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μ-σ Games

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Abstract

Risk aversion in game theory is usually modelled using expected utility, which has been critized early on leading to an extensive literature on generalized expected utility. In this paper we are first to apply μ-σ theory to the analysis of (static) games.

μ-σ theory is widely accepted in the finance literature, using it allows us to study the effect on uncertainty endogenous to the game, i.e. mixed equilibria. In particular, we look at the case of linear μ-σ utility functions and determine the best response strategy. In the case of 2×2- and N×M-games we are able to characterize all mixed equilibria.

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1 Introduction

That people differ in their attitudes towards risk and uncertainty is a well established fact in economic research. In game theoretic models of strategic interaction, the main tool to capture risk and uncertainty is expected utility theory (*von Neumann and Morgenstern* (1947)). This model has been criticized early on by *Allais* (1953) and *Ellsberg* (1963) and lead to an extended literature on generalized expected utility models that are able to accommodate the identified paradoxes. To name just few, *Quiggin* (1982) and *Yaari* (1987) propose models of rank dependend utility relaxing the independence axiom, *Segal* (1987) relaxes the reduction of compound lotteries axiom to explain the Ellsberg paradox. Contributions to this literature usually start out with relaxing axioms that are the basis for von Neumann and Morgenstern's model and show that by relaxing these axioms one can find a generalized model of expected utility keeping most of the attractiveness of the standard model and, at the same time, enriching the framework to allow for seemingly irregular behaviour with respect to the standard model.

In this paper we want to capture risk not using expected utility, but with a different approach that has gained wide acceptance in modern finance theory: μ-σ utility (see *Markowitz* (1952)). These utility functions are considered today not as a special case of expected utility, but as an entire different coverage of risk.[1] Given the wide application of the μ-σ model in finance, it seems worthwhile to try to understand its effects on game theory. Our aim is to develop basic ideas about μ-σ games in this paper. Our approach will require using monetary (material) instead of utility based payoffs in the game.[2]

Most of the above mentioned generalizations in Game Theory can be formalized using Choquet integrals. Notice that our apprach cannot be described by Choquet integrals but modern finance theory bases it on its own axiomatization.[3]

In Game Theory, the expected utilities' linear formulation (with respect to the probabilities) does not allow to capture preferences over uncertainty that is endogenous to the game. A mixed strategy of a player, whether interpreted as the belief of another player or a real randomization, causes uncertainty for the other player(s). In the standard model, due to the linear fomulation, this uncertainty is treated like it would be under the assumption of risk neutrality, i.e. in fact ignored. Looking at μ-σ utility the circumstances are different – since probabilities enter the variance σ^2 not linearly but quadratic (due to the fact that $\sigma^2 = \mathrm{E}[x^2] - \mathrm{E}[x]^2$ with x being payoffs). Hence, any mixed strategy of a player will now cause real uncertainty that will not be disregarded.

In this article we discuss how equilibrium predictions change. As we concentrate on two player games we interpret a mixed strategy as a real randomization by a player. One

[1] See, for example *Lajeri-Chaherli and Nielsen* (2000) or *Lajeri* (2002). Sometimes, μ-σ is seen as a special case of quadratic utility functions, of the form $u(x) = ax - bx^2$, with x being the payoffs of a lottery or an asset. We do not restrict ourselves to this special case.

[2] *Crawford* (1990), *Chen and Neilson* (1999), *Rabin* (1993) and *Battigalli and Dufwenberg* (2009) also consider monetary or material games.

[3] See *Löffler* (1996) for an axiomatic foundation of μ-σ utility functions based on preference axioms. That μ-σ utility functions are not a special case of a Choquet integral using some capacity can easily be seen: Choquet integrals are homogenous of degree one, a feature that many μ-σ function (for example, μ-σ^2) do not possess.

important additional aspect of this interpretation is that in the case of mixed equilibria a player's own strategy now affects his or her utility, even though it does not change the expected value of the payoff. One may interpret this in the context of repeated play as a cost to changing ones action in different rounds of the game.

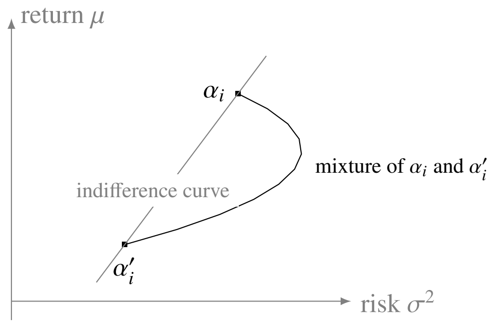
To refer back to finance theory where expected utility theory and μ-σ theory would argue that a portfolio does better than a single investment, a similar result in the application of μ-σ theory to game theory will not hold! In static games based on standard expected utility reducing the variance of (monetary) payoffs is not important because the expected value of the utility payoffs stays the same; but μ-σ theory will indicate that a pure strategy, i.e. choosing one action instead of randomizing, is preferable. By explicitely capturing variance caused by an agents own strategy choice, we may provide some reasoning why experimental players refrain frequently form using mixed strategies.

To illustrate our argument, consider the simple μ-σ utility function $V(\mu) = \mu - \frac{r}{2}\sigma^2$. This utility function is linear in expectation and variance and therefore sometimes called "linear utility", although linearity here does not refer to the material payoffs neither the probabilities of the players. In a typical μ-σ^2 diagram any indifference curves are upward sloping which follows from the fact that a higher variance needs to be compensated by a higher expected value; in our case the indifference curves are straight lines with slope $\frac{r}{2}$. We now consider a game where a player chooses his possibly mixed strategy given the possibly mixed strategy profile of other player(s), then the player faces a lottery where material payoffs depend on the specified game, and probabilities depending on the strategy profile. For a given strategy of the other player(s) each strategy of a player represents a point in the μ-σ diagram given the expected value and the implied variance of the strategy, i.e. for any strategy α this point is determined by $\mu = \mathrm{E}(\alpha)$ and $\sigma^2 = \mathrm{Var}(\alpha)$. We will show that mixing between two strategies of player i with the same utility (see figure 1, where these strategies are denoted α_i and α_i') actually leads to a loss of utility as the variance is *increased*. This is in sharp contrast to the usual "egg shaped" efficient frontier seen in almost every textbook in finance, where mixing *decreases* the variance and therefore contributes to an increase in utility.

One main difference to other application of non- or generalized expected utility functions to game theory (see for example *Dekel, Safra and Segal* (1991)) is that terminal node utilities of players are now dependend on how this terminal node is reached. In the case of random events, whether due to moves by nature or a mixed strategies of one of the players, payoffs that are identical with respect to their material (or monetary) payoff, give rise to differences in utility under the μ-σ paradigm. That terminal node utility may depend on endogenous aspects of other players' behavior has recently received some detailed attention under the heading of psychological game theory (see for example, *Geanakopolos, Pearce and Stacchetti* (1989), *Rabin* (1993) or *Battigalli and Dufwenberg* (2009)) where beliefs of players matter for the utility payoffs of players). μ-σ utility in games implies that strategies as well as the history of play, whenever random events are involved, affect the utility.

We proceed by first defining games based on μ-σ utility functions and study static 2×2 and N×N games based on linear utility functions. We then discuss nonlinear μ-σ utility functions.

Figure 1: Mixing two lotteries α_i and α'_i with the same utility decreases the utility in a μ-σ game.



2 Definition of static μ-σ games

To understand the applicability of μ-σ utility functions applied to games, we start by analysing static games with complete information. We consider games with a finite set N of players and nonempty and finite sets A_i ($i \in N$) of actions. Any profile of pure strategies $a \in \bigtimes_{i=1}^{N} A_i$ will provide player i with material (not utility) payoff $u_i(a)$; we use the notation $a = (a_1, \ldots, a_N)$.

We consider mixed strategies, that is elements of $\bigtimes_i \Delta(A_i)$. Let α be a profile of mixed strategies, that is, a vector $\alpha = (\alpha_1, \ldots, \alpha_N)$ with $\alpha_i \in \Delta(A_i)$. Furthermore, $\alpha_i(a_i)$ with $a_i \in A_i$ is the probability that player i will play the pure strategy $a_i \in A_i$ and $\alpha(a) = \prod_i \alpha_i(a_i)$ for $a \in \bigtimes_i A_i$.

Player i can expect the following material payoff from strategy combination α

$$\mathrm{E}[\alpha] := \sum_{a \in \bigtimes_i A_i} \alpha(a) u_i(a) \tag{1}$$

and a variance of

$$\mathrm{Var}[\alpha] := \sum_{a \in \bigtimes_i A_i} \alpha(a) u_i^2(a) - \mathrm{E}^2[\alpha] \tag{2}$$

For ease of notation we use $\mu = E[\alpha]$ and $\sigma^2 = \mathrm{Var}[\alpha]$.

Definition 1 *A μ-σ game is a game where the utility of player from strategy combination α is given by a μ-σ utility function*

$$U_i(\alpha) = V_i(\mu, \sigma^2).$$

V_i is strictly increasing in the first and strictly decreasing in the second variable and is strictly quasiconcave in μ and σ^2.

We assume strict quasiconcavity to ensure uniqueness of the solution to classcical maximization problems in finance. Relaxing this assumption most likely does not alter our results but makes the arguments very tedious.[4]

We first analyze the case of two players. Furthermore, we assume that the utility function is of the following simple linear form:

$$V(\mu, \sigma^2) = \mu - \frac{r}{2}\sigma^2$$

with r the parameter for the strength of the variance aversion.[5] We refer to a utility funtion of this form as *linear utility*.[6]

In the next paragraphs we want to analyse the effect of this utility model in well known examples of the literature on game theory. This allows us to show that in μ-σ games a Nash equilibrium does not always exist.

3 First results for μ-σ games

3.1 Best response with linear utility

Compared to standard game theory, μ-σ games may have different sets of equilibria. In standard game theory mixed strategies, i.e. strategies that randomize over actions that lead to the same expected (material) utility, yield the same utility payoff for a player, due to the the linearity in probabilities assumed by the expected utility framework. For μ-σ games the randomization of a mixed strategy comes at a price.

Given a behaviour of the other player(s) in the game, the following maximization problem determines the best response of a player with μ-σ utility.

$$\begin{aligned} &\max_{\alpha_i} \sum_{a \in \bigtimes_i A_i} \alpha(a) u_i(a) - \frac{r}{2}\left(\sum_{a \in \bigtimes_i A_i} \alpha(a) u_i^2(a) - \mathrm{E}^2[\alpha]\right) \\ = &\max_{\alpha_i} \sum_{a \in \bigtimes_i A_i} \alpha(a)\left(u_i(a) - \frac{r}{2}u_i^2(a)\right) - \frac{r}{2}\left(\sum_{a \in \bigtimes_i A_i} \alpha(a) u_i(a)\right)^2 \end{aligned}$$

This is a quadratic equation in α_i, where the coefficient on the quadratic term $a_i^2(a_i)$ is always negative.

Lemma 2 (best response with mixed strategy) *In equilibrium a player with μ-σ utility does not choose a mixed strategy, unless all actions chosen with positive probability are characterized by one the same expected value and the same variance, i.e. they are characterized by the same point in the μ-σ^2 diagram.*

[4]See the discussion in *Lajeri-Chaherli and Nielsen* (2000).

[5]This is analogue to the definition of the Arrow-Pratt-measure, see *Meyer* (1987); *Lajeri-Chaherli and Nielsen* (2000).

[6]Note, this utility function shares with CARA expected utility functions the characteristic that the preference over risk is independent of the wealth or income level.

Proof. We show first that all convex combinations of two (mixed) strategies lie on a convex curve in a μ-σ-Diagram. Let α_i and α_i' be two strategies of player i. Expected value and variance given the strategies of other players can be denoted as

$$(\mathrm{E}[\alpha_i], \mathrm{Var}[\alpha_i]) \text{ and } (\mathrm{E}[\alpha_i'], \mathrm{Var}[\alpha_i']).$$

Let a convex mixture choose $\lambda < 1$ times to play α_i and $1 - \lambda$ times to play α_i', we refer to this strategy as λ. The expected value can then be calculated as follows(1)

$$\begin{aligned} \mathrm{E}[\lambda] &= \lambda \mathrm{E}[\alpha_i] + (1-\lambda)\mathrm{E}[\alpha_i'] \\ &= \mathrm{E}[\alpha_i'] + \lambda \left(\mathrm{E}[\alpha_i] - \mathrm{E}[\alpha_i']\right). \end{aligned} \tag{3}$$

and the variance is given as

$$\begin{aligned} \mathrm{Var}[\lambda] &= \sum_{a \in \times_i A_i} \left(\lambda\alpha_i + (1-\lambda)\alpha_i'\right) \alpha_{-i}(a) u_i^2(a) - \mathrm{E}^2[\lambda] \\ &= \lambda \sum_{a \in \times_i A_i} \alpha_i \alpha_{-i}(a) u_i^2(a) + (1-\lambda) \sum_{a \in \times_i A_i} \alpha_i' \alpha_{-i}(a) u_i^2(a) - \left(\lambda \mathrm{E}[\alpha_i] + (1-\lambda)\mathrm{E}[\alpha_i']\right)^2 \\ &= \lambda \left(\mathrm{Var}[\alpha_i] + \mathrm{E}^2[\alpha_i]\right) + (1-\lambda)\left(\mathrm{Var}[\alpha_i'] + \mathrm{E}^2[\alpha_i']\right) - \left(\lambda \mathrm{E}[\alpha_i] + (1-\lambda)\mathrm{E}[\alpha_i']\right)^2 \\ &= \lambda \mathrm{Var}[\alpha_i] + (1-\lambda)\mathrm{Var}[\alpha_i'] + \lambda(1-\lambda)\left(\mathrm{E}[\alpha_i] - \mathrm{E}[\alpha_i']\right)^2. \end{aligned} \tag{4}$$

In the case that $\mathrm{E}[\alpha_i] \neq \mathrm{E}[\alpha_i']$

$$\begin{aligned} \mathrm{Var}[\lambda] = \frac{\mathrm{E}[\lambda] - \mathrm{E}[\alpha_i']}{\mathrm{E}[\alpha_i] - \mathrm{E}[\alpha_i']} \mathrm{Var}[\alpha_i] + \left(1 - \frac{\mathrm{E}[\lambda] - \mathrm{E}[\alpha_i']}{\mathrm{E}[\alpha_i] - \mathrm{E}[\alpha_i']}\right) \mathrm{Var}[\alpha_i'] + \\ + \frac{\mathrm{E}[\lambda] - \mathrm{E}[\alpha_i']}{\mathrm{E}[\alpha_i] - \mathrm{E}[\alpha_i']} \left(1 - \frac{\mathrm{E}[\lambda] - \mathrm{E}[\alpha_i']}{\mathrm{E}[\alpha_i] - \mathrm{E}[\alpha_i']}\right) \left(\mathrm{E}[\alpha_i] - \mathrm{E}[\alpha_i']\right)^2 \end{aligned}$$

and the second derivative is given as

$$\frac{d^2 \mathrm{Var}[\lambda]}{d\mathrm{E}[\lambda]^2} = -2.$$

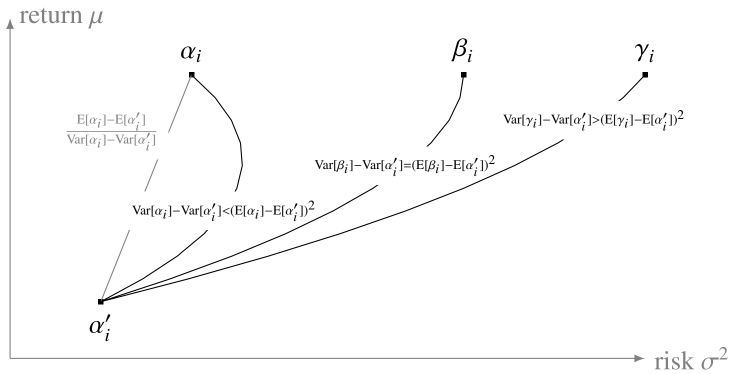
Therefore any convex combinations of two strategies lie on a convex curve. Figure 2 illustrates such curves for three different combinations of strategies.

Our result follows from the observation that indifference curves of player in the μ-σ-diagram are given by straight lines between two strategies, given the linear utility function $\mu - \frac{r}{2}\sigma^2$. For this reason any convex combination of strategies must be worse than either of the strategies that are combined. Even if a player is indifferent between two strategies α_i and α_i', he will see any mix between these strategies as inferior. ■

This implies an equilibrium in mixed strategies only exists if both strategies are represented by the same point in the μ-σ-diagram, i.e. they do not only have the same expected material payoff but also lead to the same variance.

Although this lemma seems to be related to Lemma 1 in *Chen and Neilson* (1999, Lemma 1), our model differs from them in one substantial point. The set of all pure actions in *Chen and Neilson* (1999) is convex which is not the case in static games with

Figure 2: Mixtures of three pairs of strategies (α_i, α_i'), (β_i, α_i') and (γ_i, α_i') in the μ-σ-diagram.



a finite number of pure strategies. Similarly, *Crawford* (1990) is looking at games where the players violating von Neuman-Morgenstern's independence axiom and assumes that the preferences (in terms of payments) are quasiconcave. Again, our paper differs from that work because (in terms of payments) μ-σ utility functions need not to be quasiconcave.[7]

We next study what this implies with respect to the best response towards a pure strategy as well as when an equilibrium in mixed strategies exist.

Lemma 3 (best response given pure strategies of another player) *The best response to a pure strategy is a pure strategy unless the material payoff of the player under consideration is the same for a set of at least two actions. Any (best response) mixed strategy can only randomize over this set of actions.*

Proof. Refering again to Figure 2. Given that the other player chooses a pure strategy, all strategies of the player under consideration will be a point on the μ-axis. This implies that the point higher on the axis will be chosen unless two stratgies lead to exactly the same material payoff. ■

3.2 2×2 games with linear utility

To answer the question when mixed strategy equilibria exist in μ-σ-games, we start with the case of 2×2-games. The game we consider is given in figure 3. We state our result for player 1 (the row player) and denote by q the probability that player 2 (the column player) chooses left. The following lemma characterizes the necessary condition for a best mixed strategy best response in comparison to the condition assuming standard expected utility theory.

[7]Using suitable numbers, one can already show that $\mu - \sigma^2$ has upper contour sets that are not convex.

Figure 3: Material payoffs for best response function.

	left	*right*
up	a, ·	b, ·
down	c, ·	d, ·

Lemma 4 (best response in 2×2 games) *The best response in any 2×2 game is a mixed strategy if and only if, a) the usual condition of expected utility game theory holds*

$$0 = (a - c)q + (b - d)(1 - q) \tag{5}$$

and b) the following condition is true

$$a + c = b + d. \tag{6}$$

This lemma shows that 2×2 μ-σ-games do not have more mixed equilibria than an equivalent standard 2×2 game. Being a mixed strategy equilibrium in the standard game is a necessary but not sufficient condition for being an equilibrium in the equivalent μ-σ-game. The second condition (6) has to be fullfilled as well, therefore many μ-σ-games will not have any equilibrium.

Condition (6) has an insightfull interpretation. It requires that, given any pure strategy of the other player, the sum over all material payoffs that the player is able to achieve over all his strategies is constant, i.e. independent of the pure strategy that is chosen given the randomization of the other player. Regardless the other player's choice, it is only the slice of the cake and not the size of the cake that is determined by the own player's actions.

Proof. We apply lemma 2 which implies for the 2×2-game that expected value and variance have to be the same for any variation in the probability p of player 1 to choose up. This leads to the following two conditions

$$\begin{aligned} \mathrm{E}[p] &= apq + bp(1 - q) + c(1 - p)q + d(1 - p)(1 - q) = \text{const} \\ \mathrm{Var}[p] &= a^2 pq + b^2 p(1 - q) + c^2(1 - p)q + d^2(1 - p)(1 - q) - (\mathrm{E}[p])^2 = \text{const.} \end{aligned}$$

The constant expected value implies

$$(a - c)q + (b - d)(1 - q) = 0.$$

An non-degnerate mixed equilibrium requires $0 < q < 1$, thus $a \neq c$ and $b \neq d$. Simplifying the conditon for the constant variance by solving for $\mathrm{E}[p]$ and calculating the first order condition p, gives us for the second condition so folgt

$$(a^2 - c^2)q + (b^2 - d^2)(1 - q) = 0.$$

Combining both conditions gives us the second condition for the existence of a mixed equilibrium $a + c = b + d$. ■

We next characterize the games where mixed equilibria do survive.

Theorem 5 (Mixed Equilibria in 2×2 games) *A mixed equilibrium in 2×2 μ-σ-games with linear utility functions exists if and only if*
(i) the candidate for equilibrium is a mixed equilibrium of a standard (expected utility) game with utility payoffs equal to the monetary payoffs of the μ-σ-game; and
(ii) for each strategy of the other player the sum of monetary payoffs of a player is the same for the strategies available to the player.

Proof. The two conditions imply the existence of a mixed strategy equlibrium follows directly from Lemma 4, in particular equations (5) and (6). ■

In the following we discuss special cases. We start by analyzing zero-sum-games.

Theorem 6 (2×2 zero-sum games) *The only 2×2-μ-σ-zero-sum-game with an equilibrium in non-degenerate mixed strategies is matching pennies.*

Proof. We denote the material payoffs as given by figure 4.

Figure 4: Material payoffs in the game of theorem 6.

	left	*right*
up	a, α	b, β
down	c, γ	d, δ

Given the previous results, we know that the following two conditions have to be fullfilled in a mixed strategy equilibrium

$$a + c = b + d \quad (7)$$

$$\alpha + \beta = \gamma + \delta \quad (8)$$

Given that we study zero-sum games we know

$$a + \alpha = b + \beta = c + \gamma = d + \delta = C.$$

Substituting the last equation into the previous two, gives us

$$C - \alpha + C - \gamma = C - \beta + C - \delta.$$

Figure 5: Matching pennies – the only μ-σ zero-sum game with (non-degenerate) mixed strategies.

	left	*right*
up	a, $C - a$	b, $C - b$
down	b, $C - b$	a, $C - a$

Adding this equation to (8), yields $\beta = \gamma$ and $b = c$ as well as $a = d$ and $\alpha = \delta$ therefore figure 5 represents this game. ■

Our next result concerns games that are not zero-sum. It is a well-know although disturbing result in μ-σ theory that a portfolio with higher payoffs is not necessarily prefered by an investor (preferences need not to be monotone).[8] Therefore, it is not clear that dominated strategies in μ-σ games cannot be equilibrium strategies. In 2×2 games we can show the following result.

Theorem 7 (2×2 games with dominated strategies) *If a strategy in a 2×2 game is dominated in monetary payoffs then no mixed equilibrium of the 2×2-μ-σ-games exists.*

Proof. This follows immediately from the fact the any equilibrium of the μ-σ game must be an equilibrium in the equivalent expected utility framework. In expected utility games dominated strategies never receive a positive probability weight. ■

This result seems to be obvious at first, it is less so if one considers that a player's monetary payoff choosing the – in monetary payoffs – dominant action may lead to a higher variance than the dominated action over compensating for the loss in payoff. As the result shows this cannot be the case. These results have a set of implications that are noteworthy:

1. Only coordination games and games without a pure strategy equilibrium in the expected utility framework can have mixed strategy equlibria.

   This follows from the observation that solving (5) for q and substituting (6). Given $q \in (0, 1)$ either one action dominantes the other or players prefer payoffs in two diagonal corners. Theorem 7 rules out the former. Leaving the cases where both players either prefer the same two diagonal corners (coordination games) or they prefer different corners – games without a pure strategy equilibrium in standard games.

2. Battle of sexes μ-σ games do not have a mixed strategy equilibrium unless players – in case of miscoordination – receive an additional payoff equal to the difference in their payoffs between the preferred and the alternative equilibrium.

   This can be seen from figure 6. A mixed strategy equilibrium exists iff (6) is satisfied. This is equivalent to $a = b + c$.

Figure 6: Battle of sexes with mixed strategy equilibrium ($a > b$).

	left	*right*
up	a,b	c,c
down	0,0	b,a

3.3 N×M games with linear utility

Mixed equilibria in 2×2 μ-σ games only exist if an additional constraint on payoffs holds to ensure that these strategies are a best response. In this section we show that

[8] See, for example, *Nielsen* (1987).

for N×M games a mixed equilibrium only exists if the game is degenerate. To show this, we show that any best response avoids mixing unless the material payoff for this player are constant over all possible outcomes of the game.

Theorem 8 (best response with N×M games) *In an N×M μ-σ game ($M, N > 2$) in any mixed equilibrium, players randomize at most over two pure strategies, unless the payoffs to the player are constant (independent of his choice).*

Proof. Again, we study the best responses of players. From lemma 2 we know that any action that may be chosen by a player will have to be represented by the same point in the μ-σ-diagram. To illustrate our argument, let us assume that a player randomizes over 3 actions, while the other player randomizes only over two. To show this, consider the 3x2-game with material payoffs given in figure 7.

Figure 7: Material payoffs for best response function in a 3×2 game in the proof of theorem 8.

	left	*right*
up	a, ·	b, ·
middle	c, ·	d, ·
down	e, ·	f, ·

Let p_1, p_2 and $1 - p_1 - p_2$ be the probabilities that the player chooses up, middle and down respectively. The condition for a constant expected value in this case is

$$\text{const} = \mathrm{E}[p_1, p_2] = ap_1q+bp_1(1-q)+cp_2q+dp_2(1-q)+e(1-p_1-p_2)q+f(1-p_1-p_2)(1-q).$$

This is a condition on two variables which gives us two constraints:

$$\begin{aligned} 0 &= (a-e)q + (b-f)(1-q) \\ 0 &= (c-e)q + (d-f)(1-q). \end{aligned}$$

If one combines both they imply as well $(a-c)q+(b-d)(1-q) = 0$. Any non-degenerate mixed equilibrium implies that $q \neq 0$ which immeadiately implies $a \neq e \neq c$.

Furthermore the variance needs to be constant:

$$\begin{aligned} \text{const} = \mathrm{Var}[p_1, p_2] = a^2p_1q + b^2p_1(1-q) + c^2p_2q + d^2p_2(1-q) + \\ + e^2(1-p_1-p_2)q + f^2(1-p_1-p_2)(1-q) - (\mathrm{E}[p_1, p_2])^2\,. \end{aligned}$$

This is a condition on two variables and using derivatives can be reduced to two equations

$$\begin{aligned} 0 &= (a^2-e^2)q + (b^2-f^2)(1-q) \\ 0 &= (c^2-e^2)q + (d^2-f^2)(1-q) \end{aligned}$$

and thus $0 = (a^2 - c^2)q + (b^2 - d^2)(1 - q)$. Solving implies

$$\begin{aligned} a + e &= b + f \\ c + e &= d + f \\ a + c &= b + d \end{aligned}$$

These three equations imply that $a = c = e$ and $b = d = f$ which contradicts the condition for the constant epected value. In any equilibrium where a player randomizes over more than two actions need to fullfil this condition on three of the strategies the player plays with positive probability. ■

3.4 A game with nonlinear utility functions

The results of the former section heavily depend on the fact that we restricted ourselve to linear μ-σ utility functions. If we consider other utility functions it might well be that an equilibrium in mixed strategies exists although the restrictive condition (6) is not met. In order to show this result we will consider the following utility function

$$V(\mu, \sigma^2) = -\frac{1}{\mu} - \frac{1}{2}\sigma^2.$$

Its indifference curves are convex and monotone functions in the μ-σ^2 diagram.

We now look at a game where the material payoffs are given by figure 8. We will now show that $(0.5, 0.5)$ is a mixed equilibrium of the game. Notice that in classical game theory (where utilities are given by figure 8) an equilibrium would be given by $(0.75, 0.25)$.

	left	*right*
up	3,1	0,0
down	0,0	1,3

Figure 8: Material payoffs of a game with nonlinear utility.

Assume that the row player chooses $q = 0.5$. Then the utility of the column player is given by

$$V(\mu(p), \sigma^2(p)) = -\frac{1}{8} - \frac{3}{2}p + \frac{1}{2}p^2 - \frac{1}{\frac{1}{2} + p}$$

This function has a maximum of $p = \frac{1}{2}$ in $[0, 1]$. Hence, the best response is a mixed strategy. With the same reasoning we can show that $q = \frac{1}{2}$ is the best response to the own player's strategy $p = \frac{1}{2}$.

4 Conclusions

We applied μ-σ utility theory to game theory. Using monetary games we discussed how equilibria predictions – in particular with respect to the existence of mixed equilibria in static games – changes if behaviour can be described by preferences depending on the mean and variance of random payoffs. This is an alternative to models of generalized expected utility which relax the assumption linearity in probabilities that is the basis of Von-Neuman-Morgenstern's expected utility model. While generalized expected utility models still keep the assumption that terminal utilities are independent of the way the respectiv endpoint was reached, μ-σ theory allows us to capture endogenous uncertainty caused by mixed strategies of players. In case of the 2×2-games we were able to show, that mixed strategy equilibria do survive in a μ-σ game under a set of additional restrictions. Thus, the set of mixed equilibria in a μ-σ game is a subset of the mixed equilibria of the equivalent game where the monetary payoffs are interpreted as utility payoffs.

Our analysis was based on the interpretation of mixed strategies as randomization by a player and not as the belief over the composition of a population of which the other player is chosen randomly from and this player then chooses a certain pure strategy. With the latter interpretation, selection criteria, in particular payoff dominance (*Harsanyi and Selten* (1988)) become important. Payoff dominance does not select mixed equilibria in coordination as these minimize the expected payoff and it is difficult to argue why a populations composition should yield this result. μ-σ theory applied to game theory give us in the case of coordination games a similar prediction. Mixed equilibria only survive in μ-σ games if there is a substantial gain from randomizing, for example because allowing the other player to predict ones behaviour comes at a first order cost effect, like in the case of zero sum games.

We believe this analysis can help to capture the experimentally observed aversion against mixing by players. While the μ-σ model is a very specific abstraction and somewhat arbitrary, its prominence in finance as well as its capability to capture uncertainty endogenous the the play of the game, made it for us a worthwhile starting point to reconsider equilibria when one departs from using utility function that can be characterized by Choquet integrals.

References

Allais, M. (1953) “Le Comportement de l'Homme Rationnel devant le Risque,” *Econometrica* 21, 503-546.

Battigalli, P. and M. Dufwenberg (2009) “Dynamic Psychological Games”, *Journal of Economic Theory* 144, 1-35.

Chen, H.-C. and Neilson, W.W. (1999) “Pure-strategy equilibria with non-expected utility players”, *Theory and Decision* 46, 199-209.

Choquet, G. (1953) “Theory of capacities”, *Annales lnstitut Fourier* 5, 131-295.

Crawford, V.P. (1990) ”Equilibrium without Independence”, *Journal of Economic Theory* 50, 127-154.

Dekel, E., Z. Safra and U. Segal (1991) “Existence and dynamic consistency of Nash equilibrium with non-expected utility preferences", *Journal of Economic Theory* 55, 229-246.

Ellsberg, D. (1963) "Risk, Ambiguity, and the Savage Axioms", *Quarterly Journal of Economics* 75, 643–669.

Geanakoplos, J., D. Pearce, and E. Stacchetti (1989) "Psychological Games and Sequential Rationality", *Games and Economic Behavior* 1, 60-79.

Harsanyi, J. C. and R. Selten, *A General Theory of Equilibrium Selection in Games*, MIT Press.

Lajeri-Chaherli, F., and L. T. Nielsen (2000) "Parametric characterizations of risk aversion and prudence," *Economic Theory*, 15, 469-476.

Löffler, A. (1996) "Variance aversion implies μ-σ-criterion", *Journal of Economic Theory* 69, 532-539.

Markowitz, H.M. (1952) "Portfolio Selection", *Journal of Finance* 7, 77-91.

Meyer, J. (1987) "Two Moment Decision Models and Expected Utility Maximization", *American Economic Review* 77, 421–430.

Nash, J.F. (1951) "Non-cooperative games", *The Annals of Mathmatics, Second Series* 54, 286-295.

F. Lajeri-Chaherli (2002) "More on Properness: The Case of Mean-Variance Preferences", *The Geneva Risk and Insurance Review*, 27(1), 49-60.

Nielsen, L.T. (1987) "Portfolio selection in the mean–variance–model: a note" , *Journal of Finance* 42, 1371-1376.

von Neumann, J., and O. Morgenstern (1947) *Theory of Games and Economic Behavior*, Princeton: Princeton University Press.

Quiggin, J. (1982) "A Theory of Anticipated Utility," *Journal of Economic Behavior and Organization* 3, 225-243.

Rabin, M. (1993) "Incorporating Fairness Into Game Theory and Economics", *American Economic Review* 83, 1281-1302.

Segal, U. (1987) "The Ellsberg Paradox and Risk Aversion: An Anticipated utility Approach, International Economic Review, 175-202.

Yaari, M. E. (1987) "The Dual Theory of Choice under Risk", *Econometrica* 55, 95-115.