdots & & \vdots \\ \mathcal{P}_K(x_t;\omega) & := & \left[\begin{array}{lcl} p_{K,t+1} & = & f_K(x_t;\omega) \\ p^1_{K,t+1} & = & p_{K,t} \\ p^2_{K,t+1} & = & p^1_{K,t} \\ \vdots & \vdots & \vdots \\ p^L_{K,t+1} & = & p^{L-1}_{K,t} \end{array}\right. \end{array}\right. \end{array}\right. . \tag{2.7}$$

Let Δ^K denote the K-simplex and

$$\Delta^K_c = \left\{ \mathbf{x} \in \mathbb{R}^K \,\middle|\, \sum_{k=1}^{K} x_k \leq 1 \quad \text{and} \quad x_k \geq 0,\ k = 1,\ldots,K \right\}$$

denote the K-hyper-cube corner. Name Δ^K_+ and Δ^K_{c+} their respective subsets with all positive components. Due to normalizations in (2.4-2.5) each $\mathcal{F}(\omega)$ maps the set $\mathcal{X} = \Delta^I \times (\Delta^K_{c+})^{L+1}$ in itself. The component $\mathcal{W}$ characterizes the dynamics of agents' wealth fractions, whereas $\mathcal{P}$ fixes prices using market clearing and keeps track of their past values. For any given initial state x_0, the random dynamical system representing the market dynamics is defined iterating $\mathcal{F}(\omega)$:

$$\varphi(t,\omega,x_0) = \mathcal{F}(\theta^{t-1}\omega) \circ \ldots \circ \mathcal{F}(\theta\omega) \circ \mathcal{F}(\omega)x_0. \tag{2.8}$$

Notice that, in general, price, wealth, and investment decisions defined by (2.8) are not measurable with respect to the dividends process. In our framework, contrary to markets where investment rules depend only on exogenous information (see e.g. Blume and Easley, 1992; Amir et al., 2005), the dependence on endogenously determined prices implies that the map $\mathcal{F}$ does not need to preserve the properties of the stochastic process. As a result, even if we assume an ergodic dividend process, it is not granted that the price and wealth process will be ergodic, unless by imposing restrictions on the class of rules $\mathcal{A}$. For this reason, and given the arbitrariness of the dividend process, population size I, memory span L, we acknowledge that the analysis of the global dynamics generated by (2.8) with rules in $\mathcal{A}$ cannot be performed in total generality and shall concentrate on the local dynamics.

Moreover, having a multiple agent framework with heterogeneous investment behaviors, not necessarily derived from an utility maximization given preferences and expectations, we shall not apply the traditional rational expectation approach. In the present paper, instead, our interest lies in characterizing whether long-run wealth distributions where one or many agents have gained all the wealth exist and are stable.

Because of the stationary nature of the process governing the succession of states of the world and the lack of aggregate risk due to (2.4), long-run economic equilibria are characterized by constant prices and, in accordance with Assumption 1, constant investment shares. Owing to the market dynamics and wealth normalization, constant investment decisions αs and fixed long-run asset prices imply a constant wealth distribution. Hence, we are naturally lead to identify long-run market selection equilibria with the deterministic fixed points of the random market dynamics in (2.8) as defined by the following

Definition 2.1. The state $x^* \in \mathcal{X} = (w^*, p^*)$ is a deterministic fixed point of the random dynamical system φ generated by the family of maps $\mathcal{F}(\omega)$ if, for almost all $\omega \in \Omega$, it holds

$$\mathcal{F}(\omega)x^* = x^* \,, \tag{2.9}$$

which implies

$$\varphi(t, \omega, x^*) = x^* \quad \text{for every} \quad t \in \mathbb{N} \,. \tag{2.10}$$

Intuitively a deterministic fixed point can correspond to a single investor possessing the entire wealth of the economy. In this case, according to (2.3), asset prices are equal to the vector of investment decisions of this investor. Alternatively, many investors could have positive wealth at equilibrium. In this case the constraints imposed by the wealth dynamics require that they all take the same investment decision at equilibrium prices (see Section 4).

In any case, not all deterministic fixed points represent interesting asymptotic states. Indeed, in order for the market dynamics to actually converge to a deterministic fixed point starting from a neighborhood of it, the point must be asymptotically stable.

Definition 2.2. A deterministic fixed point x^* of the random dynamical system $\varphi(t, \omega, x)$ is *asymptotically stable* if, for almost all $\omega \in \Omega$ and for all x in a neighborhood $U(\omega)$ of x^*, $\lim_{t\to\infty} ||\varphi(t, \omega, x) - x^*|| \to 0$.

For some equilibria we will make use of the weaker notion of stability, which will be, in our case, sufficient to guarantee that orbits do not diverge from deterministic fixed point when initial conditions are sufficiently close to it.

Definition 2.3. A deterministic fixed point x^* of the random dynamical system $\varphi(t, \omega, x)$ is *stable* if, for any neighborhood V of x^* and for almost all $\omega \in \Omega$, there exists a neighborhood $U(\omega) \subseteq V$ of x^* such that $\lim_{t\to\infty} \varphi(t, \omega, x) \in V$ for all x in $U(\omega)$.

Notice that in the previous definitions the neighborhood U might depend on the realization of the process ω. If a deterministic fixed point is neither asymptotically stable nor stable we shall say that it is unstable. When characterizing deterministic fixed points and their local stability the following terminology, describing the long-run wealth distribution, will be useful.

Definition 2.4. An agent i is said to *survive on a given trajectory* generated by the dynamics (2.8) if $\limsup_{t\to\infty} w_t^i > 0$ on this trajectory. Otherwise, an agent n is said to *vanish on a trajectory.* A surviving agent i is said to *dominate on a given trajectory* if she is the unique survivor on that trajectory, that is, $\liminf_{t\to\infty} w_t^i = 1$

Importantly, survival and dominance are defined only with respect to a given trajectory and not in general. The reason is that we are going to work exclusively with local stability conditions so that an agent may survive on a given trajectory (i.e., for certain initial conditions or certain realizations of the process ρ) but vanish on another. A similar definition is given in Blume and Easley (1992) for a stochastic system like ours[3] and in Anufriev and Bottazzi (2010) and Anufriev and Dindo (2010) for deterministic systems.

Applying the previous definition to a deterministic fixed point, we shall say that agent i survives at x^* if her wealth share is strictly positive, $w^{*i} > 0$, while she vanishes if $w^{*i} = 0$. Such taxonomy can be applied both to a stable or unstable deterministic fixed point, but the implications are very different in the two cases. When the fixed point is stable, all trajectories starting in a neighborhood of it will stay close to it, so that a survivor in the fixed-point will also survive on all these trajectories. If, moreover, the agent is the unique survivor and the fixed point is also asymptotically stable, the agent will dominate on all trajectories starting inside a proper neighborhood. Conversely, when the fixed point is unstable, one is not able to characterize survival and dominance for trajectories starting close to it. Both vanishing and dominating agents at an unstable fixed point may survive as well as vanish on trajectories started in any of its neighborhoods, and in absence of global results one, in general, cannot say. In the rest of the paper we shall show that the constraints on the dynamics imposed by the dividend process, the market clearing, and the wealth evolution are sufficient to uniquely characterize the level of prices in the deterministic fixed points, to describe the corresponding distributions of wealth among agents, and to derive local stability conditions.

3 A toy market

In this section we shall consider the simplest market dynamics where the implication of bringing prices into the investment rule can be fully appreciated. In the discussion we will make use of analytical results whose formal derivation is postponed to Section 4.

Even though our results apply to payoff dividend processes represented by any non-trivial matrix D, in the following example we shall restrict our attention to a square identity matrix, that is, Arrow securities. We do so for two reasons. First, it is already known that when D is not a square matrix and/or has not full rank, that is, respectively, when states of the world are not measurable with respect to the dividend process or when asset payoffs are not linearly independent so that arbitrage opportunities may arise, market selection may reward different agents at different prices and/or work against the best informed agent.[4] Showing that this is the case also when Arrow securities are traded clarifies that the ultimate source of market instability lies not in the structure of asset payoffs, but rather in the lacking of coordination on price expectations. Indeed, the only difference between the complete market case analyzed in Sandroni (2000) and Blume and Easley (2006) and the repeated market for Arrow securities

[3]Notice however that in Blume and Easley (1992) *dominance* is defined as $\liminf_{t\to\infty} w_t^i > 0$ so that, as for survival, more than one rule may dominate on a given trajectory.

[4]See e.g. Section 3.3 in Evstigneev et al. (2009) and Section 5 in Blume and Easley (2006). Notice however that, as characterized in Sandroni (2005), there exists a subset of non-trivial non-square matrices D for which market selection does work.

we investigate here is that we do not assume *a priori* that the foresight of traders is perfect.[5] Second, when trading Arrow securities, our bounds on investment rules α can be naturally interpreted as no-bankruptcy conditions.

Consider an economy with two states of the world and two agents trading according to, respectively, rules α^1 and α^2 both in $\mathcal{A}$. Two Arrow securities are traded: security $k \in \{1, 2\}$ pays 1 if state of the world k is realized and 0 otherwise. We assume that the rules adopted by agents are continuously differentiable, that is $\alpha^1, \alpha^2 \in \mathcal{C}^1$, and we initially consider investment rules which depend only on the last observed price. In this case an unique price vector is determined at each time step and the global dynamics is well defined. Using the notation of the previous section we fix $K = S = 2$, $I = 2$, $L = 1$. To simplify the discussion we further assume that agents reinvest all their wealth. Because of wealth normalization and (2.5), both agents' wealth and asset prices add up to one in every period, so that we are left with a three dimensional random dynamical system: the wealth fraction of agent 1, the price of asset 1, and its first lag. Without loss of generality we shall assume that ω is the realization of a Bernoulli process: at every period t, $\omega_t = 1$ with probability π and $\omega_t = 2$ with probability $1 - \pi$.

Given the state of the market at time t, $x_t = (w_t, p_t, p_t^1 = p_{t-1})$, and the diagonal structure of the dividend payoff matrix the random dynamical system representing the market dynamics can be written as the composition of the following map

$$\begin{cases} w_{t+1} = \begin{cases} \frac{\alpha^1(p_t^1)w_t}{p_t} & \text{if} \quad \omega_{t+1} = 1 \\ \frac{\left(1-\alpha^1(p_t^1)\right)w_t}{1-p_t} & \text{if} \quad \omega_{t+1} = 2 \end{cases} , \\ p_{t+1} = \alpha^1(p_t)w_{t+1} + \alpha^2(p_t)(1 - w_{t+1}) \, , \\ p_{t+1}^1 = p_t \, . \end{cases} \tag{3.1}$$

We are interested in characterizing long-run market equilibria. Let f_π and $f_{1-\pi}$ stand for the map in (3.1) when $\omega = 1$ and $\omega = 2$, respectively. The deterministic fixed points of the system are the states x^* such that

$$x^* = f_\pi(x^*) \quad \text{and} \quad x^* = f_{1-\pi}(x^*) \, .$$

Straightforward computations show that there are three types of deterministic fixed points, namely

$$\begin{aligned} x_1^* &= (w^* = 1, p^* = \alpha^1(p^*), p^{1*} = p^*) \, , \\ x_2^* &= (w^* = 0, p^* = \alpha^2(p^*), p^{1*} = p^*) \, , \\ x_{1,2}^* &= (w^*, p^* = \alpha^1(p^*) = \alpha^2(p^*), p^{1*} = p^*) \, . \end{aligned}$$

Either one agent has all wealth and dominates, which occurs at x_1^* and x_2^*, or both agents have some wealth and survive, which occurs at $x_{1,2}^*$. In both cases prices are fixed points of the

[5] Because of the missing assumption of rational price expectations on agents' part, the market for Arrow securities is, *strictu sensu*, incomplete. More precisely, our markets are "endogenously" incomplete rather than "exogenously" incomplete as it happens when there are fewer independent assets than realized states. See also the difference between endogenous and exogenous uncertainty in Chichilnisky (1999) and Hahn (1999). Other authors refer to the repeated market for Arrow securities as sequentially complete but not necessarily complete, see e.g. Drèze and Herings (2008).

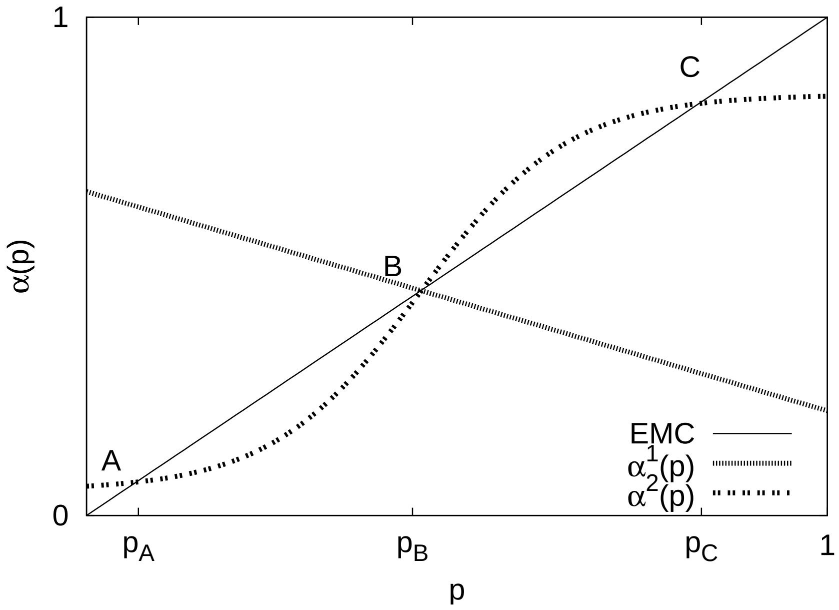

Figure 1: Investment rules $\alpha^1(p)$ and $\alpha^2(p)$ are plotted against the price of asset $k = 1$. Equilibrium prices are given by the coordinate of their intersections with the EMC, that is, points p_A, p_B and p_C.

survivor's investment rule, $p^* = \alpha^i(p^*)$, and each surviving agent receives the same earning in both states of the world.[6]

It is useful to use a plot to visualize the location of fixed points. In Fig. 1 we plot two generic investment rules as a function of the (lagged) price of the first asset p. The intersections of each investment rules (demand) with the diagonal (supply), that is, points A, B and C, are the different Walrasian equilibria corresponding to all possible deterministic fixed points of the system. Using the terminology introduced in Definition 2.4, in A and C agent 2 dominates and agent 1 vanishes. These are *single survivor* equilibria. Conversely, the existence of *multiple survivor* equilibria of the $x^*_{1,2}$ type, like the point B, in which both agents survive and neither dominates nor vanishes, requires that the first and second agent's investment rules intersect the diagonal at the same point.

No matter the shape of the investment rules, both single survivor and multiple survivors equilibria lie on the diagonal of the plot, that is, the function $f(p) = p$. For analogies with previous works (Anufriev et al., 2006; Anufriev and Bottazzi, 2010; Anufriev and Dindo, 2010) we name it the "Equilibrium Market Curve" (EMC) to stress that it is the locus of all long-run market equilibria.

The stability of the deterministic fixed points of the simple system in (3.1) depends upon

[6]If the investment rules are derived from expected utility maximization, absence of aggregate risk implies that equilibrium prices correspond, no matter the shape of the utility function, to the beliefs the surviving agent assigns to the occurrence of each state.

the values of the two quantities (c.f. Theorem 4.3 and Theorem 4.5)

$$\mu(w,p) = \left(\left(\frac{\alpha^2(p)}{\alpha^1(p)}\right)^{\pi} \left(\frac{1-\alpha^2(p)}{1-\alpha^1(p)}\right)^{1-\pi} \right)^{2w-1}, \tag{3.2}$$

$$\lambda(w,p) = w\frac{\partial \alpha^1(p)}{\partial p} + (1-w)\frac{\partial \alpha^2(p)}{\partial p}. \tag{3.3}$$

If both $\mu(w^*,p^*)$ and modulus of $\lambda(w^*,p^*)$ are smaller than one, then the single survivor equilibrium x_i^* with $i = 1, 2$ is asymptotically stable. In the case of multiple survivor equilibrium $x_{1,2}^*$, if $\lambda(w^*,p^*)$ has modulus smaller than one then the equilibrium is stable (but not asymptotically stable). Conversely, if either μ or $|\lambda|$ are grater than one, then the corresponding equilibrium is unstable. Thus, there are two different sources of instability.

Regarding μ, consider the relative entropy of the investment rule of agent i at price p, $(\alpha^i(p), 1-\alpha^i(p))$, with respect to the distribution of prices $(p, 1-p)$, defined as

$$I_\pi(\alpha^i, p) := \pi \log \frac{p}{\alpha^i(p)} + (1-\pi) \log \frac{1-p}{1-\alpha^i(p)}. \tag{3.4}$$

It is immediate to realize that $\log(\mu(w^*,p^*))$ is equal to the relative entropy of the survivor's rule minus the relative entropy of the rule of the vanishing agent, computed at equilibrium prices. Thus, the deterministic fixed point can be asymptotically stable only if the surviving agent is the one whose investment rules has, at equilibrium prices, the lowest entropy. The intuition is that in this case the surviving agent invests, on average, better than the other agent, and, consequently, her wealth share grows at an average positive rate. The fulfillment of this condition can be directly appreciated in the EMC plot. In Fig. 2 all curves are the same as in Fig. 1 with the addition on the horizontal line π equal to the probability of occurrence of state 1. The distance between this line and α^i at a given price p is monotonically related to $I_\pi(\alpha^i, p)$. Consider the point C where agent 2 dominates and 1 vanishes, that is, $w^* = 0$ and $p^* = p_C = \alpha^2(p_C)$. Since the distance from the π line is larger for $(p_C, \alpha^2(p_C))$ than for $(p_C, \alpha^1(p_C,))$, it is $I_\pi(\alpha^1, p_C) < I_\pi(\alpha^2, p_C)$ and this point is unstable. Conversely, in p_A the curve nearest to the π line is α^2, so that, at least according to this criterion, the point A is stable.

Concerning the second quantity, λ, it depends on the relation between past realized prices and present investment decisions. When $|\lambda(w^*,p^*)| > 1$ price feedbacks are too strong for the dynamics to settle down, a form of deterministic overshooting similar to the instability observed in price adjustment processes. At equilibrium, only the investment rules of the surviving agents are relevant to define stability with respect to λ. Given the slope of $\alpha^2(p)$ at p_A and p_C, both A and C are stable under past prices feedback.[7] Since A is also stable when looking at the relative entropy, it represents an asymptotically stable single survivor equilibrium and a possible outcome of the long-run market dynamics. In the example of Fig. 2 it is the unique single survivor stable equilibrium, but it is not the unique long-run equilibrium. We have still to evaluate the stability with respect to past prices feedback of B, where both agents survive. Locally, the market dynamics is equivalent to the one generated by a single agent whose investment rule is the wealth weighed average of both surviving rules. Since $|\partial\alpha^1(p_B)| < 1$ and $|\partial\alpha^2(p_B)| > 1$, if w is large enough then, for continuity, $|\lambda(w^*,p^*)| < 1$ and

[7] Given **P1-P3** in Assumption 1 and upon continuity, each rule has at least one interior intersection with the EMC with derivative lower than one in absolute value, so if one agent is alone in the market, there exists at least one stable fixed point.

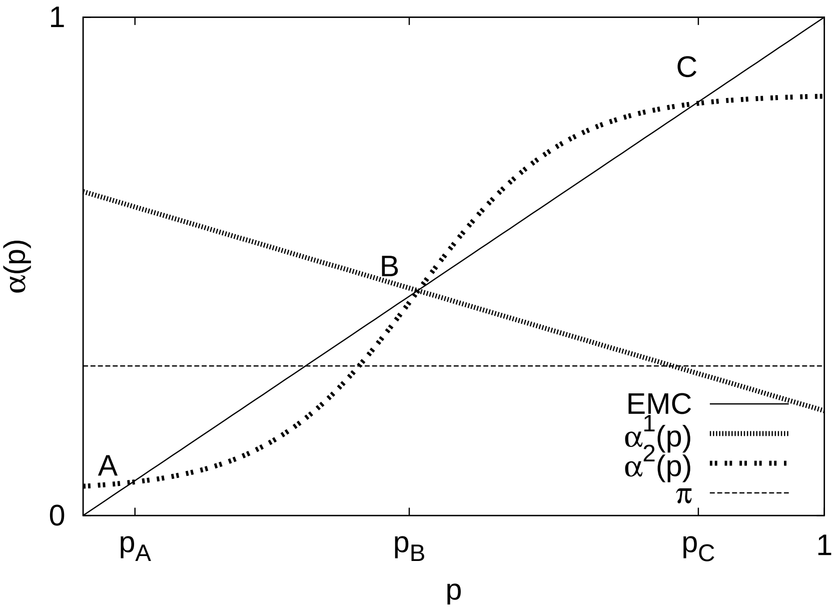

Figure 2: The local stability of deterministic fixed points A, B, C can be appraised graphically. Let i denotes the surviving agent(s) and p^* the equilibrium price, then $\mu(w^*, p^*)$ is smaller than one if $\alpha^i(p)$ is, in p^*, the nearest curve to the horizontal line π, representing the dividend payment probability, and $|\lambda(w^*, p^*)|$ is smaller than one if the slope of $\alpha^i(p^*)$ is not steeper than the EMC.

the point is stable. Conversely, for smaller values of w, the over-reaction to price movement of α^2 destabilizes the equilibrium. Notice at last that since in B investment decisions of both agents are equal, the distance of their rules in terms of relative entropy is zero and $\mu(w^*, p_B) = 1$. For this reason if $|\lambda(w^*, p^*)| < 1$ the fixed point is stable but not asymptotically stable. A perturbation can indeed generate a permanent change in the distribution of wealth. Prices will converge back to their equilibrium level p_B but the system will end up in a fixed point with a different value of w.

If investment rules depend only on current prices, the random dynamical system simplifies to

$$w_{t+1} = \begin{cases} \frac{\alpha^1(p_t) w_t}{p_t} & \omega_{t+1} = 1 \\ \frac{(1-\alpha^1(p_t)) w_t}{1-p_t} & \omega_{t+1} = 2 \end{cases}, \tag{3.5}$$

where $p_t(w_t)$ is a solution of

$$p_t = \alpha^1(p_t) w_t + \alpha^2(p_t)(1 - w_t) . \tag{3.6}$$

As already discussed in Section 2, (3.6) can possess multiple solutions, so that the global dynamics may be ill-defined. Concerning deterministic fixed points, however, they are the same of the previous case. Moreover, as long as the intersections of the investment rules with the EMC are isolated points, (3.6) possesses an unique solution in a neighborhood of any fixed point. Due to the differentiability assumption of the αs, it is sufficient to require that $w^* \partial_p \alpha^1(p^*) + (1 - w^*) \partial_p \alpha^2(p^*) \neq 1$. Stability is now only decided by the value of $\mu(w, p)$: since the investment rules do not depend on past prices there is no room for the destabilizing role of price feedbacks.

Summarizing, irrespectively of the fact that price dependence is on past or present prices, the market rewards agents whose equilibrium prices are "closest" (in entropy terms) to those of the underlying asset dividend process. Notice however that, differently from the result in Blume and Easley (1992, 2006), Sandroni (2000) or the works surveyed in Evstigneev et al. (2009), in our framework this result applies only locally. It can well happen, like in the example of Fig. 2, that multiple stable equilibria do exist or, alternatively, that none of the deterministic fixed points is stable. In the latter case, Alice is doing better at Bob's prices and Bob is doing better at Alice's prices so that neither Alice nor Bob prevail and market prices fluctuates indefinitely. In both cases it is not granted that prices reflect the best available information so that informational inefficiencies may arise.

4 Main results

This section is devoted to the formal investigation, in a more general case, of the possible sources of market instability discussed in the simple toy model of the previous section. For this purpose we consider a market populated by I investors trading K assets using investment rules depending on a vector of current and past L prices. We derive results about the existence and local stability of deterministic fixed points, or long-run market equilibria, for market dynamics described by (2.8). In presenting our findings it is convenient to treat the case of single survivor equilibria first and move to the multiple survivors case at a later stage.

4.1 Single survivor equilibria

While Theorem 2.1 guarantees, under mild conditions, the existence of a market clearing price vector, one cannot in general assume its uniqueness. Since all our results about long-run properties of the market are local, we are not very disturbed by this limitation. In what follows, our first step will be to characterize the single survivor deterministic fixed points. Then, we will provide sufficient conditions for the existence of a well-defined *local* dynamic around them and for their asymptotic stability.

If an agent possesses all wealth, prices are fixed at the intersection of her investment rule with the EMC, now the vector valued function $f(\mathbf{p}) = \mathbf{p}$, exactly as in the example in Section 3. If no other rules intersect the EMC at the same prices, we have a single survivor equilibrium.

Theorem 4.1. *Consider a market for K short-lived assets with non-trivial payoff matrix D, where I agents invest according to rules in $\mathcal{A}$ using L price lags. Assume agents' wealths and asset prices evolve according to φ in* (2.8)*. If there exists an agent $i \in \{1, \dots, I\}$ and a price vector $\mathbf{p}^*$ such that $\alpha^i(p^*) = \mathbf{p}^*$ and $\alpha^j(p^*) \neq \mathbf{p}^*$ for every $j \neq i$, where $p^* = (\mathbf{p}^*, \dots, \mathbf{p}^*)$, then $x^* = (w^*, p^*)$ with $w^{i*} = 1$ and $w^{j*} = 0$ for $j \neq i$ is a deterministic fixed point.*

The fixed point $x^* = (w^*, p^*)$ represents a single survivor equilibrium in which agent i dominates. Given x^*, a well-defined local dynamics exists for continuous differentiable investment rules provided that the excess demand function has an isolated zero in p^*. Sufficient conditions can be obtained using the implicit function theorem:

Theorem 4.2. *Under the hypothesis of Theorem 4.1, let x^* be a single survivor fixed point where, without loss of generality, agent I-th dominates. Assume further that all investment* *rules $i \in \{1, \ldots, I\}$ are continuously differentiable in a neighborhood of p^*, $\alpha^i \in \mathcal{C}^1(p^*)$. If the matrix*

$$H := \begin{pmatrix} (\alpha_1^I)^{1,0} - 1 & (\alpha_1^I)^{2,0} & (\alpha_1^I)^{3,0} & \dots & (\alpha_1^I)^{K,0} \\ (\alpha_2^I)^{1,0} & (\alpha_2^I)^{2,0} - 1 & (\alpha_2^I)^{3,0} & \dots & (\alpha_3^I)^{K,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\alpha_K^I)^{1,0} & (\alpha_K^I)^{2,0} & (\alpha_K^I)^{3,0} & \dots & (\alpha_K^I)^{K,0} - 1 \end{pmatrix}, \tag{4.1}$$

is non-singular, where

$$(\alpha_k^i)^{h,l} := \left.\frac{\partial \alpha_k^i\,(p)}{\partial p_h^l}\right|_{x^*}, \quad i = 1, \ldots, I\,, \quad l = 0, 1, \ldots, L\,, \quad k, h = 1, \ldots, K\,, \tag{4.2}$$

then the dynamics is locally well-defined, that is, for every $\omega \in \Omega$ there exists a neighborhood $U(\omega)$ of x^* where prices and wealths evolve according to (2.8).

Notice that when the investment rule of agent I does not depend on current prices it holds $H = -\mathbb{I}$ and the non-singularity condition is trivially met.

Once the local dynamics in a neighborhood of a deterministic fixed point is well defined, the crucial issue is to asses whether an agent dominating (or vanishing) in the fixed point is also dominating (or vanishing) on all trajectories started close enough to it. The next theorem provides sufficient conditions for the asymptotic stability or instability of a deterministic fixed point when locally continuous differentiable investment rules are considered

Theorem 4.3. *Under the hypothesis of Theorem 4.2, consider the fixed point $x^* = (w^*, p^*)$ of Theorem 4.1 where $w^{I*} = 1$ and $\alpha^I(p^*) = \mathbf{p}^*$ and assume that the matrix H defined in* (4.1) *is non-singular. Consider the $I - 1$ quantities*

$$\mu_i := \prod_{s=1}^{S} \left(\sum_{k=1}^{K} \frac{\alpha_k^i(p^*)}{\alpha_k^I(p^*)} D_{s,k} \right)^{\pi_s}, \quad i = 1, \ldots, I - 1\,, \tag{4.3}$$

where π_s is the probability assigned by the invariant measure to state s, and the polynomial in λ of LKth degree

$$P(\lambda) := \sum_{l_1=1}^{L} \ldots \sum_{l_K=1}^{L} \lambda^{LK - \sum_j l_j} \sum_{\sigma} sgn(\sigma) \prod_{k=1}^{K} \left((\bar{\alpha}_k^I)^{\sigma_k, l_{\sigma_k}} - \lambda\, \delta_{k,\sigma_k}\, \delta_{l_{\sigma_k},1}\,, \right) \tag{4.4}$$

where

$$(\bar{\alpha}_k^I)^{h,l} := -\sum_{k'=1}^{K} \{H^{-1}\}_{k,k'} (\alpha_{k'}^I)^{h,l} \quad l = 0, 1, \ldots, L\,, \quad k, h = 1, \ldots, K\,,$$

$(\alpha_{k'}^I)^{h,l}$ are defined in (4.2)*, and σ are the permutation of the set $\{1, \ldots, K\}$. If all μ_i and all the roots of $P(\lambda)$ have module smaller than one, then the fixed point x^* is asymptotically stable. If, for some i, $\mu_i > 1$ or if a root of $P(\lambda)$ has absolute value greater than one, then the fixed point x^* is unstable. Moreover, if the k-th component of the I-th investment rule α_k^I depends only on asset k prices, which we name no-cross-dependence condition, $P(\lambda)$ simplifies to*

$$P(\lambda) = \prod_{k=1}^{K} \left(\lambda^L - \sum_{l=1}^{L} \lambda^{L-l} (\bar{\alpha}_k^I)^{(k,l)} \right), \tag{4.5}$$

and each $(\bar{\alpha}^I_k)^{h,l}$ *to*

$$(\bar{\alpha}^I_k)^{h,l} = \frac{(\alpha^I_k)^{h,l}}{1-(\alpha^I_k)^{h,0}} \quad l = 0, 1, \ldots, L\ , \quad k, h = 1, \ldots, K\ , \ .$$

The quantity μ_i defined in (4.3) is the long-run average wealth growth rate of agent i when prices are determined by agent I.[8] If its value is greater than one, the dominance of agent I can be effectively challenged by an agent i with an infinitesimal fraction of wealth and, as a result, the fixed point is destabilized. Conversely, if its value is lower than one, the exogenous transfer of a small amount of wealth from agent I to agent i would be naturally reverted back by market forces. The origin of the μs can be understood in terms of relative entropy. When μ_i is smaller (greater) than one, the relative entropy of the dominating rule with respect to the invariant measure of the dividend process is lower (greater) than the same quantity for the competing rule i. This is basically the same condition already found by Blume and Easley (1992) and all subsequent works analyzing market selection between rules depending on assets dividends. The relevant difference is that in their case the differences in relative entropy are global, while in our case they are local and depend on prevailing prices. Quantities μs implicitly take into account different consumption patterns. In fact, if agent i invests at equilibrium proportionally to agent I, that is, if there exists a positive constant c such that $\alpha^i = c\alpha^I$, then it holds $\mu^i = c$. As a result, if two agents have the same portfolio rules, that is they split their investment across the different assets in exactly the same way, but are characterized by different consumption rates, any single survivor equilibrium in which the agent who consumes the most dominates is unstable. In other terms, with equal portfolio rules the agent who consumes the most can never dominate the economy.

The second set of stability conditions pertains to the values of λs. These are the roots of a polynomial which depends on the derivatives of the surviving investment rule. The strength of price feedback in this rule becomes a separate source of market instability, independent from the relative entropy of the adopted strategy. Even though this polynomial is heavily simplified under the no cross-dependence condition, a characterization of its root is only possible when specific investment rules are given. In any case, by continuity, it holds that if investment rules are rather flat functions of past prices, so that their partial derivatives are close to zero, the fixed point is asymptotically stable. Indeed, as a straightforward application of Theorem 4.3 one has the following

Corollary 4.1. *Under the hypothesis of Theorem 4.3, when the investment rule* α^I *depends only on current prices the asymptotic stability of* x^* *depends only on the value of of* μ*s as defined in* (4.3)*.*

4.2 Multiple survivors equilibria

Similar results can also be obtained for fixed points where more rules have positive wealth, or multiple survivors equilibria. These equilibria are associated with prices at which multiple investment rules intersect, at once, the EMC. In this case, at equilibrium, all surviving agents take the same investment decision. These equilibria are in general not isolated points, but lay on a differentiable manifold with dimension equal to the number of potential survivors minus one.

[8] When the matrix D has not full rank, arbitrage opportunities might arise and the wealth of agent i might grow or shrink deterministically at each time step. This is not possible if D has full rank.

Theorem 4.4. *Consider a market for K short-lived assets with non-trivial payoff matrix D, where I agents invest according to a rule in $\mathcal{A}$ using L price lags. Assume agents' wealths and asset prices evolve according to φ in* (2.8)*. If there exists a price vector $\mathbf{p}^*$ and M agents, say the last M, such that $\alpha^i(p^*) = \mathbf{p}^*$ for $I-M+1 \leq i \leq I$ and $\alpha^i(p^*) \neq \mathbf{p}^*$ for $i \leq I-M$, with $p^* = (\mathbf{p}^*, \dots, \mathbf{p}^*)$, then the set of non-negative wealth shares $w^{M*}, \dots, w^{I*}$ such that $\sum_{m=I-M+1}^{I} w^{m*} = 1$ defines a manifold of deterministic fixed points $x^* = (w^*, p^*)$ where the last M agents possess all the wealth and the first $I-M$ agents have zero wealth and vanish, that is $w^{i*} = 0$ for $i \leq I-M$.*

Surviving agents fix asset prices at their common intersection with the EMC. Each common intersection defines a manifold of fixed points x^* because each reallocation of wealth among surviving agents does not change the equilibrium prices and is still a fixed point. As a result, in principle, some potentially surviving agents can possess a zero wealth share. The manifold of multiple survivor equilibria is isomorphic to Δ^M. We turn now to the specification of the sufficient conditions for the stability or instability of $x^* = (w^*, p^*)$. The following theorem generalizes both Theorem 4.2 and 4.3 to the present case.

Theorem 4.5. *Consider the manifold of fixed points $x^* = (w^*, p^*)$ of Theorem 4.4 and assume that all investment rules $i \in \{1, \dots, I\}$ are continuously differentiable in a neighborhood of p^*, $\alpha^i \in \mathcal{C}^1(p^*)$. Sufficient conditions for the existence of a well-defined local dynamics in a neighborhood of x^* and for the stability or instability of x^* are the same as those specified, respectively, in Theorem 4.2 and 4.3 provided that*

(i) condition (4.3) *is checked only for the first $I-M$ rules,*

(ii) in the definition of $(\bar{\alpha}_k)^{h,l}$, H, and thus $P(\lambda)$, the expression $(\alpha^I_k)^{h,l}$ is replaced by

$$\langle \alpha_k \rangle^{h,l} := \sum_{m=I-M+1}^{M} (\alpha^m_k)^{h,l} w^{m*} \quad l = 0, 1, \dots, L\,, \quad k, h = 1, \dots, K\,, . \tag{4.6}$$

Intuitively, results for multiple survivors fixed points mimic those for a single survivor with a rule equal to the weighted average of all surviving rules, with weights equal to their equilibrium wealth shares. Notice that if at a fixed point x^* all I agents take the same investment decision, all generalized eigenvalues μ will be equal to one, so that the only binding necessary condition for local stability will be given by the roots of the polynomial $P(\lambda)$, representing the strength of the "average" price feedback. This is exactly what happened in the two agents toy market encountered in the example of Section 3 and what will happen in the case considered in Section 5.3. Notice also that while the statement in Theorem 4.3 concerns *asymptotic* stability, the conditions of Theorem 4.5 only assure stability. This is the obvious consequence of the fact that multiple survivor fixed points are non-isolated points.

5 Survival, ordering, learning: examples

In the remaining part of the paper we shall illustrate some implications of our results by considering three specific issues. Firstly, in Section 5.1 we characterize the rule α^S, or S-rule, as the rule that never vanishes when trading against any rule in $\mathcal{A}$ for which the stability analysis of Section 4 can be applied. Notwithstanding the existence of such a special rule, in Section 5.2 we show with a counter example that the survival relation on the set of rules is not transitive and thus rules cannot be ordered according to their "survivability". Last, in Section 5.3 we consider the specific class of investment rules which depend on some given statistics of past prices, as in the case in which agents use the observed average past prices and its variance to forecast future asset performances, and compare their ability to survive in a market populated by α^S investors. Before we start it is convenient to be specific about the survival relation. We consider an ecology of strategies whose behavior is sufficiently smooth for their local dynamics to be analyzed using the theorem developed in the previous section. These strategies are confronted pairwise to inspect their relative asymptotic behavior. Formally,

Definition 5.1. For any $\alpha \in \mathcal{A}$, let $E(\alpha)$ be the set of intersections of the strategy α with the EMC. Consider an ecology $\mathcal{E} \subset \mathcal{A}$ such that if $\alpha \in \mathcal{E}$ then $\alpha \in \mathcal{C}^1(\cup_{\beta\in\mathcal{E}} E(\beta))$, that is, for any intersection of any strategy in $\mathcal{E}$ with the EMC there exists a neighborhood in which all strategies in $\mathcal{E}$ are continuously differentiable. The ordered couple (α^1, α^2) belongs to the relation $\succeq \subset \mathcal{E} \times \mathcal{E}$, or $\alpha^1 \succeq \alpha^2$, if when rules α^1 and α^2 are competing alone in the market, for almost all initial conditions $x_0 \in \mathcal{X}$ and almost all $\omega \in \Omega$, an agent using rule α^1 either dominates or survives but does not vanish.

Since the set $\mathcal{E}$ is clearly not unique, many different relations can be built. Moreover, while $\mathcal{A}$ and $\mathcal{E}$ depend only on the number of assets K, price lags L, and matrix D, the relation $\succeq$ depends also on the process governing the states of the world so that different market settings are characterized by different relations.[9]

5.1 A special rule

Given the local stability analysis of the previous section, we are able to characterize the investment rule α^S that never vanishes against any given rule. On the set $\Delta^K_{c+} \times \Delta^K_{c+}$ define the function

$$I_\pi(\alpha, \mathbf{p}) = -\sum_{s=1}^{S} \pi_s \log\left(\sum_{k=1}^{K} \frac{\alpha_k}{p_k} D_{s,k}\right) ,$$

where D is a non-trivial dividend payoff matrix and π a given invariant measure. The quantity $I_\pi(\alpha, \mathbf{p})$ is the relative entropy of the (investment) vector α with respect to the (price) vector $\mathbf{p}$: a generalization of (3.4) to the case of a generic non-trivial payoff matrix D and possibly defective probabilities α and $\mathbf{p}$. Fix now the asset structure, that is, π and D. We define α^S, or S-rule, as the price dependent investment rule that minimizes the exponential of $I_\pi(\alpha, \mathbf{p})$ for each given price vector $\mathbf{p}$. Even if the S-rule cannot be in general explicitly derived, its existence and continuity is assured by the compactness and convexity properties of the minimization problem as shown by the following

Theorem 5.1. *For any given probability measure π and non-trivial matrix D, there exists a continuous vector function $\alpha^S(\mathbf{p}) : \Delta^K_+ \to \Delta^K$, also named S-rule, solving*

$$\alpha^S(\mathbf{p}) = \underset{\alpha\in\Delta^K_c}{\operatorname{argmin}} \{\exp I_\pi(\alpha, \mathbf{p})\} . \tag{5.1}$$

For internal solutions, that is, when $\alpha^S \in \Delta^K_+$, α^S is of class $\mathcal{C}^1$. Moreover, $\alpha^S(\mathbf{p}) = \mathbf{p}$ if and only if $p_k = \sum_{s=1}^S \pi_s D_{s,k}$ for every $k = 1, \dots, K$.

[9] For each given dividend process one could consider the relations on $\mathcal{E}$ as s sub-sets of an analogous relation defined over $\mathcal{A}$. The latter cannot however be analyzed in whole generality due to the differentiability conditions required by Theorems 4.3 and 4.5

As a result of the maximization, the S-rule implies zero consumption. Importantly, in order to invest according to the S-rule, an agent must possess a perfect knowledge about the invariant measures π on the states of the world. In case of a normalized diagonal matrix D (Arrow securities) the S-rule coincides with the so called Kelly rule, that is, $\alpha^S_s(\mathbf{p}) = \pi_s$. It is so in the toy market analyzed in Section 3, where the S-rule coincides with the line π plotted in Fig. 2. For more general payoff matrices D, the S-rule depends on prices and can be interpreted as a generalization of the Kelly rule.[10] For example for $D = \begin{pmatrix} 1, & 0 \\ 1/2, & 1/2 \end{pmatrix}$ and $\pi = (0.4, 0.6)$ the S-rule can be easily found to be $\alpha^S(p) = 1$ for $p \leq 5/8$ and $\alpha^S(p) = 0.4p/(2p-1)$ otherwise, where α is the fraction to be invested in the first asset and p is its price.

The central result of this section is that α^S never vanishes against any other rule for which our local stability analysis can be applied, that is,

Theorem 5.2. *Given an ecology $\mathcal{E}$ as in Definition 5.1, if $\alpha^S \in \mathcal{E}$ then $\alpha^S \succeq \alpha$ for every $\alpha \in \mathcal{E}$.*

The previous theorem exploits the local analysis of Section 4 to infer properties of the global dynamics: on every trajectory the S-rule does not vanish, because otherwise the system would converge to an equilibrium in which the other rule dominates, which is shown to be impossible. Since Theorem 5.1 guarantees that α^S is of class $\mathcal{C}^1$ when strictly positive, and in particular at its intersection with the EMC, the set of ecologies $\mathcal{E}$ where the theorem 5.2 can be applied is not trivial. However, the same reasoning cannot apply in a market with more than two rules. Still, in the latter case, the local analysis reveals that the S-rule always survives in any stable long-run equilibrium and fixes the price of assets according to their fundamental values.

Theorem 5.3. *Consider an ecology $\mathcal{E}$ as in Definition 5.1, with $\alpha^S \in \mathcal{E}$. All deterministic fixed points $x^* = (w^*, p^*)$ where α^S vanishes are unstable. Moreover, there exists at least one stable deterministic fixed point in which α^S survives and long-run asset prices are equal to $p^*_k = \sum_{s=1}^{S} \pi_s D_{s,k}$, for all $k = 1, \dots, K$.*

Notice at last that one can define an S-rule associated with any consumption level α_0 by conditioning the minimization in (5.1) with the constraints $\sum_{k=1}^{K} \alpha_k = 1 - \alpha_0$. It is easy to see that this rule will still satisfy Theorems 5.2 and 5.3 if the ecology of strategies $\mathcal{E}$ is restricted to the rules having the same (or higher) consumption rates.

5.2 Dominance and ordering

Having shown the special role of α^S for the relation $\succeq$, the next question is whether $\succeq$ induces an order relation on the set $\mathcal{E}$. To this purpose, consider a repeated market with 2 states of the world with equal probability to occur and 2 Arrow securities. Consider further the following zero-consumption investment rules expressed as fraction of wealth to be invested in the first asset (the rest is invested in the second) whose price is denoted as p:

$$\alpha^1(p) = 0.3, \quad \alpha^2(p) = \begin{cases} 0.9 & p \leq 0.2 \\ 1.5 - 3p & 0.2 < p \leq 0.3 \\ 0.6 & p > 0.3 \end{cases}, \quad \alpha^3(p) = \begin{cases} 0.2 & p \leq 0.3 \\ p - 0.1 & p > 0.3 \end{cases}.$$

[10] Notice, however, that the S-rule is different from the generalized Kelly rule, $\alpha^K_k = \sum_{s=1}^{S} \pi_s D_{s,k}$ for all $k = 1, \dots, K$, defined in e.g. Evstigneev et al. (2009). It is true, however, as shown in Th. 5.1, that the intersections of the two rules with the EMC coincide.

It is immediate to see that all these three rules belong to the same ecology $\mathcal{E}$. Let us start from the case in which only rules 1 and 2 compete on the market. The market dynamics is as in Section 3 with (3.5) updating market state variables and with prices implicitly set by (3.6). Naming w the wealth fraction of strategy 1 and solving (3.6) for market prices gives:

$$p_t = 0.6 - 0.3 w_t \,.$$

The price of asset 1 is always between 0.3 (when $w = 1$) and 0.6 (when $w = 0$). Plugging this price equation in (3.5) one obtains the 1-dimensional dynamical system describing the evolution of the market. It is straightforward to check (e.g. by plotting $\alpha^1(p)$ and $\alpha^2(p)$ on the EMC plot) that there exist two single survivor equilibria: one with $w^* = 1$ and $p^* = 0.3$, so that rule 1 dominates, and one with $w^* = 0$ and $p^* = 0.6$, so that rule 2 dominates. According to Theorem 4.3 only the second equilibrium is asymptotically stable, that is, rule 2 dominates on all trajectories starting in a neighborhood of $w = 0$. Importantly, for almost all initial conditions and for almost all realizations of the dividend process the market dynamics will never converge to $w = 1$, so that α^2 never vanishes. As a result $\alpha^2 \succeq \alpha^1$.

Next consider the case in which rules 1 and 3 are trading. Market clearing price as a function of w, the wealth fraction of rule 1, reads

$$p_t = 0.2 + 0.1 w_t \,,$$

and is bounded between 0.2 and 0.3. As in the previous case, one single survivor equilibrium, the one associated with $w^* = 1$, is asymptotically stable whereas the other is unstable. As a result we have $\alpha^1 \succeq \alpha^3$.[11]

Finally, consider the case in which rule 2 and 3 are present in the market. The dynamics is now slightly more complicated. The price of asset 1 as a function of agent 2 wealth fraction w reads

$$p_t = \begin{cases} \dfrac{0.2 + 1.3 w_t}{1 + 3 w_t} & w_t \in \left[0, \dfrac{1}{4}\right] \\[2ex] 0.7 - \dfrac{0.1}{w_t} & w_t \in \left(\dfrac{1}{4}, 1\right] \end{cases}$$

which is always between 0.2 (for $w = 0$) and 0.6 (for $w = 1$). As before, only single survivor equilibria exist and it is easily checked that the fixed point is asymptotically stable when $w = 0$ whereas it is unstable when $w = 1$. As a result it will never happen, unless for the measure zero initial condition $w = 1$, that rule 3 vanishes so that $\alpha^3 \succeq \alpha^2$.

As the previous example makes clear, the relation $\succeq$ is not transitive: $\alpha^2 \succeq \alpha^1$, $\alpha^1 \succeq \alpha^3$, but it is not true that $\alpha^2 \succeq \alpha^3$. Hence, $\succeq$ is not an order relation. This result does not depend on differences in the consumption rates, as we assumed the same consumption (zero) for all agents, nor on "exogenous" incompleteness, as agents are trading Arrow securities. Its ultimate reason is that, with price-dependent rules, only local results can be established so that long-run dominance may differ at different prices. In fact, the same relation would be transitive, and thus could be used to order investment rules, in the special case of investment rules not depending on asset prices. In this case local results become global.

[11] In these first two examples, since for all possibly realized prices one rule has a lower relative entropy than the other, we can infer that the locally stable single survivor equilibrium is also globally stable. In fact, one could also define a stricter relation $\succ$ based on the global dominance of one strategy on the other and have $\alpha^2 \succ \alpha^1$ and $\alpha^1 \succ \alpha^3$.

5.3 Learning from prices

In Section 5.1 we have established that the S-rule, not being dominated by any other rule, never vanishes. Is it the S-rule rule the unique rule having this property? The answer is negative. Indeed one can construct many different rules by working on the local stability conditions derived in Section 4. In this section we concentrate on one such example by considering a rule that, in using only market information given by past prices, "adapts" to any other rule and thus is never dominated, in particular not even by the S-rule. We first characterize the properties of this price learner and then use them to appraise its survivability when competing against the S-rule.

Consider a rule α^L of class $\mathcal{C}^1$ such that, given D and π, the ecology $\mathcal{E} = \{\alpha^L, \alpha^S\}$ is well defined. Assume further that α^L depends on some statistics, like mean or variance, computed on a finite number, say L, of past realized prices[12], satisfies the no-cross-dependence condition, prescribes the reinvestment of all the wealth, and is consistent, that is, $\alpha^L_k(p) = p_k$, $k = 1, \ldots, K$ for any constant price vector $p = (\mathbf{p}, \ldots, \mathbf{p})$. If the statistics on which the strategy α^L assigns equal weights to the L past prices, as in the case of moving averages, then all partial derivatives computed at the fixed points are equal. This implies a substantial simplification in the expression of (4.4) which in turn leads to the following

Theorem 5.4. *Consider a deterministic fixed point x^* in which only the agent using rule α^L survives. Assume that the agent investment rule does not depend on present prices, satisfies the no-cross-dependence condition, is consistent and, moreover, for every $k = 1, \ldots, K$, all partial derivatives are equal, or*

$$(\alpha^L_k)^{k,l} = (\alpha^L_k)^{k,l'}\,, \quad \text{for every} \quad l, l' = 1, \ldots, L \quad k = 1, \ldots, K\,. \tag{5.2}$$

Define $(\alpha^L_k)_{x^*}$ the common value of the partial derivative of investment rule k at the fixed point x^*. All the roots of polynomial $P(\lambda)$ defined in Theorem 4.3 are inside the unit circle provided that

$$(\alpha^L_k)_{x^*} \in \left(-1, \frac{1}{L}\right)\,, \quad \text{for every} \quad k = 1, \ldots, K\,. \tag{5.3}$$

The extension of the previous result to the multiple survivors case is straightforward: conditions are not on partial derivatives $(\alpha^L_k)^{k,l}$ but on convex combinations of partial derivatives of the type $\langle \alpha_k \rangle^{k,l}$. In this case the equilibrium could be stable for some mixtures of strategies and unstable for others. When this is the case, the stability condition can be re-written in terms of which wealth distributions among survivors guarantee stability.

We can now apply the previous result to a market populated by a price learner and an agent using the S-rule, whose wealth fraction is denoted by w. Consider a fixed point $x^* = (w^*, p^*)$, with $\mathbf{p}^* = \alpha^S(\mathbf{p}^*)$, where both agents survive. Then, under the assumptions of Theorem 5.4, one has the following

Corollary 5.1. *Given $x^* = (w^*, p^*)$, with $\mathbf{p}^* = \alpha^S(\mathbf{p}^*)$, let $(\alpha^L_k)_{x^*}$ be the partial derivatives of the k-th investment rule of the price learner with respect to price. If for every $k = 1, \ldots, K$*

$$w^* > 1 - \frac{1}{(\alpha^L_k)_{x^*} L} \quad \text{when} \quad (\alpha^L_k)_{x^*} > 0\,, \tag{5.4}$$

$$w^* > 1 - \frac{1}{|(\alpha^L_k)_{x^*}|} \quad \text{when} \quad (\alpha^L_k)_{x^*} < 0\,, \tag{5.5}$$

[12] Several so called "technical" rule of chartist inspiration, like trend detection, ceiling or floor crossing etcetera fall in this category.

then the fixed point x^* is stable.

The intuition behind this result is simple. Given a value of $(\alpha_k^L)_{x^*}$ there always exists an appropriate bound on the fraction of wealth of the S-rule agent which assures that the portfolio of a price learner asymptotically approaches the market portfolio, whose wealth is constant and thus never vanishes, at an exponential rate. As a result, a price learner never vanishes when trading with an agent using the S-rule, in that there always exists a finite wealth fraction of the former that stabilizes the deterministic fixed point. Since it is never the case that the S-rule dominates a price learner, we have established that $\alpha^L \succeq \alpha^S$.[13] The assumption of zero-consumption is essential to the proof. Indeed any rule with a positive consumption rate would vanish against the S-rule.

6 Conclusion

We have investigated wealth-driven selection and market behavior in a repeated market for short-lived assets where demands are expressed as a fraction of wealth and depend on the vector of current and past prices. We have derived local stability conditions of long-run market equilibria where a rule, or a group of rules, dominate and asset prices are fixed to a constant value. Our results show that instability of these long-run market equilibria is a common phenomenon that might lead to asset mis-pricing and informational inefficiency. We have identified two different sources of price endogenous fluctuations, namely investment rules having too strong past prices feedbacks and relative entropy of the dominating rule being too high with respect to some other rule at the equilibrium prices it determines.

Our results cast doubts on the working of market selection, and thus on the validity of the "as if" statement when applied to exchange economies with uncertainty. On the one hand our results imply that if a trader has perfect knowledge regarding the underlying dividend process and exploits it at best using the S-rule, the fixed points in which she survives are the unique stable equilibria, and prices correctly reflect, in the long run, asset's fundamental values. This is the same result found also in previous works where market selection was tested on investment rules depending on exogenous asset dividends. On the other hand, when an investor using the S-rule is not present in the market, it is not anymore the case that the market selects for the best informed trader, and informational inefficiencies due to endogenous fluctuations emerge as a generic market property.

A Appendix: Proofs

A.1 Section 2

Proof of Theorem 2.1 According to (2.3) prevailing prices $\mathbf{p}_t$ are set by the implicit equation

$$\mathbf{p}_t = \mathbf{A}(\mathbf{p}_t)$$

where $\mathbf{A}$ is the vector valued function with components $A_k = \sum_{i=1}^I w_t^i \alpha_{k,t}^i$. Due to assumptions of the theorem, $\mathbf{A}$ is a continuous function from the convex compact set $[0,1]^K$ into itself. Then

[13] In fact, along the same lines, it is straightforward to show that $\alpha^L \succeq \alpha$ for every α of class $\mathcal{C}^1$ at its equilibria.

the proposition follows from Brouwer's Theorem. Moreover, due to **P3** in Ass. 1, zero prices can never be an equilibrium.

Theorem A.1. *If for every agent $i = 1, \ldots, I$ it holds $\alpha^i \in \mathcal{A}$ and $\alpha^i \in \mathcal{C}^1$, and if for every $k = 1, \ldots, K$ α^i_k does not depend on current prices other than at most the one of the same k-th asset, then the vector $\mathbf{p}^*$ is unique provided that*

$$\left| \frac{\partial \alpha^i_k}{\partial p_k} \right| < 1 \,, \quad i = 1, \ldots, I \quad k = 1, \ldots, K \,. \tag{A.1}$$

Proof of Theorem A.1 Using the notation of Theorem 2.1, a sufficient condition for the uniqueness of the fixed point is that $\mathbf{A}$ represents a contraction mapping, that is, for each couple of prices $\mathbf{p}$ and $\mathbf{q} = \mathbf{p} + \delta\mathbf{p}$ it is

$$|\mathbf{A}(\mathbf{p}) - \mathbf{A}(\mathbf{q})| \leq |\mathbf{p} - \mathbf{q}| \ .$$

Due to the differentiability of $\mathbf{A}$, the mean value theorem implies that

$$|\mathbf{A}(\mathbf{p}) - \mathbf{A}(\mathbf{q})| = (\delta\mathbf{p})' \, Q \, \delta\mathbf{p} \ ,$$

where the matrix $Q(\mathbf{p}, \delta\mathbf{p})$ is a positive semi-definite quadratic form defined starting from the Jacobian matrix J of the function $\mathbf{A}$ as

$$Q = \int_0^1 dt_1 J'(\mathbf{p} + t_1 \, \delta\mathbf{p}) \int_0^1 dt_2 J(\mathbf{p} + t_2 \, \delta\mathbf{p}) \ ,$$

The function $\mathbf{A}$ is a contraction if for every couple $\mathbf{p}$ and $\mathbf{q}$, the matrix Q does not possess eigenvalues greater than one. This is trivially the case if the investment rules αs, and consequently the function $\mathbf{A}$, do not depend on contemporaneous prices (in this case all eigenvalues of Q are equal to zero).

If only the k-th contemporaneous price enter as a variable in the investment rules relative to asset k, the matrix Q is diagonal, with elements

$$Q_{i,i} = \left(\int_0^1 dt_1 J_{i,i}(\mathbf{p} + t_1 \, \delta\mathbf{p}) \right)^2 \ .$$

For the triangle inequality, if $|J_{i,i}(\mathbf{p})| < 1$ for any $\mathbf{p}$, then $Q_{i,i} < 1$ and the proposition follows. □

A.2 Section 4

Proof of Theorem 4.1 The result follows from substitution of x^* in (2.7). $\mathcal{W}(x^*; \omega) = w^*$ holds because, for every ω, for every $i \neq I$ $w^{i*} = 0$ is a fixed point of $\mathcal{W}^j(\cdot, \omega)$ and, since by assumption $p^*_k = \alpha^I_k(p^*)$ for every $k = 1, \ldots, K$, $w^{I*} = 1$ is a fixed point of $\mathcal{W}^I(\cdot, \omega)$. $\mathcal{P}(x^*; \omega) = p^*$ holds because, regarding the current price, it is only agent I who fixes prices (all other agents have zero wealth) and $p^*_k = \alpha^I_k(p^*)$ holds by assumption for every $k = 1, \ldots, K$. Regarding lagged prices, at any fixed point they are all equal by definition. □

Proof of Theorem 4.2 After noting that prices are implicitly defined by the set of K equations in (2.3) with $w^I_{t+1} = 1$ and $w^i_{t+1} = 0$ for $i \neq I$, the result immediately follows from the implicit function theorem. □

Proof of Theorem 4.3 Consider the reduced system in $[0,1]^{I-1} \times (0,1)^{K(L+1)}$ of dimension $I-1+K(L+1)$ obtained by substituting $w_t^I = 1 - \sum_{i=1}^{I-1} w_t^i$. With an abuse of notation we will keep using the same names for the map f, and thus also $\mathcal{F}$, even though its definition has actually changed. In particular the definition of f given in (2.6) becomes

$$f_k(x_t;\omega) = \sum_{i=1}^{I-1} \mathcal{W}^i(x_t;\omega)(\alpha_{k,t+1}^i - \alpha_{k,t+1}^I) + \alpha_{k,t+1}^I\,, \quad k = 1,\dots,K\,. \tag{A.2}$$

$\mathcal{F}$ defined in (2.7) and x^* vary accordingly, in particular $x^* = (0,\dots,0,p^*)$. The Jacobian $J(\omega,x)$ of $\mathcal{F}$ can be written as

$$J(\omega,x) = \begin{pmatrix} \frac{\partial\mathcal{W}}{\partial\mathcal{W}} & \frac{\partial\mathcal{W}}{\partial\mathcal{P}} \\ \frac{\partial\mathcal{P}}{\partial\mathcal{W}} & \frac{\partial\mathcal{P}}{\partial\mathcal{P}} \end{pmatrix}\,, \tag{A.3}$$

or, subdividing the part relative to price determination, with obvious notation,

$$J(\omega,x) = \begin{pmatrix} \frac{\partial\mathcal{W}}{\partial\mathcal{W}} & \frac{\partial\mathcal{W}}{\partial\mathcal{P}_1} & \cdots & \frac{\partial\mathcal{W}}{\partial\mathcal{P}_K} \\ \frac{\partial\mathcal{P}_1}{\partial\mathcal{W}} & \frac{\partial\mathcal{P}_1}{\partial\mathcal{P}_1} & \cdots & \frac{\partial\mathcal{P}_1}{\partial\mathcal{P}_K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\mathcal{P}_K}{\partial\mathcal{W}} & \frac{\partial\mathcal{P}_K}{\partial\mathcal{P}_1} & \cdots & \frac{\partial\mathcal{P}_K}{\partial\mathcal{P}_K} \end{pmatrix}\,. \tag{A.4}$$

The element i,j of each block matrix is the partial derivative of the i-th component of the numerator with respect to the j-th component of the denominator.

In each sub-block $\partial\mathcal{W}/\partial\mathcal{P}_k$ the first column reads

$$\left\{\frac{\delta\mathcal{W}}{\delta\mathcal{P}_k}\right\}_{i,1} = \left(\sum_{k'} \frac{(\alpha_{k'}^i)^{k,1}}{p_{k',t}} d_{k'}(\omega_{t+1}) - \frac{\alpha_{k,t}^i}{(p_{k,t})^2 d_k(\omega_{t+1})}\right) w_t^i\,, \quad i = 1,\dots,I-1\,,$$

while for $l > 1$ it is

$$\left\{\frac{\delta\mathcal{W}}{\delta\mathcal{P}_k}\right\}_{i,l>1} = \left(\sum_{k'} \frac{(\alpha_{k'}^i)^{k,l-1}}{p_{k't}} d_{k'}(\omega_{t+1})\right) w_t^i\,, \quad i = 1,\dots,I-1\,, \quad L = 2,\dots,L+1\,.$$

Since $w^{*j} = 0$ if $j \neq I$, the previous expressions at x^* reads

$$\left\{\left.\frac{\partial\mathcal{W}}{\partial\mathcal{P}}\right|_{x^*}\right\}_{i,j} = 0 \quad \text{for all} \quad i,j\,.$$

As a result, the Jacobian matrix evaluated at x^*, $J^*(\omega) = J(\omega,x^*)$, is lower block triangular and the eigenvalues of $J^*(\omega)$ are those of the left-upper, $\partial\mathcal{W}/\partial\mathcal{W}$, and right-lower, $\partial\mathcal{P}/\partial\mathcal{P}$, blocks. These blocks turn out to have a peculiar structure at x^*. Let us start from the left-upper block. Taking the partial derivatives of wealth fractions gives

$$\left\{\frac{\partial\mathcal{W}}{\partial\mathcal{W}}\right\}_{i,j} = \frac{\partial\mathcal{W}^i(x_t;\omega)}{\partial w_t^j} = \delta_{i,j} \sum_{k=1}^{K} \frac{\alpha_{k,t}^i}{p_{k,t}} d_k(\omega_{t+1}) \quad i,j = 1,\dots,I-1$$

so that the block computed in $x^* = (0, \dots, 0, p^*)$ becomes diagonal and reads

$$\left.\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right|_{x^*} = \begin{pmatrix} \mu_1(\omega_{t+1}) & 0 & \dots & 0 \\ 0 & \mu_2(\omega_{t+1}) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_{I-1}(\omega_{t+1}) \end{pmatrix}, \tag{A.5}$$

where, using the fact that prices are fixed by agent I's rule,

$$\mu_i(\omega_{t+1}) = \sum_{k=1}^{K} \frac{\alpha_k^i(p^*)}{\alpha_k^I(p^*)} d_k(\omega_{t+1}) \,. \tag{A.6}$$

Concerning the right-lower block $\partial\mathcal{P}/\partial\mathcal{P}$, in a neighborhood of the fixed point x^* it holds that

$$\begin{aligned} \left\{\frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\right\}_{1,l} &= \left.\frac{\partial f_k(x_t;\omega)}{\partial p_{h,t}^{l-1}}\right|_{x^*} \qquad l = 1, \dots, L+1\,, \\ &= -\sum_{k'=1}^{K} H_{k,k'}^{-1}(x_t) M_{k',h,l} \end{aligned} \tag{A.7}$$

where H^{-1} is the inverse of the matrix $H_{k,k'}(x_t) = \sum_{i=1}^{I} w_{t+1}^i (\alpha_k^i(p_{t+1}))^{k',l}$, which is non-singular due to the continuous differentiability of the αs and the assumption of Theorem 4.2 and

$$M_{k',h,l} = \sum_{i=1}^{I-1} \left(\left\{\frac{\partial \mathcal{W}}{\partial \mathcal{P}_h}\right\}_{1,l} + w_{t+1}^i \left(\left(\alpha_{k'}^i\right)^{h,l} - \left(\alpha_{k'}^I\right)^{h,l} \right) \right) + \left(\alpha_{k'}^I\right)^{h,l} \,. \tag{A.8}$$

Substituting in (A.7) the expression of (A.8) computed at x^* and using the matrix defined in (4.1) leads to

$$\left\{\left.\frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\right|_{x^*}\right\}_{1,l} = -\sum_{k'=1}^{K} H_{k,k'}^{-1} (\alpha_{k'}^I)^{h,l} = (\bar{\alpha}_k^I)^{h,l}\,, \quad l = 1, \dots, L \tag{A.9}$$

and $\{\partial\mathcal{P}_k/\partial\mathcal{P}_h|_{x^*}\}_{1,L+1} = 0$. The other rows are all zero but for the diagonal blocks which have a "Jordan" form, that is,

$$\left\{\frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\right\}_{i>1,l} = \frac{\partial p_{k,t+1}^l(x_t;\omega)}{\partial p_{k,t}^{l-1}} = \delta_{k,h}\delta_{i+1,i}\,, \quad i = 2, \dots, L+1 \quad l = 1, \dots, L+1\,.$$

As a result

$$\left.\frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\right|_{x^*} = \begin{pmatrix} (\bar{\alpha}_k^I)^{h,1} & (\bar{\alpha}_k^I)^{h,2} & \dots & (\bar{\alpha}_k^I)^{h,L} & 0 \\ \delta_{k,h} & 0 & \dots & 0 & 0 \\ 0 & \delta_{k,h} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \delta_{k,h} & 0 \end{pmatrix}, \quad k, h = 1, \dots, K\,. \tag{A.10}$$

The eigenvalues associated with the price blocks are obtained from the characteristic polynomial defined as the determinant

$$P(\lambda) = \begin{vmatrix} \frac{\partial \mathcal{P}_1}{\partial \mathcal{P}_1} - \lambda I & \dots & \frac{\partial \mathcal{P}_1}{\partial \mathcal{P}_K} & \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{P}_K}{\partial \mathcal{P}_1} & \dots & \frac{\partial \mathcal{P}_K}{\partial \mathcal{P}_K} - \lambda I & \end{vmatrix},$$

where I stands for the $(L+1) \times (L+1)$ identity matrix. The last zero columns in each column-block is responsible for a factor λ. This generates an eigenvalue 0 of multiplicity K. Once the associated K columns, and their corresponding rows, have been removed one remains with a residual matrix of dimension KL. This matrix has K rows filled with $\bar{\alpha}$s. Each other row is zero but for two elements 1 and $-\lambda$. Using the Laplace formula iteratively, the final expression of the characteristic polynomial of the lower-right block becomes

$$P(\lambda) = \lambda^k \sum_{l_1=1}^{L} \dots \sum_{l_K=1}^{L} \lambda^{LK - \sum_j l_j} \begin{vmatrix} (\bar{\alpha}_1^I)^{1,l_1} - \lambda\delta_{1,l_1} & (\bar{\alpha}_1^I)^{2,l_2} & \dots & (\bar{\alpha}_1^I)^{K,l_K} \\ (\bar{\alpha}_2^I)^{1,l_1} & (\bar{\alpha}_2^I)^{2,l_2} - \lambda\delta_{1,l_2} & \dots & (\bar{\alpha}_2^I)^{K,l_K} \\ \vdots & \vdots & \ddots & \vdots \\ (\bar{\alpha}_K^I)^{1,l_1} & (\bar{\alpha}_K^I)^{2,l_2} & \dots & (\bar{\alpha}_K^I)^{K,l_K} - \lambda\delta_{1,l_K} \end{vmatrix},$$

which, using the Leibniz formula for the computation of the determinant, and dropping the factor λ^K, reduces to (4.4).

Consider now the iteration, for any T, of the stochastic linear map defined by the Jacobian computed in the fixed point

$$J^*(T, \omega) = J^*(\theta^{T-1}\omega) \dots J^*(\theta\omega) J^*(\omega).$$

According to the Oseledec's multiplicative ergodic theorem (see Young, 1995, Th. 2.1.1) the eigenvalues of $J^*(T, \omega)$ can be used to compute the Lyapunov spectrum of the iterated linear map provided that the integrability condition is satisfied, that is, as long as

$$\mathrm{E} \log^+ ||J^*|| := \sum_{\omega \in \Omega} \log^+ ||J^*(\omega)|| \pi(\omega) < \infty, \tag{A.11}$$

where $\log^+ a = \mathrm{Max}\{\log a, 0\}$. In our case, the ergodic nature of the process guarantees that, component by component,

$$\mathrm{E} \log ||\{J^*\}_{i,j}|| = \sum_{\omega \in \Omega} \log ||\{J^*(\omega)\}_{i,j}|| \pi(\omega) = \sum_s \log^+ ||J^*_{i,j}(s)|| \pi_s.$$

Due to assumptions **P1-P2** the element of J^* are finite for any realization of the process, so that (A.11) immediately follows.

Since the integrability condition is satisfied, the Lyapunov spectrum of the iterated linear map reduce to

$$\lim_{T \to \infty} \frac{1}{T} \log |\{J^*(T, \omega)\}_{i,i}|.$$

Moreover, since $J^*(\omega)$ is block triangular for every ω, so it is $J^*(T,\omega)$, which can be written as

$$J^*(T,\omega) = \begin{pmatrix} \left(\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\Big|_{x^*}\right)^T & 0 \\ \blacksquare & \left(\frac{\partial \mathcal{P}}{\partial \mathcal{P}}\Big|_{x^*}\right)^T \end{pmatrix}, \tag{A.12}$$

where

$$\begin{aligned} \blacksquare &= \sum_{t=0}^{T-1} \left(\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\right)^{T-t-1} \frac{\partial \mathcal{P}}{\partial \mathcal{W}} \left(\frac{\partial \mathcal{P}}{\partial \mathcal{P}}\right)^{t}, \\ &= \sum_{t=0}^{T-1} \frac{\partial \mathcal{W}(\theta^T\omega)}{\partial \mathcal{W}} \dots \frac{\partial \mathcal{W}(\theta^{t+1}\omega)}{\partial \mathcal{W}} \frac{\partial \mathcal{P}(\theta^{t}\omega)}{\partial \mathcal{W}} \frac{\partial \mathcal{P}(\theta^{t-1}\omega)}{\partial \mathcal{P}} \dots \frac{\partial \mathcal{P}(\omega)}{\partial \mathcal{P}}. \end{aligned} \tag{A.13}$$

This implies that the eigenvalues of $J^*(T,\omega)$ are given by the union of the eigenvalues of the T-iteration of the 2 diagonal blocks of $J^*(\omega)$.

The left-upper block is diagonal and for any realization of the stochastic process it is

$$\mu_i(T,\omega) = \mu_i(\omega_{t+T}) \dots \mu_i(\omega_{t+2})\mu_i(\omega_{t+1}), \qquad i = 1\dots, I-1,$$

which, using the expression in (A.6) and the ergodic property of the process to take the limit $T \to \infty$, converges to

$$\mu_i = \prod_{s=1}^{S} \left(\sum_{k=1}^{K} \frac{\alpha_k^i(p^*)}{\alpha_k^I(p^*)} D_{s,k} \right)^{\pi_s}.$$

Concerning the right-lower block, the matrix in (A.10) does not depend upon the realization of the random variable. This implies that the eigenvalues of the T-product of right-lower block are just the T power of the eigenvalue of $\partial\mathcal{P}/\partial\mathcal{P}$.

Summarizing the list of exponential of the Lyapunov exponents of the iterated linear map is

$$\lambda(T,\omega) = \{\mu_1(T,\omega), \dots, \mu_{I-1}(T,\omega)\} \cup \left\{0, \lambda_{1,1}^T, \dots, \lambda_{1,L}^T\right\} \dots \left\{0, \lambda_{K,1}^T, \dots, \lambda_{K,L}^T\right\}, \tag{A.14}$$

where the lambdas are the LK roots of (4.4). The fact that the elements of (A.14) are, in absolute value, lower than one is a sufficient condition for the stability of the iterated linear map.

Since the random dynamical system φ is $\mathcal{C}^1$ (because $\mathcal{F}$ in (2.7) is $\mathcal{C}^1$) and we proved above that the integrability condition of the Multiplicative Ergodic Theorem is satisfied, the Local Hartman-Grobman theorem (see Coayla-Teran and Ruffino, 2004, Th. 4.2) ensures that the asymptotic stability results of the stochastic linear map $J(\omega,x)$ carry over to the system φ, and the first part of the theorem is proved.

The polynomial (4.4) is heavily simplified when the investment rule of agent I in asset k depends only on current and past prices of asset k itself. In this case all off-diagonal price/price blocks (A.10) have zero entries, and the characteristic polynomial of each diagonal block $k = 1,\dots,K$ is given by

$$P(\lambda) = \lambda \left(\lambda^L - \sum_{l=1}^{L} \lambda^{L-l} (\bar{\alpha}_k^I)^{(k,l)} \right),$$

that is, one eigenvalue is equal to zero while the other L eigenvalues are the zeros of (4.5). □

Proof of Theorem 4.5 The proof proceeds along the same lines of that of Theorem 4.3. It is still convenient to omit the state variable w_t^I by using $w_t^I = 1 - \sum_{i=1}^{I-1} w_t^i$. Consider the Jacobian $J^*(\omega)$, of $\mathcal{F}$ computed at the fixed point x^*. The components of the off-diagonal wealth/price and price/wealth blocks read

$$\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{P}_k}\Big|_{x^*} \right\}_{i,1} = \begin{cases} 0 & i = 1, \dots, I-M \\ -\frac{w^{i*}}{p_k^*} d_k(\omega_{t+1}) & i = I-M+1, \dots, I-1 \end{cases}, \tag{A.15}$$

$$\left\{ \frac{\partial \mathcal{W}}{\partial \mathcal{P}_k}\Big|_{x^*} \right\}_{i,j>1} = \begin{cases} 0 & i = 1, \dots, I-M \\ \sum_{k'} \frac{w^{i*}}{p_{k'}^*} (\alpha_{k'}^i)^{k,j-1} d_{k'}(\omega_{t+1}) & i = I-M+1, \dots, I-1 \end{cases}, \tag{A.16}$$

$$\left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{W}}\Big|_{x^*} \right\}_{1,j} = \begin{cases} \mu_j(\omega_{t+1})(\alpha_k^j(p^*) - p_k^*) & j = 1, \dots, I-M \\ 0 & j = I-M+1, \dots, I-1 \end{cases}, \tag{A.17}$$

$$\left\{ \frac{\partial \mathcal{P}_k}{\partial \mathcal{W}}\Big|_{x^*} \right\}_{i>1,j} = 0 \quad j = 1, ..., I-1\,, \tag{A.18}$$

for $k = 1, \dots, K$ and where $\mu_j(\omega_{t+1})$ is defined as in (A.6). Diagonal blocks have a similar structure to that found for the single survivor case. In particular the wealth/wealth block is

$$\frac{\partial \mathcal{W}}{\partial \mathcal{W}}\Big|_{x^*} = \begin{pmatrix} \mu_1(\omega_{t+1}) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \mu_{I-M}(\omega_{t+1}) & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}, \tag{A.19}$$

where each $\mu_i(\omega_{t+1})$ is defined in (A.6) and 1s comes from the fact that $\mu_i(\omega_{t+1}) = 1$ for all $i = I-M+1, \dots, I-1$. Price/price blocks are obtained from (A.7) with the substitution of the derivatives of the I-th investment rule with the average of the derivative of all surviving rules, weighted with the associated equilibrium wealth shares. Defining $\langle H \rangle$, $\langle M \rangle$, $\langle \bar{\alpha} \rangle$, as in, respectively, (4.1), (A.8) and (A.9) replacing $(\alpha_k^I)^{h,l}$ with $\langle \alpha_k \rangle^{h,l}$ defined in (4.6), each price/price blocks is given by

$$\frac{\partial \mathcal{P}_k}{\partial \mathcal{P}_h}\Big|_{x^*} = \begin{pmatrix} \langle \bar{\alpha}_k \rangle^{h,1} & \langle \bar{\alpha}_k \rangle^{h,2} & \dots & \langle \bar{\alpha}_k \rangle^{h,L} & 0 \\ \delta_{k,j} & 0 & \dots & 0 & 0 \\ 0 & \delta_{k,j} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \delta_{k,j} & 0 \end{pmatrix}, \quad k, h = 1, \dots, K\,, \tag{A.20}$$

The resulting Jacobian matrix has the structure

$$J^*(\omega) = \begin{pmatrix} W & 0 & 0 \\ 0 & I & A \\ B & 0 & P \end{pmatrix}, \tag{A.21}$$

where $\left(\begin{smallmatrix} W & 0 \\ 0 & I \end{smallmatrix} \right)$ is the wealth/wealth block (A.19), in particular W is the $(I-M) \times (I-M)$ upper diagonal block and I is the $(M-1) \times (M-1)$ identity matrix, P is the $K(L+1) \times K(L+1)$ price/price block built using (A.20), A is a $(M-1) \times K(L+1)$ matrix with elements defined by (A.15-A.16), B is a $K(L+1) \times (I-M)$ matrix with elements defined by (A.17-A.18), and 0 denotes, case by case, a matching null matrix. It is a trivial algebraic result that the T products of (A.21) possess the structure

$$J^*(T,\omega) = \begin{pmatrix} W^T & 0 & 0 \\ C' & I & A' \\ B' & 0 & P^T \end{pmatrix} ,$$

where the exact form of the matrices A',B',C' depend on the choice of T and is not relevant for our analysis. It then follows that the determinant of $J^*(T,\omega)$ can be easily computed as the product of the determinants of its diagonal blocks W^T and P^T. As a result, sufficient conditions for stability can be derived along the same lines of the proof of Theorem 4.3, where diagonal blocks have changed from (A.5) and (A.10) to (A.19) and (A.20), respectively.

Notice that, also in the case of multiple survivors, the stochastic component enters only in the diagonal wealth/wealth block. For multiple survivors, however, the characteristic polynomial of the wealth/wealth block possesses a unit root with multiplicity $M-1$. Consequently, the fixed point is non-hyperbolic, and thus not asymptotically stable. We shall show that each fixed point $x^* = (w^*, p^*)$ belonging to the manifold where

$$\sum_{m=1}^{M} w^{(I-M+m)*} = 1$$

is nevertheless stable. For any realization ω of the process, the direct sum of the eigenspaces associated with each unitary eigenvalue is the linear space V_I spanned by the $M-1$ vectors $\mathbf{e}_m$, $m = I-M+1, \ldots, I-1$ of the canonical base of $\mathbb{R}^{I-1+K(L+1)}$. As the direction of each vector $\mathbf{e}_m$ corresponds to a change in the relative wealths of the m-th and I-th survivor, each small enough perturbation $v \in V_I$ away from x^* push the dynamics to a new point $x'^* = x^* + v = (w'^*, p^*)$ where the wealth distribution w'^* differs from w^* for the reallocation of wealth among the M surviving agents corresponding to v^*. Since x'^* is a deterministic fixed point, when perturbations are restricted to V_I the original point x^* is stable. For the more general case notice that any perturbation h can be written as $h = h' + h^\perp$ with $h' \in V_I$, $h^\perp \in V_I^\perp$ and that x'^* is asymptotically stable for perturbations $h^\perp$ along the stable manifold and stable for perturbations h' along the center manifold. The fixed point is hence stable, but not asymptotically stable. □

B Section 5.1

Proof of Theorem 5.1 Since $I_\pi(\alpha, \mathbf{p})$ is defined for vectors $\mathbf{p} \in \Delta_+^K$, we can change variables from α_k to $x_k = \frac{\alpha_k}{p_k}$ for every $k = 1, \ldots, K$. Thus solving (5.1) is equivalent to finding

$$\mathbf{x}^S(\mathbf{p}) = \operatorname*{argmax}_{\mathbf{x} \in B(\mathbf{p})} \{\exp -I_\pi(\mathbf{x})\} \ . \tag{B.1}$$

where $B(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^K | x_k \geq 0 \text{ for every } k = 1, \ldots, K, \text{ and } \mathbf{x} \cdot \mathbf{p} \leq 1\}$. Problem (B.1) is the maximization of a continuous function on a compact set and thus has a solution for each given $\mathbf{p}$. Moreover such maxima will never be attained for those $\mathbf{x}$ where $\sum_k x_k D_{s,k} = 0$ for some s. As a result we can equally solve

$$\mathbf{x}^S(\mathbf{p}) = \operatorname*{argmax}_{\mathbf{x} \in B_+(\mathbf{p})} \{-I_\pi(\mathbf{x})\} \ . \tag{B.2}$$

where $B_+(\mathbf{p})$ is the subset of $B(\mathbf{p})$ where the function $-I_\pi(\mathbf{x})$ is well defined. Computing the Hessian matrix H of $-I_\pi(\mathbf{x})$ one finds

$$\{H\}_{n,m} = -\sum_{s=1}^{S} \pi_s \frac{D_{s,m} D_{s,n}}{\left(\sum_{k=1}^{K} x_k D_{s,k}\right)^2} , \tag{B.3}$$

so that $\mathbf{y} \cdot H\mathbf{y}$, $\mathbf{y} \in \mathbb{R}^K$, is equal to

$$\mathbf{y} \cdot H\mathbf{y} = -\sum_{n,m} y_n y_m \sum_{s=1}^{S} \pi_s \frac{D_{s,n} D_{s,m}}{\left(\sum_{k=1}^{K} x_k D_{s,k}\right)^2} = -\sum_{s=1}^{S} \frac{\pi_s}{\left(\sum_{k=1}^{K} x_k D_{s,k}\right)^2} \left(\sum_n y_n D_{s,n}\right)^2 . \tag{B.4}$$

Since the former expression is always negative for non-trivial payoff matrices D, $-I_\pi(\mathbf{x})$ is strongly concave for all vectors $\mathbf{x} \in B_+(\mathbf{p})$. Adding to this continuity and non-satiation, which are trivially proved, standard consumer theory theorems, see e.g. Proposition 2.8 in Ginsburgh and Keyzer (1997), can be used to show that $\mathbf{x}^S(\mathbf{p}) \cdot \mathbf{p} = 1$, which implies $\alpha_0^S(\mathbf{p}) = 0$, and that $x^S(\mathbf{p})$ (and thus $\alpha^S(\mathbf{p})$) is of class $\mathcal{C}^0$ in Δ_+^K and of class $\mathcal{C}^1$ when strictly positive.

Regarding the equilibria of α^S, by deriving the first order conditions of the maximization problem (B.2), it is immediate to check that $p_k = \sum_{s=1}^{S} \pi_s D_{s,k}$ for all $k = 1, \dots, K$ is the unique vector of prices where $\mathbf{x}^S(\mathbf{p}) = \mathbf{1}$, and thus where $\alpha^S(\mathbf{p}) = \mathbf{p}$.

Proof of Theorem 5.2 Let w_t be the wealth share of the S-rule and assume a trajectory φ_t exists such that along this trajectory $\lim_{t\to\infty} w_t = 0$. Then asymptotic prices converge toward a single survivor equilibrium where the rule α dominates, that is $\lim_{t\to\infty} p_{t,k} - \alpha_k(p_t) = 0$ for any $k = 1, \dots, K$ and, consequently $\lim_{t\to\infty} I_\pi(\alpha_t, \mathbf{p}_t) = 0$. Since for construction α^S minimizes $I_\pi(\alpha, \mathbf{p})$ it holds

$$\lim_{t\to\infty} I_\pi(\alpha_t^S, \mathbf{p}_t) = \lim_{t\to\infty} I_\pi(\alpha_t^S, \alpha_t) \leq 0 .$$

This implies that the quantity μ defined in (4.3) is never lower than one. When μ is greater than one, the trajectory w_t converges towards an unstable deterministic fixed point. When μ is equal to one, the long-run prices are also an equilibrium of α^S. In both cases the trajectory w_t on which the S-rule vanishes represent a zero measure set. □

Proof of Theorem 5.3 The proof replicate that of Theorem 5.2 at each deterministic fixed point where an agent, or a set of agents, survives. In particular when prices do not converge to the equilibrium of α^S the corresponding deterministic fixed point is unstable. Thus the only possible stable deterministic fixed points have prices fixed by $\alpha^S(\mathbf{p}) = \mathbf{p}$ whose unique solution is $p_k = \sum_{s=1}^{S} \pi_s D_{s,k}$ for all $k = 1, \dots, K$ as shown in Th. 5.1. Obviously at all these fixed points α^S survives and at least one of such fixed points exists, namely, the one where α^S dominates.

C Section 5.3

Proof of Theorem 5.4 Since by hypothesis the price learner rule α^L does not depend on contemporaneous prices and satisfies both the no-cross dependence condition and (5.2), the characteristic polynomial (4.5) reduces to

$$P(\lambda) = \prod_{k=1}^{K} \left(\lambda^L - (\alpha_k^L)_{x^*} \sum_{l=1}^{L} \lambda^{L-l} \right) ,$$

Notice that $P(\lambda)$ is the product of K polynomials having all one zero root and the same form namely

$$P(x;\alpha) = x^L - \alpha \sum_{l=0}^{L-1} x^l .$$

The problem of determining whether the roots of $P(\lambda)$ are all inside the unit circle can thus be solved by looking at $P(x;\alpha)$.

If $\alpha = 0$ all roots are inside the unite circle. Assume that $\alpha > 0$. On the unit complex circle, $|z| = 1$, it holds

$$|z^L - P(z;\alpha)| = |\alpha \sum_{l=0}^{L-1} z^l| \leq \alpha \sum_{l=0}^{L-1} |z^l| = L\alpha .$$

It follows that if $\alpha < 1/L$, $|z^L - P(z;\alpha)| < 1 = |z^L|$ for $|z| = 1$. The latter inequality together with Rouchè's Theorem (see e.g. Lang, 1993) imply that the polynomial $P(z;\alpha)$ and z^L have has the same number of roots inside the unit circle. Moreover notice that if $\alpha \geq 1/L$, it holds both $P(1;\alpha) \leq 0$ and $\lim_{x\to+\infty} P(x;\alpha) = +\infty$, implying the existence of a root greater or equal to one. Provided α is positive, we have proved that $\alpha < 1/L$ is both a necessary and sufficient condition for $P(x;\alpha)$ having all the roots inside the unit circle.

Take now $\alpha < 0$. The complex polynomial $P(z;\alpha)$ can be rewritten as

$$P(z;\alpha) = \sum_{l=0}^{L} z^l - (1 - |\alpha|) \sum_{l=0}^{L-1} z^l .$$

and its roots are the solutions of

$$\sum_{l=0}^{L} z^l = (1 - |\alpha|) \sum_{l=0}^{L-1} z^l .$$

Multiplying left and right hand side by $z - 1$ (remembering we are adding the root $z = 1$) and rearranging the terms leads to

$$|z - (1 - |\alpha|)| = \frac{|\alpha|}{|z|^L} ,$$

provided $z \neq 0$ which we can always assume since zero is never a root. Assume now $|\alpha| < 1$. If a root with modulus bigger or equal than one, but different from $z = 1$, exists, one could write

$$|\alpha| < |z - (1 - |\alpha|)| = \frac{|\alpha|}{|z|^L} \leq |\alpha| ,$$

which is a contradiction. We have proved that $|\alpha| < 1$ is a sufficient condition for all roots being inside the unit circle. The condition is also necessary. Indeed, since the modulus of the constant in $P(z;\alpha)$, $|\alpha|$, is given by the product of the moduli of all the roots, when $|\alpha| \geq 1$ there must exist at least a root with modulus bigger or equal to 1.

Interestingly, the role of the memory parameter L is different in the case of positive and negative prices feedbacks. In general, for consistent estimators, partial derivatives depend on the number of lags considered and scale with $1/L$: the longer the agent's memory, the lower the partial derivative. Then if $(\alpha_k^L)_{x^*} < -1$, by increasing the number of past observation, that is, the memory, it is always possible to cross the bound of -1 and thus stabilize the fixed point. Conversely, if $(\alpha_k^L)_{x^*} > 1/L$, an increase in the memory of the strategy does not improve the stability of the fixed point because the bound scales with $1/L$ as well. □

Proof of Corollary 5.1 The corollary is easily proved by using results from Theorem 5.4 and upon realizing that the characteristic polynomial now depends on the convex combination of partial derivatives, that is, $\langle\alpha\rangle_k = (1 - w^*)(\alpha_k^L)_{x^*}$ $k = 1, \ldots, K$, rather than on $(\alpha_k)_{x^*}$ $k = 1, \ldots, K$, since all partial derivatives of the S-rule are zero. □

References

Alchian, A. (1950). Uncertainty, evolution, and economic theory. *The Journal of Political Economy 58*, 211–221.

Amir, R., I. Evstigneev, T. Hens, and K. Schenk-Hoppé (2005). Market selection and survival of investment strategies. *Journal of Mathematical Economics 41*, 105–122.

Anufriev, M. and G. Bottazzi (2010). Market equilibria under procedural rationality. *Journal of Mathematical Economics 46*, 1140–1172.

Anufriev, M., G. Bottazzi, and F. Pancotto (2006). Equilibria, stability and asymptotic dominance in a speculative market with heterogeneous agents. *Journal of Economic Dynamics and Control 30*, 1787 – 1835.

Anufriev, M. and P. Dindo (2010). Wealth-driven selection in a financial market with heterogeneous agents. *Journal of Economic Behavior and Organization 73*, 327–358.

Barberis, N. and R. Thaler (2003). A survey of behavioral finance. In G. Constantinides, M. Harris, and R. Stultz (Eds.), *Handbook of the Economics of Finance*. North-Holland (Handbooks in Economics Series), Amsterdam.

Blume, L. and D. Easley (1992). Evolution and market behavior. *The Journal of Economic Theory 58*, 9–40.

Blume, L. and D. Easley (2006). If you are so smart why aren't you rich? Belief selection in complete and incomplete markets. *Econometrica 74*, 929–966.

Chiarella, C. and X. He (2001). Asset price and wealth dynamics under heterogeneous expectations. *Quantitative Finance 1*, 509–526.

Chichilnisky, G. (1999). Existence and optimality of a general equilibrium with endogenous uncertainty. In G. Chichilnisky (Ed.), *Markets, Information, and Uncertainty*. Cambridge University Press, Cambridge.

Coayla-Teran, E. and P. Ruffino (2004). Stochastic versions of Hartman-Grobman theorems. *Stochastics and Dynamics 4*, 571–593.

Drèze, J. and P. Herings (2008). Sequentially complete markets remain incomplete. *Economic Letters 100*, 445–447.

Evstigneev, I., T. Hens, and K. Schenk-Hoppé (2008). Globally evolutionary stable portfolio rules. *Journal of Economic Theory 140*, 197–228.

Evstigneev, I., T. Hens, and K. Schenk-Hoppé (2009). Evolutionary finance. In T. Hens and K. Schenk-Hoppé (Eds.), *Handbook of Financial Markets: Dynamics and Evolution*. North-Holland (Handbooks in Economics Series).

Fama, E. (1970). Efficient capital markets: a review of theory and empirical work. *Journal of Finance 25*, 383–422.

Farmer, J. (2002). Market force, ecology, and evolution. *Industrial and Corporate Change 11*, 895–953.

Friedman, M. (1953). *Essays in Positive Economics.* Univ. Chicago Press.

Ginsburgh, V. and M. Keyzer (1997). *The Structure of Applied General Equilibrium Models.* The MIT Press, Cambridge, Massachusetts.

Hahn, F. (1999). A remark on incomplete market equilibrium. In G. Chichilnisky (Ed.), *Markets, Information, and Uncertainty.* Cambridge University Press, Cambridge.

Hommes, C. (2006). Heterogeneous agent models in economics and finance. In K. Judd and L. Tesfatsion (Eds.), *Handbook of Computational Economics Vol. 2: Agent-Based Computational Economics.* North-Holland (Handbooks in Economics Series), Amsterdam.

Kelly, J. (1956). A new interpretation of information rates. *Bell System Technical Journal 35*, 917–926.

Lang, S. (1993). *Complex Analysis.* Springer, New York.

LeBaron, B. (2006). Agent-based computational finance. In K. Judd and L. Tesfatsion (Eds.), *Handbook of Computational Economics Vol. 2: Agent-Based Computational Economics.* North-Holland (Handbooks in Economics Series).

Levy, M., H. Levy, and S. Solomon (2000). *Microscopic Simulation of Financial Markets.* Academic Press, London.

Sandroni, A. (2000). Do markets favor agents able to make accurate predictions. *Econometrica 68*(6), 1303–1341.

Sandroni, A. (2005). Market selection when markets are incomplete. *Journal of Mathematical Economics 41*, 91–104.

Young, L.-S. (1995). Ergodic theory of differentiable dynamical systems. In Branner and Hjorth (Eds.), *Real and Complex Dynamics*, pp. 293–336. Kluwer Academic Publisher.