Darmstadt Discussion Papers in Economics

Jefficiency vs. Efficiency in Social Network Models

Jochen Möbert

Nr. 161

Arbeitspapiere
des Instituts für Volkswirtschaftslehre
Technische Universität Darmstadt

Applied
Research in
Economics

Jefficiency vs. Efficiency in Social Network Models *

Jochen Moebert◇

March 7, 2006

Preliminary Version

Abstract

The mainly used welfare criterion in the social network literature is Bentham´s utilitarian concept. The shortcomings of this concept are well-known. We compare the outcomes of the utilitarian concept with the Nash social welfare function. By using a Taylor approximation we deduce a formula which allows the direct comparison of both concepts. The implications of welfare considerations of important network formation models are evaluated by using the multiplicative concept. We introduce a new symmetric connection model which is related to Nash´s welfare function in the same way as the original model is related to the utilitarian function. Based on the observation that heavy tail distributions like the power law distribution and the Pareto distribution can be explained by multiplicative structures we propose to use multiplicative utility functions in social network models. Furthermore, multiplicative utility and welfare functions together exhibit favorable characteristics both in normative and positive terms. Many empirically observed social networks have structures which are better modelled by multiplicative functions. From the normative perspective, multiplicative functions might be attractive since the Nash product introduces some form of justice.

JEL Classification: A13, D11, D23, D61, D63, L14, Z13

Keywords: social networks, welfare, efficiency, Nash product, jefficiency, justice

Contact details of the author:

Darmstadt University of Technology, Department of Economics, *moebert@vwl.tu-darmstadt.de*

◇ Actually, my surname is written *Möbert*.

*I am grateful to Horst Entorf, Thomas Rupp, Ema Trifan, Oliver Schmid and all seminar participants of the TUD-seminar. Deike Primuth-Schmidt and Bernd Albrecht also gave helpful comments. The research idea was implicitly proposed by Antoni Calvó-Armengol in one of his lectures.

1 Traditional Welfare Measure

Measuring the welfare of a group of people or a society is at the heart of traditional as well as modern economics. There are several welfare measures in the literature. The standard concept among the welfarist functions is Bentham´s utilitarian function which sums up the values of individual utility functions. The classical counterpart of Bentham´s concept is the minimax criterion described by Rawls (1971) which implies under weak assumptions an egalitarian outcome. Both are extreme points of moral philosophy while more pragmatic concepts are less frequently debated. Traditionally, the success of the utilitarian as well as the egalitarian welfare function is not emphasized by applied researchers, but ideologists and theorists used the concepts. The authors of the former type use it for consumption and the latter ones for the production of scientific contributions.

Until now the standard measure in the social network literature is the utilitarian welfare function. This welfare function is inherently anchored such that the term efficiency and utilitarian concept are used synonymously. This terminology is so ingrained that it is pointless to try to change it. Hence, we follow this notion and extend it below. In social network papers it might be challenging to proof which network structures are the efficient ones. For instance, in Bala and Goyal (2000) and Calvó-Armengol (2004) some difficulties are implicitly mentioned since the authors restrict some of their results related to the efficiency criterion to specific functions respectively values and not to a general solution.[1] Also the calculation of the efficient set might be constrained by slow computer processors. However, the optimal utilitarian outcome is in many situations still one of the less complicated concepts and this is generally true in many other fields of economics where proofs and the computational task are less demanding. Hence, we believe that the use of this concept in social network papers[2] is mainly based on mathematical simplicity and not economic reasoning. The attractiveness of the utilitarian welfare function is closely related to the way utility functions are specified. Sometimes game theorists do not derive their utility functions from observing the behavior of humans but from the experience that full-fledged solutions are often obtained out of linearly specified utility functions, while the detection of properly more realistic but also more complicated specifications of utility functions are much more involved. Therefore, instead of having realistic but irresolvable models economists prefer simple models conveying some simple messages. This line of reasoning might explain the prevalence of the utilitarian concept.

[1] In the case of the job-contact network model invented by Calvó-Armengol (2004) not the deterministic but expected sum of utilities is maximized.

[2] As well as in other subfields of economics.

Many researchers who use the utilitarian welfare concept often express their discomfort about its social implications. However, they continue to use it since they claim it is better to appraise the efficiency by the utilitarian concept then to put aside the welfare problem at all. The utilitarian concept abstracts from distributional issues. It only maximizes the sum of all utilities and allows for arbitrary utility distributions across a set of players. Hence, this criterion is inappropriate for advising non-economists and it can be an important reason for the lack of communication between theorists and applied researchers. We propose a different welfare criterion and compare our results with the utilitarian concept. Thereby, we believe that the mathematical ease is obtainable if the utility functions are specified in a more realistic way.

Carayol, Roux, and Yildizoglu (2005) propose the use of a genetic algorithm to approximate the set of efficient networks. This idea might be a promising one. Of course, one shortcoming of an approximation is the lack of beauty. While beauty has a value for itself it also enhances our intuitive understanding for outcomes. Another shortcoming of approximation is that an algorithm might produce misleading results. Carayol, Roux, and Yildizoglu (2005) show that for the co-author network formation model introduced by Jackson and Wolinsky (1996) the used algorithm is exact at least for small networks. However, it is not able to catch the structure of the symmetric connection model. Even worse, it seems that the degree of exactness depends in a nonlinear way on the number of players. Thus, if the algorithm fails sometimes for small and very stylized network models what do we get for large and really existing social networks? The findings imply that a lot of additional research seems to be necessary for the application of algorithm. The use of algorithm might be especially helpful if we consider the applications of social network methods to large networks or whole economies.

Welfare functions who maximize either the allocation efficiency or the equity are less interesting from a pragmatic point of view than welfare measure which take into account both properties. An example of such a welfare function is the Nash product. This product is only one member of the constant elasticity of substitution class where the risk aversion parameter is positive and finite. In this class of welfare criterions we chose it because of its simple structure. The Nash product is also called multiplicative social welfare function and is optimal if the product of all utilities is maximized. Game theorists who construct bargaining models are familiar with the Nash product which is often defined as the product of player's utilities above their reservation utilities. Nash (1950) showed that the maximization of this product has decent properties in the context of bargaining among agents. Since it is the unique function satisfying the strong efficiency, individual rationality, scale covariance, independence of irrelevant alternatives, and symmetry axiom. Nash's work was a seminal basis for the foundation of this branch of game theory.

Of course, in this paper we take the position of a social planner and use the Nash product to evaluate the welfare of social networks. Kaneko and Nakamura (1979) as well as, for example, recently Hanany (2001) used the Nash product as a mean to measure social welfare and showed its normative implications. In the social network literature Nash is associated with some equilibrium concepts. To avoid confusion with, for instance, the Pairwise-Nash equilibrium concept and to emphasize that the Nash product imports some form of justice we call the multiplicative welfare function the "jefficiency" criterion and call a network which maximizes this criterion "jefficient". After introducing our notation in Section 2 we compare in Section 3 some properties of the efficiency and the jefficiency measure in general terms. We are also able to apply some simple results to well-known social network formation models. In Section 4 we continue the application and comparison of the efficiency and jefficiency criterion and derive the jefficient networks of the original symmetric connection model as well as the "simple multiplicative symmetric connection model". Section 5 concludes by motivating that in many situations multiplicative utility functions have attractive properties which are complementary to the jefficiency criterion.

2 Notation

Let $\mathcal{N} = \{1, ..., N\}$ be the finite set of players and the individual player is denoted by i. We assume through out the paper $N \geq 3$. We denote the utilitarian welfare concept by $W = \sum_{i=1}^{N} u_i$ where u_i is the utility level of player i. The jefficiency criterion is defined by $J = \prod_{i=1}^{N} u_i$ where $u_i \geq 0$ for all i. Otherwise J is not defined. This assumption is necessary because without this assumption negative utility functions might be multiplied by each other and yield positive outcomes. This contradicts the aim of a social planner.[3] Given a utility distribution we abbreviate the mean utility level by $\overline{u}$. Utility vectors of dimension N are denoted by u and v respectively. The link between player i and j is denoted by ij and a set of links $g = \{ij, ik, \dots\}$ describes a whole network structure. If ij is formed then also ji exists which implies that we consider undirected unweighted networks. Different networks are distinguished by a subscript g_c for $c = 1, 2$. Some considered network structures are the star g_S and the regular network of degree k $g_{R,k}$. In a star formation one player the center has links to every other player while the other players, the peripheral players, have links to the star only. In a regular network each player has the same number of links. An example for a regular network is the complete network $g_{R,N-1}$ denoted by g_N where each player i forms $N-1$ links. Also the empty network is a regular network. Another frequently debated network formation is the line

[3] An alternative specification might be to multiply positive utility functions and divide the absolute value of negative utility levels, i.e. $J = \left(\prod_i u_i\right) / \left(\prod_j u_j\right)$ where $u_i > 0$ and $u_j < 0$. However, this definition is not applicable for zero utility levels.

where $g_L = \{i_1 i_2, i_2 i_3, \ldots, i_N i_{N-1}$. Additional notation which is needed to describe the results of other papers is introduced below.

3 Efficiency vs. Jefficiency

As is known, J and W are related to each other via the class of CES-functions. The general form of these functions is

$$CES = \frac{1}{1-\rho} \sum_i [u_i(.)]^{1-\rho} \tag{1}$$

where ρ indicates the aversion towards risk. If $\rho = 0$ the Benthamite social welfare function W is attained and for $\rho = 1$ the CES-functions are completed by the Nash social welfare function. In general, the larger ρ the higher is the value of equity.

Lemma 1 (Transferable Utility) *In a transferable utility world maximizing J requires maximizing W.*

Proof: To maximize J we want distribute the maximum amount of utility units among the set of players. This implies maximizing W in a first step. □

After W is maximized the maximization of J requires to distribute all utility units equally among the player set. The reason for this is that $\prod_{i=1}^{N} \overline{u} > \prod_{i=1}^{N} u_i$ if at least one individual utility level $u_i \neq \overline{u}$ where $\sum_{i=1}^{N} u_i = N\overline{u}$ and $\overline{u}$ indicates that all players have the same utility level.

In a nontransferable utility world the maximization of J takes also into account the equity criterion. Intuitively, multiplying all utility functions of the whole player set implies that each player´s utility has some form of externality on the social welfare function. Formally, this externalities are preserved if we rewrite J as

$$J = \exp\left(\sum_{i=1}^{N} log(u_i)\right) \tag{2}$$

It is readily seen that the maximization of J yields the same outcome as maximizing $log(J) = \sum_{i=1}^{N} log(u_i)$. Here not the utility sum is maximized but the sum of logarithm utilities. The log function scales down the welfare weight of high utility levels relative to low levels. This implies that the externalities initially contained in J are translated into the concavity of the logarithm function. Therefore J puts some weight not only on the production of utility like W but takes also into account some form of equity. A Taylor approximation of $log(J)$ around the mean $\overline{u}$ yields further insights.

Proposition 1

$$\log(J) = N\left[\log(\overline{u}) + \sum_{d=2}^{D=\infty}\sum_{i=1}^{N}(-1)^{d+1}\frac{1}{d}\frac{1}{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^d\right] \tag{3}$$

where we call $\frac{1}{N}\sum_{i=1}^{N}(\frac{u_i-\overline{u}}{\overline{u}})^d$ *the* $\overline{u}$*-standardized dth-moment.*

Proof: Using the Taylor-series we can write $log(u_i) = \sum_{d=0}^{\infty}\frac{\log^{(d)}(\bar{u})}{d!}(u_i-\bar{u})^d$ where $\log^{(d)}(\bar{u})$ is the dth derivation with respect to $\bar{u}$. This can be rewritten as $log(u_i) = \log(\bar{u}) + \frac{u_i-\bar{u}}{\bar{u}} + \sum_{d=2}^{\infty}\frac{\log^{(d)}(\bar{u})}{d!}(u_i-\bar{u})^d$. Setting $\log^{(d)}\bar{u} = (-1)^{d+1}\frac{(d-1)!}{\bar{u}^d}$ and summing over each player in the player set yields the above formula. □

We call the double sum the D-series and say J is approximated up to order $D \leq \infty$. If we also rewrite $W = N\overline{u}$ then the first right-hand term of J can be compared to W. Maximizing $N\overline{u}$ yields the same economic outcome than the maximization of $Nlog(\overline{u})$. Therefore if the individual utilities are close to the mean then the D-series is close to zero and the maximization of J and W yields similar results. However, J takes also into account the distribution of the utilities. Thus, if the utility levels of the players are spread across a larger range then J yields different results than W. Taking the exponential on both sides of equation 3 might be helpful to separate the jefficiency criterion into an allocation part and an part expressing justice.

$$J = \overline{u}^N + \exp\left[\sum_{d=2}^{D=\infty}\sum_{i=1}^{N}(-1)^{d+1}\frac{1}{d}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^d\right] \tag{4}$$

Let us call $\overline{u}^N$ the allocation addend and $J-\overline{u}^N$ the distributional addend. Then the fractions $\frac{\overline{u}^N}{J}$ and $\frac{J-\overline{u}^N}{J}$ characterize how important efficiency and justice are in a specific situation. The Taylor approximation is valid for other values than $\overline{u}$. If other values are chosen then the partition into an efficiency part and a justice part depends on the choice of the value. However, the choice of $\overline{u}$ seems attractive because given this choice the justice part can be interpreted as $\overline{u}$-standardized dth-moment.

If we consider the Taylor approximation up to degree $D = 4$ then

$$\log(J) \approx N\left[\log(\overline{u}) - \frac{1}{2N}\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^2 + \frac{1}{3N}\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^3 - \frac{1}{4N}\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^4\right] \tag{5}$$

follows. This equation shows that $\log(J)$ is ceteris paribus lower if the $\overline{u}$-standardized variance and the $\overline{u}$-standardized kurtosis of the utility distribution are higher. Furthermore, $\log(J)$ is ceteris paribus higher for right skewed utility distributions than for left skewed distributions. We apply a simple example to illustrate the preference for right-skewed relative to left-skewed utility distributions. Let $N = 3$ and suppose in case (1) the utility levels are $u_1 = 2, u_2 = 2, u_3 = 5$ and in case (2) the utility levels are $u_1 = 1, u_2 = 4, u_3 = 4$. In both cases $\bar{u} = 3$. [4] Let the skewness be $S^{(case)} = \frac{1}{3}\sum_{i=1}^{N}\left(\frac{u_i-\bar{u}}{\bar{u}}\right)^3$ then $S^{(1)} = 6 > -6 = S^{(2)}$. Therefore, we can conclude that $\log(J^{(1)}) > \log(J^{(2)})$. This also holds for the not approximated equation of J since all even d-terms are identical and all odd d-terms are greater in case (1) than in case (2).

Note that the Taylor-Approximation is only reasonable if the double sum $\sum_{d=2}^{\infty}\sum_{i=1}^{N}(-1)^{d+1}\frac{1}{d}\frac{1}{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^d$ converges for large d. Otherwise the alternating elements of the sum diverge and the value of J might heavily depend on the order of the approximation or otherwise stated the approximation is not applicable.

Remark 1 *The Taylor approximation is only valid if the series converge and the series exhibits this property if and only if $u_i \leq 2\overline{u}$ for each player i.*

Proof: From $u_i \leq 2\overline{u}$ it follows that $\frac{u_i-\overline{u}}{\overline{u}} \leq 1$ and applying Leibniz' Convergence criterion completes this part of the proof. The series, however, converges only if $\frac{u_i-\overline{u}}{\overline{u}}$ does not diverge. Therefore we need $u_i \leq 2\overline{u}$. $\square$

Another helpful remark simplifies some considerations about different network characteristics.

Remark 2 *The right term of* $\log(J)$ *in Proposition 3*

$$\sum_{d=2}^{D=\infty}\sum_{i=1}^{N}(-1)^{d+1}\frac{1}{d}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^d \tag{6}$$

is always negative if the series converge.

Proof: Independent of the order of D we find to every odd addend of order $d+1$ an even addend of order d. Thus, it suffices to show that

$$-\frac{1}{d}\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^d + \frac{1}{d+1}\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^{d+1} < 0 \tag{7}$$

[4] Furthermore, convergence is guaranteed since $\max[u_i] < 2\bar{u}$ in both cases. See Remark .

holds. Which is true by the convergence of the series. □

Both welfare criteria J and W are also related to each other under specific circumstances. While it is difficult to show full-fledged results for the general model it is possible to discover some structure under special circumstances which are discussed below.

Lemma 2 (Dominant Order) *Suppose there are two networks g_1 and g_2 if $u_i(g_1) > u_i(g_2)$, for all i then it holds that both $W(g_1) > W(g_2)$ and $J(g_1) > J(g_2)$.*

The proof is straightforward and omitted.

Lemma 3 (Non-Dominant Order) *Suppose there are two networks g_1 and g_2. Suppose S_1 and S_2 are a partition of the player set $\mathcal{N}$. Suppose $u_i(g_1) > u_i(g_2)$ for all $i \in S_1$ and $u_j(g_1) < u_j(g_2)$ for all $j \in S_2$. Furthermore we assume $\sum_i [u_i(g_1) - u_i(g_2)] > \sum_j [u_j(g_2) - u_j(g_1)]$ then $W(g_1) > W(g_2)$ but it might hold that $J(g_1) < J(g_2)$.*

Proof: Let $up_{S_k} = \prod_{i=1}^N u_i$ and $us_{S_k} = \sum_{i=1}^N u_i$ for all $i \in S_k$, $k = 1, 2$. Then $J(g_1) = up_{S_1}(g_1) up_{S_2}(g_1)$ and $J(g_2) = up_{S_1}(g_2) up_{S_2}(g_2)$. Thus $J(g_1) < J(g_2)$ can be rewritten as $\alpha_1 up_{S_2}(g_1) < up_{S_1}(g_2)\alpha_2$ where $\alpha_1 = us_{S_1}(g_1) - us_{S_1}(g_2) > 0$ and $\alpha_2 = us_{S_2}(g_2) - us_{S_2}(g_1) > 0$ and $\alpha_1 > \alpha_2$. Since α_1 can be arbitrarily close to α_2 our inequality $J(g_1) < J(g_2)$ is true for large enough $up_{S_1}(g_2)$. □

Lemma 4 (Regular Networks) *(i) If players are homogenous then the efficient network among the set of regular networks is also the jefficient network among the set of regular networks and (ii) if an efficient network is regular then this network is also the jefficient network.*

Proof: (i) If players are homogenous and all players have the same position in the network then $W = \sum_{i=1}^N u_i(g) = N u_i(g)$ and $J = \prod_{i=1}^N u_i(g) = [u_i(g)]^N$. This implies that for both criteria we look for that regular network that produces the maximal utility sum for any player i since all players are identical. (ii) Lemma 1 implies that in a transferable utility world we first maximize W and then distribute the utility sum equally across all players. However, if W is maximized such that all players get the same utility the second step can be omitted and Lemma 1 is also applicable to the nontransferable utility world. □

Lemma 4 is applicable to the one-way flow model in Bala and Goyal (2000). In the one-way flow model information which spreads through networks can only flow to the player who bear the costs for the existing links. There the efficient network is either the empty network or the cycle.[5] Of course, the empty network is only efficient if the costs to form links are too high. Otherwise the cycle is the efficient network where every player forms one link and gets the whole information set available in the set. Given our Lemma 4 we know that the cycle is also the jefficient network. The same line of reasoning holds for the co-author model explained in Jackson and Wolinsky (1996). If the number of players is even then the efficient network in the co-author model is the one where each component is formed by two players only. The reason for this result is that each player gains the maximum amount of utility relative to the number of links. If a further link is added to one player then the utility of the indirectly connected players shrinks more than the direct links add to the players who form the link. Thence the welfare is reduced. Given Lemma 2 we can conclude that the jefficient network is also the network containing components of pairs of players only.

Another frequently investigated network formation is the star network. There the central player is connected to $N-1$ players while the periphery players are only connected to the center. Let us assume a convergence series as defined above and denote the utilities of both group of players by u_c for the central player and u_p for the periphery players then

$$\log(J) = N\log\left[\frac{(N-1)u_p + u_c}{N}\right] + \sum_{d=2}^{D=\infty} \frac{(-1)^{d+1}}{d}\left\{\frac{(N-1)(u_p-u_c)^d\left[1+(N-1)^{d-1}(-1)^d\right]}{[(N-1)u_p+u_c]^d}\right\}$$

which given a fixed number of players depends only on u_p and u_c. It is immediately seen that $\log(J)$ reduces to $N\log(u_i)$ if and only if $u_p = u_c$ under the assumptions stated. If $u_p \neq u_c$ then all even d-terms are negative. The odd d-terms are negative if $u_p > u_c$ because since then as described above a left-skewed utility distribution is evaluated and the d-terms are positive if $u_p < u_c$.

The J-star formula can also be reinterpreted in the following way. In a sense it relates an inequality measure to the total utility sum. Thus, we can rewrite

Lemma 5 (Star Networks) *It holds that*

$$\log(J) = N\log\left(\frac{W}{N}\right) + \sum_{d=2}^{D=\infty}\frac{(-1)^{d+1}}{d}\left(\frac{\Gamma}{W}\right)^d (N-1)\left[1+(N-1)^{d-1}(-1)^d\right]$$

where $\Gamma = u_p - u_c$ *and* $W = (N-1)u_p + u_c$.

[5]Bala and Goyal (2000) used the notion wheel while in graph theory the standard term is cycle.

The Proof requires simply replacing and solving the above equations to the original formula. Γ can be interpreted as an inequality measure while W is the sum of all utilities. It also holds that $\Gamma > W$. This inequality holds since $u_p - u_c < (N-1)u_p + u_c$ which implies $0 < (N-2)u_p + 2u_c$. Given $N \geq 3$ the inequality is always fulfilled. This formula and its interpretation is applicable to any network where only two different utility levels arise.

The results described can also be characterized graphically. The graphs in figure 1 show the functions $W(\overline{u})$ and $J(\overline{u})$. Of course, the dotted line representing W is a linear function of $\overline{u}$ while J depends not only $\overline{u}$ but also on the distribution of utilities. Therefore the mapping represented by the grey field in figure 1 shows the possible values the jefficiency criterion might take in dependence of $\overline{u}$. The maximum J is reached where the utility distribution degenerates to $u_i = \overline{u}$ for all player i. The minimum is reached if there exist at least one player i such that $u_i = 0$. The arguments also suggest that for each $\overline{u}_2 > \overline{u}_1$ it is possible to find a $J'(\overline{u}_2) \leq J(\overline{u}_1)$ where the inequality is strict if $\overline{u}_1 > 0$.

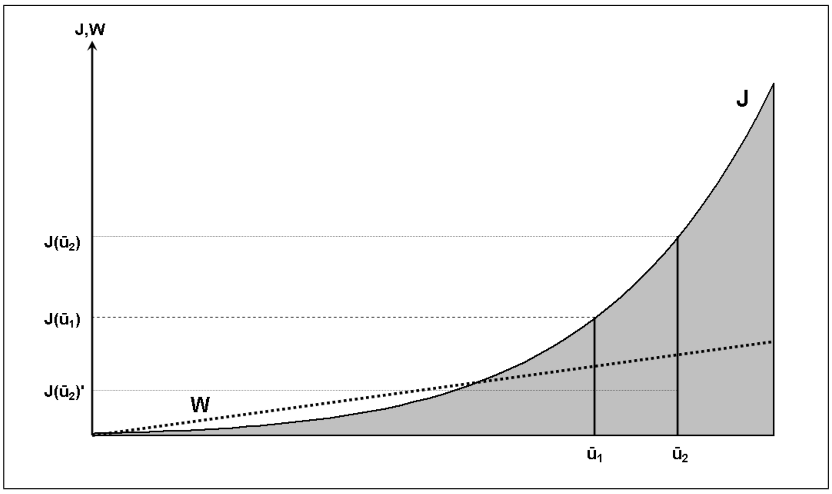

Figure 1: J and W in dependence of $\overline{u}$

Remark 3 *Let us perform a mean preserving spread such that the convergence of the series condition is still maintained. Then such a mean preserving spread reduces J.*

The proof is appended.

For a symmetric utility distribution the outcome is immediately proven since all even d-parts are decreased by a mean preserving spread and odd d-terms are zero for any symmetric utility distributions.

4 Jefficiency in the Symmetric Connection Model

A (Multiplicative) Symmetric Connection Model
Jackson and Wolinsky (1996) introduced one of the simplest network formation models available where each player gets utility out of direct and indirect links where more distant links contribute less. Costs are only paid for direct links. In this model only the empty network, the star, and the complete network are among the class of efficient networks. If we look for the jefficient networks we discover a much richer set of networks.

Proposition 2 *The set of jefficient networks of the symmetric connection model is different from the set of efficient networks.*

Proof: In the Appendix we show that for $N = 4$ the line is among the set of jefficient networks if the costs to form links c are large enough.

It can also be shown that $g_{R,2}, g_{N-ij}, g = \{12, 13, 23, 24\}, g_S$, and g_N is among the jefficient networks if the number of players is four. However, a general solution is quite involving. Given Gauss´ fundamental theorem of algebra, we conjecture that the set of jefficient network depends on the number of players.[6]

Conjecture 1 *In the symmetric connection model the set of jefficient networks is increasing with the number of players in a network.*

The result in proposition 2 shows that the set of jefficient networks has a greater cardinality than the set of efficient networks in the symmetric connection model. Above we demonstrate that J is maximal if both equity and the production of utility is taken into account. Hence, we believe that the set of jefficient networks encompasses at least some regular networks where each player gets the same utility and some minimal connected networks where the number of links and therefore the costs of forming links is minimized, and also some mixture of both network structures.

In Moebert (2006) we defined the class of all symmetric connection models – the SCM(W)-class – grounded on the utilitarian efficiency criterion[7] and showed that the structure

[6]Gauss' proof shows that in a polynomial of degree d there are d possible solutions. Some of the solutions might be complex and some of them might have multiplicity larger than one.

[7]We call this class the SCM(W)-class which means the class of symmetric connection models based on the utilitarian welfare criterion.

of the proof of the corresponding proposition in Jackson and Wolinsky (1996) can be used as a basis for the derivation of a whole class of symmetric connection models. Here we define a network formation model which is closely related to the SCM(W)-class. We call this model "simple multiplicative symmetric connection model".[8] The idea of this network formation model relies on the (original) symmetric connection model mentioned above. The "simple multiplicative symmetric connection model" exhibits the following utility function

$$u_i = \delta^{\sum_{e=1}^{\infty} \frac{l_e}{e}} c^{-l_1} \tag{10}$$

where l_e indicates the distance of player i to all other players, δ is the utility of direct respectively indirect links and c are the costs of forming direct links. It is possible to show that the maximization of J leads to a simple result which is similar to both the maximization of W in the original and in the simple additive symmetric connection model.

Proposition 3 *The jefficient network is*
(a) the complete network if $c < \delta^{0.5}$
(b) the star if $\delta^{0.5} < c < \delta^{0.5+0.25N}$
(c) the empty network if $c > \delta^{0.5+0.25N}$.

Proof: (a) If direct links are more valuable than indirect links then $\frac{\delta}{c} > \delta^{0.5}$ which implies that the fully connected network g_N is formed if $c < \delta^{0.5}$. (b) Let us call the upper bound of a connected component with m players and $k \geq m-1$ links $J_U = (\frac{\delta}{c})^{2k}(\delta^{0.5})^{[m(m-1)-2k]}$. The jefficiency measure of the star is $J_S = (\frac{\delta}{c})^{2(m-1)}(\delta^{0.5})^{(m-2)(m-1)}$. Since $J_S \geq J_U$ for $\delta^{0.5} < c$ the star is jefficient among the set of connected components. (c) The star is just restricted by the emtpy network g_0 which is jefficient if $1 > (\frac{\delta}{c})^{2(m-1)}(\delta^{0.5})^{(m-2)(m-1)}$ which can be reduced to $c > \delta^{0.5+0.25m}$ $\square$.

It might be helpful to mention that the proof here is only nearly complete. However, our proof here has the same structure as the proof who identifies the set of efficient network in the original symmetric connection model by Jackson and Wolinsky (1996). Actually, both the original proof as our result above rely on the assumption that a shortest indirect link of distance two spends more utility than any other indirect link of greater distance. Of course, in both network formation models this assumption is readily fulfilled. It might also be important to expose that there exists a close relation between the SCM(W)-class and the multiplicative network formation model here which might be a member of a whole class of multiplicative symmetric connection models.

[8] In accordance to the "simple additive symmetric connection model" introduced in Moebert (2006).

5 Conclusion

In this paper we applied a well-known welfare measure, the Nash product, also called jefficiency criterion, to social network formation models. We also derived several relationships between the standard efficiency criterion and the jefficiency criterion. In particular, by using a Taylor approximation we were able to deduce a formula which improves our understanding of the jefficiency criterion relative to the utilitarian welfare measure. We argued that the derivation of efficient outcomes is easier to perform if utility functions are linearly specified. We also showed, by means of examples, that replacing the efficiency criterion by the jefficiency criterion leads to a larger variety of network structures in the symmetric connection model. However, the results of the “simple multiplicative symmetric connection model” demonstrated that multiplicative utility functions[9] might lead to simple welfare outcomes if welfare is measured by the jefficiency criterion. Hence, the evaluation of multiplicative specified utility functions with respect to the jefficiency criterion might be a natural choice both from a mathematical point of view to get relatively simple results and from a normative point of view since the jefficiency criterion introduces some form of justice into the social network literature.

The main advantage of the use of multiplicative utility functions, however, might be that multiplicative utility functions exhibit characteristics which can explain some features of really existing social networks. For instance, physicists among other Albert, Jeong, and Barabasi (1999) have shown that a general characteristic of networks is that the distributions of links obeys the power-law or similar distributions. This kind of link distributions characterized by fat tails is also found in social networks. For instance, Goyal, van der Leij, and Moraga-González (2004) investigated that the real existing co-author network of economists also exhibits fat tails for the period 1970 to 2000. Mitzenmacher (2003) has shown that one necessity for the derivation of such distributions is the introduction of multiplicative structures. Economists who search for network formation models who describe the reality in an appropriate way might therefore prefer the modelling of multiplicative utility functions instead of the linear functions used often in today’s specifications. Also in research fields which are closely related to the social network literature like random graph theory and Markov chain theory many important results possess multiplicative structures.

Intuitively, multiplicative utility functions are directly justifiable by the simple observation that the survival of humans require the availability of several goods like air, water, food, and love. Without any of these resources human mankind is unable to survive. Hence, if only one

[9] We call utility functions “additive” respectively “multiplicative” if they exhibit the “additive separability” respectively “multiplicative separability” properties. For further details see Moebert (2006).

of these resources is not available for some players then the utility levels of players in social network theories should be zero and not above zero as in additively specified utility functions.[10]

6 Appendix

Proof (Remark 3): Let N^+ and N^- be a partition on the player set. If $i \in N^+$ then $v_i > u_i$ and if $i \in N^-$ then $v_i \leq u_i$. Therefore, the utility vector v defines our mean preserving spread of u. Independent of the order of D we find to every odd addend of order $d+1$ an even addend of order d (see Remark 6). Therefore, we just have to show that

$$-\frac{1}{d}\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^d+\frac{1}{d+1}\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^{d+1} > -\frac{1}{d}\sum_{i=1}^{N}\left(\frac{v_i-\overline{u}}{\overline{u}}\right)^d+\frac{1}{d+1}\sum_{i=1}^{N}\left(\frac{v_i-\overline{u}}{\overline{u}}\right)^{d+1} \tag{11}$$

This can be rewritten as

$$\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^d-\frac{d}{d+1}\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^{d+1} < \sum_{i=1}^{N}\left(\frac{v_i-\overline{u}}{\overline{u}}\right)^d-\frac{d}{d+1}\sum_{i=1}^{N}\left(\frac{v_i-\overline{u}}{\overline{u}}\right)^{d+1} \tag{12}$$

and

$$\sum_{i=1}^{N}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^d-\frac{d}{d+1}\left(\frac{u_i-\overline{u}}{\overline{u}}\right)^{d+1} < \sum_{i=1}^{N}\left(\frac{v_i-\overline{u}}{\overline{u}}\right)^d-\frac{d}{d+1}\left(\frac{v_i-\overline{u}}{\overline{u}}\right)^{d+1} \tag{13}$$

follows. Taking the derivative of $f(a) = a^d - d(d+1)^{-1}a^{d+1}$ with respect to a yields $f'(a) = a^{d-1} - a^d$ and we have a root at $a = 0$. If $a > 0$ then $a^{d-1} > a^d$ and if $a < 0$ then $a^{d-1} < a^d$. This implies that $f(a)$ is maximal if $a = 1$ or $a = 0$. Hence, a mean preserving spread reduces J. $\square$

Proof (Proposition 2): For $N = 4$ there are eleven possible network formations in the symmetric connection model. Four out of these networks, for example g_0, have at least one player who is not connected to anyone else. Therefore the jefficiency measure of these networks are zero. The following networks are left: $g_{R,1}, g_L, g = \{12, 13, 23, 24\}, g_S, g_{R,2}, g_{N-ij}$, and g_N where g_{N-ij} describes any network where $2^{-1}[N(N-1)] - 1$ links are formed. Let $c = \delta$ then in all networks where one player has only direct and no indirect links have a jefficiency measure of zero. Only J_L and $J_{R,2}$ are greater than zero and if $c = \delta$ then $J_L > J_{R,2}$. $\square$

[10] Of course, this statement holds not only for the social network literature but for many game theoretical models.

References

Angilella, Silvia, Salvatore Greco, Fabio Lamantia, and Benedetto Matarazzo (2004), Assessing non-additive utility for multicriteria decsision aid. European Journal of Operational Research, 158:734-744.

Bala, Venkatesh and Sanjeev Goyal (2000), A Noncooperative Model of Network Formation. Econometrica, 68(5):1181-1229.

Calvó-Armengol, Antoni (2004), Job contact networks. Journal of Economic Theory, 115:191-206.

Carayol, Nicolas, Pascale Roux and Murat Yildizoglu (2005), Efficiency of network structures: The needle in the haystack. Cahiers du GRES, (2005-06).

Goyal, Sanjeev, Marco van der Leij and José Luis Moraga-González (2004), Economics: an emerging small world, CESifo Working Paper No. 1287.

Hanany, Eran (2001), The Ordinal Nash Social Welfare Function, CMS-EMS, Northwestern University, hananye@northwestern.edu.

Jackson, Matthew O. and Asher Wolinsky (1996), A Strategic Model of Social and Economic Networks. Journal of Economic Theory, 71:44-74.

Kaneko, Mamoru and Kenjiro Nakamura (1979), The Nash Social Welfare Function. Econometrica, 47:423-435.

Mitzenmacher, Michael (2003), A Brief History of Generative Models for Power Law and Lognormal Distributions. Internet Mathematics, 1(2):226-251.

Moebert, Jochen (2006), Classification of Network Formation Models, Darmstadt Discussion Paper No. 160, http://www.bwl.tu-darmstadt.de/vwl/forsch/veroeff/index.htm.

Nash, John F. (1950), Two-Person Cooperative Games. Econometrica, 21(1):128-140.

Rawls, John (1971), A Theory of Justice. Harvard University Press, Boston.

ISSN: 1438-2733