Long term debt with hidden borrowing

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Abstract

We consider borrowers with the opportunity to raise funds from a competitive baking sector, that shares information about borrowers, and an alternative hidden lender. We highlight that the presence of the hidden lender restricts the contracts that can be obtained from the banking sector and that in equilibrium some borrowers obtain funds from both the banking sector and the (inefficient) hidden lender simultaneously. We further show that as the inefficiency of the hidden lender increases, total welfare decreases. By extending the model to examine a partially hidden lender, we further highlight the key role of information.

1 Introduction

Households have many potential sources of credit available including secured mortgages, installment loans, bank overdrafts, store credit, credit cards, payday loans, borrowing from family and borrowing from the black market. Similarly, small and large firms face a number of different financing options ranging from private placements, securitized loans, trade credit and personal loans to the owner. These alternative forms of financing differ in a number of ways. While some might be endogenous (such as the interest rate, or the term length for repayment), there are also exogenous differences, for instance, with respect to the seniority of the claims and enforcement.

An empirical puzzle with regard to these multiple sources of financing is that borrowers appear to borrow from apparently costly lenders while not fully exhausting cheaper sources. Gross and Souleles (2002) for example, report that in a large sample of credit card holders, almost 70% percent of those borrowing on bankcards have positive housing equity. Similarly, small businesses often use uncollateralized trade credit when collateral and collateralized loans are available. Finally, there is both theoretical work and empirical evidence of formal and informal sources of credit coexisting in developing countries (Bell et al (1997), Bose (1998), Jain (1999)) and where some agents may simultaneously borrow from both sources, it is unclear though whether agents need to be rationed by the formal sector in order to access the informal one.

In this paper, we suggest that an important consideration in understanding this puzzle is the observation that different lenders, or indeed different sectors might vary in the information that they have available.[1] In particular, the existence of junior lenders whose loans are hard to observe by a main senior lender may give borrowers a chance to conceal liquidity shocks that affect their creditworthiness.

As an example, missing a mortgage payment can trigger a renegotiation with the provider and lead to a higher future interest rate, reflecting the provider's belief that the borrower is a greater risk. Therefore the borrower (if she has access to alternative sources) might borrow from elsewhere to conceal her liquidity shock. In turn, this makes missing a payment even worse news. At an extreme this might lead the provider to foreclose following a missed payment.

We illustrate these ideas more formally with a two period model in which agents need to raise funds for an investment project, which produces cash flows that are correlated over the two periods. Agents are homogeneous ex-ante, but after investing, they are heterogeneous in the expected cash flows of their investment projects. Agents observe the interim cash flows produced

[1] Note that other explanations have been posited to explain this apparent puzzle; for example, Laibson et al. (2001) calibrate a model of life-cycle borrowing with time inconsistent preferences, and Haliassos and Reiter (2003) discuss a model of separate mental accounts. The results of this paper need not contradict such explanations but can be seen as complementary to them and, in particular, fail to explain the coexistence of credit card debt and liquid assets, rather they can be seen as suggesting some endogenous illiquidity of certain assets.

by their projects, which are correlated with the final cash flows.

Agents face a competitive banking sector that shares information about outstanding loans. Furthermore borrowers have the chance to repay their outstanding balance at any point in time and switch to a competitor bank. This implies that conditional on the available information at any point in time, each loan should just break even in expectation.

Under these conditions banks would like to find ways to give incentives for agents to reveal the types of their projects, as they learn them, and to use this information to try to attract or retain customers profitably. In the model, the only way to achieve both these aims is through the loan contract, by making the future interest rates of the loan contingent on the current payment, with lower interest rates associated with higher interim payments. However, the presence of an opaque lender may render such contingent contracts ineffective, as agents may seek to conceal their types by borrowing from this hidden source to make higher interim payments and enjoy the corresponding lower future interest rates. Realizing this possibility banks will increase the informational penalty associated with a low interim repayment, as only those agents with very low realizations and would prefer to pay this lower interim payment rather than borrowing from the hidden lender and paying a higher interim repayment. We show that this endogenous renegotiation cost may be so high that it precludes any possible renegotiation and the only feasible contract may be one in which borrowers face a fixed interim repayment or liquidation. In equilibrium, some agents borrow simultaneously from both sources of credit.

Our principal results are, therefore, that in the absence of the opaque sector realized contracts are complex menus, where higher levels of interim payments lead to lower final payments. This is not only to take into account that less is owed, but also since a higher interim payment reflects that the borrower is less of a credit risk.[2] However with a viable alternative hidden lender, a borrower might be tempted to borrow from that source in order to disguise her type. This possibility is anticipated by the original lender in the banking sector and in the model the unique equilibrium results in only a single level of interim payment observed in the banking sector, with some agents borrowing from the opaque sector to make this payment. Moreover, we consider how the welfare of consumers and the transparent sector vary with the cost of borrowing from the opaque source. In particular, note that while a lower cost of borrowing benefits consumers for a fixed level of borrowing, it also encourages a greater number of inefficient types to continue to seek to borrow rather than to terminate the debt contract early and more efficiently and, in addition, would lead the contract in the transparent sector to change and overall welfare falls.

A key element of the model is the observation that different lending sectors have access to

[2] This result mirrors the observation in Allen (1985) that long-term contracts allow interim payments to provide information.

different information concerning loan applicants. Empirically, this is certainly the case. For example, information sharing takes place through credit bureaus; however, there are many lenders who choose neither to pay for credit bureaus nor to provide information to these bureaus—informal and black market lending are obvious examples, though other instances include "no questions asked" payday lenders. Further, particularly, in the US historically and in developing countries and elsewhere currently, store credit, consumer credit and other sources do not participate in formal information-gathering credit bureaus. In this paper, we simply take it for granted that some lenders have access to information that others do not.[3]

Note that the banking sector cannot write contracts which make payments depend on the agent's borrowing from the alternative lender—this is a natural consequence of the assumption that the banking sector cannot observe borrowing from the hidden lender. This paper is therefore related to a growing literature on non-exclusive contracts and on hidden savings which includes Allen (1985), Cole and Kocherlakota (2001), Bisin, and Guaitoli (2004), and Doepke and Townsend (2004). The model below differs in a number of respects and in its motivation; in particular, in the model considered below we consider different lending sectors which vary in the information that they hold, we focus on an adverse selection rather than on moral hazard as much of the literature considers (Bizer and DeMarzo (1992) and Arnott and Stiglitz (1991) for example) and we model agents to be risk neutral with limited liability, rather than risk averse and seeking to smooth consumption or buy insurance, an important focus of this literature.

Note also that while Allen (1985) and other papers focus on the case where the borrowers cost of borrowing from a hidden source is equal to the social planner's rate and where Innes (1990) in order to generate monotonicity in repayment schedules considers the case where money can be repaid immediately so that essentially the cost of borrowing is zero, an important focus of our analysis is varying the cost of borrowing from the hidden source.[4]

Below, Section 2 introduces the model and elaborates the key assumptions. In Section 3, we solve for the equilibrium and characterize the principal results, in particular we discuss comparative statics with respect to the cost of borrowing from the opaque sector and in particular implications for welfare. We briefly discuss an extension to the case where borrowing from the opaque sector can be observed with some probability in Section 4. The final section briefly discusses a potential extensions and concludes. A number of the more involved proofs appear in the appendix.

[3] For further information on consumer credit reporting in the US and further references both on the theory and development of credit bureaus and reporting institution, the interested reader is referred to Hunt (2002).

[4] The general model of Doepke and Townsend (2004) as illustrated in their example in Section 7.1 allows for this more general interest rate; however, as Cole and Kocherlakota (2001) and Ljungqvist and Sargent (2003) they consider hidden saving and insurance rather than hidden borrowing and focus on numerical rather than analytical solutions.

2 The Model

We introduce a two period model to consider the interaction between alternative sources of borrowing. In the transparent sector, credit is provided by a continuum of agents that we call banks. Banks are risk neutral deep pockets and there is competition among them. Banks share information among themselves so the borrowing position of any agent with a bank is perfectly observable and verifiable among all banks. We normalize the gross riskless market interest rate of this formal sector to one. This can be summarized in the following three assumptions:

Assumption 1: Banks are deep pockets, so the total amount of loanable funds exceeds the demand for funds.

Assumption 2: A borrower can repay her outstanding balance and switch to another bank at any point in time.

Assumption 3: Banks perfectly share the information about the borrower´s outstanding loans.

These assumptions guarantee that banks do not make a profit on average and that conditional on the information known at any point in time, every contract offered must break even. In other words, there can be no observable cross subsidies between borrowers. If a set of borrowers knew, and were able to prove to a third party, that they are subsidizing other borrowers, they would switch to another bank, leaving their previous bank with only subsidized borrowers and so losses.[5]

In addition to the transparent banking sector we introduce an alternative opaque lending sector that lends at a flat rate r, which is exogenously given. A key characteristic of this alternative borrowing source is that it does not share information with the rest of the financial system. That is, the borrowing position of any agent in the opaque sector is not observable by banks. In our model lenders exogenously belong to either the banking sector or the opaque sector. Existing literature (Pagano and Japelli 1993) discusses determinants of belonging to either group as an endogenous decision.

Demand for funds come from a number of heterogeneous borrowers, who require these funds for an investment project. They are risk neutral and maximize total consumption across periods.

The timing of the project is as follows:

At $t = 0$ each borrower does not know her type. In order to raise D units of funding necessary to invest in the project, the borrower proposes a schedule of first and associated second period repayments $\{p, q(p)\}$ to a bank.

At an interim period $t = \frac{1}{2}$, each borrower learns the type of her project which is parametrized by $\alpha \sim U[0, 1]$. At this point, the borrower can either liquidate the project and fully repay the

[5] Note that the assumption of perfect competition within the banking sector is not crucial for our results. Introducing some switching costs for example would result in somewhat contingent contracts in the absence of an opaque sector which could be unwound by its presence.

loan or continue with the project.[6]

At $t = 1$ agents realize a cash flow α that corresponds to their type. They can choose to borrow d from the informal lending source. The informal lender is junior to the bank loan. A key characteristic of this informal lender is that the amount borrowed is not observable by the formal lender. Agents can use these funds to either consume or repay p to the bank. Agents consume anything leftover so residual income cannot be used to repay future debts.[7]

At $t = 2$ the project is successful and delivers $B + \alpha$ with probability ν. Otherwise the project fails and delivers only α. In both cases, seniority of debt is such that the borrower repays transparent lenders first and then repays opaque lenders up to rd. The borrower consumes all the remaining funds. The parameter α therefore represents the creditworthiness of the borrower as that the expected final cash flow of the project is positively correlated with its interim cash flow.

We make the following assumption, which ensures that banks in the transparent sector break even both at the *ex-ante* and interim stages, as stated in Lemma 1.[8]

Assumption 4: Some infinitesimal proportion of borrowers already know their own types at t=0.

Lemma 1 *: The set of contracts $p, q(p)$ is renegotiation proof and breaks even at all future possible stages.*

Proof. Competition between banks ensures that banks break even at the *ex-ante* stage.

Assumption 4 guarantees that the initial schedule of contracts is already contingent on all the future public information about the borrower´s type so borrowers will effectively not switch (or renegotiate on the threat of switching) to another bank. This follows since if, contingent on some possible future information, a contract would contain cross subsidies then among the small subset of agents that already know their type, subsidizing agents would propose alternative contracts to other banks while subsidized agents would stay.[9] Thus, those banks offering contracts which allowed for cross subsidies would suffer losses. ■

As a transversality condition we make the following assumption.

[6] We model this option to stop the project as a costless liquidation in a very early stage but supposing that the agent was able to recover a sufficiently large salvage value at an early stage would generate similar qualitative results.

[7] In fact consuming everything is optimal in the model even when the consumption good is costlessly storable.

[8] The renegotiation proof condition is effectively equivalent to an exclusivity proof contract. That is, a contract that guarantees that at any point in time the borrower does not want to switch to another bank (see Rampini and Bisin 2004).

[9] While one might consider that such offers are off-equilibrium offers and so might be rendered meaningless by a modeller with judicious use of off-equilibrium beliefs. We focus on equilibria robust to the intuitive criterion.

Assumption 5: An agent cannot borrow from the opaque lender if the nominal value of the loan is higher than the highest possible residual income.

This assumption can be understood as a "no fraud" condition. For example, it might be appropriate if agents can be punished beyond limited liability (personal liability, non pecuniary punishments, prison) if it were found (perhaps with some probability) that they did not intend to repay in any possible state of the world. This is a sensible borrowing limit as most legislations allow for punishment above limited liability (i.e. prison or personal liability) whenever a borrower takes a loan that she does not intend to repay even in the best possible situation. Alternatively if one thinks of the alternative lender as either an informal lender in development economies, friends and family or black market, one could think that while limited liability may still hold, it would not apply to agents that use the system without any intention of repayment. This limit leads to the result that borrowing from the hidden source in order to consume does not pay if the agent repays in the good state because hidden borrowing is inefficient, that is $r > \frac{1}{\nu}$. It would only pay if the agent intends to default for sure.

We make parametric restrictions, which preclude some trivial and uninteresting cases.

Assumption 6: $D > 2$ and $2 > D - \nu B > 0$

The first restriction ensures that no borrower can repay for sure, the next that all types of borrowers will default to a different extent if the project is unsuccessful (so from the point of view of lenders they really are different types) and in particular, some projects are efficient and some are not. We define $z = D - \nu B$ as an inverse measure of the common profitability of the project. In particular $z = 0$ implies that all projects are efficient and should be funded, while $z = 2$ implies that no projects should be funded. With intermediate values of z, only projects with $\alpha \geq \frac{z}{2}$ are efficient.

3 Equilibrium

The feasible strategies for the borrower are any offer of schedules $\{p, q(p)\}$ at $t = 0$. Furthermore the borrower has to decide whether to pursue the project at $t = \frac{1}{2}$ or liquidate. Finally the borrower has to decide on how much to borrow from the hidden source. The bank has to choose whether to agree to menu of contracts. As discussed above, in Lemma 1, any meaningful contract on the schedule, that is any contract that is ever taken up in equilibrium, will break even at all the stages of the contract and so will not contain any observable cross subsidies.

In order to solve for equilibrium, we can draw on the revelation principle and think of the borrower's choice of repayment from the schedule $\{p, q(p)\}$ as a function of her type that is we could think of offering a schedule $\{p(\alpha), q(\alpha)\}$. As discussed above the schedule must satisfy break even at all stages. Moreover, since there is competition among banks and borrowers make offers, it will maximize the *ex-ante* welfare of consumers (that is before they know their types) and maximizing the total expected surplus is equivalent to maximizing the expected surplus of the borrower (the average surplus over all types of borrowers). Finally, associated with each of the payments that are ever made in equilibrium, incentive compatibility must be satisfied (that is, once a borrower has learned her type α, she prefers to pay $p(\alpha)$ than any other $p(\alpha')$).

Note that the equilibrium configuration crucially depends on whether the interest rate at which the informal sector lends r is above or below the threshold $\frac{2-\nu}{\nu}$. We separate between these two cases in the discussion below.

Throughout, the exogenous interest rate r can be thought of as a measure of the degree of inefficiency of the opaque sector. Given the seniority of bank debt, the size of the project and Assumption 5, we know that the break even rate for r is $\frac{1}{\nu}$. This is because regardless of the amount borrowed, the opaque lender will always be repaid if the good state is realized and will always face default in the bad state. However, whether we think of the opaque lender as a credit card, payday lender, informal lender or a criminal organization, it is reasonable to believe that the interest rate charged could be above this break even rate, either because the opaque lender is not as specialized as the bank in lending money, in enforcing its claim (collecting money) or because informal credit is also used for purposes other than concealing your type for this type of project and that these other purposes require a higher interest rate.

3.1 Very inefficient informal sector

In this section we explore the implications of a very inefficient opaque sector. In particular we explore the resulting equilibrium when $r > \frac{2-\nu}{\nu}$. We begin by characterizing an equilibrium where there is full separation among those types that borrow and go on to briefly discuss other equilibria and argue that they are somewhat unreasonable.

Proposition 1 *When the opaque sector lends at an interest rate above $r > \frac{2-\nu}{\nu}$, then there exists an equilibrium where consumers offer the schedule $\{p(\alpha), q(\alpha)\}$ with $p(\alpha) = \alpha$ and $q(\alpha) = \frac{D-\alpha-(1-\nu)\alpha}{\nu}$ and all types $\alpha < \frac{z}{2}$ liquidate at $t = \frac{1}{2}$.*

Proof. The banks equilibrium beliefs are consistent with the borrower behaviour, that is a type that pays $p = \alpha$ is an α-type (note that some types will simply prefer to liquidate at the $t = \frac{1}{2}$ stage).

This fully contingent contract has to fulfill the break even and incentive compatibility conditions.

Break even condition: Given that the first payment $p = \alpha$ reveals the type of the agent as α, the break even condition for the second payment is $D = \alpha + \nu q + (1 - \nu)\alpha$. This determines that the break even second payment $q = \frac{D-p-(1-\nu)p}{\nu}$.

We analyze the incentive compatibility condition by considering two deviations: imitating a lower type and imitating a higher type.

Incentive compatibility condition 1: The contract needs to guarantee that no agent wants to imitate a lower quality agent. Let´s assume that an agent of quality α claims to be a lower quality agent $\alpha' < \alpha$ by paying a first payment $p = \alpha'$ her total utility would be $(\alpha - \alpha') + \nu(B - \frac{D-\alpha'-(1-\nu)\alpha'}{\nu})$. Note that $(\alpha - \alpha')$ is the additional consumption at t=1 from reporting a lower type, while $(B - \frac{D-\alpha'-(1-\nu)\alpha'}{\nu})$ is the net consumption in the good state (which occurs with probability ν) after repaying $q(\alpha')$. Instead, by revealing her own type she would get $\nu(B - \frac{D-\alpha-(1-\nu)\alpha}{\nu})$. The difference between these two terms is $-(1 - \nu)(\alpha - \alpha') < 0$ so it does not pay to claim to be an agent of a lower type.

Incentive compatibility condition 2: The contract also needs to guarantee that no agent wants to imitate a higher quality agent by borrowing form the hidden source and paying a first payment $p > \alpha$. Suppose (for contradiction) that an agent claims to be a higher quality agent by paying a first payment $p = \alpha'' > \alpha$ by borrowing from the hidden source. The total utility of the agent would be $\nu(B - \frac{D-\alpha''-(1-\nu)\alpha''}{\nu} - r(\alpha'' - \alpha))$ instead of $\nu(B - \frac{D-\alpha-(1-\nu)\alpha}{\nu})$. The difference between the two is $(2 - \upsilon - \upsilon r)(\alpha'' - \alpha)$ which is negative if and only if $r > \frac{2-\nu}{\nu}$ so this is the necessary and sufficient condition for this incentive compatibility condition to hold.

Notice that off-equilibrium beliefs only apply to $p > 1$ and even assigning the most optimistic beliefs to such offers (that is $\alpha = 1$) agents prefer their equilibrium contracts. ■

Lemma 2 *The above equilibrium achieves first best.*

Proof. This is almost immediate, in the first best, a borrower should be funded iff they generate sufficient expected revenues that is if and only if $\alpha + \upsilon(B + \alpha) + (1 - \nu)\alpha \geq 0$. This is precisely the marginal borrower in the equilibrium described above. ■

Corollary 1 *In the absence of an opaque sector, the first best can be achieved.*

Proof. The absence of an opaque sector is equivalent to $r \longrightarrow \infty$ and so the results above apply. ■

First note that, formally, beyond the equilibrium described in Proposition 1, there are many other equilibria. First there are some which are essentially observationally equivalent in the sense that many other redundant $(p, q(p))$ contracts could be included in the offered schedule which are never taken up and which have no effect on outcomes. Henceforth we ignore such equilibria.

A more substantive source of multiplicity of equilibria arises from the private information on the part of a (small) proportion of borrowers. As is common, in these sorts of games, this opens the possibility to equilibria where there is no borrowing, for example, supported by the beliefs that the only offers are from those borrowers who know that their own types are $\alpha = 0$ (and such beliefs are never challenged because offers are off-equilibrium). We note that such equilibria exist but could be refined away assuming trembles or other equilibrium refinements. Most importantly, we find them unreasonable, and rather than dwell on such technicalities, instead we focus on more efficient equilibria—indeed we highlight above an equilibrium that achieves first best.

3.2 Relatively Efficient Informal Sector

In the previous section, we supposed that the opaque sector was so inefficient, or equivalently that the cost of borrowing from the opaque sector was so high, that in fact it had no effect on outcomes and on the contracts taken up in the transparent sector. In this section, we explore the equilibrium outcome when the opaque sector is more efficient, that is when $r < \frac{2-\nu}{\nu}$. In the proof of Proposition 1, we argued that in the case where types were fully separating in their payments and paid exactly their period 1 incomes, then no type (at this interim stage) would want to imitate a higher type if and only if $r \geq \frac{2-\nu}{\nu}$. In particular, this implies that the outcomes described in Proposition 1 can no longer be an equilibrium. Instead, there will be some pooling among different types of agents with regard to their borrowing from the transparent sector.

As described below, in fact the outcome will be full pooling in the sense that all types that borrow from the transparent sector will choose the same contract from the schedule. In equilibrium only one level of borrowing from the transparent sector will be observed. Rather than the menu of contracts actually taken up in the previous section, when $r < \frac{2-\nu}{\nu}$, borrowing from the transparent sector will take the simple form of either liquidation at $t = \frac{1}{2}$ or else the same payment p at $t = 1$ for all types and the same remaining debt q due at $t = 2$ (which will be fully repaid in the good state and only partially repaid -depending on type- in the bad state). Although the intuition for some pooling is clear, the proof of this full pooling result is somewhat involved and relies on the distributional assumptions of the model, as discussed below. Before getting to the result, we first introduce a couple of preliminary results, first a "continuity of pools" lemma and then as a corollary a result on the weak monotonicity of payments with type. These results rely primarily on the incentive compatibility constraints of borrowers and in particular, require no assumptions about the distribution of types.

Lemma 3 *For every three borrowers with types α, β and γ such that $\alpha > \beta > \gamma$ where $p(\alpha) = p(\gamma)$ it must be the case that $p(\alpha) = p(\beta) = p(\gamma)$.*

Proof. See Appendix. ■

Corollary 2 *For every type $\alpha > \beta$ that does not liquidate $p(\alpha) \geq p(\beta)$.*

Proof. If not, that is if $p(\alpha) < p(\beta)$ then it must be that $q(\alpha) > q(\beta)$ or else the incentive compatibility for β would be violated, but for this to be the case then by Lemma 1 then it must be the case that there is some $\gamma < \beta$ such that $p(\alpha) = p(\gamma)$ but then lemma 3 suggests a contradiction. ■

We proceed to a central result of the paper, that with r sufficiently low, the only bank contract taken up by borrowers is a single contract that requires the same first payment for all borrowing types.

Proposition 2 *For every α that does not liquidate the interim payment is the same, that is $p(\alpha) = p$.*

The proof which appears in the appendix is somewhat notationally involved and requires analysis of a number of cases. At the core is a proof-by-contradiction argument. If the result is false then there must be at least two types that pay different amounts. We focus on the highest two payments (and by Corollary 2 these will correspond to the highest differing types). We find that borrowers, at the ex-ante stage where the contracts are determined, would rather that the top two pools be combined as a single pool, in order to maximize their anticipated surplus. Since they propose these contracts, this will be the equilibrium outcome. Then an induction argument for finite number of pools will imply that one overall pool appears as the equilibrium contract. Note that in the case where $r > \frac{2-\nu}{\nu}$ an induction argument would be inappropriate because there could be an infinite number of pools.

It is worth noting that the assumption that types are uniformly distributed induces linearity into the model which might be a driving force towards corner solutions, which lead to this stark result of a single pool as the equilibrium outcome.

The proposition states that the equilibrium contract has only one possible first payment p and a second payment q that is the one that makes the bank break even on average, considering the pool of agents that do not liquidate their project. We go on to further characterize the equilibrium.

Let l denote the type that is "just indifferent" between liquidating and continuing the project with the (p, q) contract. Even though the optimal l under perfect information is $l = \frac{z}{2}$, limited liability and the cross subsidies between agents inside the pool (from higher quality to lower quality ones) will imply that $l < \frac{z}{2}$. This observation highlights an important externality in our model. Whenever there is some pooling between agents, there will be cross subsidies from agents of higher quality to the agents of lower quality. This generates an inefficient liquidation policy, as some inefficient projects are not liquidated due to this implicit subsidy.

We begin by arguing that in this equilibrium, there is no loss in supposing that the first period payment p will be chosen so that $l \leq p$.

Lemma 4 *In equilibrium when $r < \frac{2-\nu}{\nu}$ then without loss of generality we can focus on $l \leq p$.*

Proof. Suppose that $l \geq p$

The conditions that determine l are the indifference of the marginal agent who decides to liquidate. This can be expressed as $0 = \nu(B + l - q) + (l - p)$.

A further condition is the break even condition of the bank $D = p + \nu q + (1-\nu)\frac{1+l}{2}$. This can be rewritten again as $q = \frac{D-p-(1-\nu)\frac{1+l}{2}}{\nu}$.

Using these two conditions, we obtain that $l = \frac{2z-1+\nu}{3+\nu}$.[10] In particular, therefore, so long as $l \geq p$, p is undetermined, since it will also have no effect on overall welfare then without loss of generality we can think about $l = p$. ■

We proceed by focusing on the case $l \leq p$. First note that in case that $l = 0$, it is trivial that the optimal choice of p is $p = 0$ and overall welfare in this case is $W = 1 - z$, this is simply the average surplus generated by a project, given that all types of projects will be pursued.

Alternatively, it may be optimal to choose an interior l. In this case we can characterize l by noting that a couple of conditions must be satisfied. First by definition, an agent of type l must be indifferent between liquidating or continuing with the project, that is

$$0 = \nu(B + l - q - r(p - l)). \tag{1}$$

In addition, banks need to break even on average and so

$$D = p + \nu q + (1-\nu)\frac{1+l}{2}. \tag{2}$$

Note that the indifference condition (1) implies that $B + \alpha > q$ for every $\alpha > l$ and so it is appropriate to write the break even condition as above, being sure that the loan will be fully repaid if the contract is successful for every borrowing type. Substituting for q from (2) into (1), we obtain the following expression for l:

$$l = \frac{2z + \nu + 2p(r\nu - 1) - 1}{\nu + 2r\nu + 1}. \tag{3}$$

[10] In particular this means that conditional on $l > p$ the type of the marginal liquidating agent l is constant and smaller than $\frac{z}{2}$. This is intuitive since for a type α, with $\alpha > p$ then a slightly lower or higher p makes no difference—for this α-type borrower with $\alpha > p$, changes in p merely transfer utility 1-for-1 between periods 1 and 2. Thus conditional on $l > p$, it is unsurprising that the marginal indifferent l is indepent of p.

We characterize the equilibrium p, under the assumption that both the optimal p^* and l are interior. Having done so, it is easy to verify conditions under which this is indeed the case and then go on to consider outcomes when these conditions fail.

Continuing under the assumption that l is interior, it remains to consider the first order condition and to maximize total welfare in order to find the equilibrium contract offered in the optimal equilibrium (other equilibria exist but as discussed at the end of Section 3.1, we focus attention on the most efficient equilibria. We begin with the expression of total welfare, that is the net (positive or negative) welfare from each project financed subtracted by the welfare loss out of inefficient borrowing:

$$W = \int_l^1 (2x + \upsilon B - D)dx - (\nu r - 1) \int_l^p (p - x)dx \tag{4}$$

Note that the above condition supposes that $p < 1$, which it will be easy to verify is true in equilibrium.

The above expression is equivalent to:

$$W = lz - z - l^2 + 1 - \frac{1}{2}(\nu r - 1)(p - l)^2. \tag{5}$$

We can substitute for l from Equation (3) and maximize total welfare with respect to p.

Let p^* denote the optimal p that maximizes total welfare. The first order condition that characterizes p^* is:

$$\frac{dW}{dp} = (z - 2l)\frac{dl}{dp} - (\nu r - 1)(p - l)(1 - \frac{dl}{dp}) = 0, \tag{6}$$

where

$$\frac{dl}{dp} = \frac{2(r\nu - 1)}{2r\nu + 1 + \nu}. \tag{7}$$

Therefore the expression that implicitly defines the optimal first payment p^* is:

$$\frac{dW}{dp} = (z - 2l)\frac{2(r\nu - 1)}{2r\upsilon + 1 + \nu} - (\nu r - 1)(p - l)(\frac{3 + \nu}{2r\upsilon + 1 + \nu}) = 0 \tag{8}$$

Simple algebraic manipulation, yields the following equilibrium expression for p^*.

$$p^* = \frac{2z - l + \nu l}{3 + \nu}. \tag{9}$$

We solve simultaneously for p^* and l from this equation and equation (3) to obtain:

$$l = \frac{2z + 2\nu + 2z\nu + 4rz\nu + \nu^2 - 3}{6\nu + 8r\nu + \nu^2 + 1} \tag{10}$$

$$p^* = \frac{4z\nu - 2\nu + 4rz\nu + \nu^2 + 1}{6\nu + 8r\nu + \nu^2 + 1} \tag{11}$$

and

$W_I = (\frac{2(z-2)(\nu+2r\nu+1)}{6\nu+8r\nu+\nu^2+1})z - (\frac{2z+2z\nu+4rz\nu+2\nu+\nu^2-3}{6\nu+8r\nu+\nu^2+1})^2 + 1 - \frac{1}{2}(\nu r - 1)(\frac{2(z-2)(\nu-1)}{6\nu+8r\nu+\nu^2+1})^2$

The notation W_I is intended to highlight that this is the welfare under the optimal interior solution when it is feasible. However this need not be the global optimum since choosing $p = 0 = l$ and generating an expected surplus of $1 - z$ is always feasible.

The equilibrium expression for p^*, together with the break even condition (2) and the expression for l (3) determine the equilibrium value for the second payment q.

Lemma 5 *Both l and p^* are interior when $z \geq \frac{(3+\nu)(1-\nu)}{2(1+\nu+2r\nu)}$ and $W_I \geq 1 - z$.*

Proof. Note that p is linear in z it is sufficient therefore to consider the two extremes $z = 0$ and $z = 2$.

For $z = 2$, $p^* = 1$.

For $z = 0$, $p^* = \frac{(1-\nu)^2}{6\nu+8r\nu+\nu^2+1}$ which is greater than 0 and less than 1.

At the same time.

For $z = 2$, $l = 1$

And $l > 0$ as long as $z > \frac{(3+\nu)(1-\nu)}{2(1+\nu+2r\nu)}$

For values of z below $\frac{(3+\nu)(1-\nu)}{2(1+\nu+2r\nu)}$ including $z = 0$ the optimal contract is to set $p^* = 0$ that leads to $l = 0$.

Thus $z \geq \frac{(3+\nu)(1-\nu)}{2(1+\nu+2r\nu)}$ is required for an interior l and p^* to be feasible. An interior l would generate more surplus and so would be the equilibrium outcome when the welfare generated is higher than the next best alternative–choosing $l = 0$ and $p^* = 0$, or equivalently:

$W_I \geq 1 - z$.

For $z > 1$, first note that the condition $z \geq \frac{(3+\nu)(1-\nu)}{2(1+\nu+2r\nu)}$ always holds since $z \geq \frac{(3+\nu)(1-\nu)}{2(1+\nu+2)} > \frac{1}{2}$ (since $r \geq \frac{1}{\nu}$). When $z > 1$ then rather than fund all projects, ensuring that no projects are funded (by choosing a high p with $l = 1$) would be preferred and would generate welfare of 0. However, the interior solution is always preferred to this outcome, first note that since the condition on z is met it is feasible and second note that since the maximum of Equation (5) is obtained when $p \leq 1$ then this maximum is greater than or equal to the maximum of the following expression in the range $p \leq 1$ (note that the upper limit of the second integral is 1 rather than p): ■

$$W' = \int_l^1 (2x + vB - D)dx - (\nu r - 1) \int_l^1 (p - x)dx \tag{12}$$

Proof. Note that the maximum of W' is greater than or equal to 0 since $l = 1$ (and choosing some p that ensures this) can always be chosen and so W_I is always greater than or equal to 0. In particular this implies that for $z > 1$ the interior solution is always optimal. ■

Note that for $z = 2$ both $z \geq \frac{(3+\nu)(1-\nu)}{2(1+\nu+2r\nu)}$ and $W_I \geq 1 - z$ hold as strict inequalities and so in particular for large enough z, both l and p^* will be interior.

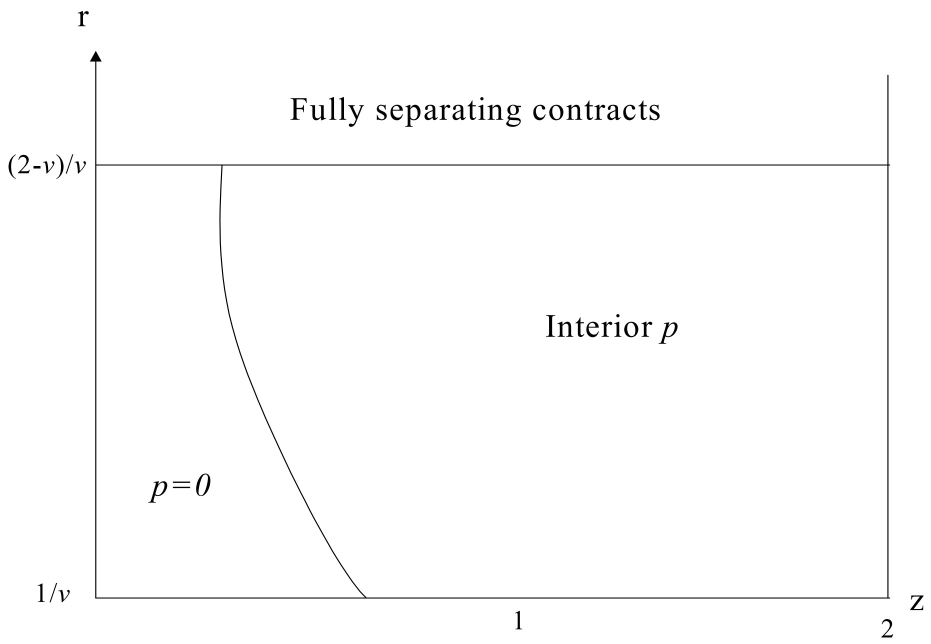
Equations (10) and (11) show that in the parameter range where l is interior then both l and p are linear and increasing in z. That they should be increasing is quite intuitive. As z goes up, projects become less and less attractive, so the optimal first payment p goes up to increase the liquidation threshold l. On the other hand, as z goes down, more projects should be funded so p^* goes down. Further, as z increases the parameters are more likely to be such that the optimal choice of l is interior and in particular this is always the case when $z = 2$.[11]

3.3 Equilibrium summary

There are three equilibrium regimes. When $r > \frac{2-\nu}{\nu}$ there is full separation. $p = \alpha$ and $q = \frac{D-\alpha-(1-\nu)\alpha}{\nu}$ firms gets financed as long as $\alpha > \frac{z}{2}$. When $r < \frac{2-\nu}{\nu}$ and $z > \frac{(3+\nu)(1-\nu)}{2(1+\nu+2r\nu)}$ and $W_I > 1-z$ there is full pooling with $p^* = \frac{1-2\nu+4(r+1)\nu(D-\nu B)+\nu^2}{6\nu+8r\nu+\nu^2+1}$ and $q = \frac{D-p^*-(1-\upsilon)\frac{l+1}{2}}{\upsilon}$ and all types below $l = \frac{2\nu+(4r\nu+2+2\nu)(D-\nu B)+\nu^2-3}{6\nu+8r\nu+\nu^2+1}$ liquidate, and finally, otherwise, no types liquidate, $p^* = 0$ and $q = \frac{2D-1+\nu}{2\nu}$.

All three regions are non-trivial as illustrated in the diagram below which illustrates these three equilibrium regions for a general ν.

[11] See Lemma 8 in the appendix.



3.4 Comparative statics on welfare

Having obtained explicit characterizations of l and p^* in terms of the exogenous parameters of the model and noting that welfare is given by Equation (5), it is relatively simple to consider comparative statics of welfare. It is of particular interest, to consider how welfare changes (and the channels through which it changes) as r, the exogenous rate of interest in the opaque sector, varies. We focus attention on the range $r < \frac{2-\nu}{\nu}$ since outside this range the first best can be achieved.

First note that in the range $r < \frac{2-\nu}{\nu}$ and for $z > \frac{(3+\nu)(1-\nu)}{2(1+\nu+2r\nu)}$ and $W_I \geq 1-z$.

$$\frac{dl}{dr} = \frac{4\nu(3+\nu)(1-\nu)(2-z)}{(6\nu+8r\nu+\nu^2+1)^2} > 0. \tag{13}$$

In particular, this suggests that one source of inefficiency is reduced, since as r increases l rises and so fewer inefficient projects are conducted.

Note also that

$$\frac{dp^*}{dr} = -\frac{4\nu(1-\nu)^2(2-z)}{(6\nu+8r\nu+\nu^2+1)^2} < 0. \tag{14}$$

As the interim payment falls, and since the lowest type borrowing rises, then the amount of borrowing from the opaque sector falls; however, since the cost of borrowing from the opaque sector rises, the welfare consequences may be ambiguous.[12] By examining welfare directly we can see that the first of these two effects always dominates, specifically:

Lemma 6 *$\frac{dW}{dr} > 0$ in the range $r < \frac{2-\nu}{\nu}$, $z > \frac{(3+\nu)(1-\nu)}{2(1+\nu+2r\nu)}$ and $W_I > 1 - z$ (and $\frac{dW}{dr} = 0$ outside this range)*

Proof. See Appendix. ∎

4 Partially Hidden Borrowing

In this section we modify our model slightly to allow for a partially hidden lender. We introduce the possibility that the banking sector observes the level of hidden borrowing of the lender with some probability $(1 - h)$. With probability h borrowing from the non-banking sector remains hidden. A rationale for this modelling assumption is that the banking sector investigates each of its borrowers with some probability $(1 - h)$ and once a borrower is investigated its borrowing position with all possible alternative lenders is perfectly known by the whole banking sector. On the contrary, if a particular borrower is not investigated, the banking sector cannot observe any borrowing outside the pool of competitive creditors and is aware of the possibility of some additional lending.

If a borrower is investigated, we assume that the borrower is aware of it, and that she has the opportunity to repay the opaque sector immediately. Early repayment entails a cost sd, where d is the amount borrowed from the opaque sector.

If an agent is investigated and repays early, then we know that full separation holds. We know from Proposition 1 that if the full separation contract is offered there are no incentives to imitate downwards. With observable payoffs there is no feasible way to imitate upwards as banks would take into account any inefficient borrowing and discount that to calculate the agent´s true type. Given that the incentive compatibility constraints for the fully revealing equilibrium hold and that it achieves first best, this is the only equilibrium once an agent has been investigated.

Assumption 7 $r - s > \frac{1}{\nu}$.

The role of this assumption is to guarantee that early repayment is the optimal strategy of the borrower once the alternative lender becomes transparent.

[12] Note that the welfare as defined in Equation (5) does not take into account surplus gained by the alternative sector. Including this surplus into the welfare calculation, would suggest that the *only* source of inefficiency would be inefficient liquidation and so only the first effect would apply and the qualitatitve results would apply—welfare increases in r. The analysis here would still be of interest as Equation (5) captures consumer surplus.

Lemma 7 *Once the hidden borrowing is observed, early repayment is the optimal strategy for the agent.*

Proof. Borrowing from a hidden source gives no concealment benefit, so the only benefit from that borrowing comes from either keeping, investing or consuming it. Keeping the money is unproductive, investing it at the gross market interest rate of 1 or consuming it gives a (negative) expected utility of $1 - \nu r$. This loss has to be compared with the cost of early repayment $-\nu s$. Early repayment is therefore the optimal strategy as long as $-\nu s > 1 - \nu r$ which is guaranteed by Assumption 7. ■

The model with probabilistic observability of the hidden borrowing is therefore like a switching model in which, with probability $(1-h)$ full separation is achieved for sure and with probability h the model looks like in the previous sections. In this latter case the only difference is that, from the point of view of the borrower, the costs and benefits of the hidden borrowing need to be recalculated as with probability $(1-h)$ hidden borrowing is useless and entails a cost s.

In fact, once the alternative borrowing remains hidden, the rest of the model with probabilistic observation of the hidden borrowing can be fully solved by realizing that the cost of borrowing from the hidden source is now $\frac{hr+(1-h)s}{h}$ instead of just r. Borrowing one unit from the hidden source costs r with probability h and costs s with probability $(1-h)$. It only produces some benefit to the borrower with probability h so the whole cost has to be re-scaled by h.

We write $r_h = \frac{hr+(1-h)s}{h}$ as the effective interest rate when borrowing from the opaque sector remains hidden with probability h, the rate of interest is r when borrowing remains hidden, and s for early repayment in case that the banking sector observes the borrowing. With this notation, similar results to those in the fully hidden case arise:

Proposition 3 *When the opaque sector lends at an interest rate above* $r_h > \frac{2-\nu}{\nu}$*, then there exists an equilibrium where consumers offer the schedule* $\{p(\alpha), q(\alpha)\}$ *with* $p(\alpha, d) = \alpha$ *and* $q(\alpha) = \frac{D-\alpha-(1-\nu)\alpha}{\nu}$ *and there is no borrowing from the hidden sector.*

Proof. The proof is almost identical to the one in Proposition 1 except that now borrowing from the hidden source entails higher costs and so the proof is omitted. ■

The functional form of the welfare equation and the incentive compatibility conditions are similar to those of the basic model, so a similar result to Proposition 2 still holds. Thus, if $r_h > \frac{2-\nu}{\nu}$ the only possible equilibrium is one of full pooling and if the optimal solution is interior then the optimal first payment is:

$$p^* = \frac{4z\nu - 2\nu + 4zr_h\nu + \nu^2 + 1}{6\nu + 8r_h\nu + \nu^2 + 1}$$

and the type of borrower who is just indifferent between liquidating the project and continuing it is:

$$l = \frac{2z + 2\nu + 2z\nu + 4r_h + \nu^2 - 3}{6\nu + 8r_h\nu + \nu^2 + 1}$$

Total welfare can be expressed as:

$$W = h\left[\int_l^1 (2x + \upsilon B - D)dx - (\nu r - 1)\int_l^p (p-x)dx\right] + (1-h)\left[\int_{\frac{z}{2}}^1 (2x + \upsilon B - D)dx - \nu s\int_l^p (p-x)dx\right]$$

The first term corresponds with the welfare when the hidden sector remains hidden, while the second ones is related to when it becomes observable. The expression can be rearranged as

$$W = h\left[\int_l^1 (2x + \upsilon B - D)dx - (\upsilon r_h - 1)\int_l^p (p-x)dx\right] + (1-h)\left[\int_{\frac{z}{2}}^1 (2x + \upsilon B - D)dx\right].$$

Note that the expression in the first bracket is identical to the expression in Section 3 with a change of the social cost of borrowing from r to $r_h$and that the second bracket is constant in p and l.

The welfare implications of the changes in the probability of the hidden sector becoming transparent $(1-h)$ are as follows. A higher $(1-h)$ implies higher welfare in a couple of ways. First is the automatic switching from the pooling equilibrium to the full separation equilibrium whenever the banking sector observes the hidden lending. Second, increasing $(1-h)$ increases r_h and so the results on welfare increasing in r from Section 3 apply. Similarly an increase in s raises r_h and so also raises welfare. This analysis, therefore, is related to the literature on the interactions between direct screening of lenders through active investigation and the indirect screening that can be achieved by offering them a menu of contracts, as in Manove et al. (2001). While in most models these are seen as substitutes, in our model they are complements. That is, an increase in $(1-h)$ leads to more information about some borrowers directly and also to a more informative equilibrium with respect to the other borrowers (who may have loans from the alternative sector which remain hidden).[13]

[13] Even though so far we have considered h as an exogenous parameter, endogenizing it seems relatively straightforward. Higher transparency (lower h) would be more costly and competition among banks should equalize the marginal cost of additional monitoring (reducing h) with its marginal gain in terms of welfare in equilibrium W_l.

5 Conclusions

In this paper we have presented a model in which a banking sector and an alternative opaque source of lending coexist. The banking sector is competitive at all the stages of the loan, this imposes that contracts have to break even, conditional on publicly available information. The results show that if the alternative source of borrowing is sufficiently inefficient, the banking sector will offer contracts that perfectly reveal the credit quality of the borrowers and will achieve first best. The optimal contract gives incentives to borrowers to reveal their intermediate cash flows through lower future payments. However if the alternative source of borrowing is relatively efficient, then the fully contingent contract is not sustainable. In this case, agents may want to conceal their types by borrowing from the hidden source and repaying a larger part of their loans early. In the model, the optimal contract is not contingent on the interim payments of the loans. There is only one possible first payment and one second payment. The contract does not achieve first best, since a higher number of inefficient projects are funded and some borrowers, in equilibrium borrow from the inefficient alternative sector.

Moreover we show that overall welfare is increasing the higher is the cost of borrowing from the opaque sector. This result is in contrast to some conventional wisdom in discussions of developing economies which focuses on the role that informal sector may play in alleviating the financing constraints. The informal sector lends to firms and households when the formal sector is rationing them.[14] However in our model, as the informal sector gets more efficient, the banking sector has to offer a less contingent contract and total welfare falls. This approach, considering the interaction between formal and informal sectors from an informational point of view, provides a counter-argument to traditional literature by showing how an efficient informal sector may reduce welfare.

Relaxing the assumption that the alternative sector is not entirely opaque makes it less appealing for a borrower to use the costly alternative sector to disguise her type as this may turn out to be ineffective. In this case, the qualitative results outlined above carry through in this richer environment and moreover, welfare is decreasing in the opacity of the informal sector.

Our results highlight that one of the possible reasons for long term debt contracts not being contingent on interim payments is that the information that long term lenders would extract from these interim payments would be corrupted by additional borrowing from hidden sources of funds. They also highlight that we may observe in equilibrium simultaneous borrowing from different

[14] Jain (1999) who also discusses related literature, provides an explanation for the observation of borrowers in both sectors based on a trade-off between the formal sector's lower opportunity cost of funds and the informal sector's better information. Instead we assume that the formal sector is unambiguously more efficient and the informal sector has no informational advantage.

sources even when there is a clear pecking order among them and there is available borrowing on the cheaper one (for example mortgages and credit card borrowing). Finally we also show that the existence of an alternative opaque source of borrowing may be welfare diminishing because it may distort the set of contracts that the competitive lending sector may offer.

References

[1] Arnott, R. J. and J. E. Stiglitz (1991): "Equilibrium in Competitive Insurance Markets with Moral Hazard," NBER Working Paper No. W3588.

[2] Allen, F. (1985): "Repeated Principal-Agent Relationships with Lending and Borrowing," *Economics Letters*, **17**, 27-31.

[3] Bell, Clive ; T. N. Srinivasan; Christopher Udry Rationing, (1997) Spillover, and Interlinking in Credit Markets: The Case of Rural Punjab *Oxford Economic Papers,* New Series, Vol. 49, No. 4. (Oct., 1997), pp. 557-585

[4] Bisin, A. and D. Guaitoli (2004): "Moral Hazard and Nonexclusive Contracts Production: Theories and Evidence from Joint Ventures," *RAND Journal of Economics,* 35, 306-328.

[5] Bisin, Alberto; Adriano A. Rampini (2000) Exclusive Contracts and the Institution of Bankruptcy *New York University* Department of Finance Working Paper No. 270

[6] Bizer, D. S. and P. M. DeMarzo (1992): "Sequential Banking," *Journal of Political Economy*, 100, 41-61.

[7] Bose, Pinaki, (1998). Formal-informal sector interaction in rural credit markets, *Journal of Development Economics*, Elsevier, vol. 56(2), 265-280

[8] Cole, H. L. and N. R. Kocherlakota (2001): "Efficient Allocations with Hidden Income and Hidden Storage," *Review of Economic Studies*, **68**, 523-542.

[9] Doepke, M. and R. M. Townsend (2004): "Dynamic Mechanism Design with Hidden Income and Hidden Actions," *working paper*, UCLA and University of Chicago

[10] Gross, D. and N. S. Souleles (2002): "Do Liquidity Constraints and Interest Rates Matter for Consumer Behavior? Evidence from Credit Card Data," *Quarterly Journal of Economics*, 149-185.

[11] Haliassos, M. and M. Reiter (2003): "Credit Card Debt Puzzles," working paper University of Cyprus and Universitat Pompeu Fabra

[12] Hunt, R. M. (2002): "The Development and Regulation of Consumer Credit Reporting in America," working paper, Federal Reserve Bank of Philadelphia

[13] Innes, R. (1990) "Limited Liability and Incentive Contracting With Ex-Ante Action Choices." Journal of Economic Theory 52, 45-68

[14] Jain, Sanjay (1999) Symbiosis vs. crowding-out: the interaction of formal and informal credit markets in developing countries *Journal of Development Economics,* vol. 59, issue 2, pages 419-444

[15] Laibson, D., Repetto, A. and J. Tobocman (2001) "A Debt Puzzle" in eds. Philippe Aghion, Roman Frydman, Joseph Stiglitz, Michael Woodford *Knowledge, Information, and Expectations in Modern Economics: In Honor of Edmund S. Phelps*, forthcoming

[16] Manove, M., A. J. Padilla and M. Pagano (2001), "Collateral vs. Project Screening: A Model of Lazy Banks," *Rand Journal of Economics*, 32 (4), 726-744

[17] Ljungqvist, L. and T. Sargent (2004) *Recursive Macroeconomic Theory,* MIT Press

[18] Pagano, M. and T. Japelli (1993): Information Sharing in Credit Markets, Journal of Finance 43(5),1693-1718.

6 Appendix

Proof of lemma 3

Proof. Suppose that borrowers face the choice between two generic contracts a and b and without loss of generality, we label them so that $p_a > p_b$. There are a number of cases to consider:

(i) $\alpha > p_a > \beta > p_b > \gamma$

(ii) $p_a > p_b > \alpha > \beta > \gamma$

(iii) $\alpha > \beta > \gamma > p_a > p_b$

(iv) $\alpha > p_a > p_b > \beta > \gamma$

(v) $\alpha > \beta > p_a > p_b > \gamma$

We examine each in turn:

(i) $\alpha > p_a > \beta > p_b > \gamma$

Agent α will go for contract a whenever

$\alpha - p_a + \nu(B + \alpha - p_a - q_a - r(p_a - \alpha)1_{p_a>\alpha}) \geq \alpha - p_b + \nu(B + \alpha - p_b - q_b - r(p_b - \alpha)1_{p_b>\alpha})$

since $\alpha > p_a > p_b$ this can be rewritten as

$\alpha - p_a + \nu(B + \alpha - p_a - q_a) \geq \alpha - p_b + \nu(B + \alpha - p_b - q_b)$

$-p_a(1 + \nu) - \nu q_a \geq -p_b(1 + \nu) - \nu q_b$

$\nu(q_b - q_a) \geq (p_a - p_b)(1 + \nu)$.

The inequality is independent of α so all the agents with type above p_a will make the same choice.

For an agent of type γ she would the a contract over the b contract whenever

$\gamma - p_a + \nu(B + \gamma - p_a - q_a - r(p_a - \gamma)1_{p_a>\gamma}) \geq \gamma - p_b + \nu(B + \gamma - p_b - q_b - r(p_b - \gamma)1_{p_b>\gamma})$

or

$\gamma - p_a + \nu(B + \gamma - p_a - q_a - r(p_a - \gamma)) \geq \gamma - p_b + \nu(B + \gamma - p_b - q_b - r(p_b - \gamma))$

$\gamma - p_a + \nu(B + \gamma - p_a - q_a - r(p_a - \gamma)) - (\gamma - p_b + \nu(B + \gamma - p_b - q_b - r(p_b - \gamma))) = p_b - p_a - \nu p_a + \nu p_b - \nu q_a + \nu q_b - r\nu p_a + r\nu p_b$

$\nu(q_b - q_a) \geq (p_a - p_b)(1 + \nu + r\nu)$

Finally an agent of type β will go for contract a over contract b whenever

$\beta - p_a + \nu(B + \beta - p_a - q_a - r(p_a - \beta)1_{p_a>\beta}) \geq \beta - p_b + \nu(B + \beta - p_b - q_b - r(p_b - \beta)1_{p_b>\beta})$

or

$\beta - p_a + \nu(B + \beta - p_a - q_a - r(p_a - \beta)) \geq \beta - p_b + \nu(B + \beta - p_b - q_b)$

$\nu(q_b - q_a) \geq (p_a - p_b)(1 + \nu) + r\nu(p_a - \beta)$

Well $(p_a - p_b)(1 + \nu) + r\nu(p_a - \beta) > (p_a - p_b)(1 + \nu) + r\nu(p_a - p_b)$

and so since γ prefers contract a to contract b, an agent of type β must also prefer contract a to contract b.

(ii) $p_a > p_b > \alpha > \beta > \gamma$

In this case the condition for an α type agent to prefer the a contract over the b contract is $\nu(q_b - q_a) \geq (p_a - p_b)(1 + \nu + r\nu)$

which is the identical condition for the β type agent

(iii) $\alpha > \beta > \gamma > p_a > p_b$

The condition which ensures that the β type agent prefers the a contract over the b contract is identical to the condition for the α-type agent and for the γ-type agent and so will hold true in this case.

(iv) $\alpha > p_a > p_b > \beta > \gamma$

The condition which ensures that the β type agent prefers the a contract over the b contract is identical to the condition for the γ-type agent and so will hold true in this case.

(v) $\alpha > \beta > p_a > p_b > \gamma$

The condition which ensures that the β type agent prefers the a contract over the b contract is identical to the condition for the α-type agent and so will hold true in this case.

Thus if the preferred contract for both an α agent and a γ agent is a then this is also the preferred contract for any β-type agent where $a > \beta > \gamma$. ■

Proof of Proposition 2

Proof. We prove by contradiction. Suppose that this result is false, then there must be at least two types that pay different amounts. We focus on the highest two payments (and by Corollary 2 these will correspond to the highest differing types). We will find that in equilibrium, the top two pools would rather be combined as a single pool. Then an induction argument for finite number of pools will imply that one overall pool appears as the equilibrium contract. Note that in the case where $r > \frac{2-\nu}{\nu}$ an induction argument won't work because we could have infinite number of pools.

We continue by considering the top two pools of types that do not liquidate.

First note that if any type α strictly prefers not to liquidate then all types $\beta > \alpha$ would prefer to mimic α than to liquidate. Thus in restricting attention to the highest two payments $p_1 < p_2$ and associated types (and by corollary 2 we know that higher types are associated with higher payments) then we can be sure that there are some $\alpha_1 \leq \alpha_2$ such that types $(\alpha_1, \alpha_2]$ pay p_1 in the first period (with the associated q_1) and $(\alpha_2, 1]$ pay p_2 in the first period (with the associated q_2 in the second period).

The proof is somewhat involved but the structure is as follows. First we highlight a number of possible cases. In each case we seek to determine the optimal choice of α_2 (and associated p_1 and p_2) given that there are two pools in the range $(\alpha_1, 1]$ and keeping α_1 indifferent (and so all other types below may also remain with their existing contracts and there are no changes to equilibrium or welfare consequences from types below α_1).[15]

By definition $p_2 > p_1$. There are a number of cases to consider:

I $\quad p_2 > 1$ and $p_1 > \alpha_2$

II $\quad p_2 > 1$ and $\alpha_2 > p_1 > \alpha_1$

III $\quad p_2 > 1$ and $\alpha_1 > p_1$

IV $\quad 1 > p_2 > \alpha_2$ and $p_1 > \alpha_2$

V $\quad 1 > p_2 > \alpha_2$ and $\alpha_2 > p_1 > \alpha_1$

VI $\quad 1 > p_2 > \alpha_2$ and $\alpha_1 > p_1$

We focus on each in turn and show that the optimum outcome in all cases pushes α_2 into a corner.

Recall also that we have $q_1 = \frac{D-p_1-(1-\nu)\frac{\alpha_1+\alpha_2}{2}}{\nu}$ and $q_2 = \frac{D-p_2-(1-\nu)\frac{1+\alpha_2}{2}}{\nu}$

CASE I $p_2 > 1$ and $p_1 > \alpha_2$

Well the IC condition for α_2 is

$\nu(B + \alpha_2 - q_2 - r(p_2 - \alpha_2)) = \nu(B + \alpha_2 - q_1 - r(p_1 - \alpha_2))$

which yields

$q_1 - q_2 = rp_2 - rp_1$

and substituting in for q_1 and q_2 and simplifying yields

$p_2 = \frac{1-\nu}{\nu r-1}\frac{1-\alpha_1}{2} + p_1$

in particular note that $\frac{dp_2}{d\alpha_2} = \frac{dp_1}{d\alpha_2}$.

The participation constraint for α_1 is

$\nu(B + \alpha_1 - q_1 - r(p_1 - \alpha_1)) = k$

$p_1 = \frac{1}{\nu r-1}(-k + \nu B - D + (1-\nu)\frac{\alpha_1+\alpha_2}{2} + \nu(1+r)\alpha_1)$

and note in particular that $\frac{dp_2}{d\alpha_2} = \frac{dp_1}{d\alpha_2} = \frac{1-\nu}{2(\nu r-1)}$

Welfare

Now let's look at overall welfare

$W = \int_{\alpha_1}^{1}(\nu B - D + 2x)dx - (\nu r-1)\int_{\alpha_1}^{\alpha_2}(p_1 - x)dx - (\nu r-1)\int_{\alpha_2}^{1}(p_2 - x)dx + cst$

$W = (\nu B - D)(1-\alpha_1) + 1 - \alpha_1^2 - (\nu r-1)p_1(\alpha_2-\alpha_1) + \frac{1}{2}(\nu r-1)(\alpha_2^2-\alpha_1^2) - (\nu r-1)p_2(1-\alpha_2) + \frac{1}{2}(\nu r-1)(1-\alpha_2^2) + cst$

and in particular

$W = (\nu B - D)(1-\alpha_1) + 1 - \alpha_1^2 - (\nu r-1)p_1(\alpha_2-\alpha_1) + \frac{1}{2}(\nu r-1)(1-\alpha_1^2) - (\nu r-1)p_2(1-\alpha_2) + cst$

so

$\frac{dW}{d\alpha_2} = (\nu r-1)(p_2 - p_1 - \frac{1-\nu}{2(\nu r-1)}(1-\alpha_1))$

[15] While noting that sufficiently bizarre off-equilibrium beliefs could justify a wide range of equilibria, we focus on the most efficient equilibria (which would also be the one preferred by the borrowers).

This expression cannot be easily signed but note that it is independent of α_2 and so without loss of generality we can maximize welfare taking $\alpha_2 = p_1$ and case II applies.

CASE II $p_2 > 1$ and $\alpha_2 > p_1 > \alpha_1$

Well the IC condition for α_2 is

$\nu(B + \alpha_2 - q_2 - r(p_2 - \alpha_2)) = (\alpha_2 - p_1) + \nu(B + \alpha_2 - q_1)$

substituting in the expressions for q_1 and q_2 gives

$\nu(B + \alpha_2 - \frac{D - p_2 - (1-\nu)\frac{1+\alpha_2}{2}}{\nu} - r(p_2 - \alpha_2)) = (\alpha_2 - p_1) + \nu(B + \alpha_2 - \frac{D - p_1 - (1-\nu)\frac{\alpha_1+\alpha_2}{2}}{\nu})$

which yields $p_2 = \alpha_2 + \frac{(1-\nu)(1-\alpha_1)}{2(r\nu-1)}$

in particular note that $\frac{dp_2}{d\alpha_2} = 1$.

The participation constraint for α_1 is

$\nu(B + \alpha_1 - q_1 - r(p_1 - \alpha_1)) = k.$

Substituting in the expression for q_1

$\nu(B + \alpha_1 - \frac{D - p_1 - (1-\nu)\frac{\alpha_1+\alpha_2}{2}}{\nu} - r(p_1 - \alpha_1)) = k$

yields $p_1 = \frac{1}{\nu r - 1}(-k + \nu B - D + (1-\nu)\frac{\alpha_1+\alpha_2}{2} + \nu(1+r)\alpha_1)$

and note in particular that $\frac{dp_1}{d\alpha_2} = \frac{1-\nu}{2(\nu r-1)}$

Welfare

Now let's look at overall welfare

$W = \int_{\alpha_1}^{1}(\nu B - D + 2x)dx - (\nu r - 1)\int_{\alpha_1}^{p_1}(p_1 - x)dx - (\nu r - 1)\int_{\alpha_2}^{1}(p_2 - x)dx + cst$

$W = (\nu B - D)(1 - \alpha_1) + 1 - \alpha_1^2 - (\nu r - 1)p_1(p_1 - \alpha_1) + \frac{1}{2}(\nu r - 1)(p_1^2 - \alpha_1^2) - (\nu r - 1)p_2(1 - \alpha_2) + \frac{1}{2}(\nu r - 1)(1 - \alpha_2^2) + cst.$

and in particular

$\frac{dW}{d\alpha_2} = -(\nu r - 1)\frac{1-\nu}{2(\nu r-1)}(p_1 - \alpha_1) - (\nu r - 1)p_1\frac{1-\nu}{2(\nu r-1)} + (\nu r - 1)p_1\frac{1-\nu}{2(\nu r-1)} - (\nu r - 1)(1 - \alpha_2) + (\nu r - 1)p_2 - (\nu r - 1)\alpha_2$

$\frac{dW}{d\alpha_2} = (\nu r - 1)(p_2 - 1 - \frac{1-\nu}{2(\nu r-1)}(p_1 - \alpha_1))$

$\frac{d^2W}{d\alpha_2^2} = (\nu r - 1)(1 - (\frac{1-\nu}{2(\nu r-1)})^2)$

now at $\frac{dW}{d\alpha_2} = 0$ it's a maximum so long as $\frac{d^2W}{d\alpha_2^2} < 0$ that is so long as $1 < (\frac{1-\nu}{2(\nu r-1)})^2$

$1 < \frac{1-\nu}{2(\nu r-1)}$

$2\nu r < 3 - \nu$

$r < \frac{3-\nu}{2\nu}$

and so when $r > \frac{3-\nu}{2\nu}$ then $1 > \frac{1-\nu}{2(\nu r-1)}$ and so setting $\frac{dW}{d\alpha_2} = 0$ defines a minimum and so the maximum is at a corner.

What happens in the $r < \frac{3-\nu}{2\nu}$ case (and FOC defines a maximum)?

recall that $p_1 < \alpha_2 < 1$

well we have that α_1 satisfies $p_1 = \frac{1}{\nu r-1}(-k + \nu B - D + (1-\nu)\frac{\alpha_1+\alpha_2}{2} + \nu(1+r)\alpha_1)$

so

$p_1 - \frac{1}{\nu r-1}(-k+\nu B-D) - \frac{1-\nu}{(\nu r-1)}\frac{\alpha_2}{2} = \alpha_1 \frac{2\nu r+1+\nu}{2(\nu r-1)}$.

Now $p_1 < \alpha_2$ and $(\nu B - D) < 0$

so $\alpha_1 \frac{2\nu r+1+\nu}{2(\nu r-1)} < \alpha_2 - \frac{1-\nu}{(\nu r-1)}\frac{\alpha_2}{2}$

so $\alpha_1 < (\frac{2(\nu r-1)-1+\nu}{1+\nu+2\nu r})\alpha_2$

note that $2(\nu r-1)-1+\nu < 0$ when $r < \frac{3-\nu}{2\nu}$

so $\alpha_1 < 0$ which is impossible.

CASE III $p_2 > 1$ and $\alpha_1 > p_1$

Well the IC condition for α_2 is

$\nu(B+\alpha_2-q_2-r(p_2-\alpha_2)) = (\alpha_2-p_1)+\nu(B+\alpha_2-q_1)$

which yields $p_2 = \alpha_2 + \frac{(1-\nu)(1-\alpha_1)}{2(r\nu-1)}$

The participation constraint for α_1 is

$\alpha_1 - p_1 + \nu(B+\alpha_1-q_1) = k$

$\alpha_1 + \nu B - D + (1-\nu)\frac{\alpha_1+\alpha_2}{2} = k$

here p_1 is unconstrained

$\alpha_2 = \frac{k+D-B\nu-\alpha_1-\frac{1}{2}\alpha_1(-\nu+1)}{-\frac{1}{2}\nu+\frac{1}{2}}$

which yields $\alpha_2 = \frac{2}{1-\nu}(k+D-\nu B) - \alpha_1\frac{3-\nu}{2}$

So here α_2 is not a choice variable and so also p_2 cannot be freely chosen, the only choice variable is p_1. Well it's clear that from the welfare perspective p_1 is irrelevant since here

$W = \int_{\alpha_1}^{1}(\nu B - D + 2x)dx - (\nu r-1)\int_{\alpha_2}^{1}(p_2-x)dx + cst$

and so p_1 does not arise, without loss of generality therefore we can take $p_1 = \alpha_1$ and we're back in case II.

Case IV $1 > p_2 > \alpha_2$ and $p_1 > \alpha_2$

Well the IC condition for α_2 is

$\nu(B+\alpha_2-q_2-r(p_2-\alpha_2)) = \nu(B+\alpha_2-q_2-r(p_2-\alpha_2))$

$p_2 = \frac{1-\nu}{\nu r-1}\frac{1-\alpha_1}{2} + p_1$

in particular note that $\frac{dp_2}{d\alpha_2} = \frac{dp_1}{d\alpha_2}$.

The participation constraint for α_1 is

$\nu(B+\alpha_1-q_1-r(p_1-\alpha_1)) = k$

substituting in the expression for q_1

$\nu(B+\alpha_1 - \frac{D-p_1-(1-\nu)\frac{\alpha_1+\alpha_2}{2}}{\nu} - r(p_1-\alpha_1)) = k$

yields $p_1 = \frac{1}{\nu r-1}(-k+\nu B-D+(1-\nu)\frac{\alpha_1+\alpha_2}{2}+\nu(1+r)\alpha_1)$

and note in particular that $\frac{dp_1}{d\alpha_2} = \frac{1-\nu}{2(\nu r-1)}$

Welfare

Now let's look at overall welfare

$W = \int_{\alpha_1}^{1} (\nu B - D + 2x)dx - (\nu r - 1) \int_{\alpha_1}^{\alpha_2} (p_1 - x)dx - (\nu r - 1) \int_{\alpha_2}^{p_2} (p_2 - x)dx + cst$

$W = (\nu B - D)(1-\alpha_1) + 1 - \alpha_1^2 - (\nu r - 1)p_1(\alpha_2 - \alpha_1) + \frac{1}{2}(\nu r - 1)(p_2^2 - \alpha_1^2) - (\nu r - 1)p_2(p_2 - \alpha_2) + cst$

$\frac{dW}{d\alpha_2} = -(\nu r - 1)p_1 - \frac{1-\nu}{2(\nu r - 1)}\alpha_1 + (\nu r - 1)p_2(2 - \frac{1-\nu}{2(\nu r - 1)})$

Note that this is independent of α_2 and so without loss of generality we can maximize α_2 by setting $\alpha_2 = \min\{p_1, p_2\}$ which would be covered by case V (Note that in case $p_2 = \alpha_2$ the α_2 type strictly prefers the higher contract).

Case V $\quad 1 > p_2 > \alpha_2$ and $\alpha_2 > p_1 > \alpha_1$

Well the IC condition for α_2 is

$\nu(B + \alpha_2 - q_2 - r(p_2 - \alpha_2)) = (\alpha_2 - p_1) + \nu(B + \alpha_2 - q_1)$

substituting in the expressions for q_1 and q_2 gives

$\nu(B + \alpha_2 - \frac{D - p_2 - (1-\nu)\frac{1+\alpha_2}{2}}{\nu} - r(p_2 - \alpha_2)) = (\alpha_2 - p_1) + \nu(B + \alpha_2 - \frac{D - p_1 - (1-\nu)\frac{\alpha_1 + \alpha_2}{2}}{\nu})$

which yields $p_2 = \alpha_2 + \frac{(1-\nu)(1-\alpha_1)}{2(r\nu - 1)}$

in particular note that $\frac{dp_2}{d\alpha_2} = 1$

The participation constraint for α_1 is

$\nu(B + \alpha_1 - q_1 - r(p_1 - \alpha_1)) = k$

substituting in the expression for q_1

$\nu(B + \alpha_1 - \frac{D - p_1 - (1-\nu)\frac{\alpha_1 + \alpha_2}{2}}{\nu} - r(p_1 - \alpha_1)) = k$

yields $p_1 = \frac{1}{\nu r - 1}(-k + \nu B - D + (1-\nu)\frac{\alpha_1 + \alpha_2}{2} + \nu(1+r)\alpha_1)$

and note in particular that $\frac{dp_1}{d\alpha_2} = \frac{1-\nu}{2(\nu r - 1)}$.

Welfare

Now let's look at overall welfare

$W = \int_{\alpha_1}^{1} (\nu B - D + 2x)dx - (\nu r - 1) \int_{\alpha_1}^{p_1} (p_1 - x)dx - (\nu r - 1) \int_{\alpha_2}^{p_2} (p_2 - x)dx + cst$

$W = (\nu B - D)(1 - \alpha_1) + 1 - \alpha_1^2 - (\nu r - 1)p_1(p_1 - \alpha_1) + \frac{1}{2}(\nu r - 1)(p_1^2 - \alpha_1^2) - (\nu r - 1)p_2(p_2 - \alpha_2) + \frac{1}{2}(\nu r - 1)(p_2^2 - \alpha_2^2) + cst.$

and in particular

$\frac{dW}{d\alpha_2} = -(\nu r - 1)\frac{1-\nu}{2(\nu r - 1)}(p_1 - \alpha_1) - (\nu r - 1)p_1\frac{1-\nu}{2(\nu r - 1)} + (\nu r - 1)p_1\frac{1-\nu}{2(\nu r - 1)} - (\nu r - 1)(p_2 - \alpha_2) + (\nu r - 1)(p_2 - \alpha_2)$

$\frac{dW}{d\alpha_2} = -(\nu r - 1)\frac{1-\nu}{2(\nu r - 1)}(p_1 - \alpha_1)$

so pushed to a corner where $\alpha_2 = p_1$.

Case VI $\quad 1 > p_2 > \alpha_2$ and $\alpha_1 > p_1$

Well the IC condition for α_2 is

$\nu(B + \alpha_2 - q_2 - r(p_2 - \alpha_2)) = (\alpha_2 - p_1) + \nu(B + \alpha_2 - q_1)$

which yields $p_2 = \alpha_2 + \frac{(1-\nu)(1-\alpha_1)}{2(r\nu - 1)}$

The participation constraint for α_1 is

$\alpha_1 - p_1 + \nu(B + \alpha_1 - q_1) = k$

$\alpha_1 + \nu B - D + (1-\nu)\frac{\alpha_1+\alpha_2}{2} = k$

and so here p_1 is unconstrained but

$\alpha_2 = \frac{k+D-B\nu-\alpha_1-\frac{1}{2}\alpha_1(-\nu+1)}{-\frac{1}{2}\nu+\frac{1}{2}}$

which yields $\alpha_2 = \frac{2}{1-\nu}(k + D - \nu B) - \alpha_1 \frac{3-\nu}{2}$

So here α_2 is not free and so also p_2 is not freely determined, the only free variable is p_1, well it's clear that from the welfare perspective p_1 is irrelevant since here

$W = \int_{\alpha_1}^{1}(\nu B - D + 2x)dx - \int_{\alpha_2}^{1}(p_2 - x)dx + cst$

and so since p_1 does not arise, without loss of generality therefore we can take $p_1 = \alpha_1$ and we're back in case V

Checking corners

Finally, it remains to consider what it means for α_2 to be pushed into a corner in each of these cases and whether it necessarily leads to a contradiction.

There are a number of possibilities, first $\alpha_2 = p_1$ and $1 > \alpha_2 > \alpha_1$ with either $p_2 > 1$ or $1 > p_2 > \alpha_2$. We show that this cannot be the case unless either $\alpha_2 > 1$ or $\alpha_2 = \alpha_1$.

In either case the IC for α_2 is

$p_2 = \alpha_2 + \frac{1-\nu}{2(\nu r-1)}(1-\alpha_1)$

and the IC for α_1 is

$\nu(B + \alpha_1 - q_1 - r(p_1 - \alpha_1)) = k$

substituting in the expression for q_1 and that $p_1 = \alpha_2$ yields

$\nu(B + \alpha_1 - \frac{D-p_1-(1-\nu)\frac{\alpha_1+\alpha_2}{2}}{\nu} - r(\alpha_2 - \alpha_1)) = k$

simplifying we obtain

$\alpha_2 = \frac{1}{\nu+2r\nu-3}(2B\nu - 2D - 2k + \alpha_1 + \nu\alpha_1 + 2r\nu\alpha_1)$

Well the case is not degenerate, that is in equilibrium these top two pools do not collapse into one, as long as α_2 is interior which it needs to be. In particular, it must be that both $\alpha_2 > \alpha_1$ and $1 > \alpha_2$. We begin by considering the first of these two conditions

Well, $\alpha_2 > \alpha_1$ if and only if

$\frac{1}{\nu+2r\nu-3}(2B\nu - 2D - 2k + \alpha_1 + \nu\alpha_1 + 2r\nu\alpha_1) > \alpha_1$

iff $2B\nu - 2D - 2k + \alpha_1 + \nu\alpha_1 + 2r\nu\alpha_1 - (\nu + 2r\nu - 3)\alpha_1 > 0$

iff $-2 - 2k + \alpha_1 + \nu\alpha_1 + 2r\nu\alpha_1 - (\nu + 2r\nu - 3)\alpha_1 > 0$

iff $4\alpha_1 - 2k - 2 > 0$

iff $\alpha_1 > \frac{1}{2}(1+k)$

Now consider the second of the two conditions that are required for the top two pools to survive and not degenerate into a single pool, namely that $\alpha_2 < 1$. This requires that

$\frac{1}{\nu+2r\nu-3}\left(2B\nu-2D-2k+\alpha_1+\nu\alpha_1+2r\nu\alpha_1\right)<1$
which is true if and only if
$2B\nu-2D-2k+\alpha_1+\nu\alpha_1+2r\nu\alpha_1<\nu+2r\nu-3$
iff $-2-2k+\alpha_1+\nu\alpha_1+2r\nu\alpha_1-(\nu+2r\nu-3)<0$
iff $\alpha_1-\nu-2r\nu-2k+\nu\alpha_1+2r\nu\alpha_1+1<0$
iff $\alpha_1(1+\nu+2r\nu)<\nu+2r\nu-1+2k$
iff $\alpha_1<\frac{\nu+2r\nu-1+2k}{(1+\nu+2r\nu)}$
and so in particular, using that $\alpha_1>\frac{1}{2}(1+k)$, we require
$\frac{\nu+2r\nu-1+2k}{(1+\nu+2r\nu)}>\frac{1}{2}(1+k)$
so $2(\nu+2r\nu-1+2k)-(1+\nu+2r\nu)(1+k)>0$
iff $3k+\nu-k\nu+2r\nu-2kr\nu-3>0$
iff $k(3-\nu-2r\nu)>3-2r\nu-\nu$
iff $k>1$
which is impossible—the utility for the best possible type when recognized as the best possible type is for type 1 recognized as such who gets a profit of $\nu B-D+1+1=1$ and so it cannot be that k, which is the profit for the α_1 type, is greater than 1.

Finally it remains to consider the cases when either $p_2>1>\alpha_2>\alpha_1=p_1$ (and here pushed to a corner means α_2 goes to 1 or to α_1 either way there is just one pool rather than two) or $1>p_2>\alpha_2>\alpha_1=p_1$. In this final case being pushed to a corner means that α_2 goes to α_1 and so there is a single pool and the requisite contradiction, otherwise α_2 is pushed to p_2 then $1>p_2=\alpha_2>\alpha_1=p_1$.

Well in this last case, the IC condition for α_2 is

$$\nu(B+\alpha_2-q_2)=(\alpha_2-\alpha_1)+\nu(B+\alpha_2-q_1) \tag{15}$$

Substituting for q_1 and q_2 we obtain

$$\frac{1}{2}\nu\alpha_1-\frac{1}{2}\alpha_1-\frac{1}{2}\nu+\frac{1}{2}=0 \tag{16}$$

which requires $\alpha_1=1$.

Note that it is possible to have full separation with $\alpha_1=1$ only in the case that the top pool is infinitesimally thin—the full separation case, but with a finite number of pools such an outcome is ruled out.

This completes the proof. ■

Lemma 8 *The condition $W_I\geq 1-z$ is more likely to hold the larger is z.*

Proof. First note $W_I \geq 1 - z$ if and only if

$A = (2z + 2\nu + 2z\nu + 4rz\nu + \nu^2 - 3)(6\nu + 8r\nu + \nu^2 + 1)2z - 2(2z + 2\nu + 2z\nu + 4rz\nu + \nu^2 - 3)^2 - (\nu r - 1)(4z\nu - 2\nu + 4rz\nu + \nu^2 + 1 - (2z + 2\nu + 2z\nu + 4rz\nu + \nu^2 - 3))^2 \geq 0$

$A = 2z - 8\nu - 16r\nu + 8z\nu + 16rz\nu + 20\nu^2 - 8\nu^3 - 2\nu^4 + 32r\nu^2 - 16r\nu^3 - 20z\nu^2 + 4z^2\nu + 8z\nu^3 + 2z\nu^4 - 32rz\nu^2 + 4rz^2\nu + 16rz\nu^3 + 24z^2\nu^2 + 4z^2\nu^3 + 56rz^2\nu^2 + 4rz^2\nu^3 + 32r^2z^2\nu^2 - 2$

Taking the derivative with respect to z yields

$\frac{dA}{dz} = 8\nu(1-\nu) + 16r\nu(1-\nu)^2 + 8z\nu + 8rz\nu - 12\nu^2 + 8\nu^3 + 2\nu^4 + 48z\nu^2 + 8z\nu^3 + 112rz\nu^2 + 8rz\nu^3 + 64r^2z\nu^2 + 2$

note that this latter is linear in z and in r and takes minimal value when $z = 0$ then

$\frac{dA}{dz} = 8\nu - 20\nu^2 + 8\nu^3 + 2\nu^4 + 16r\nu(1-\nu)^2 + 2$

which is linear in r and takes it's minimal value when $r = \frac{1}{\nu}$

then $\frac{dA}{dz} = 8\nu^3 - 4\nu^2 - 24\nu + 2\nu^4 + 18$

which it can be easily verified is non-negative in the range ν is in $(0, 1)$. ■

Proof of Lemma 6

Proof. First note that in this parameter range $W = W_I$.

Note that $p^* - l = \frac{2(1-\nu)(2-z)}{6\nu+8r\nu+\nu^2+1}$ and so $\frac{d(p^*-l)}{dr} = -\frac{16\nu(1-\nu)(2-z)}{(6\nu+8r\nu+\nu^2+1)^2}$. Next, by taking the derivative of W with respect to r from Equation 5, we obtain:

$$\frac{dW}{dr} = \frac{dl}{dr}(1 - 2l) - \frac{1}{2}\nu(p^* - l)^2 - (\nu r - 1)(p^* - l)\frac{d(p^* - l)}{dr}. \tag{17}$$

Substituting in the expressions for $\frac{dl}{dr}$ and $\frac{d(p^*-l)}{dr}$ and simplifying

$$\frac{dW}{dr} = \frac{4\nu + 16\nu^2 - 40\nu^3 + 16\nu^4 + 4\nu^5 + 32r\nu^2 - 64r\nu^3 + 32r\nu^4}{2(6\nu + 8r\nu + \nu^2 + 1)^3}(2 - z). \tag{18}$$

The denominator of this expression is positive and $(2 - z) > 0$ and so $\frac{dW}{dr}$ has the same sign as the numerator of the fraction

$$\begin{aligned} sign(\tfrac{dW}{dr}) &= sign(4\nu + 16\nu^2 - 40\nu^3 + 16\nu^4 + 4\nu^5 + 32r\nu^2 - 64r\nu^3 + 32r\nu^4) \\ &= sign(1 + 4\nu - 10\nu^2 + 4\nu^3 + \nu^4 + 8r\nu - 16r\nu^2 + 8r\nu^3) \end{aligned} \tag{19}$$

where the second equality holds since $4\nu > 0$.

It follows that $\frac{dW}{dr} > 0$ if and only if $1 + 4\nu - 10\nu^2 + 4\nu^3 + \nu^4 + 8r\nu - 16r\nu^2 + 8r\nu^3 > 0$, which is true if and only if:

$$\frac{1 + 4\nu - 10\nu^2 + 4\nu^3 + \nu^4}{16\nu^2 - 8\nu - 8\nu^3} > r. \tag{20}$$

Note that $\frac{2-\nu}{\nu} > r \geq \frac{1}{\nu}$ and so $\frac{dW}{dr} > 0$ requires

$$\frac{1 + 4\nu - 10\nu^2 + 4\nu^3 + \nu^4}{16\nu^2 - 8\nu - 8\nu^3} > \frac{1}{\nu}, \tag{21}$$

or equivalently,

$$4\nu^3 - 2\nu^2 - 12\nu + \nu^4 + 9 > 0 \tag{22}$$

which is always true for ν in the range $(0, 1)$.

The second part of the lemma follows trivially, since W is independent of r outside the range.

■