UNIVERSITÀ DELLA CALABRIA

Dipartimento di Economia e Statistica
Ponte Pietro Bucci, Cubo 0/C
87036 Arcavacata di Rende (Cosenza)
Italy
http://www.ecostat.unical.it/

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AN EXHAUSTIVE COEFFICIENT OF RANK CORRELATION

Agostino Tarsitano
Dipartimento di Economia e Statistica
Università della Calabria
Ponte Pietro Bucci, Cubo 1/C
Tel.: +39 0984 492465
Fax: +39 0984 492421
e-mail: agotar@unical.it

Rosetta Lombardo
Dipartimento di Economia e Statistica
Università della Calabria
Ponte Pietro Bucci, Cubo 0/C
Tel.: +39 0984 492466
Fax: +39 0984 492421
e-mail: r.lombardo@unical.it

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An exhaustive coefficient of rank correlation

Agostino Tarsitano[a,*], Rosetta Lombardo[b]

[a]*Dipartimento di Economia e Statistica - Università della Calabria*
Via Pietro Bucci, cubo 1C, 87036 Rende (CS) - Italy
[b]*Dipartimento di Economia e Statistica - Università della Calabria*
Via Pietro Bucci, cubo 0C, 87036 Rende (CS) - Italy

Abstract

Rank association is a fundamental tool for expressing dependence in cases in which data are arranged in order. Measures of rank correlation have been accumulated in several contexts for more than a century and we were able to cite more than thirty of these coefficients, from simple ones to relatively complicated definitions invoking one or more systems of weights. However, only a few of these can actually be considered to be admissible substitutes for Pearson's correlation. The main drawback with the vast majority of coefficients is their "resistance-to-change" which appears to be of limited value for the purposes of rank comparisons that are intrinsically robust. In this article, a new nonparametric correlation coefficient is defined that is based on the principle of maximization of a ratio of two ranks. In comparing it with existing rank correlations, it was found to have extremely high sensitivity to permutation patterns. We have illustrated the potential improvement that our index can provide in economic contexts by comparing published results with those obtained through the use of this new index. The success that we have had suggests that our index may have important applications wherever the discriminatory power of the rank correlation coefficient should be particularly strong.
JEL classification codes: C14, C18, A12

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*Corresponding author
Email addresses: agotar@unical.it (Agostino Tarsitano),
r.lombardo@unical.it (Rosetta Lombardo)

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1. Introduction

Measuring agreement between two sets of rankings is a frequently encountered issue in economic researches because, among other things, methods based on ranks of data are a common approach when the values themselves are of less interest than their relative ordering. For example, this would be the case when the features represent counts, ratings, rankings, or otherwise dimensionless quantities. Ranking methods are often recommended when variables used for analysis are scaled over a different range or the absolute distance between their values is nonlinear, unknown or cannot be measured for practical or theoretical reasons. A rank transformation may also be employed in order to avoid distortion because the actual data are contaminated with errors, inconsistencies or outliers. On different occasions (*e.g.* Croux & Dehon, (2010)), there is a possibility for any reported income to deviate from the true one, which may yield a large error, meanwhile, the ranks may still be reliable even after some volatility. Rank association is a less restrictive measure of gauging relationships between variables because it does not impose any assumptions of linearity. Nevertheless, it should be born in mind that ranks are not as powerful as original values when linear associations between variables are dominant.

The present article begins with an excursus on the concept of rank correlation: from the axioms that underlie the association between ordinal data, to the relationships between nonparametric measures of concordance and distances between permutations. After isolating a few "admissible" rank correlations which could be considered plausible alternatives to Pearson's correlation, we conclude that they all share the same drawback: a "resistance-to-change" that appears to be of limited value for the purposes of rank comparisons. We are convinced that ranks are intrinsically robust and do not need any additional robustification; on the contrary, it is necessary to have a rank correlation coefficient which can assume a wide range of values over the interval $[-1, 1]$.

We admit that, for ordinal data, it is perhaps paradoxical to postulate that there is a different value of the coefficient for each different pair of patterns, but it is at least equally paradoxical to suppose that permutation comparisons should be restricted to just a handful of different basic types. In this article, a new nonparametric correlation coefficient is defined which is based on the principle of maximization of a ratio of two ranks. In comparing it with existing indices it was found to have great discriminatory power in the identification of groups of permutations. The sensitivity of our coefficient is investigated with a complete enumeration of all permutations for a number of ranks less than or equal to 13; for a larger number of ranks, the distribution is determined by evaluating the statistic for a wide random sample of permutations. In order to clarify the significance and the usefulness of the new measure of correlation between ranks, we have re-analyzed the data from a number of earlier reports which are to be found in the literature, and concluded that these results are generally improved by using the new index. The last section summarizes the main results of the paper and suggests several extensions.

2. Rank correlations

Throughout this paper we will examine situations of the following type. Consider a fixed set of n distinct items ordered by $\mathbf{J}$ judges according to the different degree to which they possess $\mathbf{K}$ attributes $X_1, X_2, \ldots, X_{\mathbf{K}}$ consisting of a host of intangibles that can be ranked but not necessarily measured. Judges assign numerical, admittedly arbitrary values (at least within a certain range), to items which are essentially qualitative, but do not assign any other numerical values, such as scores or points. This scenario can also be invoked when one considers ranks as manifestations of an underlying absolutely continuous random variable whose observed values are transformed into a ranking by discretizing the variable according to a set of thresholds. Essentially, we assume that the two sets of ranked data have something directly in common. This is less obvious than it may appear since both rankings could be based on a set of measurements of overlapping but not directly correlated items.

Let us suppose that the evaluations for each attribute are expressed in terms of an ordinal scale of n ranks: $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ where $\boldsymbol{\sigma}$ is an element of S_n, the set of all $n!$ permutations of integers $\{1, 2, \ldots, n\}$ without omissions or repetitions (each of the integers appears exactly once in each ranking). Hereafter we restrict our discussion to the simplest case $\mathbf{J} = 1$ and $\mathbf{K} = 2$. The judge ranks the items in X_1 in the order $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ while the items in X_2 will be ordered as $\boldsymbol{\pi} = \{\pi_1, \pi_2, \ldots, \pi_n\}$.

A rank correlation $r(\boldsymbol{\sigma}, \boldsymbol{\pi})$ is a coefficient summarizing the degree of agreement/ disagreement between the two rankings $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$. Obviously, there are as many possible measures of association as there are different aspects of the structure of dependence between variables and not all of these can be picked up by a single coefficient. Therefore, rank correlations are only partially comparable (see Zayed & Quade, (1997)). This is why no rank correlation can be selected as the "best" choice.

Gideon & Hollister, (1987) pointed out that reasonable coefficients need to possess certain properties and gave a list of postulates for nonparametric measures of dependence based on Rényi, (1959) and Schweizer & Wolff, (1981). On this specific issue, see also Scarsini, (1984), King & Chinchilli, (2001) and Genest & Plante, (2003). In short, the properties that any index of rank correlation should satisfy are the following

a) $r(\boldsymbol{\sigma}, \boldsymbol{\pi})$ is defined for any pair of permutations $(\boldsymbol{\sigma}, \boldsymbol{\pi})$
b) Comparability. $-1 \leq r(\boldsymbol{\sigma}, \boldsymbol{\pi}) \leq 1$ with $r(\boldsymbol{\sigma}, \boldsymbol{\sigma}) = 1$ and $r(\boldsymbol{\sigma}, \boldsymbol{\sigma}^*) = -1$ where $\boldsymbol{\sigma}^* = n + 1 - \boldsymbol{\sigma}$.
c) Symmetry. $r(\boldsymbol{\sigma}, \boldsymbol{\pi}) = r(\boldsymbol{\pi}, \boldsymbol{\sigma})$
d) Zero expected value under independence. $E_{\boldsymbol{\sigma}, \boldsymbol{\pi} \in S_n}[r(\boldsymbol{\sigma}, \boldsymbol{\pi})] = 0$
e) Right-invariance. $r(\boldsymbol{\sigma} \cdot \boldsymbol{\theta}, \boldsymbol{\pi} \cdot \boldsymbol{\theta}) = r(\boldsymbol{\sigma}, \boldsymbol{\pi})$ for all $\boldsymbol{\sigma}, \boldsymbol{\pi}, \boldsymbol{\theta} \in S_n$
f) Antisymmetry under reversal. $r(\boldsymbol{\sigma}, \boldsymbol{\pi}^*) = r(\boldsymbol{\sigma}^*, \boldsymbol{\pi}) = -r(\boldsymbol{\sigma}, \boldsymbol{\pi})$ where $\boldsymbol{\pi}^* = n + 1 - \boldsymbol{\pi}$ and $\boldsymbol{\sigma}^* = n + 1 - \boldsymbol{\sigma}$.

Comparability

Rank correlation coefficients are usually constructed to vary between -1 and 1. Their directionality indicates a positive or negative link, while their absolute value indicates the strength of the association. In particular, the value of +1 should be obtained if the two rankings coincide and the value of -1 should be obtained if one ranking is the absolute reverse of its respective other. The magnitude of $r(\boldsymbol{\sigma}, \boldsymbol{\pi})$ generally increases as the association increases; generally, but not always because the maximum value of $+1$ is found in the case of perfect agreement, whereas the minimum value may be lower than the threshold -1. Furthermore, the minimum value does not necessarily imply complete reverse order. The value of zero is indicative of no association, but does not necessarily imply independence.

Many indices that have appeared in literature do not achieve their extremes in correspondence with perfect agreement or perfect disagreement between the two rankings. One example is the coefficient derived from the Hamming distance (Hamming, (1950)), *i.e*, the number of unmatched ranks. Other examples can be drawn from the Cayley distance based on the minimum number of transpositions required to transform $\boldsymbol{\sigma}$ into $\boldsymbol{\pi}$ (Diaconis & Graham, (1977)[p. 117]); from the Lee distance (Cameron & Wu, (2010)) and from the Chebyshev distance (Stoimenova, (1996)).

Symmetry

This is an essential condition of any rational discussion regarding dependence and association. Nonetheless, measures that do not verify the contraint $r(\boldsymbol{\sigma}, \boldsymbol{\pi}) = r(\boldsymbol{\pi}, \boldsymbol{\sigma})$ are found in literature. For example, the index proposed by Mango, (1997) which places emphasis on the relative importance of low ranks and a very similar index proposed by Blest, (2000), which favors high ranks, are not symmetric.

Zero expected value under independence

In the theory of ranking, a situation in which all possible permutations of the first n positive integers are equally likely, is often the basis for testing the hypothesis of independence of two rankings. In setting up a test of significance in such circumstances, one should consider statistics that have an expected value of zero given independent rankings. If unaccounted for, latent correlation would make statistical tests less stringent; *i.e*, one could incorrectly reject the null hypothesis $H_0 : r(\boldsymbol{\sigma}, \boldsymbol{\pi}) = 0$ with a probability greater than the nominal significance level.

It should be recognized that several rank correlations do not satisfy this requirement. One case is Spearman's footrule (Spearman, (1906)). Other measures that fail this criterion have been derived from a metric for permutations: Okazaki *et al.*, (2004) which was constructed to assess the degree of sentence continuity in reading; the coefficient used by Gordon, (1979) which is based on the length of the longest monotone sub-sequence and the coefficient developed by Bhat & Nayar, (1997) which was devised for establishing visual correspondence in images. The index presented by Pinto da Costa & Soares, (2005) (see also, Genest & Plante, (2003)) is a little skewed under the null hypothesis of independence, although the authors argue that there is no compelling reason for a coefficient of correlation to be necessarily symmetric.

Right-invariance

Let $(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})$ be the composition $(\sigma \cdot \pi)_i = \boldsymbol{\pi}[\sigma_i], i = 1, 2, \ldots, n$ so that π_1 is mapped to σ_1, π_2 is mapped to σ_2 and so on. The identity permutation $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ is the permutation that leaves the permutation $\boldsymbol{\sigma}(e_i) = \sigma_i, i = 1, 2, \ldots, n$ unchanged. Every permutation $\boldsymbol{\sigma} \in S_n$ has a symmetrical element $\boldsymbol{\sigma}^{-1} \in S_n$ such that $(\boldsymbol{\sigma}^{-1} \cdot \boldsymbol{\sigma}) = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{-1} = \mathbf{e}$. In other words, $\boldsymbol{\sigma}^{-1}$ is the permutation obtained by sorting the elements of $\mathbf{e}$ with respect to the elements of $\boldsymbol{\sigma}$. Essentially, if $\boldsymbol{\sigma}$ mixes up e, then $\boldsymbol{\sigma}^{-1}$ will unmix it. A measure of rank correlation is right-invariant, if

$$r\left(\boldsymbol{\sigma}\cdot\boldsymbol{\theta}, \boldsymbol{\pi}\cdot\boldsymbol{\theta}\right) = r\left(\boldsymbol{\sigma}, \boldsymbol{\pi}\right) \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\pi}, \boldsymbol{\theta} \in S_n. \tag{1}$$

This is in line with the intuitive principle that the rank correlation between $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ does not change when the sequence of the n pairs of ranks $\left(\sigma_i, \pi_i\right), i = 1, 2, \ldots, n$ is modified. With no essential loss of generality one ranking can be held constant, and the position of the same item in a second ranking relative to the constant rank can be determined. For instance, we may assume that $\boldsymbol{\sigma}$ contains the ranks of X_1 after that the items of X_2 have been arranged in natural order *i.e.* $\boldsymbol{\pi} = \mathrm{e}$. In practice, $\boldsymbol{\pi}$ acts as a reference permutation. Alternatively, we may assume that $\boldsymbol{\pi}$ contains the ranks of X_2 after $\boldsymbol{\sigma}$ has been transformed into the identity permutation e (now $\boldsymbol{\sigma}$ acts as the reference permutation). Right-invariance implies that the rank correlation between $\boldsymbol{\sigma}$ and $\boldsymbol{\pi}$ remains the same in both cases:

$$r\left(\boldsymbol{\sigma}, \boldsymbol{\pi}\right) = r\left(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}^{-1}, \boldsymbol{\pi}\cdot\boldsymbol{\sigma}^{-1}\right) = r\left(\mathrm{e}, \boldsymbol{\pi}^{-1}\cdot\boldsymbol{\sigma}\right) = r\left(\boldsymbol{\pi}\cdot\boldsymbol{\sigma}^{-1}, \mathrm{e}\right). \tag{2}$$

It should be pointed out that various coefficients still in current use fail to satisfy (2). For instance, the index discussed in Knuth, (1973)[p. 12] which is based on the square of the major index of a permutation and coincides with the gamma rank correlation measure advocated by Goodman & Kruskal, (1954); the monotone cover correlation introduced by Dallal & Hartigan, (1980) as an index of association insensitive to outliers. The index derived from the median of the slopes between all combinations of two ranks (see Theil, (1950)) suffers from the same drawback. The index described by Salvemini, (1951) should be added to this list together with Fechner's index (see Salvemini, (1951)) which is equivalent to the test of randomness devised by Moore & Wallis, (1943) and to the rank correlation statistics based on rises discussed by Salama & Quade, (1997).

Antisymmetry under reversal

Rankings can be organised as a classification with 1 assigned to the most preferred item, 2 to the next-to-most preferred and so forth. If an opposite orientation of one of the two arrangements is applied, then a rank correlation coefficient that changes its sign, but not its absolute value, is said to be antisymmetric under reversal.

$$r\left(\boldsymbol{\sigma}, \boldsymbol{\pi}^*\right) = r\left(\boldsymbol{\sigma}^*, \boldsymbol{\pi}\right) = -r\left(\boldsymbol{\sigma}, \boldsymbol{\pi}\right). \tag{3}$$

The values of $r(\boldsymbol{\pi}, \boldsymbol{\sigma})$ are symmetrical about the value of zero because, to each value of $r(\boldsymbol{\pi}, \boldsymbol{\sigma})$, there corresponds another value of equal magnitude but opposite in sign. Consequently, rank correlation coefficients which are antisymmetric under reversal have a mean (and median) value of zero over the set of all permutations S_n. In addition, the odd moments are zero.

Property (3) is incompatible with the left-invariance *i.e.* the requirement, often made in the literature (see, for example, Critchlow, (1992) and Deza & Huang, (1998)), that distances between permutations do not depend on how the items are labelled. More explicitly, a distance function $\delta(.)$ on S_n is defined as left-invariant if

$$\delta(\boldsymbol{\theta}\cdot\boldsymbol{\sigma}, \boldsymbol{\theta}\cdot\boldsymbol{\pi}) = \delta(\boldsymbol{\sigma}, \boldsymbol{\pi}) \ \forall\, \boldsymbol{\sigma}, \boldsymbol{\pi}, \boldsymbol{\theta} \in S_n. \tag{4}$$

If such a constraint is acceptable for distances, it is not applicable to rank correlations for which (3) may be viewed as a pre-requisite for a meaningful interpretation. Two of the coefficients proposed in literature which do not completely fulfill the requirement of antisymmetry are the index discussed in Borroni & Cazzaro, (2006), based on Gini's mean difference being computed on the total ranks $\sigma_i + \pi_i, i = 1.2., \ldots, n$ and the index recommended in Mango, (2006), based on the average determinant of the second order minors with a constant sum of elements from a data matrix of two ordinal variables. An analogous problem arises with rank correlation coefficients for which the extreme value -1 can only be achieved for large values of n. This is the case, for example, with the index proposed by Salama & Quade, (1982) as a measure for comparing the rankings of the regressors in two populations. The weighted version of the Kendall rank correlation coefficient introduced by Shieh, (1998) and the average precision correlation advanced by Yilmaz *et al.*, (2008) are not antisymmetric under reversal either. The same is true for the coefficients proposed by Iman & Conover, (1987) and by Maturi & Abdelfattah, (2008). The weighted rank correlation given by Genest & Plante, (2003) is no better in this sense.

3. Admissible rank correlations

In Table (1) we have collected a selection of admissible rank correlations (in the sense that they have the desirable properties described in the previous section). The expressions are given in terms of the distance between the composition $\boldsymbol{\theta} = \boldsymbol{\pi}\cdot\boldsymbol{\sigma}^{-1}$ and the identity permutation $\mathbf{e}$ (exploiting the right-invariance properties). The indices r_1 and r_3 are well-known. The cograduation coefficient r_2 was proposed by Gini, (1914) as an improvement upon the Spearman's footrule.

Table 1: Admissible rank correlations.

Name	Formula	
Spearman	$r_1 = 1 - 6\dfrac{\sum_{i=1}^{n}\left(i-\theta_i\right)^2}{n^3-n}.$	
Gini	$r_2 = 2\dfrac{\sum_{i=1}^{n}\lvert i-\theta_i^*\rvert - \sum_{i=1}^{n}\lvert i-\theta_i\rvert}{\left(n^2-k_n\right)},$	$k_n = n \mod 2$
Kendall	$r_3 = 2\dfrac{\sum_{i<j} sgn\left(\theta_j-\theta_i\right)}{n\left(n-1\right)}$	

The main drawback of the coefficients included in Table (1) is their "resistance-to-change" which appears to be of limited value for the purposes of rank comparisons. For coherence, we have excluded from this review two other admissible and accepted indices: 1) the rank correlation given by Gideon & Hollister, (1987) which originates from the principle of greatest deviation between the observed $\boldsymbol{\theta}$ and the identity permutation $\mathbf{e}$; 2) the quadrant correlation (Mosteller, (1946); Blomqvist, (1950)) which is computed by dividing the plane into four quadrants and making use of concordance/discordance in the pairs belonging to the various quadrants. Both indices can take on an extremely small number of different values.

It is plain that a given value of a rank correlation coefficient does not, in general, define a specific pair of permutations, except perhaps for the extreme values of coefficients. Nevertheless, a rank correlation that concentrates the permutation comparisons into too few values would underestimate the number of differentiated elements that belong to a category or class. When the value of any categorizing method for $r\left(\boldsymbol{\sigma},\boldsymbol{\pi}\right)$ is assessed, the two main characteristics that need to be considered are robustness and sensitivity. The former determines the degree of rank order inconsistency that can be withstood by the coefficient before mismatches begin to occur. Robustness is a valuable characteristic if rank correlation does not change greatly when data are changed slightly. In so doing, robust correlation coefficients ensure stability of the estimates of association characteristics in the case of deviations from the Gaussian model, outliers and disturbances in the tails of the distributions. However, since robustness is achieved at the cost of a loss in precision, it can become a problem if the same value is applied to describe very different patterns. The sensitivity of $r\left(\boldsymbol{\sigma},\boldsymbol{\pi}\right)$ refers to its ability to differentiate between rankings. Such quality is laudable, but it probably reduces the accuracy of the classification where substantially similar permutations are mapped onto very distant values in the coefficient.

Robustness and sensitivity are antithetical requirements because more robust indices give greater stability against random changes in rankings, whereas more sensitive coefficients offer a richer source of information regarding the association structure. Therefore, in choosing a good index of association, some balancing of conflicting objectives is required. A reasonable solution may be obtained by considering that ranking is an intrinsically robust procedure and rank correlation measures have been well regarded as robust measures in many performance evaluation schemes. Since these methods rely on the relative ordering of elements, they are very tolerant of noise and interference that do not affect the actual order. Thus, in choosing a coefficient, particular consideration should be given to its discriminatory power rather than its robustness. From this point of view, the rank correlation coefficients of Table 1 are largely insufficient for use in screening for permutation comparisons when the range of possible relationships between the underlying variables is wide.

3.1. An exhaustive coefficient of rank correlation

In this section we propose a new measure of rank correlation which has high resolution over the set of all permutations. The formula of the new index is

$$r_4(\boldsymbol{\sigma}, \boldsymbol{\pi}) = \frac{a_{\theta,\mathbf{e}} d_{\theta,\mathbf{e}} - b_{\theta,\mathbf{e}} c_{\theta,\mathbf{e}}}{\max_{\boldsymbol{\theta} \in S_n} \{a_{\theta,\mathbf{e}} d_{\theta,\mathbf{e}} - b_{\theta,\mathbf{e}} c_{\theta,\mathbf{e}}\}} \tag{5}$$

where $\boldsymbol{\theta} = \boldsymbol{\pi} \cdot \boldsymbol{\sigma}^{-1}$, $\mathbf{e}$ is the identity permutation and $\mathbf{e}^*$ and $\boldsymbol{\theta}^*$ are the reverse permutations of $\mathbf{e}$ and $\boldsymbol{\theta}$ respectively. Furthermore

$$a_{\theta,\mathbf{e}} = \frac{1}{n} \sum_{i=1}^{n} \left\lceil \frac{\theta_i}{e_i^*} \right\rceil, b_{\theta,\mathbf{e}} = \frac{1}{n} \sum_{i=1}^{n} \left\lceil \frac{\theta_i^*}{e_i^*} \right\rceil, c_{\theta,\mathbf{e}} = \frac{1}{n} \sum_{i=1}^{n} \left\lceil \frac{\theta_i}{e_i} \right\rceil, d_{\theta,\mathbf{e}} = \frac{1}{n} \sum_{i=1}^{n} \left\lceil \frac{\theta_i^*}{e_i} \right\rceil. \tag{6}$$

The symbol appearing in (6) denotes the majorization ratio of two numbers

$$\left\lceil \frac{x}{y} \right\rceil = \max\left\{ \left[\frac{x}{y} \right], \left[\frac{y}{x} \right] \right\}, \qquad \text{for } x, y > 0. \tag{7}$$

Of course

$$1)\ \left\lceil \frac{x}{y} \right\rceil = \left\lceil \frac{y}{x} \right\rceil; \quad 2)\ \left\lceil \frac{x}{y} \right\rceil \geq 1; \quad 3)\ \left\lceil \frac{x}{y} \right\rceil = \exp\{|\log(x) - \log(y)|\}. \tag{8}$$

Other properties of the majorization operator are discussed in Brizzi, (1992). To illustrate the behavior of the index, consider that $a_{\theta,\mathbf{e}}, b_{\theta,\mathbf{e}}, c_{\theta,\mathbf{e}}, d_{\theta,\mathbf{e}}$ are positive and each of them is an average of n majorization ratios. Together they express the four possible comparisons between the ranks and the reverse ranks of $\mathbf{e}$ and $\boldsymbol{\theta}$. Let $\mathbf{A}_{\boldsymbol{\theta},\mathbf{e}}$ be the matrix formed with the elements in (6).

$$\mathbf{A}_{\boldsymbol{\theta},\mathbf{e}} = \begin{pmatrix} a_{\boldsymbol{\theta},\mathbf{e}} & b_{\boldsymbol{\theta},\mathbf{e}} \\ c_{\boldsymbol{\theta},\mathbf{e}} & d_{\boldsymbol{\theta},\mathbf{e}} \end{pmatrix}. \tag{9}$$

The denominator of $r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})$ is fixed with respect to $\boldsymbol{\theta}$. The numerator is the determinant of the matrix $\mathbf{A}_{\boldsymbol{\theta},\mathbf{e}}$ and it can be interpreted as the orientated area of the parallelogram spanned by the vectors of $\mathbf{A}_{\boldsymbol{\theta},\mathbf{e}}$. The area of the parallelogram increases (in absolute value) as its acute angle increases towards $90\,^\circ$ and hence, the determinant increases with the vectors increasing orthogonality. In other words, the more the vectors point in different directions, the larger the area is. When the angle is $90\,^\circ$, the parallelogram becomes a rectangle, *i.e.* the parallelogram of maximum area. Here, $|r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})| = 1$. As the acute angle between the spanning vectors in (9) tends toward $0\,^\circ$, the parallelogram becomes a straight line and the area shrinks to zero. It follows that, the determinant $|\mathbf{A}_{\boldsymbol{\theta},\mathbf{e}}|$ and, consequently, $r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})$, is zero if and only if the two vectors of $\mathbf{A}_{\boldsymbol{\theta},\mathbf{e}}$ are linear dependent. This is the case when comparing, for example, the identity permutation $\mathbf{e}$ with a permutation $\boldsymbol{\theta}$ consisting of an alternate sequence of high and low ranks determining an equal-value sum in all the four elements: $a_{\theta,\mathbf{e}}, b_{\theta,\mathbf{e}}, c_{\theta,\mathbf{e}}, d_{\theta,\mathbf{e}}$.

The orientation of the area or (it is the same) the sign of the determinant, is positive if low (high) ranks of one permutation tend to be matched by low (high) ranks in the other because, in this case, $b_{\boldsymbol{\theta},\mathbf{e}}$ and $c_{\boldsymbol{\theta},\mathbf{e}}$ move toward 1 (their minimum value). Simultaneously, $a_{\boldsymbol{\theta},\mathbf{e}}$ and $d_{\boldsymbol{\theta},\mathbf{e}}$ tend to their maximum value which, as it is easily verified, in the case of untied ranks, is given by

$$\frac{1}{n}(k_n + 2h_n) \qquad \text{with} \quad h_n = \sum_{i=1}^{\lfloor n/2 \rfloor} \frac{e_i^*}{e_i} \tag{10}$$

where k_n is zero if n is even, or one if n is odd and $\lfloor n/2 \rfloor$ is the integer part of $n/2$. Conversely, the orientation of the area (and the determinant) are negative if low (high) ranks of one permutation tend to be matched by high (low) ranks in the other, because now the terms $a_{\boldsymbol{\theta},\mathbf{e}}$ and $d_{\boldsymbol{\theta},\mathbf{e}}$ converge towards the minimum value (which is always 1) whereas the other two terms, converge to the maximum (10). The index $r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})$ meets all the conditions given in section 3. Firstly, it is clear that $r_4(\mathbf{e},\mathbf{e}) = 1$, $r_4(\mathbf{e}^*,\mathbf{e}) = -1$. Secondly, thanks to the properties of (7), we have: $r_4(\boldsymbol{\theta},\mathbf{e}) = r_4(\mathbf{e},\boldsymbol{\theta})$ and $r_4(\boldsymbol{\theta},\mathbf{e}) = -r_4(\boldsymbol{\theta}^*,\mathbf{e})$. Thirdly, when $\boldsymbol{\theta} = \mathbf{e}$, $a_{\theta,\mathbf{e}}$ and $d_{\theta,\mathbf{e}}$ become equal, achieve their maximum value and are greater than $b_{\theta,\mathbf{e}}$ and $c_{\theta,\mathbf{e}}$. At the same time, $b_{\theta,\mathbf{e}}$ and $c_{\theta,\mathbf{e}}$ become equal and reach their minimum value. The opposite is true when $\boldsymbol{\theta} = \mathbf{e}^*$. Consequently $-1 \leq r_4(\boldsymbol{\theta},\mathbf{e}) \leq 1$. Lastly, under the hypothesis that each pairing of ranks contained in $\boldsymbol{\sigma}$ with any ranks of $\boldsymbol{\pi}$ is equally likely, then the property of antisymmetry under reversal ensures that $E_{\boldsymbol{\sigma},\boldsymbol{\pi}\in S_n}[r_4(\boldsymbol{\theta},\mathbf{e})] = 0$. In general, the proofs are rather long and tedious and, hence, the details are not given here.

Table 2 reports the Pearson's correlation between the four rank correlation coefficients considered in the present section. For $n \leq 13$ the values are obtained by systematically enumerating all of the possible permutations. For $n > 13$ we generated a random sample of five million of all the possible permutations by using the "shuffle" algorithm proposed in Durstenfeld [1964].

Table 2: Correlation between rank correlations coefficients.

n	Cor(r_1, r_4)	Cor(r_2, r_4)	Cor(r_3, r_4)
8	0.9875	0.9558	0.9677
9	0.9840	0.9485	0.9685
10	0.9804	0.9413	0.9630
11	0.9767	0.9347	0.9603
12	0.9729	0.9282	0.9575
13	0.9692	0.9221	0.9546
14	0.9655	0.9163	0.9517
15	0.9618	0.9108	0.9487
20	0.9441	0.8859	0.9337
25	0.9279	0.8648	0.9193
30	0.9131	0.8466	0.9058
50	0.8638	0.7903	0.8594
100	0.7822	0.7049	0.7801

The indices move in the same direction and are, therefore, concordant. The greatest similarity occurs between r_4 and Spearman's coefficient r_1 which is generally considered to be one of the most sensitive indices of rank correlation. However, the degree of linear dependence between r_4 and r_1, r_2, r_3, decreases as n

increases. We interprete this as an indication that r_4 describes a distinct although not necessarily distant aspect of the association between two permutations.

Table 3 gives a selected set of critical values of r_4 for $n = 6, \ldots, 13$. For $n > 13$, we used a random sample of five million permutations to estimate the "true" distribution.

Table 3: Critical values for r_4.

	Level of significance for two-tailed test								
n	0.0005	0.0010	0.0025	0.0050	0.0100	0.0250	0.0500	0.1000	0.2500
6	0.9999	0.9999	0.9999	0.9542	0.9051	0.8358	0.8049	0.7013	0.5340
7	0.9768	0.9768	0.9416	0.9243	0.8900	0.8190	0.7545	0.6654	0.5019
8	0.9451	0.9296	0.9021	0.8673	0.8271	0.7638	0.7033	0.6154	0.4593
9	0.9281	0.9088	0.8729	0.8410	0.8001	0.7347	0.6684	0.5849	0.4333
10	0.8992	0.8763	0.8397	0.8053	0.7650	0.6981	0.6328	0.5517	0.4063
11	0.8776	0.8549	0.8166	0.7821	0.7408	0.6733	0.6088	0.5288	0.3873
12	0.8540	0.8294	0.7905	0.7555	0.7141	0.6466	0.5832	0.5052	0.3686
13	0.8364	0.8110	0.7715	0.7359	0.6942	0.6269	0.5642	0.4875	0.3545
14	0.8174	0.7909	0.7505	0.7145	0.6729	0.6066	0.5451	0.4697	0.3406
15	0.7555	0.7295	0.6914	0.6579	0.6186	0.5560	0.4985	0.4291	0.3104
20	0.7293	0.7025	0.6618	0.6265	0.5861	0.5230	0.4667	0.3992	0.2864
25	0.6788	0.6509	0.6109	0.5766	0.5376	0.4773	0.4243	0.3618	0.2583
30	0.6373	0.6118	0.5723	0.5388	0.5011	0.4443	0.3937	0.3350	0.2383
50	0.5336	0.5103	0.4762	0.4467	0.4136	0.3645	0.3217	0.2727	0.1930
100	0.4267	0.4064	0.3764	0.3521	0.3253	0.2855	0.2511	0.2121	0.1495

The entries in the table are the smallest values of r_4 (to four decimal places) which correspond to two-tail probabilities. The observed value is meaningful if it is equal to, or greater than, the value in Table 3. For example, if $n = 12$ and a value of $r_4 = 0.6$ is observed, then there would be a probability of between 5% and 10% that it had occurred by chance. In other words, the observed value $r_4 = 0.6$ cannot be considered a highly significant rank correlation between $n = 12$ pairs of rankings.

3.2. Sensitivity analysis

Let us suppose that the values of $r_i(\boldsymbol{\sigma}, \boldsymbol{\pi})$ are rounded off after the m-th decimal place

$$\frac{\lfloor r_i(\boldsymbol{\sigma}, \boldsymbol{\pi})\, 10^m + 0.5 \rfloor}{10^m} \quad i = 1, 2, \ldots, 4. \tag{11}$$

The discriminatory power of $r_i(\boldsymbol{\sigma}, \boldsymbol{\pi})$ for rankings of a given size can be quantified by the values assumed by (11) as a fraction of the maximum number.

$$\psi = \frac{\nu}{\min\left\{{}_nP_n, 2(10^m)+1\right\}} \tag{12}$$

where ν is the number of distinct values that ψ takes on over S_n. Thus $\psi = 1$ would indicate that $r_i(\boldsymbol{\sigma}, \boldsymbol{\pi})$ has the minimum number of repeated values at the given level of approximation. Put differently, given a value between -1 and 1, there is at least one pair of permutations whose $r_i(\boldsymbol{\sigma}, \boldsymbol{\pi})$, rounded off after the m-th decimal place, assumes that value. Conversely, $\psi \approxeq 0$ would indicate that, from the point of view of $r_i(\boldsymbol{\sigma}, \boldsymbol{\pi})$, virtually all members of S_n are considered to be of an identical type. A ψ value of around 0.50 would mean that if one ranking is chosen at random then there would be a 50% probability that the next randomly chosen ranking would be indistinguishable from the first.

A summary of (12) for $r_i(\boldsymbol{\sigma}, \boldsymbol{\pi}), i = 1, 2, 3, 4$ is given in Table 1 for $n = 9, \ldots, 13, 20, 50, 100$. In particular, columns 2 and 3 show the standard deviation and the standardized coefficient of kurtosis γ_2. The 3-rd column reports the ratio [12] where the values have been rounded off to the 4-th decimal place to keep computations to a feasible level. Columns 5-7 show the proportion of the total frequencies which fall outside the indicated ranges. The expected values under the hypothesis of a normal distribution are 0.3173, 0.0455, 0.0027.

The null distribution of the four coefficients was determined for $n \leq 13$ by combinatorial enumeration, *i.e.*, explicitly computing the statistics for every permutation in S_n and counting the number of rankings giving rise to specific values of $r_i(\boldsymbol{\sigma}, \boldsymbol{\pi}), i = 1, 2, 3, 4$. For a larger number of ranks $(20, 50, 100)$, rather than treating all permutations exhaustively, we looked at a random sample of five million permutations. In this case, we have omitted the value of (12) because it loses its ability to capture the resolution of the coefficients efficiently.

Table 4: Summary statistics for four admissible rank correlations.

n	Coefficient	σ	γ_2	ψ	$\pm\sigma$	$\pm 2\sigma$	$\pm 3\sigma$
9	r_1	0.3536	2.48	6.05	0.3363	0.0369	0.0000
	r_2	0.2958	2.62	2.05	0.3706	0.0478	0.0002
	r_3	0.2664	2.76	1.85	0.3585	0.0446	0.0009
	r_4	0.3570	2.37	94.36	0.3545	0.0313	0.0000
10	r_1	0.3333	2.54	8.30	0.3487	0.0390	0.0004
	r_2	0.2749	2.66	2.55	0.3607	0.0459	0.0005
	r_3	0.2485	2.78	2.30	0.2912	0.0466	0.0025
	r_4	0.3363	2.41	97.16	0.3505	0.0334	0.0000
11	r_1	0.3162	2.59	11.05	0.3414	0.0402	0.0100
	r_2	0.2625	2.69	3.05	0.3551	0.0456	0.0005
	r_3	0.2335	2.80	2.80	0.3587	0.0405	0.0016
	r_4	0.3220	2.46	98.86	0.3472	0.0349	0.0000
12	r_1	0.3015	2.61	14.35	0.3210	0.0399	0.0001
	r_2	0.2479	2.71	3.65	0.3542	0.0468	0.0008
	r_3	0.2210	2.82	3.35	0.3108	0.0447	0.0018
	r_4	0.3073	2.49	99.36	0.3447	0.0361	0.0000
13	r_1	0.0000	0.00	00.00	0.3177	0.0403	0.0001
	r_2	0.2479	2.71	3.65	0.3471	0.0456	0.0007
	r_3	0.2210	2.82	3.35	0.3269	0.0432	0.0014
	r_4	0.3073	2.49	99.36	0.3415	0.0345	0.0000
20	r_1	0.2295	2.77	–	0.3278	0.0428	0.0011
	r_2	0.1878	2.83	–	0.3322	0.0441	0.0014
	r_3	0.1623	2.89	–	0.3190	0.0468	0.0019
	r_4	0.2423	2.66	–	0.3341	0.0405	0.0053
50	r_1	0.1601	2.89	–	0.3220	0.0444	0.0019
	r_2	0.1308	2.91	–	0.3193	0.0450	0.0022
	r_3	0.1100	2.95	–	0.3241	0.0458	0.0024
	r_4	0.1807	2.79	–	0.3267	0.0430	0.0013
100	r_1	0.1005	2.95	–	0.3192	0.0451	0.0024
	r_2	0.0821	2.97	–	0.3180	0.0451	0.0025
	r_3	0.0678	2.98	–	0.3193	0.0455	0.0026
	r_4	0.1288	2.88	–	0.3263	0.0442	0.0019

The sensitivity of all the indices increases as the number of ranks increases, but r_4 has a considerably greater power to discriminate between pairs of permutations than the other coefficients. In other words, r_4 is more efficient because it seems to take more account of the item diversification than do the other coefficients. It may also be observed that, as n becomes large, the standard deviations decrease and the coefficients of kurtosis increase for all the rank correlations included in Table 4. The new coefficient r_4 has a slightly larger standard deviation than r_1, r_2, r_3 and a moderately platikurtic distribution (thin tail and broad peak). The approach of r_4 to normality is appreciably slower than that of r_2 (Gini) or r_3 (Kendall). However, for large n, all the distributions are nearly normal with zero mean.

4. Rank correlation in economic contexts

A number of important problems in economic research logically require an accurate measurement of the agreement between ranks rather than the correlation between two or more variables, particularly when behavior is being modelled in contexts where, in practice, measurements are subjective or difficult. The purpose of this section is to discuss the application of rank correlation techniques by focusing on issues relevant to economists and to indicate the importance of a responsive measure of ordinal association, such as $r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})$, studied as a methodological concept in its own right. An emblematic example is the assumption of a Gaussian error distribution, even though such an hypothesis is dubious for many experimental data. To attenuate the negative impact of a refutation, it is often assumed that some order preserving transformation exists: $h(X)$ so that $h(X)$ is Gaussian. Distribution-free procedures have been proposed where $h()$ is left unspecified, provided that it is order preserving (Friedman, (1937)). Most of these procedures are based directly on the ranks of observations (see Pettitt, (1982)). Since rank values are scale free, *i.e.* they do not change when data are monotonically transformed, rank correlations are robust to non-Gaussianity of micro- and macro-economic indicators which are usually heavily skewed, have heavy tails and/or are affected by outliers.

The use of rank transformed variables may be desirable if association between original variables does not follow a mathematically predictable pattern or is thought to be non-linear. Relationships suggested in economic theory are, in many cases, of a nonlinear type, for example production functions or the Phillips curve. However, when theory cannot provide a precise specification of the functional form, it is advisable to have alternative tools for estimation and inference which avoid imposing linear assumptions on nonlinear and often contradictory relationships. For example, Gapen *et al.*, (2008) used rank association instead of the conventional Pearson's coefficient of correlation because this implicitly assumes linear relationships between variables, an assumption that contradicts the nonlinear links between variables found in their paper. Various examples can be reported to highlight the usefulness of rank correlation in economics. Inequality indices are statistical estimators which measure distinct aspects of an income distribution between individuals, countries, regions, factors, etc. They are often presented in the form of ranks to measure their relative magnitudes. However, a rank ordering may produce misleading inference, because ranks necessarily exclude ties between lower and higher values of the indices, which is in contrast to the deterministic outcome that countries at the extreme ends of the ordering are best and worst (Horrace *et al.* [2008]). Rankings are also helpful to test the equality of opportunity (Peragine [2004]). Axiomatic approaches to rank correlation are applied for ranking opportunity sets more generally (Davidson & Duclos [2000]).

Studies of the intergenerational association in the occupation of fathers and sons often construct an ordered ranking of occupational status prestige and then regress the rank of the son on the rank of the father (see, for example, Checchi [1997], Majumder [2010]). A set of countries might be rank ordered according to their distance from the efficiency frontier. This ranking might be useful for detecting the existence of trends in efficiency performance over time (Brockett *et al.* [1998]). Rather than continuing the list of economic issues to which rank correlation is usefully applicable, we prefer to test the measure proposed in this paper. To this end, we have selected published reports that are intended to be representative of the available economic literature concerning the use of rank correlation in a multivariate scenario which might require a highly-sensitive coefficient and, above all, render data freely and easily available for evaluation purposes.

4.1. Comparing price levels

In cases where data are affected by uncertainty and imprecision, it is common practice (see *e.g.* Conover & Iman [1982]) to transform the observed values into ranks, so long as their ranking is possible. In this transformation the smallest sample value is transformed into 1, the second smallest into 2 and so on until the largest value, which is transformed into n. Of course, such a choice is not without cost. The use of the first n positive integers presupposes equal interval spacing between items: two values separated by several orders of magnitude may have contiguous ranks, which are the same as two other very close values. This results in a loss of the information offered from ratio or interval variables and jeopardizes the validity of subsequent analyses. For instance, the measure of association between rank-transformed variables will give us a correlation coefficient, but it will be the correlation of the ranks of the variables, not the correlation of the variables themselves.

Hill [1999] on bilateral comparison of price level across a group of countries, illustrates a case in which there is a need to resort to an appropriate and effective use of rank correlation. The author shows how a comparison of price levels across a group of countries can be made by chaining bilateral price indices across a spanning tree. The article presents a table on ICP (international comparison program) per capita income rankings for $n = 30$ countries. The entries refer to three different methods of computing multilateral transitive price indices. The data are sorted according to per capita income values for each method.

To assess the type and strength of the relationship between the methods, the author adds a table of the rank correlation coefficients between the three methods, but omits to specify which particular rank correlation coefficient has been used. In Table 5 we report the values computed by the author together with the values obtained using $r_1, \ldots, r_4$. Furthermore, in the last row, we show the value of Pearson's product moment correlation between the observed values of per capita income. The matrix reported in the article is very dissimilar from the others. In particular, Hill [1999] obtains a relatively low rank correlation (0.532) between "Penn" and "ExR" methods in 1980 which has not been found in the other matrices. The correlation is highest between the "MST" and "Penn" rankings both in 1980 and 1985: 0.762 and 0.901. The author explains this through the volatility of exchange rates. The same pattern is seen for the other matrices in the table. In general, the methods of computing multilateral transitive price indices appear significantly more concordant according to any of $r_1, \ldots, r_4$ than reported in the article.

The author states that the rank correlation between methods was greater in 1985 than in 1980. Our findings confirm this assertion, though the links between 1980 and 1985 methods are stronger than shown by the author. The column labelled "F norm" presents the values of the Frobenius distance between the correlation matrices of 1980 and 1985.

$$PF = \sqrt{\text{Trace}\left(\mathbf{R}_{1980}, \mathbf{R}^t_{1985}\right)}. \tag{13}$$

It can be noted that the Frobenius norm between $\mathbf{R}_{1980}$ and $\mathbf{R}_{1985}$ reported by the author is the lowest in the column.

Table 5: Rank correlations coefficients.

Ranr correlation	1980	MST	ExR	1985	MST	ExR	F norm
Hill (1999)	Penn	0.762	0.532	Penn	0.901	0.881	2.528
	MST		0.621	MST	0.870		
Spearman	Penn	0.997	0.961	Penn	0.988	0.972	2.950
	MST		0.959	MST	0.973		
Gini	Penn	0.968	0.829	Penn	0.931	0.877	2.776
	MST		0.824	MST	0.878		
Kendall	Penn	0.971	0.847	Penn	0.988	0.972	2.792
	MST		0.835	MST	0.973		
r_4	Penn	0.989	0.904	Penn	0.989	0.954	2.902
	MST		0.906	MST	0.959		
Pearson	Penn	0.997	0.965	Penn	0.996	0.973	2.959
	MST		0.969	MST	0.976		

4.2. Securities market line

A criterion that is widely employed in the financial sector for assessing portfolio performance is the securities market line: the linear relation between mean returns on assets and the betas of these assets or portfolios calculated against a market index. To illustrate the ambiguity of this criterion, Roll, (1978) considered an idealized contest with $n = 15$ contestants. Each contestant selected a portfolio from a four-asset universe. After the portfolios were selected, a sample period was observed. Because of its wide acceptance, the securities market line criterion is used by two hypothetical judges of the contest in order to distinguish winners from losers. Judge 1 ranks the contestants from best (largest positive deviation from his securities market line) to worst as follows

$$\boldsymbol{\sigma} = (2, 15, 14, 5, 10, 6, 13, 9, 8, 7, 12, 4, 11, 3, 1)\,.$$

The second judge has a different set of assessments

$$\boldsymbol{\pi} = (4, 15, 11, 9, 14, 10, 8, 13, 1, 12, 7, 5, 2, 6, 3)\,.$$

Although some contestants were similarly rated by both judges (*e.g.* contestant 15 was ranked second by both and contestant 3 was ranked 14-th by judge no.1

and 15-th by judge no. 2), other contestants were rated quite differently (*e.g.*, the number one winner according to judge no.1 was a loser and ranked 13-th out of 15 by judge no. 2). The plot in Figure 1 shows clear evidence for a bilinear ascending structure.

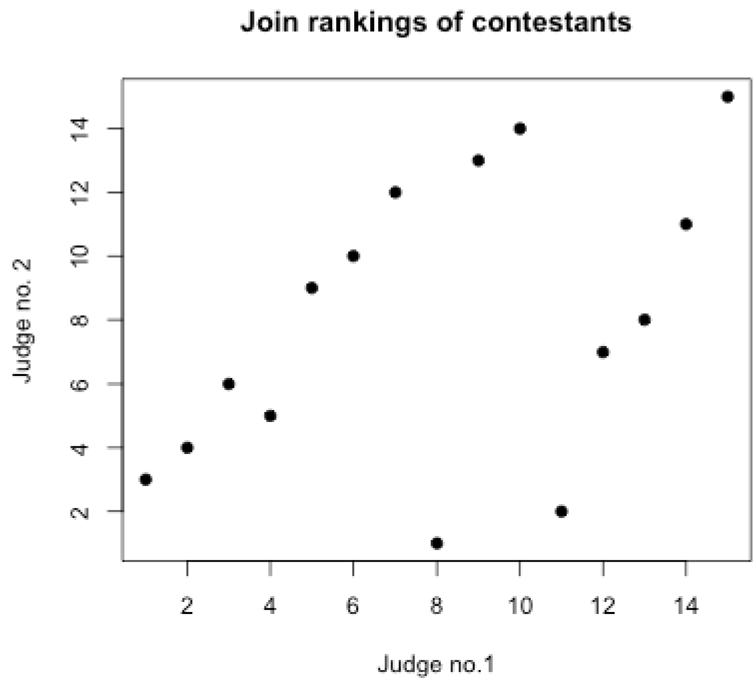

Figure 1: Positions as perceived by the two contest judges.

Increasing ranks of judge no.1 are combined with increasing ranks of judge no.2, but the mean of the ranks to the left for judge no.1 is significantly higher than the mean of the ranks on the opposite side. This situation may occur, for instance, when evaluators tend to separate items under consideration into two distinct groups with all the items in one group being considered superior, in some sense, to all the items in the other group and each group being ordered within itself independently of the other group. The rank correlations assume values

$$r_1(\boldsymbol{\sigma}, \boldsymbol{\pi}) = 0.471;\ r_2(\boldsymbol{\sigma}, \boldsymbol{\pi}) = 0.339;\ r_3(\boldsymbol{\sigma}, \boldsymbol{\pi}) = 0.429;\ r_4(\boldsymbol{\sigma}, \boldsymbol{\pi}) = 0.519.$$

Coefficients of rank correlation usually provide a value for overall association without giving explicit information about the pattern of the relationship between the permutations under comparison. The event described in this example is one

Table 6: Rank correlation matrices for r_3 and r_4

r_3				r_4			
X_1	X_2	X_3	X_4	X_1	X_2	X_3	X_4
1.0000	0.5205	0.5556	0.4620	1.0000	0.5886	0.7971	0.7279
0.5205	1.0000	0.4737	0.4269	0.5886	1.0000	0.6011	0.3956
0.5556	0.4737	1.0000	0.5322	0.5886	1.0000	0.6011	0.3956
0.4620	0.4269	0.5322	1.0000	0.5886	1.0000	0.6011	0.3956

in which a coefficient should take into account the contextual factors that affect judgment. Our index r_4 achieves a value superior to 0.5 which, if nothing else, might serve to alert us to the probability that some degree of association might be present.

4.3. Ordinal principal components

Korhonen & Siljamäki, (1998) reports ranks of $n = 19$ hypermarkets ordered according to 4 performance indicators: profit before taxes (X_1), sales profit (X_2), netprofit/staff hours (X_3), net profit/sales space (X_4). The authors define the first principal component (PC) as the ranking of the n items for which the sum of the squared rank correlation coefficient between the ordinal PCs and each of the original variables is maximized. In this analysis, the correlation matrix using Kendall's rank correlation coefficient is computed first. Assuming that management considers all the variables to be of equal importance, the first ordinal PC provides a quite acceptable rank order for the items. Table 6 shows the rank correlation matrices for both Kendall and r_4.

The two matrices are positive so that the Perron-Frobenius theorem (see, for example, Lin, (1977)) ensures that there is a single eigenvalue for both the matrices, which is positive and greater than or equal to all other eigenvalues in modulo, and that there is a strictly positive eigenvector corresponding to the largest eigenvalue. Furthermore, due to the orthogonality requirement, all the others PCs have elements of different signs. The results are reported in Table 7.
The first PC can be interpreted as the best one-dimensional "summary" of the linear relationships between the original variables *i.e.* as a general measure of the management evaluation. The other components can be interpreted as bipolar

Table 7: The results of principal component analysis

	r_3				r_4			
Variable	PC_1	PC_2	PC_3	PC_4	PC_1	PC_2	PC_3	PC_4
X_1	0.513	0.271	-0.543	-0.606	0.531	0.074	-0.808	-0.243
X_2	0.484	0.644	0.560	0.195	0.425	-0.841	0.264	-0.208
X_3	0.519	- 0.240	-0.418	0.706	0.541	0.105	0.116	0.827
X_4	0.484	-0.674	0.465	-0.309	0.495	0.526	0.513	-0.463
Variance	2.488	0.596	0.496	0.421	2.962	0.625	0.231	0.181
%	62.20	14.90	12.40	10.53	74.05	15.63	5.78	4.53

components because of high positive weights for some variables as opposed to high negative weights for other variables.

The findings based on r_4 are more intelligible than those based on r_3 used in the paper by Korhonen & Siljamäki, (1998) because the first principal component explains a greater portion of the total variance; hence, the use of just the first principal component is more legitimate. Moreover, each successive principal component on the right hand side of Table 7 has only one high loading factor and is almost coincident with one of the original variables.

4.4. Dimensions of structure in effective organizations

The research presented in this section (Reimann, (1974)) was designed to examine the relationship between underlying dimensions of structure and organizational performance. The data set of the study consisted of 19 North East Ohio industrial organizations of a wide range of sizes, manufacturing a variety of products. Organization structure was conceptualized in a multidimensional framework including $n = 11$ variables: functional specialization, functional dispersion, formalization of roles, lack of autonomy, functional specificity, delegation of authority, vertical span, staff density, administrative density, hierarchical control, centralization index. The author provides the scores of the 19 firms for the above variables.

A nonmetric cluster analysis was chosen since several of the structural variables had essentially ordinal scales. The clustering procedure employed by the author requires Spearman's rank correlation coefficients to be computed between all the 11 variables for each firm. Then Johnson's hierarchical clustering algorithm is applied to the absolute value of these rank correlation coefficients.

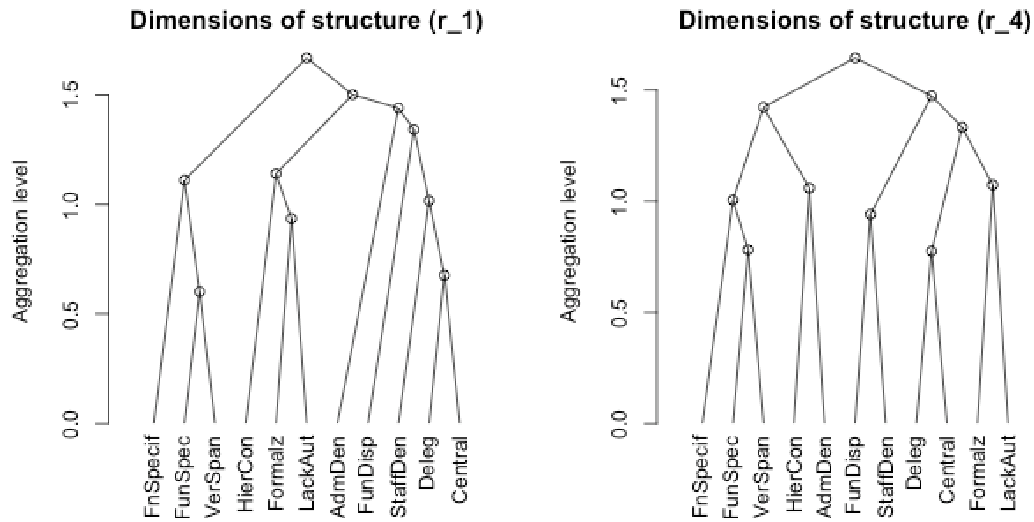

Figure 2: Dendrograms based on the complete link

In re-analyzing the work of Reimann, (1974), we have to make some preliminary considerations. First, for most of the variables, the scores of some firms are identical which means they must have the same rank, but the author does not specify how the ranks are allocated with regards to tied items. We followed the customary practice of averaging all the ranks covered by these identical items, and then giving each of them the value of the mean of the ranks (the maximum in (10) is correspondingly changed). The second minor problem encountered is that the author used rank correlation as a measure of similarity, but we have transformed it into a distance function in the interval $[0, 1]$ by using $\delta(\boldsymbol{\sigma}, \boldsymbol{\pi}) = 1 - |r(\boldsymbol{\sigma}, \boldsymbol{\pi})|$. The main feature of this distance is that a strong correlation, whatever its sign, corresponds to a small distance which may be true in almost every case. It also requires, though, the opposite assumption that a lower correlation between two other rankings means that they are less closely related, and this may not be true at all.

In addition to the remarks above, there is another fact that militates against a complete comparability. This lies in the fact that the author does not specify which link is used to cluster the structural variables. We have used the complete link. The results for the clustering of all firms are shown in Figure 2
The author determines a three-cluster solution

1. Decentralization: delegation of authority, centralization index;

2. Specialization: functional specialization, vertical span, functional specificity, hierarchical control;
3. Formalization: formalization of roles, lack of autonomy;

The three variables: functional dispersion, staff density and administrative density were left out because they correlated weakly with the other variables. This configuration is clearly described by the dendrogram on the left of Figure 2 which depicts the results obtained with r_1. The solution achieved by using r_4 confirms the limited degree to which functional dispersion and staff density are related, but aggregates administrative density with hierarchical control, something which appears very reasonable.

5. Conclusion

The basic concept underlying many techniques for estimating the degree of dependence between any two variables is that one should take advantage of all the available data if it is physically possible to do so. If one uses as much of the available data as possible, the conclusions drawn from the analysis will be more informative and more reliable. Although this guiding principle is enormously important, it is far, far from enough. If the data are of low quality, the results will be unsatisfactory and the experimental results will not be reliable, regardless of the qualities of the methods and quantities of data. The more flexible coefficient of rank correlation seems much more suitable for measuring the association between variables, if the data quality is known to be poor.

This article proposes a new rank correlation coefficient that has a remarkably high resolution over the $[-1, 1]$ interval while satisfying the requirements of symmetry, antisymmetry under reversal, zero expected values in case of independence, and right-invariance. The same properties are ascribed to some classical coefficients of rank correlation: Spearman's, Kendall's, Gini's, and a few other coefficients. It appears therefore that the coefficient $r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})$ has a good claim to serious consideration as a measure of rank correlation which is particularly useful when a high discriminatory ability is needed to differentiate between permutations. The examples presented in the section on applications in economic contexts give some experimental evidence to support this point of view, but they cannot replace a formal investigation of the sampling and asymptotic properties of $r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})$. We have analysed the convergence behavior of the new coefficient and have computationally shown its tendency towards the Gaussian distribution. Nevertheless, formal proof is still the subject of ongoing work. Future research in this area should attempt to investigate the possibility of using $r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})$ as a statistic for a distribution free test of the independence of two random variables, with a more rigorous discussion of how it works in the presence of ties.

Spearman's rank correlation is computed as a Pearson's product moment correlation coefficient on ranks, so it will assess linear association in the rank scale. This can be the starting point for a fruitful development which helps identify situations in which $r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})$ may be more appropriate than other coefficients. The extensions of $r_4(\boldsymbol{\sigma}, \boldsymbol{\pi})$ to multiple and partial rank correlation are lines of investigation to be pursued.

Concerns regarding relative positions are an important aspect of many economic problems. People, for instance, do not only care about their wealth, but also about their relative position in the wealth distribution. This affects individuals' consumption of ordinary goods (Neumark & Postlewaite, (1998)). Rankings of countries (or sets of countries) with respect to inequality, for example, are important social indicators for measuring relative well-being at a point in time and over time. Additionally, they may serve to improve our understanding of growth and equality relations, equity-efficiency tradeoffs, etc. (Horrace *et al.*, (2008)).

On the basis of experience matured during the development of the present paper, we are now convinced that, although appropriate techniques for handling ranking methods and measuring rank correlation are available, they have mainly been used in non-economic fields. Phelps Brown, (1972) had already observed that most of the conspicuous developments in economics have been in the direction of quantification, at the expense of the understanding of qualitative differences. This contradiction is resolved by the fact that the ranks allow statistical science to be applied to constructs which cannot be measured. Hence, a fruitful research area would be to critically examine the role ranks and measures of association play in econometric applications and explore how and to what extent they were or might be converted into advantages both from an empirical and methodological point of view.

References

Bhat, D.N. and Nayar, S.K. (1997). Ordinal measures for visual correspondence. *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, San Francisco (CA, USA), 18–20 Jun 1996, 351-357.

Blest, D.C. (2000). Rank correlation - An alternative measure. *Australian and New Zealand Journal of Statistics*, **42**, 101–111.

Blomqvist, N. (1950). On a measure of dependence between two random variables. *The Annals of Mathematical Statistics*, **21**, 593–600.

Borroni, C. G. and Cazzaro, M. (2006). Some developments about a new nonparametric test based on Ginis mean difference. *Statistica & Applications*, **4**, 33–47.

Brizzi, M. (1992). Misure di variabilit, concentrazione e dissomiglianza come sintesi di rapporti. *Statistical Techniques Research Group*, **2**, Dipartimento di Scienze Statistiche, Universitị Bologna.

Brockett, P.L. and Golany, B. and Li, S. (1998). Analysis of intertemporal efficiency trends using rank statistics with an application evaluating the macro economic performance of OECD nations. *Journal of Productivity Analysis*, **11**, 169–182.

Cameron, P. J. and Wu, T. (2010). The complexity of the weight problem for permutation and matrix groups. *Electronic Notes in Discrete Mathematics*, **28**,109–116.

Checchi, D. (1997). Education and intergenerational mobility in occupations: A comparative study. *The American Journal of Economics and Sociology*, **56**, 331–351.

Conover, W.J. and Iman, R.L. (1982). Analysis of covariance using the rank transformation. *Biometrics*, **38**, 715–724.

Critchlow, D. E. (1992). On rank statistics: an approach via metrics on the permutation group. *Journal of Statistical Planning and Inference*, **32**, 325–346.

Croux, C. and Dehon, C. (2010). Influence functions of the Spearman and Kendall correlation measures. *Statistical Methods & Applications*, **19**, 497–515.

Dallal, G.E. and Hartigan, J.A. (1980): Note on a test of monotone association insensitive to outliers. *Journal of the American Statistical Association*, **75**, 722–725.

Davidson, R. and Duclos, J.Y. (2000). Statistical inference for stochastic dominance and for the measurement of poverty and inequality. *Econometrica* **68**, 1435–1465.

Diaconis, P. and Graham, R.L. (1977) Spearmans footrule as a measure of disarray. *Journal of the Royal Statistical Society, B*, **39**, 262–268.

Deza, M. and Huang, T. (1998). Metrics on permutations, a survey. *Journal of Combinatorics, Information and System Sciences*, **23**, 173–185.

Durstenfeld, R. (1964). Algorithm 235: random permutation. *Communications of the ACM*, **7**, 420–420.

Friedman, M. (1937). The use of ranks to avoid the assumption of normality implicit in the analysis of variance. *Journal of the American Statistical Association*, **32**, 675–701.

Gapen, M. and Gray, D.F. and Lim, C.H. and Xiao, Y. (2008). Measuring and analyzing sovereign risk with contingent claims. *IMF Staff Papers*, **55**, 109–148.

Genest, C. and Plante, J.-F. (2003). On Blests measure of rank correlation. *The Canadian Journal of Statistics*, **31**, 35–52.

Gideon, R.A. and Hollister, A. (1987). A rank correlation coefficient resistant to outliers. *Journal of the American Statistical Association*, **82**, 656–666.

Gini, C. (1914). *Di una misura delle relazioni tra le graduatorie di due caratteri*. Tipografia Cecchini, Roma.

Goodman, L. A. and Kruskal, W. H. (1954). Measures of association for cross classifications. *Journal of the American Statistical Association*, **49**, 732–764.

Gordon, A.D. (1979). A measure of the agreement beween rankings. *Biometrika*, **66**, 7–15.

Hamming, R. W. (1950). Error detecting and error correcting codes, *Bell System Technical Journal*, **29**, 147–160.

Hill, R.J. (1999). Comparing price levels across countries using minimum-spanning trees. *The Review of Economics and Statistics*, **81**, 135–142.

Horrace, W. C. and Marchand, J.T. and Smeeding, T. M. (2008). Ranking inequality: Applications of multivariate subset selection. *Journal of Economic Inequality*, **6**, 5–36.

Iman, R.L. and Conover, W.J. (1987). A measure of top-down correlation. *Technometrics*, **29**, 351–357.

King, T.S. and Chinchilli, V.M. (2001). Robust estimators of the concordance correlation coefficient. *Journal of Biopharmaceutical Statistics*, **11**, 83–105.

Knuth D.E. (1973). *The art of computer programming, 2nd edition. Vol 3: Sorting and Searching*. Addison-Wesley Publishing Company, Reading Mas.

Korhonen, P. and Siljamäki, A. (1998). Ordinal principal components analysis. Theory and applications. *Computational Statistics & Data Analysis*, **26**, 411–424.

Lin, K. Y. (1977). An elementary proof of the Perron-Frobenius theorem for non-negative symmetric matrices. *Chinese Journal of Phisics*, **15**, 283–285.

Majumder, R. (2010). Intergenerational mobility in educational and occupational attainment. A comparative study of social classes in India. *The Journal of Applied Economic Research*, **4**, 463–494.

Mango A. (1997). Rank correlation coefficients: A new approach. *Computing Science and Statistics. Computational Statistics and Data Analysis on the Eve of the 21st Century. Proceedings of the Second World Congress of the IASC*, **29**, 471–476.

Mango, A. (2006). A distance function for ranked variables: a proposal for a new rank correlation coefficient. *Metodološki Zvezki*, **3**, 9–19.

Maturi, T.A. and Abdelfattah, E.H. (2008). A new weighted rank rorrelation. *Journal of Mathematics and Statistics* **4**, 226–230.

Moore, G.H. and Wallis, A. W. (1943). Time series significance test based on sign differences. *Journal of the American Statistical Association*, **38**, 153–165.

Mosteller, F. (1946). On some useful ineffcient statistics. *Annals of Mathematical Statistics*, **17**, 377–377.

Neumark, D. and Postlewaite, A. (1998). Relative income concerns and the rise in married women's employment. *Journal of Public Economics*, **70**, 157–183.

Okazaki, N. and Matsuo, Y. and Ishizuka, M. (2004). TISS: An integrated summarization system for TSC-3. Working note of the 4th NTCIR Workshop Meeting. Tokio, april 2003.

Peragine, V. (2004). Ranking income distributions according to equality of opportunity. *Journal of Economic Inequality*, **2**, 11–30.

Pettitt, A.N. (1982). Inference for the linear model using a likelihood based on ranks. *Journal of the Royal Statistical Society, Series B*, **44**, 234–243.

Phelps Brown, E. H. (1972). The underdevelopment of economics, *The Economic Journal*, **82**,1–10.

Pinto da Costa, J. and Soares, C. (2005). A weighted rank measure of correlation. *Australian and New Zealand Journal of Statistics*, **47**, 515–529.

Reimann, B.C. (1974). Dimensions of structure in effective organizations: some empirical evidence. *The Academy of Management Journal*, **17**, 693–708.

Rényi, A. (1959). On measures of dependence. *Acta Mathematica Hungarica*, **10**, 441–451.

Roll, R. (1978). Ambiguity when performance is measured by the securities market line. *Journal of Finance*, **33**, 1051–1069.

Salama, I.A. and Quade, D. (1982). A nonparametric comparison of two multiple regressions by means of a weighted measure of correlation. *Communications in Statistics, Part A-Theory and Methods*, **11**,1185–1195.

Salama, I.A. and Quade, D. (1997). The asymptotic normality of a rank corrrelation statistic based on rises. *Statistics and Probability Letters*, **32**, 201–205.

Salvemini, T. (1951). Sui vari indici di concentrazione. *Statistica*, **2**, 133-154.

Scarsini, M. (1984). On measures of dependence. *Stochastica*, **8**, 201–218.

Schweizer, B. and Wolff, E.F. (1981). On nonparametric measures of dependence for random variables. *Annals of Statistics*, **9**, 879–885.

Shieh, G.S. (1998). A weighted Kendalls tau statistic. *Statistics and Probability Letters*, **39**, 17–24.

Spearman, C. (1906). The proof and measurement of association between two things. *British Journal of Psichology*, **2**, 89–108.

Stoimenova, E. (1996). On statistical properties of Chebyshev's norm. *Statistics & Probability Letters*, **29**, 89–93.

Theil, H. (1950). A rank-invariant method of linear and polynomial regression analysis. Part 3. *Proceedings of Koninklijke Nederlandse Akademie van Wetenenschappen*, **53**, 1397–1412.

Yilmaz, E. and Aslam, J. and Robertson, S. (2008). A new rank correlation coefficient for information retrieval. In: *Proceedings of the ACM International Conference on Research and Development in Information Retrieval (SIGIR)*, 587–594.

Zayed. H. and Quade, D. (1997). On the resistance of rank correlation. *Journal of Statistical Computation and Simulation*, **48**, 59–81.