Biased Quantitative Measurement of Interval Ordered Homothetic Preferences

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Abstract

We represent interval ordered homothetic preferences with a quantitative homothetic utility function and a multiplicative bias. When preferences are weakly ordered (i.e. when indifference is transitive), such a bias equals 1. When indifference is intransitive, the biasing factor is a positive function smaller than 1 and measures a threshold of indifference. We show that the bias is constant if and only if preferences are semiordered, and we identify conditions ensuring a linear utility function. We illustrate our approach with indifference sets on a two dimensional commodity space.

Keywords: weak order, semiorder, interval order, intransitive indifference, independence, homothetic, representation, linear utility

JEL Classification Numbers: D000, D110, D210, O220.

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1 Introduction

We introduce an algebraic approach to represent interval ordered homothetic preferences with a homothetic utility function and a multiplicative factor that "distorts" of "biases" this utility function. This biasing factor is a positive function smaller or equal to 1 which is uniquely characterized by the underlying preferences. The utility function is unique up to multiplication by a positive number, hence allowing for meaningful comparisons of both differences and ratios of utility.

The notion of homotheticity is familiar to several branches of economics and captures the idea of scale-invariance. Homotheticity is a key assumption of the theory of demand and is commonly used to reflect constant economies of scale in the theory of production. Besides studies of aggregation properties (e.g. Chipman [6]), theoretical works have been mainly concerned with conditions under which homothetic preferences can be represented by a continuous utility function (Dow and Werlang [7], Candeal and Indurain [2], Bosi et Al. [3]). In these approaches, the measurement of homothetic utility is ordinal, in the sense that the utility function is unique up to increasing transformations. This lack of uniqueness in the measurement of utility does not allow for a meaningful comparison of differences and ratios of utility (Stevens [16]).

The importance of intransitivity of indifference has been early recognized in economics, for instance with the concept of semiorders and the interpretation that individuals may lack discrimination (e.g. Luce [13]). In a seminal paper, Scott and Suppes [15] prove that finite semiorders can be represented by a function and a constant additive threshold, naturally interpreted as a threshold of indifference. However, the lack of uniqueness of this representation impedes any genuine measurement of this threshold of indifference, i.e. the extent to which indifference is intransitive.

In the more general case of interval ordered preferences, conditions of ordinal type have been identified to guarantee a representation of preferences by two functions (Fishburn [8],[9], Bridges [4], Chateauneuf [5], Oloriz et Al. [14], Bosi [1]). Such a pair of functions provides for a utility interval, i.e. a lower and upper value for the utility of each object (hence the name "interval order"). In these approaches, no utility *per se* is assigned to each object. Again, the lack of uniqueness property of the representation hampers the measurement of this interval and of the indifference threshold.

In this paper, we assign a fully quantitative utility to each object and we measure thresholds of indifference. We can recover the interval representation with two functions, with uniqueness conditions that ensure a tight measurement of the utility interval. Moreover, we show that it is possible to construct a series of progressively finer relations that approximate the equivalence relation among objects of identical utility. As for semiordered homothetic preferences, we show that they correspond to the case of a constant biasing factor. The threshold of indifference is then proportional to the level of utility. We naturally recover the representation with a constant additive threshold by applying any logarithmic transformation. Such a transformed utility function is then cardinal, in the sense that it preserves the differences of levels of utility but not their ratios. When we consider the possibility of combining different objects, we can clarify the conditions under which a linear utility function is obtained. We show that the linearity of the utility function requires homothetic preferences to be semiordered and to verify a condition that we call pseudo-independence. This last condition is weaker than the traditional condition of independence assumed in order to construct a linear utility when indifference is transitive.

A most immediate economic application of this approach lies in isoutility curves and indifference sets. We hence illustrate our main results and concepts with examples on a two dimensional commodity space where objects are bundles of goods. When indifference is transitive, the quantitative measurement of utility justifies the notion of equally-spaced isoutility curves. When indifference is not transitive, two indifferent goods can have different utility and indifference sets differ from isoutility curves. Thresholds of indifference then measure the "thickness" of these indifference sets. In general, different goods with identical utility have different indifference sets, except when homothetic preferences are semiordered. Naturally, in this two-dimensional commodity space, linear homothetic preferences lead to straight lines isoutility curves.

We realized after completion that our result for the representation of weakly ordered homothetic preferences (Theorem 1 below) was very similar to Theorem 9 of Krantz & Al. [10, p. 104]. Also, a more detailed mathematical treatment of homothetic interval orders can be found in Lemaire & Le Menestrel [11]. In the present paper, we provide for a simpler formulation of our results and we emphasize their economic significance. In particular, we have omitted here the consideration of objects with null utility, which induces significant technical complications in the formulations and in the proofs. Finally, an intermediary result that intuitively corresponds to the case of a one-dimensional commodity space was published in Le Menestrel and Lemaire [12]. The rest of the paper is structured as follows.

In section 2, we present the algebraic setting of our approach and prove a useful lemma. In section 3, we treat the case of homothetic preferences that are weakly ordered. In section 4, we relax the assumption of transitivity of indifference and introduce the biased representation of interval ordered homothetic preferences. In section 5 we recover (in our homothetic context) the more familiar representation of interval order with two functions. In section 6, we deal with homothetic preferences that are semiordered, and in section 7 we consider the case of linear homothetic semiordered preferences. We conclude in a brief section 8.

2 Preliminaries and a Key Lemma

Let A denote a non-empty set of objects, $x, y, z, t...$ the elements of A, and $\mathbb{N}^*$ the set of positive integers. Our basic algebraic structure consists of the set A together with a map $\mathbb{N}^* \times A \to A, (m, x) \longmapsto mx$ such that $(mm')x = m(m'x)$ and $1x = x$. Such a structure is called a $\mathbb{N}^* - set$. In this manner, objects can be replicated with themselves and we interpret mx as the quantity m of object x. Note that the results we obtain for $\mathbb{N}^*$-sets are true (mutatis mutandis) for $\mathbb{R}_+^*$-sets, where $\mathbb{R}_+^*$ denotes the set of positive real numbers. Hence, they are also true for a "vector space" $A = (\mathbb{R}_+^*)^L$ of any dimension L, which is the more traditional structure used in economics to represent preferences (our examples further illustrate this point).

We model preferences by a binary relation $\succ$ on A. The indifference relation on A is noted $\sim$ and is defined by $x \sim y \Leftrightarrow x \not\succ y$ *and* $y \not\succ x$. We note $\succsim$ the relation on A defined by $x \succsim y \Leftrightarrow x \succ y$ *or* $x \sim y$.

For all $x, y, z, t \in A$, the relation $\succ$ is said to be *asymmetric* if $x \succ y \Rightarrow y \not\succ x$; *transitive* if $x \succ y \succ z \Rightarrow x \succ z$, *strongly transitive* if $x \succ y \succsim z \succ t \Rightarrow x \succ t$ and *negatively transitive* if $x \not\succ y \not\succ z \Rightarrow x \not\succ z$. Note that the relation $\succ$ is asymmetric if and only if $x \not\succ y \Leftrightarrow y \succsim x$. The relation $\succ$ is called an *interval order* if it is asymmetric and strongly transitive; a *semiorder* if it is an interval order and, for all $x, y, z \in A$, we have $x \succ y \succ z \Rightarrow (t \succ z$ *or* $x \succ t)$; a *weak order* if it is asymmetric and negatively transitive. So we have the implications

$$\text{weak order} \Rightarrow \text{semiorder} \Rightarrow \text{interval order}.$$

Note that indifference may fail to be transitive for both an interval order and a semiorder.

We now introduce the following axioms for a relation $\succ$ on a $\mathbb{N}^*$-set A $(x, y, z, t \in A; m, m', m'' \in \mathbb{N}^*)$:

Axiom 1 (*homotheticity*) $\forall(x, y, m)$ we have $x \succ y \Leftrightarrow mx \succ my$;

Axiom 2 (*strong separability*) $\forall(x, y, z)$ such that $x \succ y, \exists(m, m', m'')$ such that $mx \succ m'z \succsim m''z \succ my$;

Axiom 3 (*archimedean*) $\forall(x, y)$ such that $x \succ y$, $\exists(m, m')$ such that $m < m'$ and $mx \succ m'y$;

Axiom 4 (*positivity*) $\forall(x, y, m, m')$ such that $m > m'$, we have $x \succ y \Rightarrow mx \succ m'y$;

Axiom 5 (*strongly non-empty*) $\forall(x, y)$ such that $y \succsim x$, $\exists(m, m')$ such that $mx \succ m'y$.

Note that axiom 5 excludes the objects that would be assigned a null utility. With a slight reformulation of axiom 2, this axiom is omitted in Lemaire and Le Menestrel [11].

Suppose $\succ$ is strongly non-empty. Then $\succ$ is called a *homothetic structure* if it verifies axioms 1 to 4. A homothetic structure is called a *homothetic interval order*, a *homothetic semiorder* and a *homothetic weak order* if $\succ$ is respectively an interval order, a semiorder and a weak order.

We now introduce the basic tools of our algebraic approach.

Let A be a homothetic structure and define the (non-empty) subsets of $\mathbb{Q}_+^*$, where $\mathbb{Q}_+^*$ denotes the set of positive rational numbers:

$$\begin{aligned}\mathcal{Q}_{x,y} &= \{mn^{-1} : m, n \in \mathbb{N}^*, mx \succsim ny\},\\ \mathcal{P}_{x,y} &= \{mn^{-1} : m, n \in \mathbb{N}^*, mx \succ ny\}.\end{aligned}$$

Let $r_{x,y} = \inf_{\mathbb{R}_{\geq 0}} \mathcal{Q}_{x,y}$ and $s_{x,y} = \inf_{\mathbb{R}_{\geq 0}} \mathcal{P}_{x,y}$. For non-empty subsets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^*$, let $\mathcal{X}^{-1} = \{x^{-1}, x \in \mathcal{X}\}$ and $\mathcal{X}\mathcal{Y} = \{xy, x \in \mathcal{X}, y \in \mathcal{Y}\}$. If $\succ$ is asymmetric, we have the partitions of $\mathbb{Q}_+^*$: $\mathbb{Q}_+^* = \mathcal{Q}_{x,y} \cup \mathcal{P}_{x,y}^{-1} = \mathcal{Q}_{x,y}^{-1} \cup \mathcal{P}_{x,y}$ with $\mathcal{Q}_{x,y} \cap \mathcal{P}_{x,y}^{-1} = \mathcal{Q}_{x,y}^{-1} \cap \mathcal{P}_{x,y} = \varnothing$.

We now prove the following useful lemma.

Lemma 1: *If $\succ$ is a homothetic interval order, then for all $x, y, a \in A$ we have $\mathcal{P}_{x,y} = \mathcal{Q}_{>s_{x,y}}$ with $s_{x,y} > 0, \mathcal{Q}_{y,x} = \mathbb{Q}_{\geq r_{y,x}}$ with $r_{y,x} = s_{x,y}^{-1}$, and $\mathcal{P}_{x,y} = \mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y}$.*

Proof. Let $x, y \in A$. Put $s = s_{x,y}$ and $r = r_{x,y}$. If $q \in \mathcal{P}_{x,y}$, we have (axiom 4) $\mathbb{Q}_{\geq q} \subset \mathcal{P}_{x,y}$. If $q \in \mathbb{Q}_{>s}$, then by definition of s, there exists $q' \in \mathcal{P}_{x,y}$ such that $s \leq q' < q$. Hence, we have $\mathbb{Q}_{>s} \subset \mathcal{P}_{x,y}$. From axiom 3, we have $s \in \mathbb{Q} \Rightarrow s \notin \mathcal{P}_{x,y}$. Thus $\mathcal{P}_{x,y} = \mathbb{Q}_{>s}$. Since $\mathcal{Q}_{y,x}^{-1} = \mathbb{Q}_+^* \setminus \mathcal{P}_{x,y}$ is non-empty, we have $s > 0$. Since $\mathcal{Q}_{y,x} = \mathbb{Q}_+^* \setminus \mathcal{P}_{x,y}^{-1} =]0, s^{-1}[$, we have $\mathcal{Q}_{y,x} = \mathbb{Q}_{s^{-1}}$ and $r = s^{-1}$. From strong transitivity of $\succ$ and axiom 1, we have the inclusion $\mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y} \subset \mathcal{P}_{x,y}$; and from axioms 2 and 1, we have the inclusion $\mathcal{P}_{x,y} \subset \mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,y}$. ■

3 Homothetic Weak Orders

In this section, we show that homothetic weak orders can be represented by a utility function unique up to multiplication by a positive number, i.e. by a fully quantitative utility function. This allows for meaningful comparisons of differences and ratios of utility, hence refining ordinal and cardinal approaches to the measurement of utility.

Theorem 1: *Let A be a non-empty $\mathbb{N}^* -$ set endowed with a strongly non-empty binary relation $\succ$. The two following conditions are equivalent $(x, y \in A; m \in N^*)$:*

(i) There exists a function $u : A \to \mathbb{R}_+^*$ such that $\forall (x, y, m)$ we have

$$\left\{\begin{array}{c} x \succ y \Leftrightarrow u(x) > u(y) \\ u(mx) = mu(x) \end{array}\right. .$$

(ii) The relation $\succ$ is a homothetic weak order.

Moreover, if $\succ$ is a homothetic weak order, the function u of (i) is unique up to multiplication by a positive number.

Proof. The implication $(i) \Rightarrow (ii)$ is clear. Suppose $\succ$ is a homothetic weak order. From Lemma 1, we have $x \succ y \Leftrightarrow s_{x,y} < 1$; and using negative transitivity, we obtain $s_{x,x} = 1 = r_{x,x}$. Choose an element $a \in A$ and let $u : A \to \mathbb{R}_+^*$ be the function defined by $u(x) = r_{a,x}$. Clearly, we have $u(mx) = mu(x)$. Let us prove that $x \succ y \Leftrightarrow u(x) > u(y)$. From the equality $\mathcal{P}_{x,x} = \mathcal{P}_{x,a}\mathcal{Q}_{a,a}\mathcal{P}_{a,x}$, we obtain $s_{x,x} = s_{x,a}r_{a,a}s_{a,x}$, that is $s_{a,x} = s_{x,a}^{-1} = r_{a,x}$. Hence $s_{x,y} = s_{x,a}r_{a,a}s_{a,y} = r_{a,x}^{-1}r_{a,y}$. Since $x \succ y \Leftrightarrow s_{x,y} < 1$, we have $x \succ y \Leftrightarrow y(x) > u(y)$.

Now let $v : A \to \mathbb{R}_+^*$ be another function verifying (i). Let $\lambda : A \to \mathbb{R}_+^*$ be the function defined by $\lambda(x) = u(x)^{-1}v(x)$. Suppose there exist two elements $x, y \in A$ such that $\lambda(x) \neq \lambda(y)$. By symmetry, we can assume $\lambda(x) > \lambda(y)$. Let $\alpha = \lambda(y)\lambda(x)^{-1} < 1$. Then by density, there exists a $q \in \mathbb{Q}_+^*$ such that $\alpha u(y)u(x)^{-1} < q < u(y)u(x)^{-1}$. In other words, we have $v(y) < qv(x)$ and $qu(x) < u(y)$, which is impossible. Hence, λ is a constant map. ■

Let $\succ$ be a strongly non-empty homothetic weak order on a $\mathbb{N}^*$-set A. We chose a function verifying condition (i) of Theorem 1 and we say that u *represents* $\succ$. We call u a *utility function* and $u(x)$ the *utility* of x.

For $x \in A$, we note U_x the *isoutility curve containing* x, defined by $U_x = \{y \in A : u(y) = u(x)\}$. Because of the uniqueness of u, U_x does not depend on u.

We now illustrate the quantitative measurement of homothetic utility with three equally-spaced isoutility curves. In this illustration, like in all other examples in this paper, our basic structure is a two-dimensional commodity space that consists in bundles of goods made of a quantity x_1 of good X_1 and x_2 of good X_2. In order to ease the graphical and numerical illustrations, which may feature non integer values, we consider real-valued quantities of goods. Formally, A is the set $\{(x_1X_1, x_2X_2) : x_1, x_2 \in \mathbb{R}_+^*\}$ endowed with the structure of $\mathbb{R}_+^*$-set given by the map $\mathbb{R}_+^* \times A \to A, (\lambda, (x_1X_1, x_2X_2)) \mapsto (\lambda x_1X_1, \lambda x_2X_2)$.

Example 1: Consider the utility function $x = (x_1X_1, x_2X_2) \mapsto u(x) = x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}$ and define $\succ$ by $x \succ y \Leftrightarrow u(x) > u(y)$ for all $x, y \in A$. With the quantity x_1 in abscisse and the quantity x_2 as ordinate, Figure 1 plots three equally-spaced isoutility curves $U_5 = \{x : u(x) = 5\}, U_{10} = \{x : u(x) = 10\}, U_{15} = \{x : u(x) = 15\}$. In this manner, there is as much utility difference between goods belonging to U_5 and U_{10} than between goods belonging to U_{10} and U_{15} (comparison of differences of utility). Moreover, goods belonging to U_{10} have twice as much utility than goods belonging to U_5 (comparison of ratios of utility). We illustrate this with the goods $a \in U_5, 2a \in U_{10}$, and $3a \in U_{15}$.$\triangle$

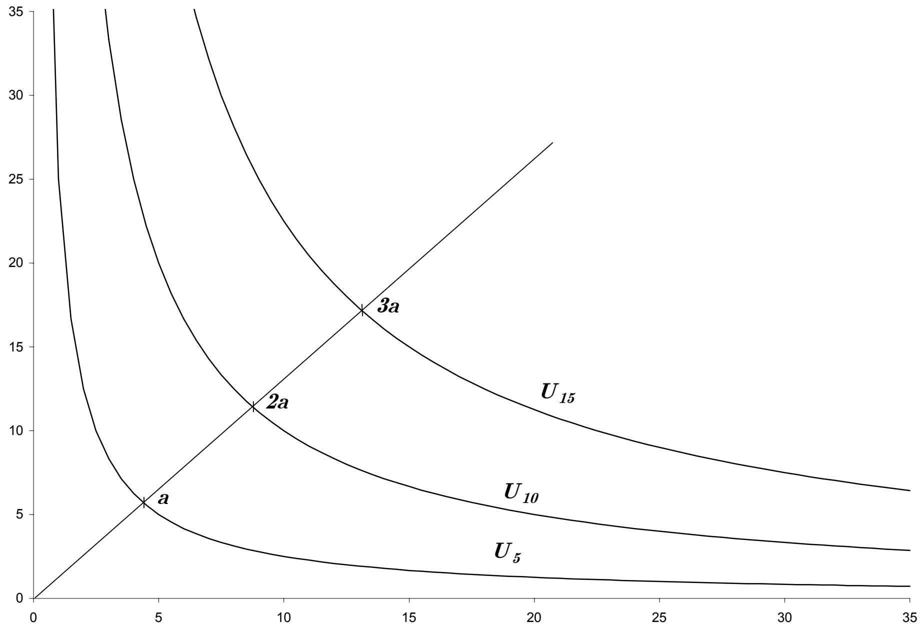

Figure 1: Three Equally-spaced Isoutility Curves of a Homothetic Weak Order

4 Homothetic Interval Orders, Biased Representation

Relaxing the transitivity of indifference, we now prove that homothetic interval orders can be represented by a utility function unique up to multiplication by a positive number and a biasing function. This function is positive and smaller or equal to 1, and is uniquely characterized by the underlying homothetic interval order.

Theorem 2: *Let* A *be a non-empty* $\mathbb{N}^* -$ *set endowed with a strongly non-empty binary relation* $\succ$ *. The two following conditions are equivalent* $(x, y \in A; m \in \mathbb{N}^*)$:

(i) There exist two functions $u : A \to \mathbb{R}_+^*$ *and* $\gamma : A \to \,]0, 1]$ *such that* $\forall(x, y, m)$ *we have*

$$\begin{cases} x \succ y \Leftrightarrow \gamma(x)u(x) > \gamma(y)^{-1}u(y) \\ u(mx) = mu(x) \\ \gamma(mx) = \gamma(x) \end{cases}.$$

(ii) The relation $\succ$ *is a homothetic interval order.*

Moreover, if $\succ$ *is a homothetic interval order, the pair* (u, γ) *of (i) is unique up to multiplication of* u *by a positive number.*

Proof. The implication $(i) \Rightarrow (ii)$ is easy to verify. Suppose $\succ$ is a homothetic interval order. From Lemma 1, we have $x \succ y \Leftrightarrow s_{x,y} < 1$. Let $\succ_0$ be the binary relation on A defined by $x \succ_0 y \Leftrightarrow s_{y,x} > s_{x,y}$. Since $s_{y,mx} = ms_{y,x}$ and $s_{mx,y} = m^{-1}s_{x,y}, \succ_0$ is strongly non-empty. We now prove that $\succ_0$ is a homothetic weak order. Let $\sim_0$ be the indifference relation associated with $\succ_0$. Thus we have $x \sim_0 y \Leftrightarrow s_{x,y} = s_{y,x} \Leftrightarrow \mathcal{P}_{x,y} = \mathcal{P}_{y,x}$. By using the equalities $s_{x,z} = s_{x,y}r_{y,y}s_{y,z}$ and $s_{z,x} = s_{z,y}r_{y,y}s_{y,z}$, we obtain the transitivity of $\sim_0$: if $x \sim_0 y$ and $y \sim_0 z$, then $x \sim_0 z$. Hence $\succ_0$ is negatively transitive; in particular, it is a weak order. It remains to verify that $\succ_0$ satisfies axiom 2. Let $x, y \in A$ such that $x \succ_0 y$. Since $s_{x,y} = s_{x,z}r_{z,z}s_{z,y}$ and $s_{y,x} = s_{y,z}r_{z,z}s_{z,x}$, we have $s_{x,z}s_{z,y} < s_{y,z}s_{z,x}$. Hence there exist $p, m, n \in \mathbb{N}^*$ with $m > n$ such that $(\frac{m}{n})^2 \frac{s_{x,z}}{s_{z,x}} < 1 < (\frac{n}{p})^2 \frac{s_{y,z}}{s_{z,y}}$ and $(pn^{-1})^2 s_{z,y} < s_{y,z}$. Then we have $s_{px,mz} < s_{mz,px}$ and $s_{nz,py} < s_{py,nz}$, i.e. $px \succ_0 mz \succsim_0 nz \succ_0 py$.

Then, by Theorem 1, we can choose a function $u : A \to \mathbb{R}_+^*$ such that $u(mx) = mu(x)$ and $x \succ_0 y \Leftrightarrow u(x) > u(y)$. Let $\sigma(x, y) : A \times A \to \mathbb{R}_+^*$ be the function defined by $\sigma(x, y) = r_{y,x}u(x)^{-1}u(y)$. Since $x \succ y \Leftrightarrow r_{x,y} > 1$, by construction we have $x \succ y \Leftrightarrow \sigma(x, y)u(x) > u(y)$. And returning to the definition of $\succ_0$, we obtain

$$u(x) > u(y) \Leftrightarrow \sigma(x, y)^{\frac{1}{2}}u(x) > \sigma(y, x)^{\frac{1}{2}}u(y).$$

This implies $\sigma(x, y) = \sigma(y, x)$. Hence, we have $\sigma(x, y) = \gamma(x)\gamma(y)$ with $\gamma(x) = \sigma(x, x)^{\frac{1}{2}}$. Since $\sigma(mx, m'y) = \sigma(x, y)$, we have $\gamma(mx) = \gamma(x)$. The uniqueness of u up to multiplication by a positive number (Theorem 1) implies the uniqueness of γ. ■

Let $\succ$ be a strongly non-empty homothetic interval order on a $\mathbb{N}^*$-set A. We chose a pair (u, γ) verifying condition (i) of Theorem 2 and we say that (u, γ) *represents* $\succ$. When $\gamma = 1$ (i.e. the constant function $x \mapsto 1$), we recover Theorem 1: indifference is transitive and $\succ$ is a weak order. Consistently with this special case, we call u a *utility function* and $u(x)$ the *utility* of x.

For $x \in A$, we note I_x the *indifference set containing* x, defined by $I_x = \{y \in A : y \sim x\}$. Because of the uniqueness of u, I_x does not depend on u. And because of the symmetry of $\sim$ (i.e. $x \sim y \Leftrightarrow y \sim x$), we have $y \in I_x \Leftrightarrow x \in I_y$. Moreover, if $\succ$ is a weak order (i.e. if $\gamma = 1$), then I_x coincides with the isoutility curve $U_x = \{y \in A : u(y) = u(x)\}$. When indifference is not transitive, indifference sets show a threshold of indifference. We have

$$y \in I_x \Leftrightarrow \gamma(x)\gamma(y)u(x) \leqslant u(y) \leqslant \gamma(x)^{-1}\gamma(y)^{-1}u(x).$$

Note that two objects with the same utility may have different indifference sets.

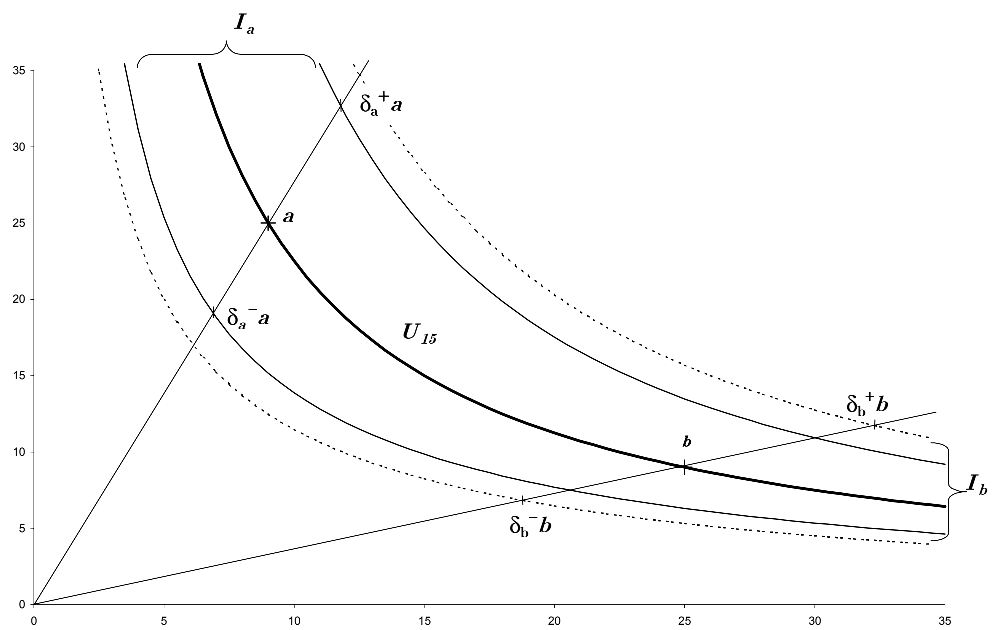

Figure 2: Two Indifference Sets of a Homothetic Interval Order

For $x \in A$, we note $\mathfrak{I}_x$ the subset of $\mathbb{Q}_+^*$ defined by $\mathfrak{I}_x = \{\frac{m}{n} : mx \sim nx\}$. Because of the symmetry of $\sim$, we have $\mathfrak{I}_x^{-1} = \mathfrak{I}_x$. Let $\bar{\mathfrak{I}}_x$ denote the closure of $\mathfrak{I}_x$ in $\mathbb{R}$ for the usual topology. We deduce from Theorem 2 that $\bar{\mathfrak{I}}_x$ coincides with the closed interval $[\gamma(x)^2 : \gamma(x)^{-2}]$. We let $\delta_x^+ = \gamma^{-2}(x)$ and $\delta_x^- = (\delta_x^+)^{-1}$. Thus we have $\bar{\mathfrak{I}}_x = [\delta_x^-, \delta_x^+]$. We propose to call δ_x^+ the *upper indifference threshold at* x, and δ_x^- the *lower indifference threshold at* x. We illustrate these concepts in the following example.

Example 2: Consider the utility function $x = (x_1X_1, x_2X_2) \mapsto u(x) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$ and consider a factor $\gamma(x) = \frac{\lambda x_1 + \mu x_2}{x_1 + x_2}$ with $\lambda, \mu \leq 1$ that biases bundle x depending on the relative quantities of goods X_1 and X_2. The binary relation $\succ$ defined by $x \succ y \Leftrightarrow \gamma(x)u(x) > \gamma(y)^{-1}u(y)$ for all $x, y \in A$ is a homothetic interval order. Letting $\lambda = 0.95$ and $\mu = 0.80$, Figure 2 shows goods $a = (25X_1, 9X_2)$ and $b = (9X_1, 25X_2)$ with $u(a) = u(b) = 15$. Their identical isoutility curve U_{15} appears in bold line and their distinct indifference sets I_a and I_b are delimited by the plain and dotted lines respectively. Note that since $\gamma(a) > \gamma(b)$, we have $I_a \subset I_b$. We also show the lower and upper indifference thresholds of goods a and b. Since A is a $\mathbb{R}_+^*$-set; for $x \in A$, δ_x^+ coincides with the $\sup\{\lambda \in \mathbb{R}_+^* : \lambda x \sim x\}$, and δ_x^- coincides with the $\inf\{\lambda \in \mathbb{R}_+^* : \lambda x \sim x\}$. Numerically, we have $\delta_a^- \simeq -17\%$ and $\delta_a^+ \simeq +21\%$. Also, $\delta_b^- \simeq -29\%$ and $\delta_b^+ \simeq +42\%$.$\triangle$

Remark that the biasing factor of a given object does not depend on its quantity. Hence, if the set A is *homogeneous* in the sense that for all $x, y \in A$, there exist $m, n \in \mathbb{N}^*$ such that $mx = ny$, then the bias is a constant function and homothetic preferences are semiordered. This is the restricted case studied in Le Menestrel and Lemaire [12]. In section 6, we will show that being semiordered is a sufficient condition for homothetic structures to be represented with a constant biasing function.

5 Homothetic Interval Orders, Representation with Two Functions

The classical representation of an interval order $\succ$ consists in finding two real-valued functions u_1 and u_2, with $u_1 \leq u_2$, such that $x \succ y \Leftrightarrow u_1(x) > u_2(y)$. From Theorem 2, it is easy to recover such a two-function formulation by letting, for instance, $u_1(x) = \gamma(x)u(x)$ and $u_2(x) = \gamma^{-1}(x)u(x)$.

The purpose of this section is to construct, for a given homothetic interval order $\succ$, a canonical two-function representation (u_1, u_2) directly from two weak orders associated with $\succ$. This ensures the uniqueness of the pair (u_1, u_2) up to multiplication by a positive number. Such a two-function representation is tighter than the one of Theorem 2 in a sense made precise below; in particular, we can recover one formulation from the other.

Let A be a non-empty set $\mathbb{N}^*-$set endowed with a strongly non-empty homothetic interval order $\succ$. We define the following three binary relations:

- $x \succ_0 y \Leftrightarrow u(x) > u(y)$ for one (i.e. for any) pair (u, γ) verifying condition (i) of Theorem 2,

- $x \succ_1 y \Leftrightarrow (mx \succ z \succsim my, \exists (z, m) \in A \times \mathbb{N}^*)$,

- $x \succ_2 y \Leftrightarrow (mx \succsim z \succ my, \exists (z, m) \in A \times \mathbb{N}^*)$.

Note that $x \succ y \Rightarrow (x \succ_1 y$ and $x \succ_2 y)$. Hence, since $\succ$ is strongly non-empty, so is $\succ_i$ $(i = 1, 2)$. The relation $\succ_0$ is clearly a homothetic weak order. The following corollary shows it is the same for both $\succ_1$ and $\succ_2$.

Corollary 1: *Let A be a non-empty $\mathbb{N}^* -$ set endowed with a strongly non-empty homothetic interval order $\succ$. Then for $i = 1, 2, \succ_i$ is a homothetic weak order.*

Proof. Choose an element $a \in A$. By Theorem 2, it is easily shown that the two functions $v_1, v_2 : A \to \mathbb{R}_+^*$ defined by $v_1(x) = r_{a,x}$ and $v_2(x) = s_{a,x}$ represent $\succ_1$ and $\succ_2$: for $i = 1, 2$, we have $x \succ_i y \Leftrightarrow v_i(x) > v_i(y)$. Since we clearly have $v_i(mx) = mv_i(x)$ $(i = 1, 2)$, we conclude by applying Theorem 1. ■

We can now prove the following theorem:

Theorem 3: *Let A be a non-empty $\mathbb{N}^* -$ set endowed with a strongly non-empty binary relation $\succ$. The two following conditions are equivalent $(x, y \in A; m \in \mathbb{N}^*)$:*

(i) There exist two functions $u_1, u_2 : A \to \mathbb{R}_+^*$ such that $u_1 \leq u_2$ and $\forall (x, y, m)$ we have

$$\begin{cases} x \succ y \Leftrightarrow u_1(x) > u_2(y) \\ u_1(mx) = mu_1(x) \\ u_2(mx) = mu_2(x) \end{cases}.$$

(ii) The relation $\succ$ is a homothetic interval order.

Moreover, if $\succ$ is a homothetic interval order, the pair (u_1, u_2) of (i) can be chosen in such a way that for $i = 1, 2$, u_i represents $\succ_i$: $x \succ_i y \Leftrightarrow u_i(x) > u_i(y)$. In this case, the pair (u_1, u_2) is unique up to multiplication of u_1 and u_2 by a positive number (i.e., up to replacing it by $(\lambda u_1, \lambda u_2)$ for a *$\lambda > 0$) and the pair $(u, \gamma) = ((u_1 u_2)^{\frac{1}{2}}, (\frac{u_1}{u_2})^{\frac{1}{2}})$ verifies the condition (i) of Theorem 2.*

Proof. The implication $(i) \Rightarrow (ii)$ is easy to verify. Suppose $\succ$ is a homothetic interval order. Choose an element $a \in A$ and let $u_1, u_2 : A \to \mathbb{R}_+^*$ be the functions defined by $u_1(x) = s_{a,a} r_{a,x}$ and $u_2(x) = s_{a,x}$. For $i = 1, 2$, we clearly have $u_i(mx) = mu_i(x)$. Since $x \succ y \Leftrightarrow s_{x,y} < 1$ and $s_{x,y} = r_{a,x}^{-1} s_{a,a}^{-1} s_{a,y}$, we have $x \succ_i y \Leftrightarrow u_i(x) > u_i(y)$. Moreover, we have

$$\begin{aligned} s_{y,x} > s_{x,y} &\Leftrightarrow s_{y,a} r_{a,a} s_{a,x} > s_{x,a} r_{a,a} s_{a,y} \\ &\Leftrightarrow r_{a,x} s_{a,x} > r_{a,y} s_{a,y} \\ &\Leftrightarrow (u_1 u_2)(x) > (u_1 u_2)(y) \\ &\Leftrightarrow u(x) > u(y) \end{aligned}$$

with $u = (u_1 u_2)^{\frac{1}{2}}$. Clearly, we have $u(mx) = m(u(x)$. And from the proof of Theorem 2, we have $\gamma(x)^2 = r_{x,x} = (\frac{u_1}{u_2})(x)$. ■

Let $\succ$ be a strongly non-empty homothetic interval order on a $\mathbb{N}^*$-set A. We choose two functions (u_1, u_2) verifying condition (i) of Theorem 3 and we say that (u_1, u_2) *represents* $\succ$. We put $(u, \gamma) = ((u_1 u_2)^{\frac{1}{2}}, (\frac{u_1}{u_2})^{\frac{1}{2}})$.

For $x \in A$, we note J_x the set $\{y : u_1(x) \leqslant u(y) \leqslant u_2(x)\}$, and we propose to call it the *tight indifference set containing* x. Because of the uniqueness property of Theorem 2, J_x does not depend on the choice of the pair (u_1, u_2) .

For $x \in A$, we have the inclusion (in general strict) $J_x \subset I_x$. By construction, if $y, z \in J_x$, then $y \sim z$; a property that is not verified by indifference sets. Note also that we may have $y \in J_x$ but $x \notin J_y$. More precisely, for $x, y \in A$, we have $x \sim y$ if and only if $J_x \cap J_y \neq \emptyset$, i.e. if and only if the intersection of the two closed utility intervals $u(J_x)$ and $u(J_y)$ is nonempty.

Let $\approx$ be the binary relation on A defined as follows: $x \approx y$ if and only if $x \in J_y$ and $y \in J_x$. It is clearly symmetric, and we call it the *tight indifference* relation associated with $\succ$. For $x \in A$, we note $\mathfrak{J}_x$ the subset of $\mathbb{Q}_+^*$ defined by $\mathfrak{J}_x = \{\frac{m}{n} : mx \approx nx\}$. We have $\mathfrak{J}_x = \mathfrak{J}_x^{-1}$. Let $\bar{\mathfrak{J}}_x$ denote the closure of $\mathfrak{J}_x$ in $\mathbb{R}$ for

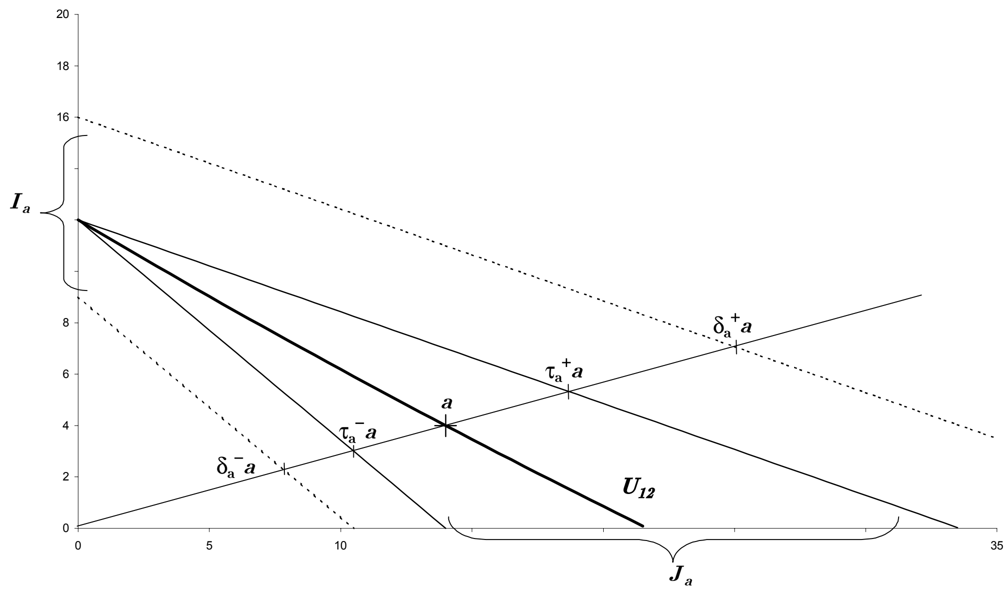

Figure 3: A Tight Indifference Set and an Indifference Set of a Homothetic Interval Order

the usual topology. We deduce from Theorem 2 (or Theorem 3) that $\bar{\mathfrak{J}}_x$ coincide with the closed interval $[\gamma(x) : \gamma(x)^{-1}]$. We put $\tau_x^+ = \gamma(x)^{-1}$ and $\tau_x^- = (\tau_x^+)^{-1}$. Thus we have $\bar{\mathfrak{J}}_x = [\tau_x^-, \tau_x^+]$. We call τ_x^+ the *upper tight indifference threshold at x, and* τ_x^- the *lower tight indifference threshold at x.* We illustrate these concepts in the following example.

Example 3: Consider the two functions $x = (x_1X_1, x_2X_2) \mapsto u_1(x) = \lambda x_1 + x_2$ and $x = (x_1X_1, x_2X_2) \mapsto u_2(x) = \mu x_1 + x_2$ where $0 < \lambda \leq \mu$ and define $\succ$ by $x \succ y \Leftrightarrow u_1(x) > u_2(y)$. The relation $\succ$ is a homothetic interval order and we recover the formulation $x \succ y \Leftrightarrow \gamma(x)u(x) > \gamma^{-1}(y)u(y)$ of Theorem 2 with $\gamma(x) = (\lambda x_1 + x_2)^{\frac{1}{2}}(\mu x_1 + x_2)^{-\frac{1}{2}}$ and $u(x_1X_1, x_2X_2) = (\lambda x_1 + x_2)^{\frac{1}{2}}(\mu x_1 + x_2)^{\frac{1}{2}}$. Letting $\lambda = \frac{5}{14}$ and $\mu = \frac{6}{7}$, Figure 3 shows good $a = (14, 4)$ of utility $u(a) = 12$ and its isoutility curve $U_{12} = \{x : u(x) = 12\}$ in bold line. In plain lines appears the tight indifference set J_a and in dotted lines appears the indifference set I_a. We also depict the lower and upper indifference and tight indifference thresholds of good a. Since A is a $\mathbb{R}_+^*$-set; for $x \in A$, τ_x^+ coincides with the $\sup\{\lambda \in \mathbb{R}_+^* : \lambda x \approx x\}$, and τ_x^- coincides with the $\inf\{\lambda \in \mathbb{R}_+^* : \lambda x \approx x\}$. Numerically, we have $\delta_a^- = \frac{9}{16}$ and $\delta_a^+ = \frac{16}{9}$. Also, $\tau_a^- = \frac{3}{4}$ and $\tau_a^+ = \frac{4}{3}$.$\triangle$

Remark that a series of progressively tighter and tighter indifference relations can be constructed until we eventually reach the equivalence relation among objects with identical utility. For a homothetic interval order $\succ$ represented by a pair (u, γ) (Theorem 2) and for $k \in \mathbb{N}^*$, we define the homothetic interval order $\succ^k$ by the pair $(u, \gamma^{\frac{1}{k}})$ and note $\sim^k$ its associated indifference relation. We have $\succ^1 = \succ$ and $\succ^k \subset \succ^{k+1}$ (i.e. $\succ^{k+1}$ is finer than $\succ^k$) : $x \succ^{k+1} y \Rightarrow x \succ^k y$. If $x \sim^k y$, we say that x and y are *k-indifferent*.

Proposition 1: *Let A be a non-empty $\mathbb{N}^* -$ set endowed with a strongly non-empty homothetic interval order $\succ$. Then for $x, y \in A$ such that $x \sim y$, either there exists $k \in \mathbb{N}^*$ such that $x \sim^k y$ and $x \not\sim^{k+1} y$, either x and y have the same utility (i.e. $y \in U_x$).*

Proof. If $\succ$ is a weak order, then there is nothing to prove: $x \sim y \Leftrightarrow y \in U_x$, and $\succ^k = \succ$ for all $k \in \mathbb{N}^*$. So we can suppose that $\succ$ is not a weak order. For $x \in A$ and $k \in \mathbb{N}^*$, we note I_x^k the k-indifference set containing x (Cf. section 4). Since $\succ^k \subset \succ^{k+1}$, we have $I_x^{k+1} \subset I_x^k$. Moreover, we have $U_x \subset I_x^k$ for all $k \in \mathbb{N}^*$, and since $\gamma^{\frac{1}{k}}$ tends to the constant function $x \mapsto 1$ when k tends to $+\infty$, we have $\cap_k I_x^k = U_x$. So if $x \sim y$, either $y \in U_x$, either there exists $k \in \mathbb{N}^*$ such that $y \in I_x^k \setminus I_x^{k+1}$. ■

6 Homothetic Semiorders

Introduced by Luce [13], semiorders are a special case of interval orders known as leading to a representation with a constant additive threshold. For a semiorder $\succ$ on a finite set A, the seminal result due to Scott and Suppes [15] proves the existence of a function u such that $x \succ y \Leftrightarrow u(x) > u(y) + 1$. No uniqueness property is specified and the "1" value for the additive threshold should not be interpreted as a genuine measurement of the threshold since this value could be replaced by any other arbitrary positive value. Also, the interpretation that semiorders have a constant additive threshold may be misleading as it is indeed not a necessary formulation of their representation.

The purpose of this section is to show that homothetic semiorders can be represented by a utility function unique up to multiplication by a positive number and a constant biasing factor. Hence, this representation of homothetic semiorders with a strong uniqueness property leads to a constant *multiplicative* threshold.

Theorem 4: *Let A be a non-empty $\mathbb{N}^* -$ set endowed with a strongly non-empty binary relation $\succ$. The three following conditions are equivalent ($x, y \in A; m \in \mathbb{N}^*$) :*

(i) There exists a function $u : A \to \mathbb{R}_+^*$ and a number $\alpha \in]0, 1]$ such that $\forall (x, y, m)$ we have

$$\begin{cases} x \succ y \Leftrightarrow \alpha u(x) > u(y) \\ u(mx) = mu(x) \end{cases}.$$

(ii) The relation $\succ$ is a homothetic interval order such that $\succ_1 = \succ_2$.

(iii) The relation $\succ$ is a homothetic semiorder.

Moreover, if $\succ$ is a homothetic semiorder, then we have $\succ_0 = \succ_1 = \succ_2$ and the pair (u, α) of (i) is unique up to multiplication of u by a positive number.

Proof. The implication $(i) \Rightarrow (iii)$ is easy to verify, while the implication $(ii) \Rightarrow (i)$ is implied by Theorem 2, Corollary 1 and Theorem 3.

Suppose $\succ$ is a homothetic interval order such that $\succ_1 = \succ_2$. Choose an element $a \in A$. Then, from Corollary 1, there exists a constant $\beta > 0$ such that $r_{a,x} = \beta s_{a,x}$. By Theorem 3, this implies that $\succ_0 = \succ_1 = \succ_2$ and the function γ of Theorem 2 condition (i) is constant on A. The implication $(iii) \Rightarrow (i)$ is left to the reader (if γ is not constant on A, it is easy to produce four elements $x, y, z, t \in A$ such that $x \succ y \succ z$ and $z \succsim t \succsim x$). ■

Let $\succ$ be a strongly non-empty homothetic semiorder. We choose a pair (u, α) verifying condition (i) of Theorem 4. Then we have

$$y \in I_x \Leftrightarrow \alpha u(x) \leqslant u(y) \leqslant \alpha^{-1} u(x).$$

Hence, these indifference thresholds do not depend on x. For all x, we have $\delta_x^+ = \delta^+ = \alpha^{-1}$ and $\delta_x^- = \delta^- = \alpha$. This is illustrated in the following example.

Example 4: Consider the function $x = (x_1 X_1, x_2 X_2) \mapsto u(x) = x_1^{\frac{2}{5}} x_2^{\frac{3}{5}}$ and define $\succ$ by $x \succ y \Leftrightarrow \alpha u(x) > u(y)$ for all $x, y \in A$. Letting $\alpha = 0.9$, Figure 2 shows the isoutility curves $U_{10} = \{x : u(x) = 10\}$ with $a, b \in U_{10}$ and $U_{20} = \{x : u(x) = 20\}$ with $a', b' \in U_{20}$ in bold lines. We depict the corresponding indifference sets I_{10} and I_{20} in plain lines. We have $\delta^+ = \frac{10}{9}$ and $\delta^- = \frac{9}{10}$.△

Remark that we have $x \succ y \Leftrightarrow u(x) > u(y) + \frac{1-\alpha}{\alpha} u(x)$. Hence, the *additive* indifference threshold $\varepsilon_{u(x)} = \frac{1-\alpha}{\alpha} u(x)$ depends on $u(x)$. However, it maintains a constant ratio $\sigma = \frac{1-\alpha}{\alpha}$ with respect to the utility level $u(x)$.

Remark that we can reformulate the biased representation with a constant multiplicative threshold as a representation with a constant additive threshold by taking any logarithmic transformation of the representing function. If (u, α) represents a homothetic semiorder $\succ$, then we have $x \succ y \Leftrightarrow v(x) > v(y) + \epsilon$ where (for instance) $v(x) = \log(u(x))$ and $\epsilon = -\log \alpha$. However, if $u' = \lambda u$ with $\lambda > 0$ is another function verifying condition (i) of Theorem 4, letting $v'(x) = \log u'(x)$, we do have $v(x) - v(y) = v'(x) - v'(y)$ but we have $\frac{v(x)}{v(y)} = \frac{v'(x)}{v'(y)}$ if and only if $\lambda = 1$. Hence, the transformed function v preserves the differences of utility levels but does not preserve the ratios of utility. It is cardinal.

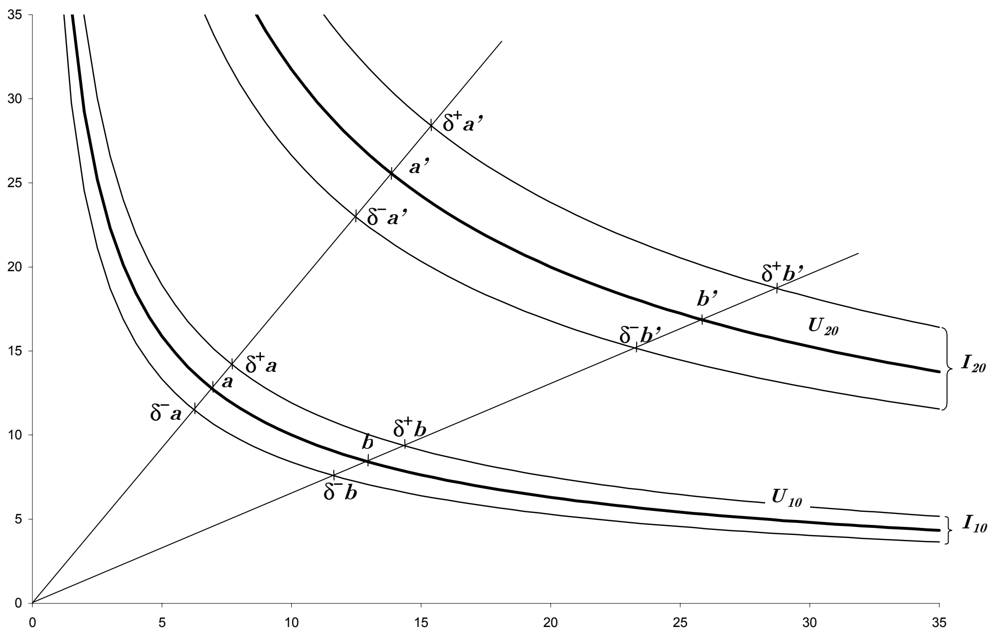

Figure 4: Two Indifference Sets of a Homothetic Semiorder

7 Linear Homothetic Semiorders

Homothetic structures introduced in section 2 model the replication of objects in order to construct quantities. It is natural to ask what happens when we combine objects among themselves. In the examples above, this amounts to model the good formed by a quantity $x_1 + x_1'$ of X_1 and a quantity $x_2 + x_2'$ of X_2 as the natural combination of the goods (x_1X_1, x_2X_2) and $(x_1'X_1, x_2'X_2)$. More generally, an object x and an object y combine to form an object z. A natural question is then to identify properties ensuring that the utility of z be the sum of the utility of x and y. A utility function verifying such a property is called linear. The purpose of this section is to identify conditions under which a homothetic interval order $\succ$ can be represented with a linear utility function.

We introduce the structure of a *commutative semigroup*, that is a non-empty set A endowed with a map $A \times A \to A, (x, y) \mapsto x \circ y$ such that for all $x, y, z \in A$, we have $x \circ (y \circ z) = (x \circ y) \circ z$ (associativity) and $x \circ y = y \circ x$ (commutativity). Note also that a commutative semigroup A is a $\mathbb{N}^*$-set for the operation of $\mathbb{N}^*$ defined by $\mathbb{N}^* \times A \to A, (m, x) \mapsto mx = x \circ ... \circ x$ (m times). A real-valued function on a commutative semigroup A is then called *linear* if and only if, for all $x, y \in A$, $u(x \circ y) = u(x) + u(y)$. Now, consider the following properties for a binary relation $\succ$ on A :

Independence: $\forall (x, y, z \in A)$ we have $x \succ y \Leftrightarrow x \circ z \succ y \circ z$;

Pseudo-independence: $\forall(x, y, z, t \in A)$ we have

$$\begin{cases} (x \succ y, z \succ t) \Rightarrow x \circ z \succ y \circ t \\ (x \succsim y, z \succsim t) \Rightarrow x \circ z \succsim y \circ t \end{cases}.$$

When the relation $\succ$ is not a weak order, pseudo-independence is weaker than independence. For instance, the relation $\succ$ of example 3 is pseudo-independent. However, it is independent if and only if $\lambda = \mu$. In this special case, the utility function u is linear. Note also that the relations $\succ$ of examples 1, 2 and 4 are not independent nor pseudo-independent. We prove the following theorem.

Theorem 5: *Let A be a commutative semigroup endowed with a strongly non-empty homothetic interval order $\succ$. The three following conditions are equivalent*:
(i) There exists a function $u : A \to \mathbb{R}_+^*$ and a number $\alpha \in]0, 1]$ such that $\forall(x, y \in A)$ we have

$$\begin{cases} x \succ y \Leftrightarrow \alpha u(x) > u(y) \\ u(x \circ y) = u(x) + u(y) \end{cases}.$$

(ii) The relation $\succ_0$ is independent.
(iii) The relation $\succ$ is a semiorder pseudo-independent.

Proof. The implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are easy to verify.
From Theorem 1, a strongly non-empty homothetic weak order $\succ'$ is independent if and only if for all (i.e. for one) functions $u' : A \to \mathbb{R}_+^*$ such that $u'(mx) = mu'(x)$ and $x \succ' y \Leftrightarrow u'(x) > u'(y)$, we have $u'(x \circ y) = u'(x) + u'(y)$.
Then, by using the explicit functions u_1, u_2, u of the proof of Theorem 3, we prove that if $\succ_0$ is independent, then $\succ$ is a semiorder. Since for one (i.e. for all) pair (u, α) as in the condition (i) of Theorem 4, we have $x \succ_0 y \Leftrightarrow u(x) > u(y)$, the implication $(ii) \Rightarrow (i)$ is verified.
Finally, $\succ$ is pseudo-independent if and only if $\succ_1$ and $\succ_2$ are independent. Joint with Theorem 4, this proves the implications $(iii) \Rightarrow (ii)$. ∎

We illustrate this result with a two-dimensional commodity space $A = \mathbb{R}_+^* X_1 \times \mathbb{R}_+^* X_2$ endowed with the operation $\circ$ defined by $(x_1X_1, x_2X_2) \circ (x_1'X_1, x_2'X_2) = ((x_1 + x_1')X_1, (x_2 + x_2')X_2)$ with $x_1, x_2 \in \mathbb{R}_+^*$.

Example 5: Consider the function $x = (x_1X_1, x_2X_2) \mapsto u(x) = \lambda x_1 + \mu x_2$ and define $\succ$ by $x \succ y \Leftrightarrow \alpha u(x) > u(y)$ for all $x, y \in A$. Letting $\alpha = 0.8$, $\lambda = 0.7$ and $\mu = 0.9$, Figure 2 shows the isoutility straight lines $U_{20} = \{x : u(x) = 20\}$ in bold, with the corresponding indifference set I_{20} delimited by plain straight lines.

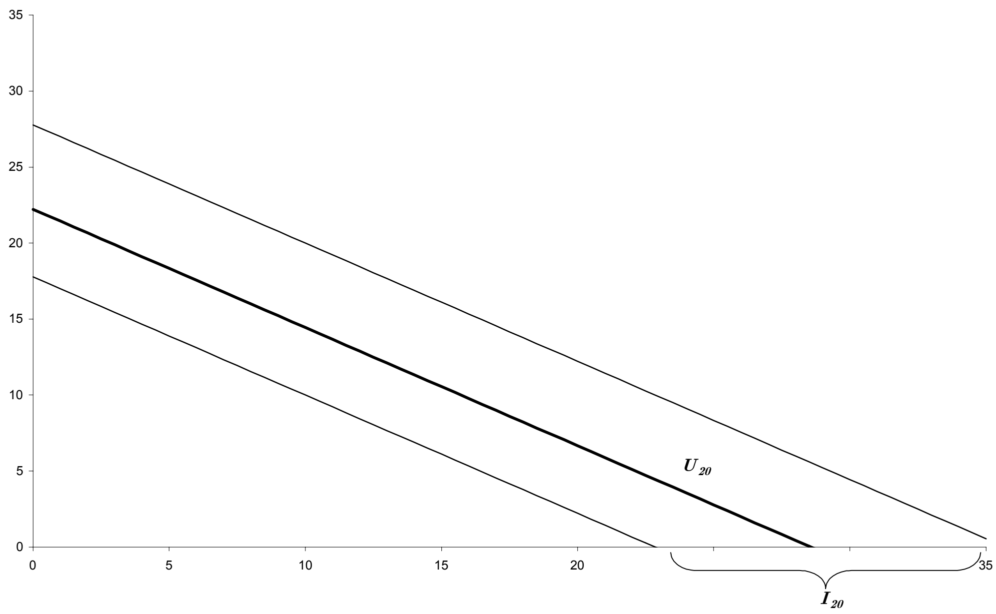

Figure 5: One Indifference Set of a Linear Homothetic Semiorder

8 Conclusion

Using an algebraic approach to represent interval ordered homothetic preferences, we have separated the assignment of utility to the objects of preferences from other considerations that may, or may not, influence these preferences. These considerations are modelled as a biasing function that combines mutiplicatively with the utility function. The utility function is fully quantitative and the biasing function is unique. Departures from the maximization of the utility function, as modelled by intransitive indifference, are then precisely measured. Overall, a broader class of phenomena is modelled while improving the numerical properties of the model. Several developments of this approach could be contemplated.

We could model heterogenous preferences with identical homothetic utility functions while reflecting the diversity of individuals by a distribution of biasing factors. This should lead to interesting developments for the treatment of aggregation problems in the theory of demand. The question of ordering these factors is also of theoretical interest and may provide foundations for the comparisons of preferences among individuals, or across time in order to model evolution of preferences. Also, we could identify the conditions under which biasing factors are not necessarily smaller than 1, hence covering an even wider class of preferences and of biases.

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