INCENTIVES FOR EXPERIMENTING AGENTS

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Incentives for Experimenting Agents*

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Abstract. We examine a repeated interaction between an agent, who undertakes experiments, and a principal who provides the requisite funding for these experiments. The agent's actions are hidden, and the principal, who makes the offers, cannot commit to future actions. We identify the unique Markovian equilibrium (whose structure depends on the parameters) and characterize the set of all equilibrium payoffs, uncovering a collection of non-Markovian equilibria that can Pareto dominate and reverse the qualitative properties of the Markovian equilibrium. The prospect of lucrative continuation payoffs makes it more expensive for the principal to incentivize the agent, giving rise to a dynamic agency cost. As a result, constrained efficient equilibrium outcomes call for nonstationary outcomes that front-load the agent's effort and that either attenuate or terminate the relationship inefficiently early.

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1 Introduction

1.1 Experimentation and Agency

Suppose an entrepreneur has a potential project that may or may not be profitable. The project's profitability can be investigated and potentially realized only through a series of costly experiments. For example, the project may involve new technological developments that require building and testing a sequence of prototypes, until either achieving a breakthrough or abandoning the project in discouragement. Alternatively, the project may involve a consumer product that requires successive marketing campaigns until it is either caught up in the latest fashion trend or abandoned to obscurity.

For an entrepreneur with sufficient financial resources, the result is a conceptually straightforward programming problem. He funds a succession of experiments until either realizing a successful outcome or becoming sufficiently pessimistic as to make further experimentation unprofitable. But what if he lacks the resources to support such a research program? What if the project can only be realized through the joint efforts of an entrepreneur, supplying the requisite technical expertise but no capital, and a technically unable but financially endowed venture capitalist? What constraints does the need for outside funding place on the experimentation process? What is the nature of the contract between the venture capitalist and entrepreneur?

This paper addresses these questions. The answers are relatively simple if the entrepreneur can sell the project to the venture capitalist. The venture capitalist will then duplicate the optimal experimentation process that the entrepreneur would have undertaken in the absence of financial constraints. Suppose, however, that the entrepreneur *cannot* sell the project to the venture capitalist. The entrepreneur and venture capitalist may have different information about the project, leading to a prohibitive lemons problem. For example, the venture capitalist may be unable to ascertain whether the blueprints spread in front of her really describe a new energy technology. Alternatively, an interpretation we adopt here, it may be that the entrepreneur's participation in the experimentation process is essential. There may be no amount of explaining sufficient to equip the venture capitalist with the skills required to design the new fashion line or perform the new music that the entrepreneur is convinced will sweep the world.

The venture capitalist must then provide funding for the entrepreneur's experimentation, in return for a payoff in the event of a success. In the absence of any contractual difficulties, the problem is still relatively straightforward. Suppose, however, that the experimentation requires costly effort on the part of the entrepreneur that the venture capitalist cannot monitor. It may require hard work to develop either a new super-efficient battery or a new pop act. The venture capitalist may be able to verify whether the entrepreneur has been successful, but unable to discern whether a string of failures represents the unlucky outcomes of earnest experimentation or the product of too much time spent playing computer games. We now have an incentive problem that significantly complicates the relationship. In particular, the entrepreneur continually faces the temptation to simply pocket the funding provided for experimentation, explaining the resulting failure as an unlucky draw from a good-faith effort, and hence must receive sufficient rent to forestall this possibility.

The problem of providing incentives for the entrepreneur to exert effort is complicated by the assumption that the venture capitalist cannot commit to future contract terms. Perhaps paradoxically, one of the advantages to the entrepreneur of a failure is that the entrepreneur may then be able to extract further rent from future experiments, while a success terminates the rent stream. The venture capitalist may be able to reduce the cost of current incentives by committing to a string of less lucrative future contracts (perhaps terminating experimentation altogether) in the event of failure. We allow the venture capitalist to alter future contract terms or terminate the relationship only if doing so is sequentially rational.

Our exercise is motivated by quite real considerations. As summarized by Hall [8], the literature on venture capital emphasizes the importance of the following key features of our model: (i) moral hazard (hidden actions); (ii) asymmetric information (hidden information); (iii) learning over time;[1] and (iv) rates of return for the venture capitalist above those normally used for conventional investment. The latter feature, which distinguishes our analysis from Bergemann and Hege [2], is well-documented in the empirical literature (see, for instance, Blass and Yosha [4]). Funding for project development is scarce: technology managers often report that they have more projects they would like to undertake than funds to spend on them.[2] Our results resonate with a key empirical finding in the literature: investors often wish to downscale or terminate projects that entrepreneurs are anxious to continue.[3]

[1]In the words of Hall [8, p. 411], "An important characteristic of uncertainty for the financing of investment in innovation is the fact that as investments are made over time, new information arrives which reduces or changes the uncertainty. The consequence of this fact is that the decision to invest in any particular project is not a once and for all decision, but has to be reassessed throughout the life of the project. In addition to making such investment a real option, the sequence of decisions complicates the analysis by introducing dynamic elements into the interaction of the financier (either within or without the firm) and the innovator."

[2]See Peeters and van Pottelsberghe [10].

[3]See Cornelli and Yosha [5].

1.2 Optimal Incentives: A Preview of Our Results

Our analysis begins (Section 3) with the case in which the entrepreneur and the venture capitalist know the (fixed and constant) probability with which a particular experiment yields a breakthrough success that obviates the need for further experimentation. We refer to this as the case of a *development* project, since a success may still require a potentially long sequence of experiments or development stages.

As one would expect, agency is costly. The need to provide the entrepreneur with the incentives to undertake experiments not only places a lower bound on the share of the surplus going to the entrepreneur, but can lead to inefficiency. In particular, there are success probabilities for which it would be optimal to experiment until achieving a success in the absence of the agency problem, but under which no experimentation can be achieved in the presence of agency.

We first consider stationary (i.e. Markovian) equilibria of the agency relationship. If the project is sufficiently unprofitable, the only equilibrium features a (possibly inefficient) complete lack of experimentation. At the other end, if the development project is sufficiently lucrative (i.e., the benefit-cost ratio of the expected payoff of an experiment to its cost is sufficiently high), then the unique stationary equilibrium calls for the entrepreneur to work until a success is achieved. Surprisingly, however, there is a gap between these two outcomes, a range of development projects of intermediate profitability in which experimentation must occur in equilibrium, but in which the entrepreneur cannot be induced to always experiment. Here, the entrepreneur works, but not at the maximal rate.

To see what lies behind this "partial work" equilibrium, note that the entrepreneur bears two costs whenever working. One is that the entrepreneur does not divert for personal use the financing provided by the venture capitalist, and the other is that the entrepreneur risks a success, thereby eliminating the chance to earn future experimentation rents. Slowing the pace at which the entrepreneur works reduces these future rents, making it cheaper to provide current incentives. For development projects of intermediate value, this is the only way the venture capitalist can support experimentation.

The development project is inherently stationary—a failure leaves the players facing precisely the situation with which they started. One might then expect that the set of equilibrium payoffs is exhausted by considering equilibria with stationary outcomes (though not necessarily stationary equilibria), but this is the case only if the benefit-cost ratio of the development project is very high. We show that the set of all (weak perfect Bayesian) equilibrium payoffs is spanned by a simple class of equilibria, including a worst equilibrium and a collection of efficient equilibria. In the latter, the entrepreneur is induced to experiment at the maximal rate for some initial segment of time, after which experimentation is either terminated completely (if the benefit-cost ratio of the development project is intermediate, in which case such a threat is credible) or the entrepreneur switches permanently to exerting partial effort (for higher benefit-cost ratios). Front-loading the entrepreneur's effort reduces the entrepreneur's continuation value and hence reduces the cost of current incentives, in the process increasing the venture capitalist's payoff. Somewhat surprisingly, this front-loading can be better (than any stationary-outcome equilibrium) for the entrepreneur as well as the venture capitalist.

The next step is to consider the more general case of *research* projects (Section 4), in which the (fixed) success probability characterizing an experiment is unknown. The problem is now inherently nonstationary, as each failure makes players a bit more pessimistic about the project. We start the analysis by considering Markovian equilibria, in which the belief about the project serves as one obvious state variable. Because the entrepreneur's action is hidden, his private belief may differ from the public belief held by the venture capitalist, introducing a second, hidden state variable. Part of our contribution is accordingly methodological, as we explicitly solve for the unique Markovian equilibrium of this hidden-action, hidden-information problem. We then characterize the entire set of (Markovian or not) equilibrium payoffs, and prove that, as for development projects, there is a class of simple equilibria spanning this payoff set.

Not surprisingly, the structure of Markovian equilibria is more complex here than in the case of development projects, although the underlying logic is similar. Depending on whether the cost-benefit ratio and the players' rate of learning are low or high, four possible configurations emerge. In all of them, experimentation always takes place until the venture capitalist's belief reaches a threshold, at which point the project is abandoned. In two cases, however, this time is never actually reached in equilibrium, with experimentation instead gradually grinding to a halt. Again, slowing the pace of experimentation allows the venture capitalist to profitably economize on incentive costs, and may be necessary in order to cost-effectively induce effort. Now, however, the pace of experimentation can vary as does the posterior probability of a success, and it may be that the pace of experimentation is low early in the process when the venture capitalist is optimistic, but high for subsequent, lower beliefs. In this case, the venture capitalist is better off when the project is less promising, to the point that the venture capitalist would *prefer* a less promising experiment.

Even though the evolution of beliefs can provide implicit commitment power within the framework of a Markovian equilibrium, the venture capitalist can still reap gains from non-Markovian equilibria. In the case of a development project, a non-Markovian equilibrium can never reduce the venture capitalist's payoff below that of the Markovian equilibrium, and in some cases non-Markovian equilibria do not expand the set of equilibrium payoffs at all. In the case of a research project, non-Markovian equilibria offer the potential for reducing the venture capitalist's payoff, and always open the prospect of increasing her payoff. The venture capitalist's favorite equilibrium, which may make both players better off, calls for the entrepreneur to experiment at the maximal rate for some duration that is deliberately kept short, after which the equilibrium switches to a low pace of experimentation, or even to a premature abandonment of the project. Unlike in the

Markovian case, in this equilibrium, more promising projects are *always* more profitable for the venture capitalist than less promising ones.

1.3 Implications

What do we learn from this analysis? The development project appears to be inherently stationary, and in the research project, the evolution of beliefs appears to capture all of the relevant dynamic aspects of the project. Our results show that intertemporal links in incentives generate additional dynamics, giving rise to efficient equilibria whose structure belies the seemingly simple structure of the environment.

- Agency is costly, in the sense that experimentation may be efficient in the absence of agency and impossible within the agency relationship. Dynamic agency is yet more costly, in the sense that full experimentation may be possible with a single agency interaction, but impossible in a repeated interaction.

- Under dynamic agency, the constrained-efficient outcome[4] front-loads the entrepreneur's effort. The venture capitalist supplies full funding and the entrepreneur exerts full effort for an initial phase, after which *no* funding is ever offered again. In particular, this front-loading of effort reduces the entrepreneur's continuation values and hence relaxes his incentive constraints.

- Sequential rationality may preclude the existence of a constrained-efficient equilibrium by making it impossible for the venture capitalist to terminate funding. However, there are non-Markovian equilibria that achieve *some* degree of front-loading: the initial phase of full funding is followed by a worst (again, non-Markovian) equilibrium in which funding is either reduced or actually terminated. The resulting dynamics can completely reverse the qualitative properties of Markovian equilibria.

- The efficient equilibrium reproduces a key feature of the venture capital market: investors routinely terminate projects early.[5]

1.4 Related Literature

Our paper is most directly related to Bergemann and Hege [2].[6] Bergemann and Hege [2] examine a model differing primarily from ours in that their entrepreneur makes an

[4]That is, the surplus-maximizing outcome, dropping the venture capitalist's sequential rationality constraint, but retaining individual rationality constraints and the entrepreneur's incentive constraints.

[5]Efficient equilibrium outcomes call for terminating or attenuating effort, when the equilibrium in a static agency problem would call for full effort. From a practical point of view, investors often reduce funding while entrepreneurs insist on the viability of their project (cf. Cornelli and Yosha [5]).

[6]Bergemann, Hege and Peng [3] present an alternative model of sequential investment in a venture capital project, without an agency problem, which they then use as a foundation for an empirical analysis

offer to the venture capitalist in each period, reversing the bargaining positions from ours (in which the venture capitalist makes offers).[7]

Perhaps seemingly a minor detail, our results show that switching the bargaining power to the venture capitalist has a surprisingly significant impact. In particular, Bergemann and Hege find an array of Markovian equilibria analogous to ours, including four regions of parameter values, each with a Markovian equilibrium whose qualitative properties match those of the current paper. However, in Bergemann and Hege, these Markovian equilibria account for *all* of the equilibrium possibilities—there are no non-Markovian equilibria. The link between constrained efficiency and front-loaded effort, the rich structure of non-Markovian equilibria (and sharp contrast with Markovian equilibria), and the optimality of early abandonment are all absent from Bergemann and Hege. We return to the forces behind this difference in Section 5.

Our analysis combines elements of optimal experimentation and learning, venture capital provision, and dynamic contracting, giving rise to a vast collection of potentially related papers. We mention here only the most directly related papers in which an agent undertakes experimentation on behalf of a principal. Gerardi and Maestri [7] examine a model that differs in that the principal need not provide funding to the agent in order for the latter to exert effort, the length of the relationship is fixed (though the principal can end the relationship by making the decision early), the outcome of the agent's experiments is unobservable (and so the agent must be given incentives to report that outcome), and the principal can ultimately observe and condition payments on the state. Mason and Välimäki [9] examine a model in which the probability of a success is known and the principal need not advance the cost of experimentation to the agent, instead making a single payment to the agent upon completion of the contract.

Finally, our paper incorporates both hidden action and hidden information. Such models are notoriously challenging when the uninformed party (here, the principal) makes the offers. In this sense, this paper is related to the literature on repeated moral hazard with unmonitored wealth. In both cases, the agent takes a hidden action (here, how much to divert funds; there, how much to save income) that affects his future attitudes towards risk-taking (here, it affects his optimism; there, his actual risk-aversion). See, for instance, Werning [11] and Doepke and Townsend [6].

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of venture capital projects. Bergemann and Hege [1] examine a model in which the players can sign long-term contracts governing the financing of the experimentation process, removing the sequential rationality considerations that play a key role in Bergemann and Hege [2] and in the current paper.

[7]Bergemann and Hege allow the entrepreneur to choose effort levels from the interval $[0, \overline{e}] \subset [0, 1)$, with effort level e incurring cost ce and generating success probability e. Choosing an effort level $e < \overline{e}$ effectively slows the pace of experimentation, an effect we achieve below by having the players observe a public randomization device, some realizations of which lead to an offer to the entrepreneur from the venture capitalist and some of which do not. Bergemann and Hege consider the case of "relationship financing," in which the venture capitalist observes the effort expended by the entrepreneur, as well as "arm's length" financing, where (as in our model) the venture capitalist cannot do so.

2 The Model

2.1 The Agency Relationship

We consider an entrepreneur, hereafter called the agent or he, and a venture capitalist, hereafter the principal or she. The agent has access to a project that is either good or bad. The project's type is potentially unknown, with principal and agent initially assigning probability $\overline{q} \in (0, 1]$ to the event that it is good. In each period, the agent can conduct an experiment, at cost c. If the project is bad, the experiment is inevitably a failure, yielding no payoffs in that period but leaving open the possibility of conducting further experiments in future periods. If the project is good, the experiment yields a failure with probability $1 - p$ and a success with probability $p \in (0, 1)$. A success represents a breakthrough that obviates the need for further experimentation, ending the process with a payoff of $\pi > 0$ representing the future value of the successful project. The agent is unable to fund his experimentation, and in order to conduct an experiment, must obtain the requisite funding $c > 0$ from the principal.

The principal and agent interact for potentially an infinite number of periods, discounting at the common rate $\delta \in (0, 1)$. We will often be interested in the limit as δ approaches 1, viewed as representing the case in which time periods become arbitrarily short, so that any commitment power vanishes. To keep things in proportion, the cost c and success probability p will be proportional to the period length, while the lump-sum success payoff π and the interest rate per unit of time will remain fixed.

In each period t, the players first observe the outcome of a random variable with continuous cumulative distribution, i.e. players have access to a public correlation device. We shall briefly explain as we proceed how we could dispense with the resulting possibility for correlated actions, at the cost of more cumbersome strategies. The principal then either offers no contract to the agent, in which case we proceed to the next (discounted) period with no further ado, or the principal advances the cost c of experimentation to the agent and fixes a sharing rule $s_t \in [0, 1]$. If offered such a contract, the agent can either work (or, equivalently, experiment, or exert effort) or shirk. In the former case, the agent spends c on the experiment, leading to a success with probability p if the project is good (0 if not) and failure otherwise. In the latter case, the agent expropriates the advance c and conducts no experiment. The principal cannot observe the agent's action, observing only a success (if the agent experiments and draws a favorable outcome) or failure (otherwise). In the event of a success, the principal receives (normalized) payoff $(1 - \delta)(s_t\pi - c)$ and the agent retains $(1 - \delta)(1 - s_t)\pi$. A more formal description of the model is relegated to Appendix A. The timing is illustrated in Figure 1.

Note that, if $\overline{q} < 1$, the agent's hidden action gives rise to hidden information: if the agent deviates, he will update his belief unbeknownst to the principal, and this will affect his future incentives to work and hence his payoff from deviating. In turn, the principal

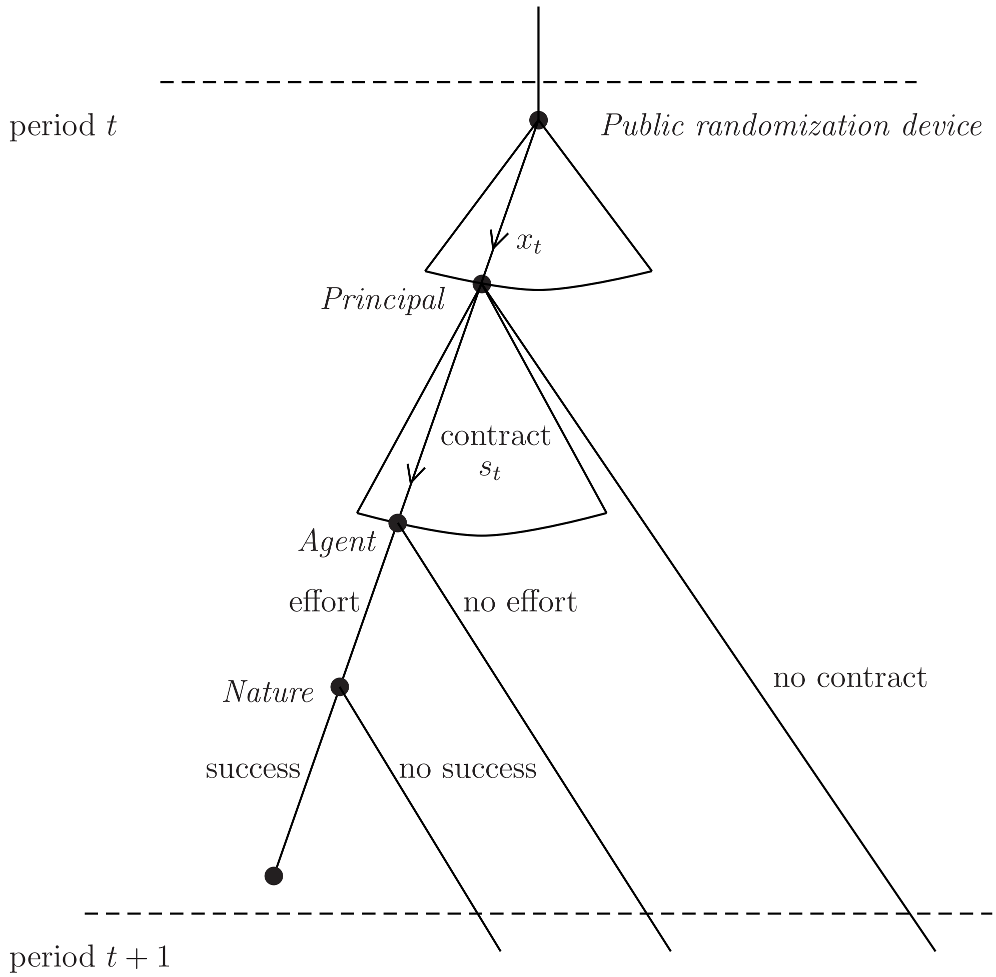

Figure 1: Timing of the period-t stage game. A success ends the game with payoffs $(1-\delta)(s_t\pi - c)$ to the principal and $(1-\delta)(1-s_t)\pi$ to the agent. If no contract is offered, an event observed by both players, the game proceeds to period $t+1$ with no current payoffs. If a contract is offered and the agent exerts no effort, current payoffs to the agent and principal are $(c, -c)$ and the game continues to period $t+1$. If the agent exerts effort and the outcome is not a success, current payoffs are $(0, -c)$ and the game continues to period $t+1$. Only the agent observes whether he exerted effort.

must compute this payoff in order to determine which offers will induce the agent to work. We thus have a potentially quite difficult hidden-information problem. We accordingly first focus on the simpler case in which $\overline{q} = 1$ (Section 3). We refer to this as the case of a development project, since the project is known to be good, but nonetheless requires a series of development stages before yielding a success. We then turn our attention to the case in which $\overline{q} < 1$ (Section 4). We refer to this as a research project, since the players must now ascertain whether the project is good. In this latter case, every failure is bad news, leading to a more pessimistic posterior expectation that the project is good.

We examine weak perfect Bayesian equilibria of this game in which, in equilibrium, the agent (but not necessarily the principal) plays a pure strategy. This ensures that along the equilibrium path (though not out of equilibrium) the principal and agent have identical beliefs about the value of the project.

2.2 The First-Best Policy

Suppose first that there is no agency problem—either the principal can conduct the experiments (or equivalently the agent can fund the experiments himself), or there is no monitoring problem and hence the agent necessarily experiments whenever asked to do so by the principal.

2.2.1 Development Projects

If the project is known to be good ($\overline{q} = 1$), then the value of conducting an experiment is given by

$$\begin{aligned} V &= (1-\delta)(p\pi - c) + \delta(1-p)V \\ &= \frac{(1-\delta)(p\pi - c)}{1-\delta(1-p)}. \end{aligned}$$

The optimal action is to experiment if and only if $V \geq 0$, or

$$p \geq \frac{c}{\pi}. \tag{1}$$

The optimal strategy thus either never conducts any experiments, or relentlessly conducts experiments until a success is realized, depending on whether $p < c/\pi$ or $p > c/\pi$.

2.2.2 Research Projects

A principal facing a project that may or may not be good will experiment until either achieving a success, or being rendered sufficiently pessimistic by a string of failures as to deem further experimentation unprofitable. Let $\varphi(q_t)$ be the posterior probability the project is good, given prior q_t and having observed a failure. We have

$$\varphi(q_t) = \frac{(1-p)q_t}{1-pq_t} < q_t \qquad \text{and hence} \qquad \frac{q_t}{\varphi(q_t)} = \frac{1-pq_t}{1-p}. \tag{2}$$

The value of the project to the principal, given a current probability q_t that the project is good, is given by

$$V(q_t) = \max\{0, (1-\delta)(q_t p\pi - c) + \delta(1-q_t p)V(\varphi(q_t))\}.$$

Since $V(\varphi(q_t)) \leq V(q_t)$, this is strictly positive if and only if $q_t p\pi - c > 0$. Hence, the principal experiments if and only if

$$q_t > \frac{c}{p\pi}, \qquad \text{or} \qquad q_t p > \frac{c}{\pi}. \tag{3}$$

Notice that the experimentation criterion for the development project (given by (1)) and the research project (3) are equivalent, with the relevant success probability p in the first case giving way to $q_t p$ in the second.

3 Development Projects: Fixed Success Probability

We begin our inquiry by stripping away learning considerations to examine development projects. We start by considering stationary equilibria, that is, equilibria in which the actions specified in any period t are independent of the period t *and* of the history up to t. Because of stationarity, we omit the time subscripts throughout.

3.1 Stationary Full-Effort Equilibrium: Lucrative Projects

We first investigate a particularly simple and intuitive candidate for behavior—a stationary equilibrium in which the principal extends funding and the agent exerts effort in every period. If the principal offers share s, she receives an expected payoff in each period of

$$(1-\delta)[ps\pi - c].$$

The agent's payoff solves, by the principle of optimality,

$$W = \max\{(1-\delta)c + \delta W,\ (1-\delta)p(1-s)\pi + \delta(1-p)W\},$$

or

$$W = \max\left\{c, \frac{(1-\delta)p(1-s)\pi}{1-\delta(1-p)}\right\}. \tag{4}$$

Therefore, such an equilibrium will exist if and only if the principal finds it optimal to fund the project and the agent finds it optimal to work, or (respectively)

$$(1-\delta)ps\pi \geq (1-\delta)c, \quad \text{and} \quad ((1-s)(1-\delta)\pi - \delta c)p \geq (1-\delta)c.$$

Combining, this is equivalent to

$$p \cdot \min\{(1-\delta)s\pi, (1-s)(1-\delta)\pi - \delta c\} \geq (1-\delta)c.$$

There is some value of $s \in [0,1]$ rendering the second term in the minimum positive, a necessary condition for the agent to work, only if $(1-\delta)\pi > \delta c$. If this is the case, then since the arguments of the minimum vary in opposite directions with respect to s, the lowest value of p or lowest ratio π/c for which such an equilibrium exists is attained when the two terms are equal, that is, when

$$s = \frac{1}{2}\left(1 - \delta \frac{c}{(1-p)\pi}\right), \tag{5}$$

in which case the constraint reduces to

$$p \geq \frac{2(1-\delta)c}{(1-\delta)\pi - \delta c} \;\equiv\; \overline{p}, \qquad \text{or} \qquad \frac{\pi}{c} \geq \frac{2}{p} + \frac{\delta}{1-\delta}, \tag{6}$$

which implies $(1-\delta)\pi > \delta c$. Hence, necessary and sufficient conditions for the existence of a full-effort stationary equilibrium are that the project be sufficiently lucrative to satisfy (6).

The principal will choose s to make the agent indifferent between working and shirking, giving equality of the two terms in (4) and hence an agent payoff of $W^* = c$. This is expected—by always shirking, the agent can secure a payoff of c. In a stationary equilibrium, this must also be his unique equilibrium payoff, since the principal has no incentive to offer him more than the minimal share that induces him to work (the continuation play being independent of current behavior).

The total surplus S of the project satisfies

$$S = (1-\delta)(p\pi - c) + \delta(1-p)S, \qquad \text{or} \qquad S = \frac{(1-\delta)(p\pi - c)}{1-\delta(1-p)}.$$

The principal's payoff is then

$$\frac{(1-\delta)(p\pi - c)}{1-\delta(1-p)} - c = \frac{(1-\delta)(p\pi - 2c) - \delta pc}{1-\delta(1-p)} \equiv V^*,$$

which is positive if and only if $p \geq \overline{p}$.

3.2 Stationary Equilibria for Other Parameters

What happens if the success probability falls below the boundary $\overline{p}$ for the existence of a stationary equilibrium in which the agent always works?

3.2.1 Unprofitable Projects: Null Equilibrium

If the agent is going to work, it must be that

$$(1-\delta)p(1-s)\pi + \delta(1-p)W \geq (1-\delta)c + \delta W,$$

where W is the agent's continuation value in the event the project is not a success (whether this is the result of shirking or working coupled with an unlucky draw). The left side is the payoff from working and the right from shirking.

This condition indicates that inducing effort is more expensive, in the sense that the minimum value of $1-s$ satisfying this equation is higher when the continuation value W is higher. Shirking ensures the continuation value is realized, while working runs the risk of a game-ending success. If we consider the case in which effort is cheapest, namely $W = 0$, we have

$$(1-\delta)(1-s)p\pi \geq (1-\delta)c,$$

or

$$p\pi - c \geq sp\pi. \tag{7}$$

The principal's payoff is $(1-\delta)[sp\pi - c]$ (recall that the principal must provide c to the agent if the agent is to undertake the experiment). Since (7) implies that

$$sp\pi - c \leq p\pi - 2c,$$

the principal will never find it optimal to induce work if

$$p < \underline{p} \equiv \frac{2c}{\pi}, \quad \text{or} \quad \frac{\pi}{c} < \frac{2}{p}.$$

In this case, the unique equilibrium involves the principal never funding the project. If an (out-of-equilibrium) offer is made, the agent works if and only if the offer is at least $c/(p\pi)$.

The requirement $p \geq \underline{p}$ for experimentation is a more demanding bound on p than when the principal conducts the experiments himself (cf. (1)): agency is costly. In addition,

$$\underline{p} = \frac{2c}{\pi} < \frac{2(1-\delta)c}{(1-\delta)\pi - \delta c} = \overline{p},$$

which means that there remains a region of values for π/c for which we have not yet constructed equilibria. This reflects a dynamic agency cost. If there were only one opportunity to experiment, the principal would induce such experimentation from the agent whenever $p > 2c/\pi$. In the presence of repeated opportunities, the principal can elicit consistent experimentation only if p exceeds $\overline{p} > 2c/\pi$. In the intermediate range, the dynamics of the agent's incentive constraint place an upper bound on how much effort the principal can induce from the agent.

3.2.2 Intermediate Projects: Mixed, Partial-Work Equilibria

We now consider the remaining case in which $p \in (\underline{p}, \overline{p})$, or equivalently

$$\frac{2}{p} < \frac{\pi}{c} < \frac{2}{p} + \frac{\delta}{1-\delta}.$$

The only remaining possibility is a stationary mixed-strategy equilibrium. In any given period, with probability z, the principal offers a share $s < 1$ that makes the agent exert effort, while with complementary probability, the project is not funded. The agent is willing to shirk whenever offered a nontrivial contract, and so his payoff is $(1-\delta)(zc + \delta zc + \delta^2 zc + \cdots) = zc$.[8]

The principal is indifferent in each period between offering the contract $s < 1$ and offering no contract, and so it must be that she just breaks even: $ps\pi = c$. On the other hand, since the agent is indifferent between shirking and not, we must have

$$(1-\delta)c + \delta zc = (1-\delta)p(1-s)\pi + \delta(1-p)zc.$$

This gives

$$z = \frac{1-\delta}{\delta}\left(\frac{\pi}{c} - \frac{2}{p}\right) \in [0,1), \quad \text{and} \quad s = \frac{c}{p\pi}.$$

The payoff of the principal in this equilibrium is 0, and the agent's payoff is

$$W = \frac{(1-\delta)(p\pi - 2c)}{\delta p}.$$

As p increases from $\underline{p}$ to $\overline{p}$, the agent's payoff in this equilibrium increases from 0 to c.

Since the parameter restrictions for the three possible types of equilibria partition the space of parameters, Sections 3.1–3.2 provide a complete characterization of stationary equilibria, yielding payoffs that are summarized in Figure 2.

3.3 Nonstationary Equilibria

We now extend our analysis to a characterization of all equilibria, including nonstationary ones. We will find no additional equilibria for either unprofitable projects or for

[8] More formally, the agent's strategy specifies that he works if and only if $s \geq c/(p\pi)$.

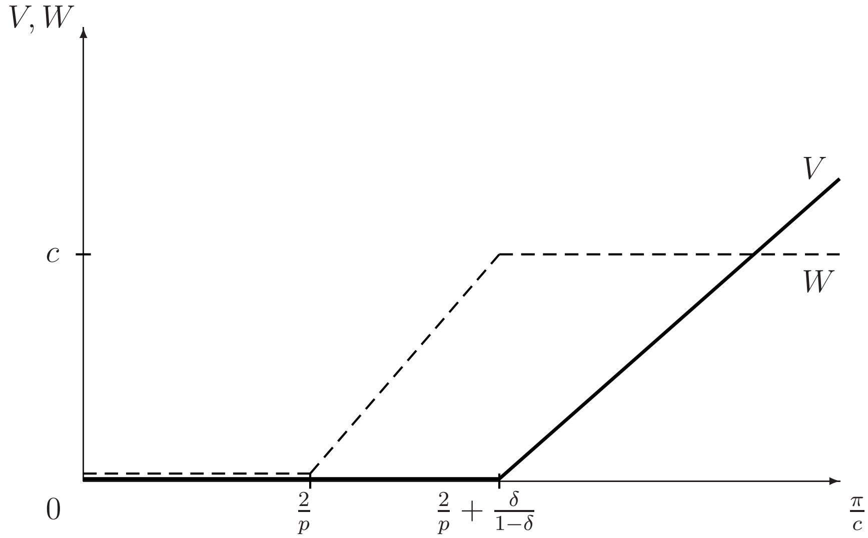

Figure 2: Payoffs from the stationary equilibrium of a development project, as a function of the "benefit-cost" ratio π/c, fixing c (so that we can identify c on the vertical axis). Both players obviously earn zero in the null equilibrium of an unprofitable project. The principal's payoff is fixed at zero for intermediate projects, while the agent's increases as does π. The agent's payoff is fixed at c for lucrative projects, while the principal's payoff increases in π.

very lucrative projects. However, we will find additional equilibria for both intermediate projects and for projects that are lucrative, but not too lucrative. In each case, we first find equilibria with stationary outcomes backed up by the threat of out-of-equilibrium punishments, and then use these to construct a family of equilibria with nonstationary outcomes.

Our first step is the following lemma, proved in Appendix B.1.

Lemma 3.1 *The agent's equilibrium payoff never exceeds c.*

We then need to consider the three cases identified in Sections 3.1–3.2.

3.3.1 Lucrative Projects

Suppose first that $\frac{\pi}{c} \geq \frac{2}{p} + \frac{\delta}{1-\delta}$. Section 3.1 established that there then exists a stationary equilibrium in which the agent always works on the equilibrium path, with payoffs

$$(W^*, V^*) \equiv \left(c, \frac{(1-\delta)(p\pi - 2c) - \delta pc}{1-\delta(1-p)} \right).$$

It is immediate that V^* puts a lower bound on the principal's payoff in *any* equilibrium. In particular, the share s offered by the principal in this equilibrium necessarily induces the agent to work, since it does so when the agent expects his maximum continuation payoff of W^* (cf. Lemma 3.1), and hence when it is hardest to motivate the agent. By continually offering this share, the principal can then be assured of payoff V^*.

We begin our search for additional equilibrium payoffs by constructing a family of potential equilibria with stationary equilibrium paths. For the first time in the analysis, we use the public randomization device.[9] In each period, this device yields the outcome "work" with probability z and "wait" with probability $1-z$. In the former case, the principal provides the capital c and a sharing rule s to the the agent, making the agent indifferent between working and not working. In the latter, nothing happens until the next period.

Why doesn't the principal make an offer to the agent anyway? Doing so prompts an immediate switch to the full-effort equilibrium with payoffs (W^*, V^*) (with the agent shirking unless offered a share at least as large as in the full-effort equilibrium). We will then have an equilibrium as long as the principal's payoff exceeds V^*, and δ is sufficiently close to one.[10]

The agent is indifferent between working and shirking, whenever offered a nontrivial contract, and so his payoff is $(1-\delta)(zc+\delta zc+\delta^2 zc+\cdots)=zc$. Using this continuation value, the agent's incentive constraint is

$$(1-\delta)p(1-s)\pi+\delta(1-p)zc=(1-\delta)c+\delta zc,$$

or

$$(1-\delta)(p\pi-2c)-\delta pzc=(1-\delta)(p\pi s-c).$$

Using this for the second equality, the principal's value is then

$$\begin{aligned} V &= z(1-\delta)(ps\pi-c)+\delta(1-zp)V \\ &= z[(1-\delta)(p\pi-2c)-\delta pzc]+\delta(1-zp)V \\ &= z\frac{(1-\delta)(p\pi-2c)-\delta pzc}{1-\delta(1-zp)}. \end{aligned}$$

This gives us a value for the principal that equals V^* when $z=1$, in which case we have simply duplicated the stationary full-effort equilibrium. However, these strategies may give equilibria with a higher payoff to the principal, and a lower payoff to the agent, when $z<1$. In particular, as we decrease the probability z of funding and work, we decrease

[9] The public randomization device is simply a convenient way to smooth out the description of strategies, and can be replaced by considering strategies that depend on calendar time.

[10] This ensures that, conditional on the outcome "wait," the principal's continuation payoff is still arbitrarily close to her expected payoff, so that she prefers not to fund the project rather than to receive V^*.

both the total surplus and the rent that the agent can guarantee by shirking. This implies that the principal might be better off scaling down the project from $z = 1$, if the cost of the rent is large relative to the profitability of the project, i.e., if π/c is relatively low. Indeed, this returns us to the intuition behind the existence of mixed-strategy stationary equilibria for intermediate projects, where π/c is too low for the existence of an equilibrium with $z = 1$: by scaling down the project, the principal's payoff might increase (and become nonnegative).[11]

Let $V(z)$ denote the principal's payoff as a function of z. We have $V(0) = 0$ when $z = 0$, giving the expected result that there is no payoff when no effort is invested. Are there any values for which $V(z) > V(1)$? The function $V(\cdot)$ is concave, and $V(z) = V(1)$ admits a unique root $z^\dagger \neq 1$ equal to

$$z^\dagger = \frac{1-\delta}{\delta pc} V^*,$$

which is less than one if and only if

$$\frac{\pi}{c} \leq \frac{2}{p} + \frac{\delta}{1-\delta}\left(2 + \frac{\delta}{1-\delta}p\right).$$

Note that this inequality is compatible with the restriction $\frac{\pi}{c} \geq \frac{2}{p} + \frac{\delta}{1-\delta}$ defining a lucrative project, but is not implied by this restriction.

We must then split our analysis of lucrative projects into two cases. If π/c is large (i.e., $\frac{\pi}{c} > \frac{2}{p} + \frac{\delta}{1-\delta}\left(2 + \frac{\delta}{1-\delta}p\right)$), then $z^\dagger > 1$. Since V is concave, this means that $V(z) < V^*$ for all $z < 1$. Therefore, our search for nonstationary equilibria has not yet turned up any additional equilibria. Indeed, Lemma (3.2) shows that there are *no* other equilibria in this case. Alternatively, if $z^\dagger < 1$, i.e. if π/c is not too large ($\frac{2}{p} + \frac{\delta}{1-\delta} \leq \frac{\pi}{c} < \frac{2}{p} + \frac{\delta}{1-\delta}\left(2 + \frac{\delta}{1-\delta}p\right)$), then as the agent's work probability z in our construction drops below unity, the principal's payoff initially increases. We have then constructed an entire family of stationary-outcome equilibria, one for each value $z \in [z^\dagger, 1]$ (recalling again that V is concave). These nonstationary (but stationary-outcome) equilibria give the agent a payoff less than $W^* = c$ and the principal a payoff larger than V^*.

The following lemma, proven in Appendix B.2, states that these equilibria yield the lowest equilibrium payoff to the agent.

Lemma 3.2

[11] We were considering stationary equilibria when examining intermediate projects, and hence the optimality of the principal's mixture required that the principal be indifferent between offering a contract and not offering one, which in turn implied that the principal's payoff was zero. Here, we are using the public randomization device to essentially make the principal's mixture observable (though see note 9) and then the threat of a reversion to payoff V^* to enforce the resulting action, and hence the principal need not be indifferent and can earn a positive payoff.

[3.2.1] [Very Lucrative Projects] If

$$\frac{\pi}{c} \;\geq\; \frac{2}{p} + \frac{\delta}{1-\delta}\left(2 + \frac{\delta}{1-\delta}p\right),$$

then the lowest equilibrium payoff $\underline{W}$ to the agent is given by $W^* = c$. Hence, there is then a unique equilibrium with payoffs (W^*, V^*).

[3.2.2] [Moderately Lucrative Projects] If

$$\frac{2}{p} + \frac{\delta}{1-\delta} \;\leq\; \frac{\pi}{c} \;<\; \frac{2}{p} + \frac{\delta}{1-\delta}\left(2 + \frac{\delta}{1-\delta}p\right),$$

then the infimum over equilibrium payoffs $\underline{W}$ to the agent (as $\delta \to 1$) is given by $W(z^\dagger) = \frac{1-\delta}{\delta p}V^* \leq c$.

In the latter case, the limit of the equilibria corresponding to $z = z^\dagger$, as $\delta \to 1$, gives the principal payoff V^*, and so gives both players their lowest equilibrium payoff. We accordingly refer to this as the *worst* equilibrium and denote the corresponding payoffs by $(\underline{W}, \underline{V}) = (\underline{W}, V^*)$. To be clear, references here and below to this "worst equilibrium" are an abuse of language (see footnote 10): this simultaneous lower bound on the players' equilibrium payoffs is not achieved by any equilibrium, but for any $\varepsilon > 0$, there is $\overline{\delta} < 1$ such that for all $\delta > \overline{\delta}$, there exists an equilibrium with payoffs within ε of $(\underline{W}, \underline{V})$. We simplify the exposition by proceeding as if we can actually obtain this worst equilibrium, recognizing that the set of payoffs determined below is then actually the closure of the equilibrium payoff set as $\delta \to 1$. Avoiding this simplification would change none of the results, but would replace various statements with more cumbersome "$\epsilon - \delta$" counterparts.

We have now established (W^*, V^*) as the unique equilibrium payoffs for very lucrative projects. For moderately lucrative projects, we have bounded the principal's payoff below by V^* and bounded the agent's payoff below by $W(z^\dagger)$ and above by W^*. To characterize the complete set of equilibrium payoffs for moderately lucrative projects, we must consider equilibria with nonstationary outcomes.

Appendix B.3 establishes the following technical lemma:

Lemma 3.3 *Let the parameters satisfy $\frac{2}{p} + \frac{\delta}{1-\delta} \leq \frac{\pi}{c} < \frac{2}{p} + \frac{\delta}{1-\delta}\left(2 + \frac{\delta}{1-\delta}p\right)$ and let (W, V) be an arbitrary equilibrium payoff. Then*

$$\frac{V - \underline{V}}{W - \underline{W}} \;\leq\; \frac{\delta p}{1-\delta} \;=\; \frac{\underline{V}}{\underline{W}}.$$

The geometric interpretation of this lemma is immediate: the ratio of the principal's to the agent's payoff is maximized by the worst equilibrium.

Any equilibrium payoff can be achieved by an equilibrium in which, in each period t, the worst equilibrium is played with probability $1 - x_t$ (for some x_t) and with probability

x_t (relying on the public randomization device) an equilibrium is played that maximizes the principal's payoff, conditional on the payoff of the agent.[12] In addition, the latter equilibrium calls for the principal to offer some share s_t to the agent that induces the agent to work.[13] Bearing this in mind, given W, consider the supremum over values of V among equilibrium payoffs, and say that such a payoff is on the frontier of the equilibrium payoff set. Using Lemma 3.3, Appendix B.4 completes our characterization of equilibria by showing:

Lemma 3.4 *In an equilibrium whose payoff is on the frontier of the equilibrium payoff set, it cannot be that both $x_t \in (0,1)$ and $x_{t+1} \in (0,1)$. More precisely, x_t is weakly decreasing in t, and there is at most one value of t for which x_t is in $(0,1)$.*

This lemma tells us that the equilibria on the frontier can be described as follows: for some $T \in \mathbb{N} \cup \{\infty\}$ periods, the project is funded by the principal, and the agent exerts effort, being indifferent between doing so or not. From period T onward, the worst (or, more precisely, an equilibrium arbitrarily close to the worst) equilibrium is played. We have already seen the two extreme points of this family: if $T = \infty$, the project is always funded, resulting in the payoff pair (W^*, V^*). If $T = 0$, the worst equilibrium is obtained. For very lucrative projects, all these equilibria are equivalent (since the full-effort equilibrium *is* then the worst equilibrium), and only the payoff vector (W^*, V^*) is obtained. For moderately lucrative projects, however, this defines a sequence of points (one for each possible value of T), the convex hull of which defines the set of all equilibrium payoffs. Any payoff in this set can be achieved by an equilibrium that randomizes in the initial period between the worst equilibrium, and an equilibrium on the frontier.

An analytical determination of the set of equilibrium payoffs is obtained in the continuous-time limit described in Section 3.3.4, and illustrated in the two right panels of Figure 3.

[12]Because the set of equilibrium payoffs is bounded and convex, any equilibrium payoff can be written as a convex combination of two extreme payoffs. One of these extreme payoffs can be chosen freely, and hence can be taken to be the worst equilibrium payoff $(\underline{W}, \underline{V})$, and because this is the worst equilibrium payoff for both players, the other extreme payoff must maximize the principal's payoff, conditional on that of the agent.

[13]If this latter equilibrium, with payoffs (W, V), calls for the principal to offer no contract with some probability, then (W, V) is itself the convex combination of two payoffs, one corresponding to the case in which a contract is offered and one corresponding to offering no contract. But the latter is an interior payoff, given by δ times the accompanying continuation payoff, and hence (W, V) cannot be extreme. Should the principal be called upon to offer a contract that induces the agent to shirk, it is a straightforward calculation (cf. footnote 26) that it increases the principal's payoff, while holding that of the agent constant, to increase the share s_t just enough to make the agent indifferent between working and shirking, and to have the agent work, again ensuring that (W, V) is not extreme.

3.3.2 Unprofitable Projects

When

$$\frac{\pi}{c} < \frac{2}{p},$$

we have seen that the agency cost exceeds the surplus that can be generated. Therefore, in this case, the unique equilibrium payoff vector is $(W, V) = (0, 0)$, whether or not attention is restricted to stationary equilibria.

3.3.3 Intermediate Projects

Consider now the case in which

$$\frac{2}{p} \leq \frac{\pi}{c} < \frac{2}{p} + \frac{\delta}{1-\delta}.$$

The stationary equilibria in this region involve a zero payoff for the principal. This means, in particular, that we can construct an equilibrium in which both players' payoff is zero: on the equilibrium path, the principal makes no offer to the agent; if he ever deviates, both players play the stationary equilibrium from that point on, which for those parameters also yields zero profit to the principal. Since this equilibrium gives both players a payoff of zero, it is trivially the worst equilibrium.

Lemma 3.4 is valid here as well,[14] and so the equilibrium payoffs on the frontier are again obtained by considering the strategy profiles indexed by some integer T such that the project is funded for the first T periods, and effort is exerted (the agent being indifferent doing so), after which the worst equilibrium is played. Unlike in the case of a lucrative project, we now have a constraint on T. In particular, as $T \to \infty$, the value to the principal of this strategy profile becomes negative. Since the value must remain nonnegative in equilibrium, this defines an upper bound on the values of T that are consistent with equilibrium. While the sequence of such payoffs can be easily computed, and the upper bound implicitly defined, the analysis is crisper once we consider the continuous-time limit, and the set is illustrated on the second panel of Figure 3.

3.3.4 A Continuous-Time Description

Section 3.3 explicitly describes the set of equilibrium payoffs. However, this description is not easy to use, as the difference equations describing the boundaries of the equilibrium payoff set are rather unwieldy. We consider here the limit of these difference equations, and hence of the payoff set, as we let the length dt of a period tend to 0. Note that in doing so, we are not defining and examining a game in continuous time. Instead, we are

[14] In this range of parameters, $\underline{W} = \underline{V} = 0$, and upon inserting these values, the proof of Lemma 3.4 continues to hold. From (25), the counterpart of Lemma 3.3 in this case is $\frac{V}{W} \leq \frac{p\pi - 2c}{c}$.

describing the limit of the difference equations defining the equilibrium payoff set of the discrete-time game.

We interpret $p > 0$ and $c > 0$ as the *rate* at which a success arrives and the *flow* cost of effort, when the agent exerts effort. That is, given that the length of time between two successive periods is given by dt and that effort is exerted in period t, the probability that a success arrives in the interval $[t, t + dt]$ is given by $p \cdot dt$, and the cost of this effort to this agent is $c \cdot dt$. We let r denote the interest rate. The lump-sum in case of success, π, remains unchanged.

Given an equilibrium in which the agent invariably exerts effort, the value V_t at time t to the principal solves (up to terms of order dt^2 or higher)

$$V_t = ps_t\pi dt - cdt + (1 - (r + p)dt)(V_t + \dot{V}_t dt),$$

or, in the limit as $dt \to 0$,

$$0 = ps_t\pi - c - (r + p)V(t) + \dot{V}(t), \tag{8}$$

where s_t is the share to the principal in case of success, and $\dot{V}$ is the time derivative of V (whose differentiability is easy to derive from the difference equations).[15] Similarly, if the agent is indifferent between exerting effort or not, we must have (up to terms of order dt^2 or higher)

$$W_t = p(1 - s_t)\pi dt + (1 - (r + p)dt)(W_t + \dot{W}_t dt) = cdt + (1 - rdt)(W_t + \dot{W}_t dt),$$

where W_t is the agent's continuation payoff from time t onwards. In the limit as $dt \to 0$, this gives

$$0 = p(1 - s_t)\pi - (r + p)W_t + \dot{W}_t = c - rW(t) + \dot{W}_t. \tag{9}$$

We may use these formulae to obtain closed-forms in the limit for the boundaries of the payoff sets described above. For this description, it is useful to introduce

$$\sigma \equiv p/r \quad \text{and} \quad \xi \equiv (p\pi - c)/c.$$

Unsurprisingly, the different cases arising in discrete time translate into as many cases in the limit. To understand the necessary restrictions on parameters, let us first ignore the terminal condition and study the stationary case in which $\dot{V}_t = \dot{W}_t = 0$ for all t. Then

$$W_t = W^* \equiv \frac{c}{r}, \qquad V_t = V^* \equiv \frac{\xi - \sigma - 1}{\sigma + 1}\frac{c}{r},$$

which are positive provided $\xi \geq 1 + \sigma$. If instead $\xi < 1 + \sigma$, the principal's payoff is zero in the unique stationary equilibrium. It is easy to check that if in addition $\xi < 1$, it is

[15] Throughout, given a function f, $\dot{f}$ denotes the time derivative of f.

not possible to have the agent exert effort in any equilibrium, and the unique equilibrium payoff vector is $(0, 0)$. This provides us with two of the relevant boundaries, between unprofitable and intermediate projects, and between intermediate and lucrative projects. The derivation of the boundary between moderately lucrative and very lucrative projects is more involved, and available along with the proof of Proposition 1 in Appendix B.8 (where we take advantage of of some intermediate results developed in the course of analyzing research projects, of which the development project is a limiting case). There, straightforward computations translate our results into the following:

Proposition 1 *The set of equilibrium payoffs for a development project, in the limit as period length becomes short, is given by:*

> **Unprofitable Projects ($\xi < 1$).** *No effort can be induced, and the unique equilibrium payoff is $(W, V) = (0, 0)$.*
>
> **Intermediate Projects ($1 \leq \xi < 1 + \sigma$).** *The set of equilibrium payoffs is given by the pairs (W, V), where $W \in [0, W^\dagger]$, and*
>
> $$0 \ \leq \ V \leq \frac{\xi}{\sigma+1}\left[1 - \left(1 - \frac{rW}{c}\right)^{\sigma+1}\right]\frac{c}{r} - W,$$
>
> *where $W^\dagger$ is the unique positive value for which the upper extremity of this interval is equal to zero. In the equilibria achieving payoffs on the frontier, the project is always funded, and the agent always exerts effort, until some time $T < \infty$ at which funding stops altogether. Such equilibria exist for all T below some parameter-dependent threshold $\overline{T}$.*
>
> **Moderately Lucrative Projects ($1 + \sigma \leq \xi < (1 + \sigma)^2$).** *The set of equilibrium payoffs is given by the pairs (W, V), for $W \in [\underline{W}, \frac{c}{r}]$, and*
>
> $$V^* \leq V \leq \left[\frac{\xi}{\sigma+1} - \left(\frac{(1+\sigma)^2 - \xi - 1}{\sigma(\sigma+1)}\right)^{-\sigma}\left(1 - \frac{rW}{c}\right)^{\sigma+1}\right]\frac{c}{r} - W,$$
>
> *where $V^* = \frac{\xi-\sigma-1}{\sigma+1}\frac{c}{r}$ and $\underline{W} = V^*/\sigma$. In the equilibria achieving payoffs on the frontier, the project is always funded, and the agent exerts effort, until some time $T \leq \infty$ from which point on the project is only funded a fraction of the time, with continuation payoff $(\underline{W}, V^*)$.*
>
> **Very Lucrative Projects ($\xi \geq (1 + \sigma)^2$).** *The unique equilibrium payoff involves the principal always funding the project, and the agent exerting effort: $(W, V) = (\frac{c}{r}, V^*)$.*

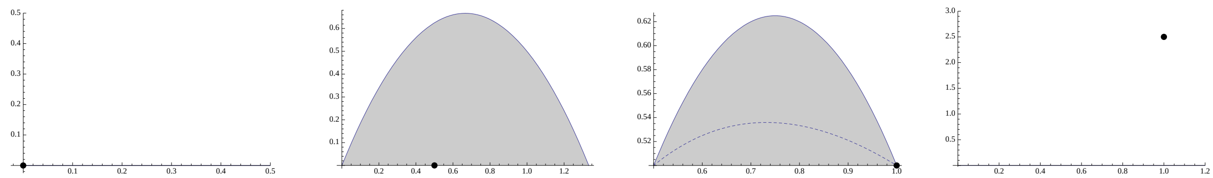

Figure 3: Set of equilibrium payoffs for a development project, measuring the agent's payoff W on the horizontal axis and the principal's payoff V on the vertical axis. To obtain concrete results, we set $c/r = 1$ and $\sigma = 1$ and, from left to right, $\xi = 0$ (unprofitable project), $\xi = 3/2$ (intermediate project), $\xi = 3$ (moderately lucrative project), and $\xi = 7$ (very lucrative project). The point in each case identifies the payoffs of stationary equilibria. The dotted line in the case of a moderately lucrative project identifies the payoffs of the equilibria with stationary outcomes, and the shaded areas identify the sets of equilibrium payoffs. Note that neither axis in the third panel starts at 0.

3.3.5 Summary

Figure 3 summarizes our characterization of the set of equilibrium payoffs for a development project. In each case, the stationary equilibrium puts a lower bound on the principal's payoff. For either very lucrative or (of course) unprofitable projects, there are no other equilibria. It is not particularly surprising that, for moderately lucrative projects, there are equilibria with stationary outcomes backed up by out-of-equilibrium punishments that increase the principal's payoff. The principal has a commitment problem, preferring to reduce the costs of current incentives by reducing the pace and hence the value of continued experimentation. The punishments supporting the equilibrium path in the case of moderately lucrative projects effectively provide such commitment power, allowing the principal to increase her payoff at the expense of the agent. It is somewhat more surprising that for intermediate and moderately lucrative projects the principal's payoff is maximized by an equilibrium whose outcome is nonstationary, coupling an initial period of maximal experimentation with a future in which the project is either scaled back or abandoned. Moreover, in the case of an intermediate project, such equilibria can increase the payoffs of both agents.

4 Research Projects: Unknown Success Probability

We now turn our attention to research projects ($\overline{q} < 1$), in which the players' posterior probability that the project is good evolves over the course of the agent's experimentation. Given this evolution of beliefs, we can no longer expect *any* equilibrium to be stationary.

Instead, the obvious place to start looking for equilibria is among the class of *Markovian* equilibria, in which the prescribed actions depend only on the posterior probability that the project is good.

4.1 A Candidate Markovian Full-Effort Equilibrium

Paralleling our investigation of the development project, we begin by considering a full-effort equilibrium in which the principal asks the agent to work in every period, and the agent does so, until the posterior falls below a threshold (in the event of continued failure) after which no further experimentation occurs.

Let $\underline{q}$ be the threshold (to be determined) below which the project is abandoned. A successive string of failures generates a corresponding sequence of posterior probabilities $\overline{q}, \varphi(\overline{q}), \varphi(\varphi(\overline{q})), \ldots$, until the posterior falls below $\underline{q}$. It is convenient to number periods backwards, letting $q_0 < \underline{q}$ be the posterior produced by the final failure, after which experimentation ceases, $q_1 \geq \underline{q}$ be the preceding posterior, at which the final experiment is undertaken, and so on.

Because of the agent's hidden action, the agent's behavior and payoff depend both on the public belief about the project, derived from the public history of offers (along with the equilibrium strategies), and on his privately held belief, given his history of actual effort choices. On the equilibrium path, both beliefs coincide. But to identify the agent's optimal action, we must determine his payoff from deviating, at which point those beliefs would differ. Thus, define $W(q, q')$ as the agent's payoff when the public belief is q and his private belief is q', in the candidate full-effort equilibrium. Since we are considering a full-effort equilibrium, the only deviations that are available to the agent lead to $q' \geq q$, as shirking by the agent when it is not expected by the principal leads the agent to be more optimistic than the principal.

The principal's offer s_t in period t must suffice to induce effort on the part of the agent, and hence must satisfy

$$\begin{aligned}(1-\delta)(1-s_t)q_t p\pi + \delta(1-q_t p)W_{t-1}(q_{t-1}, q_{t-1}) &\geq (1-\delta)c + \delta W_{t-1}(q_{t-1}, q_t) && (10)\\ &= (1-\delta)c + \delta\frac{q_t}{q_{t-1}}W_{t-1}(q_{t-1}, q_{t-1}). && (11)\end{aligned}$$

The first inequality is the agent's incentive constraint, with the left side being the expected payoff from exerting effort. This effort brings an immediate payoff of $(1-s_t)q_t p\pi$, being the agent's share $(1-s_t)$ of the expected payoff $q_t p\pi$ of an experiment, plus the probability $1-q_t p$ of a failure multiplied by the discounted continuation value $W(q_{t-1}, q_{t-1})$. On the right side, the current payoff to shirking is the value c of expropriating the experimentation funding (ensuring a failure). The continuation payoff is now $W(q_{t-1}, q_t)$, as the principal operates under the equilibrium hypothesis that the agent has worked and hence enters the next period with posterior q_{t-1}, while the agent knows that no such work has been done and retains posterior q_t.

The second equality in (10)–(11) rests on the following key observation, proven in Appendix B.5, that allows us to reduce the dimension of the state space.

Lemma 4.1 *In any equilibrium in which the agent never shirks,*

$$\forall q' \geq q : W(q, q') = \frac{q'}{q} W(q, q). \tag{12}$$

Hence, the agent's expected payoffs from the asymmetric beliefs that potentially arise off the equilibrium path are linear in the agent's beliefs. This relationship is intuitive, as higher beliefs simply scale up all the success probabilities involved in the agent's expected payoff calculation.

Two comments on this candidate equilibrium are in order. First, because the equilibrium is Markovian, the principal will invariably offer a share s_t causing the incentive constraint (10)–(11) to hold with equality, and will offer such a contract only if the result is a nonnegative payoff. Given this, the agent's strategy is to work if and only if he is offered at least this share.[16] In (the final) period 1, the incentive constraint is

$$(1-\delta)[(1-s_1)q_1 p\pi - c] \geq 0.$$

The principal's payoff is $(1-\delta)(s_1 q_1 p\pi - c)$. Using the incentive constraint, this is nonnegative only if $q_1 p\pi - 2c \geq 0$, and hence only if

$$q_1 \geq \underline{q} = \frac{2c}{p\pi}.$$

Here again, we see the cost of agency. The unconstrained optimal solution experiments until the posterior drops to $c/p\pi$, while the agency cost forces experimentation to cease at $2c/p\pi$. Intuitively, the principal must pay the cost of the experiment c, and must also provide the agent with a rent of at least c, to ensure the agent does not shirk and appropriate the experimental funding.

To verify that the proposed strategies are indeed an equilibrium, we must show that the principal is earning a nonnegative payoff. The share s_t that the principal can retain while satisfying the agent's incentive constraint may be so low (indeed, may be negative) that it does not cover the principal's outlay of the experimentation cost c. Using (11), we can write the agent's incentive constraint as

$$(1-\delta)[(1-s_t)q_t p\pi - c] \geq \delta p \frac{q_t}{q_{t-1}} W(q_{t-1}, q_{t-1}).$$

[16] That is, as long as he has not deviated in the past himself; otherwise, his acceptance threshold depends on both beliefs; given the equilibrium offers, he is more optimistic than the principal after a deviation, and therefore has a strict incentive to work.

The principal's share must at least cover the cost of her expenditure c, or $s_t q_t p\pi \geq c$, giving

$$(1-\delta)[q_t p\pi - 2c] \geq \delta p \frac{q_t}{q_{t-1}} W(q_{t-1}, q_{t-1}). \tag{13}$$

Could this inequality fail? If so, then there is no way to satisfy the agent's incentive constraint and still cover the principal's experimentation cost. The key observation here is that as the agent's continuation value becomes more lucrative, it becomes more expensive to provide incentives for the agent. Experimenting exposes the agent to the risk that the project may be a success now, eliminating future returns. Shirking now ensures an immediate payment of c (the diverted funds) plus the prospect of future experimentation. If the latter is too tempting, the principal cannot always work.

The principal may thus prefer, or indeed the existence of a Markovian equilibrium may require, an outcome in which the agent does *not* always exert effort. This is not to suggest that the principal will find it optimal to provide experimental funding to the agent knowing that the agent will abscond with it, but rather that the principal may prefer to sometimes attenuate the pace of experimentation, to ensure that her payoff is nonnegative. If her payoff is zero for some belief, the principal is indifferent between funding the project or not; in that case, by randomizing between those two choices, she delays the agent's opportunities to divert the funds, and so reduces his outside option. This makes it cheaper to induce the agent to work, and we can then determine the amount of delay that would make the principal indifferent between funding the project or not. Alternatively, for some beliefs, the aforementioned constraint might not be binding; in that case, there can be no delay (in a Markovian equilibrium), since the principal strictly prefers to fund the project.

It is then clear how to solve for both (mutually exclusive) possibilities arising at any point in time in a Markovian equilibrium: either the principal's payoff is zero, allowing us to solve for the delay; or the delay is zero, allowing us to solve for the principal's payoff. In either case, if an offer is made, it is such that the agent is indifferent between shirking or not. The next section uses these two possibilities to identify the unique Markovian equilibrium, as a function of the parameters of the research project.

4.2 The Markovian Equilibrium

4.2.1 Building Blocks

Analyzing the research project is more intricate than its development-project counterpart, and so we move more quickly to the continuous-time limit. We study the limit as the period length goes to zero, to determine for which parameters, if any, condition (13) holds. Once again, we emphasize that we are not defining or examining a continuous-time game, but summarizing the equilibrium behavior of our discrete-time game, insisting on the discrete-time context for substantive proofs. In continuous time, it is less confusing to assume that higher values of t correspond to later times, and we denote by $T \in \mathbb{R} \cup \{\infty\}$ the time at which experimentation stops altogether, if ever.

As in Section 3.3.4, we let p denote the flow rate of success if the project is good, c the flow cost of working, and r the discount rate. Then the payoff to the principal solves, approximately,

$$V_t = (q_t p s_t \pi - c)dt + (1 - r\lambda_t dt)(1 - pq_t dt)V_{t+dt},$$

or, in the limit,

$$0 = q_t p s_t \pi - c - (r\lambda_t + pq_t)V_t + \dot{V}_t. \tag{14}$$

Elaborating on the presentation in Section 3.3.4, the variable $\lambda_t \geq 1$ captures the possibility that the principal may offer the agent a contract with probability less than one.[17] In particular, we interpret $\lambda_t > 1$ as indicating that the principal offers the agent a contract at less than the maximal rate, but still at a rate of the order dt. When taking the continuous-time limit, we can rescale so as to encompass this probability in the discount factor, making the effective discount factor $r\lambda_t$, with offering a contract with non-unitary probability causing the discount factor to increase (hence $\lambda_t > 1$ in this case).

Similarly, whenever the agent is indifferent between shirking and not (as must be the case in a Markovian equilibrium), the payoff to the agent, W_t, must solve, approximately,

$$W_t = q_t p(1 - s_t)\pi dt + (1 - r\lambda_t dt)(1 - q_t p dt)W_{t+dt} = cdt + (1 - r\lambda_t dt)(W_{t+dt} + X_t dt),$$

where

$$X_t = \int_t^T e^{-\int_t^u r\lambda_\tau d\tau}(-\dot{q}_u)p(1 - s_u)\pi du$$

is the gain from $t + dt$ onward from not exerting effort at t (given that effort is then optimal at all later dates as long as it would have been in the absence of a deviation, since the off-the-equilibrium-path relative optimism of the agent makes the agent more likely to accept the principal's offer).[18] Using (12), we obtain

$$X_t dt = W_{t+dt}(q_t) - W_{t+dt} = \left(\frac{q_t}{q_{t+dt}} - 1\right) W_{t+dt} = p(1 - q_t)W_t dt.$$

Expanding and taking limits, the agent's payoff satisfies

$$0 = q_t p\pi(1 - s_t) + \dot{W}_t - (r\lambda_t + q_t p)W_t = c - r\lambda_t W_t + \dot{W}_t + p(1 - q_t)W_t. \tag{15}$$

[17] This notational convention allows us to view the rate of experimentation as constant, but the rate of discounting as variable, which turns out to be a convenient representation. For example, it simplifies the evolution of beliefs (cf. (16)). Note, however, that this means that the variable t should not literally be interpreted as real time. We return to this distinction in Section 4.2.7.

[18] To understand the expression for X_t, note that, at any later time u, the agent gets her share $(1 - s_u)$ of π with a probability that is increased by a factor $-q'(u)$, relative to what it would have been had she not deviated. Of course, even if the project is good, it succeeds only at a rate p, and this additional payoff must be discounted.

It follows from Bayes' rule that the belief q follows the familiar law of motion

$$\dot{q}_t = -pq_t(1-q_t). \tag{16}$$

We now have two possible regimes, for a given t, that might arise in a Markovian equilibrium:

1. No delay, and positive principal value. Here, we have $\lambda_t = 1$, and we must confirm that $V_t \geq 0$.

2. Delay and zero principal value. Here, $V_t = 0$, and so $q_t p s_t \pi = c$. For this to be an equilibrium, we must confirm that $\lambda_t \geq 1$.

Indeed, for given parameters, both situations might arise at different values of t, and it may be necessary to "paste" the corresponding value functions.

Since we are considering Markovian equilibria (for now), it will often prove useful to change variables and express V and W in terms of the posterior belief q rather than in terms of t. Let $w(q) = W_t$ and $v(q) = V_t$ be those payoffs to the agent and principal, respectively. With an abuse of notation, we shall write $\lambda(q)$ and $s(q)$ when referring to delay and shares as a function of the belief q.

As with development projects, it is useful to introduce

$$\sigma \equiv p/r \quad \text{and} \quad \xi \equiv (p\pi - c)/c.$$

With this notation, the lower threshold below which there is no experimentation, $\underline{q} = 2c/(p\pi)$, is now $2/(\xi+1)$, and so we restrict attention to $\xi > 1$. In our former terms, we hereafter ignore unprofitable projects.

The detailed analysis of both regimes is carried out in Appendices B.6 and B.7, respectively. When there is never any delay until $q = \underline{q}$, the differential equations have as solutions

$$v(q) = \left[\frac{2-\sigma-q}{\sigma-1} + \frac{q\xi}{\sigma+1} + \left(\frac{1-q}{1-\underline{q}}\frac{\underline{q}}{q}\right)^{\frac{1}{\sigma}}\left(1 - \frac{q(1-\underline{q})}{(\sigma-1)\underline{q}} - \frac{\xi}{\sigma+1}\frac{\underline{q}(1-q)}{1-\underline{q}}\right)\right]\frac{c}{r}, \tag{17}$$

and

$$w(q) = \left[\frac{q\sigma-1}{\sigma-1} - \left(\frac{1-q}{1-\underline{q}}\frac{\underline{q}}{q}\right)^{\frac{1}{\sigma}}\frac{q}{\underline{q}}\frac{\underline{q}\sigma-1}{\sigma-1}\right]\frac{c}{r}. \tag{18}$$

4.2.2 Classifying Research Projects

Using the tools developed in Appendices B.6 and B.7, we now describe the unique Markovian equilibrium of the research project. The nature of this equilibrium depends on the parameters of the research project, with the analysis in Appendices B.6 and B.7 directing our attention to the following four cases:

	$\xi > 3$	$\xi < 3$
$\xi > 1+\sigma$	Lucrative Fast Learning	Lucrative Slow Learning
$\xi < 1+\sigma$	Intermediate Fast Learning	Intermediate Slow Learning

In each of these four cases, the Markovian equilibrium is unique, with details presented in the following four subsections. One of these classifications is familiar from the development-project case, where we identified projects for which $\xi > 1+\sigma$ as lucrative projects and those with $\xi < 1+\sigma$ as intermediate projects. In the case of a research project, another distinction appears, involving ξ alone. To control for the information already contained in the designation of a project as lucrative or intermediate, we rewrite ξ as $\xi - \sigma + p/r$. Those cases in the left column then involve high values of p/r (adjusting parameters so as to keep $\xi - \sigma$ fixed), while those in the right column involve relatively low values of p/r. The higher is p, the more rapidly does the players' posterior probability that the project is good evolve (conditional on failed experiments), while the lower is r, the less steeply are the future periods in which this information is available discounted. We thus characterize a high value of p/r as a "fast-learning" environment and a small value of p/r as a "slow-learning" environment.

4.2.3 Lucrative, Fast-Learning Projects: Full Effort

In this case, the full-effort candidate equilibrium developed in Section 4.1 is the unique Markovian equilibrium: there is no delay until the belief reaches $\underline{q}$ (in case of repeated failures). At this stage, the project is abandoned.

To say that the agent learns quickly is to say that the terminal belief at which the project is abandoned arrives relatively quickly. As a result, future payoffs are not especially valuable to the agent. There is then little scope for reducing incentive costs by reducing the agent's future payoffs. Hence, if it is ever going to be profitable to induce effort from the agent, it will be profitable to induce effort for all subsequent beliefs. Because the project is lucrative, it is optimal to induce effort for all beliefs. This is the most profitable environment, combining a lucrative project with fast learning (which tends to reduce incentive costs), and so it is unsurprising that this environment supports consistent full effort.

4.2.4 Intermediate, Fast-Learning Projects: Delay for High Beliefs

In this case, the unique Markovian equilibrium is characterized by some belief $q^* \in [\underline{q}, 1)$. (The value of q^* is the unique root in $(\underline{q}, 1)$ of v, as defined by (17)). For higher beliefs, there is delay, i.e. the project is only undertaken a fraction of the time, and the principal's payoff is zero. As the belief reaches q^*, delay disappears (in fact, delay is discontinuous at q^*), and the project is then funded at a maximal rate, until the project is abandoned when the belief reaches $\underline{q}$. The equilibrium specifies the minimal amount of delay that drives the agent's payoff down to the point where the principal just breaks even when inducing the agent to work.

Once again, the fast learning ensures that if effort is to be induced for any given belief, it will be induced for all smaller beliefs. It is indeed possible to induce effort for relatively low beliefs, where continuation payoffs are relatively unimportant and hence effort relatively cheap. Because this is only an intermediate project, however, as beliefs get large, incentive costs become so large that full effort is no longer compatible with a nonnegative principal payoff. At this point, the equilibrium requires delay and hence a zero principal payoff. The fast learning here again tends to induce low incentive costs, but the project is only intermediate and not lucrative, precluding full effort for high beliefs (where incentive costs are relatively high).

4.2.5 Lucrative, Slow-Learning Projects: Delay for Low Beliefs

In this case, the unique Markovian equilibrium is characterized by some belief $q^{**} \equiv (2-\sigma)/(\xi+1-2\sigma) \in (\underline{q}, 1)$. When beliefs are higher than q^{**}, there is no delay, as the project gets funded at maximal rate. However, from the point at which the belief reaches q^{**}, funding starts being delayed (here, delay is continuous at q^{**}).

The agent is slow learner here, and hence continuation payoffs are relatively important. As a result, incentive costs are relatively high, and there is both considerable scope and considerable gain to be had from reducing incentive costs by reducing continuation values. We thus have delay for low probabilities that the project is good. Because the project is lucrative, we can capitalize on the resulting relatively small incentive costs by supporting full effort for higher beliefs.

4.2.6 Intermediate, Slow-Learning Projects: Perpetual Delay

In this case, the unique Markovian equilibrium involves delay for all values of $q \in [\underline{q}, 1]$. Once again, slow learning implies that if the agent is to exert effort, the cost-effective way to provide the appropriate incentives involves continuation-payoff-reducing future delay. In this case, however, the combination of an intermediate project and slow learning makes the incentives sufficiently expensive as to preclude any possibility of full effort. This is the least profitable environment (at least as far as Markovian equilibria are concerned), combining an intermediate project with the high incentive costs associated with slow learning, and hence we have no possibility of full effort.

4.2.7 Summary

We collect the highlights of our characterization of Markovian equilibria in Proposition 2 and Figure 4.[19]

Proposition 2 *The research project admits a unique Markovian equilibrium, whose form depends on the project's parameters as follows:*[20]

- *Lucrative, Fast-Learning Projects ($\xi > 1+\sigma$ and $\xi > 3$): The agent exerts full effort until either achieving a success or until the posterior probability of a good project drops below $\underline{q}$. The principal's payoff is positive for all posteriors exceeding $\underline{q}$.*

- *Intermediate, Fast-Learning Projects ($\xi < 1+\sigma$ and $\xi > 3$): The principal initially induces partial effort from the agent by mixing between offering an effort-inducing contract and offering no contract, until the posterior probability drops to a threshold $q^* > \underline{q}$. The principal subsequently induces full effort until the posterior hits $\underline{q}$. The principal's payoff is zero for $q > q^*$ and positive for $q \in (\underline{q}, q^*)$.*

- *Lucrative, Slow-Learning Projects ($\xi > 1+\sigma$ and $\xi < 3$): The principal initially induces full effort, enjoying a positive payoff, until the posterior drops to a threshold $q^{**} > \underline{q}$, at which point the principal elicits partial effort and commands a payoff of zero.*

- *Intermediate, Slow-Learning Projects ($\xi < 1+\sigma$ and $\xi < 3$): The principal induces only partial effort from the agent, for every posterior, with a zero payoff.*

Appendix B.9 provides the proof. Perhaps the most interesting case here is that of an intermediate, fast-learning project. The principal's payoff is zero when she is quite optimistic about the project, and then becomes positive when the principal becomes pessimistic. Hence, the principal would actually actually prefer to be pessimistic about the project. Given the chance to "burn" some probability that the project is good, the principal would do so.

Development and research projects appear to be fundamentally different, with the stationarity of the former contrasting with the backward induction from the terminal

[19] Bergemann and Hege [2] use an analogous four-fold classification to describe the Markovian equilibria in their model, with high-return and low return taking the place of our lucrative and intermediate classification, and with high discount factor and low discount factor taking the place of our fast learning and slow learning designations.

[20] Of course, in case $\overline{q}$ is lower than the relevant threshold, part of the description does not apply.

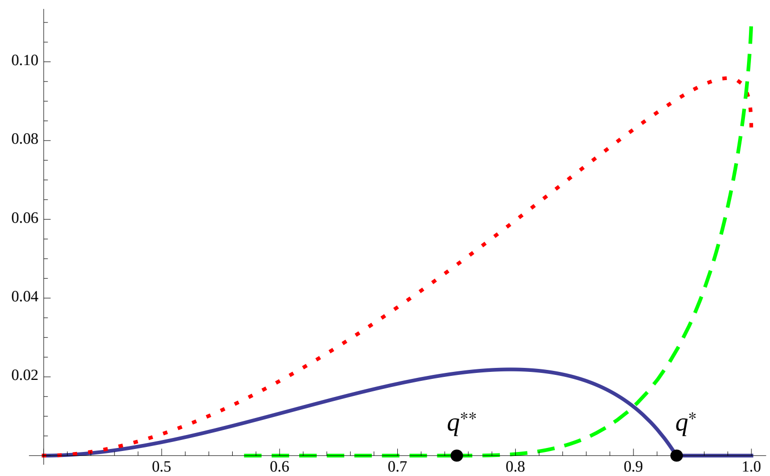

Figure 4: The principal's payoff (vertical axis) from the Markovian equilibrium, as a function of the probability q that the project is good (horizontal axis). The parameters are $c/r = 1$ for all curves. For the dotted curve, $(\xi, \sigma) = (4, 27/10)$, giving a lucrative, fast-learning project, with no delay and the principal's value positive throughout. For the dashed curve, $(\xi, \sigma) = (5/2, 5/4)$, giving a lucrative, slow-learning project, with delay and a zero principal value below the value $q^{**} = 0.75$. For the solid curve, $(\xi, \sigma) = (4, 4)$, giving an intermediate, fast-learning project, with delay and a zero principal value for $q > q^* \approx .94$. We omit the case of an intermediate, slow-learning project, where the principal's payoff is 0 for all q. Notice that the principal's payoff need not be monotonic in the probability the project is good.

belief $\underline{q}$ of the latter. However, in the two possible cases of a slow-learning research project, there is delay in equilibrium for beliefs that are low enough. There might be sufficient delay that the possibility of beliefs reaching $\underline{q} = 2/(\xi+1)$ is essentially irrelevant in determining payoffs. Indeed, it is straightforward to verify that not only does the "rate of delay" $\lambda(q)$ diverge as $q \searrow \underline{q}$, but so does the cumulative delay, i.e.,

$$\lim_{q \searrow \underline{q}} \int_q^{\overline{q}} \lambda(u) du = \infty. \tag{19}$$

That is, the event that the project is abandoned is entirely discounted away in the case of a slow-learning research project. Remembering that we have modeled delay in terms of the discount factor (see footnote (17)), this means that in real time, the belief $\underline{q}$ is only reached asymptotically—the pace of experimentation slows sufficiently fast that the belief $\underline{q}$ is never actually reached and hence the project is never actually abandoned. The contrast between research and development projects is then not as sharp as it initially appears.

4.3 Non-Markovian Equilibria

We now study the other perfect Bayesian equilibria of the game. Our first step is to understand how severely players might be credibly punished for a deviation, and thus, each player's lowest equilibrium.

4.3.1 Lowest Equilibrium Payoffs: Slow-Learning Projects ($\xi < 3$)

We first discuss the relatively straightforward case of a slow-learning project. In the corresponding unique Markovian equilibrium, there is delay for all beliefs that are low enough, i.e., for all values of q in a lower interval I, where $I = [\underline{q}, q^{**}]$ in the lucrative case and $I = [\underline{q}, 1]$ in the intermediate case. When there is delay, the principal's equilibrium payoff is zero. This in turn implies that, for beliefs $q \in I$, there exists a trivial non-Markovian equilibrium in which the principal offers no funding on the equilibrium path, and so both players get a zero payoff. This is supported by the convention that if the principal deviates and makes an offer, players revert to the strategies of the Markovian equilibrium, thus ensuring that the principal has no incentive to deviate. Let us refer to this equilibrium as the "full-stop equilibrium."

The existence of the full-stop equilibrium implies that, at least for q in the interval I, both players' minmax (and lowest equilibrium) payoffs are 0. We claim that this is also the case for *all* lucrative projects. (The interval I already exhausts the set of beliefs for intermediate projects.) First, we can construct a non-Markovian "no-delay" equilibrium in which there is no delay from the prior $\overline{q}$ until the belief reaches some given $\hat{q} \in I$ with $\hat{q} > \underline{q}$, at which point players revert to the full-stop equilibrium (with the contract s_t making the agent indifferent between shirking and not for all intermediate beliefs $q \in (\hat{q}, \overline{q}]$). This no-delay equilibrium gives a positive payoff to the principal only if $\overline{q}$ is not too large relative to the fixed belief $\hat{q}$. That is, there exists $\overline{q}(\hat{q})$ at which this equilibrium gives the principal a payoff of zero. Note that it must be that $\lim_{\hat{q} \to \underline{q}} \overline{q}(\hat{q}) = \underline{q}$, since otherwise the payoff function $v(q)$ defined by (17) would not be negative for values of q close enough to $\underline{q}$. This implies that $\overline{q}(I)$, the image of I under this map $\hat{q} \mapsto \overline{q}(\hat{q})$, intersects I, and hence $I_1 \equiv I \cup \overline{q}(I)$ is an interval of length strictly greater than I. Hence, for every initial belief $\overline{q}$ in I_1, we can choose $\hat{q} \in I$ such that $\overline{q} = \overline{q}(\hat{q})$ and construct a no-delay equilibrium, giving the principal a value of 0. We can then in turn construct an equilibrium for every such belief in which no offers are ever made, with any offer prompting the agents to switch immediately to the no-delay equilibrium, in the process obtaining an equilibrium for $\overline{q}$ in which both agents receive their minmax payoffs of zero.

We may now repeat the entire argument, beginning with the interval I_1. For each such belief $\hat{q} \in I_1$, we can use this belief as the point at which experimentation stops in a no-delay equilibrium for some initial belief $\overline{q}(\hat{q})$ chosen to be sufficiently large as to give the principal a payoff of zero. This allows us to obtain a collection of no-delay equilibria giving the principal a payoff of zero, and hence of full-stop equilibria giving both agents payoffs of zero, for an interval I_2 which is a superset of I_1. Continuing, we have a sequence of intervals $I_{n+1} \equiv I_n \cup \overline{q}(I_n)$.

Plainly, every $\overline{q}$ in $[\underline{q}, 1)$ is in I_n for n large enough. We have thus shown that for every prior belief, there is an equilibrium giving each agent the minmax payoff of 0. We shall refer to this equilibrium payoff as the worst equilibrium payoff. Summarizing:

Lemma 4.2 *For slow-learning projects and for any probability that the project is good, there exists an equilibrium giving both players their minmax payoffs of zero.*

4.3.2 Lowest Equilibrium Payoffs: Fast-Learning Projects ($\xi > 3$)

This case is considerably more involved, as the unique Markovian equilibrium features full effort for initial beliefs that are low enough, i.e., for all values of q in a lower interval I, where $I = [\underline{q}, q^*)$ for intermediate projects and $I = [\underline{q}, 1]$ for lucrative projects. For intermediate fast-learning projects and relatively *high* prior beliefs ($q \geq q^*$), the same arguments as above yield that there is a worst equilibrium with a zero payoff for both players. But what about beliefs $q \in I$?

Because the principal's payoff in the corresponding Markovian equilibrium is not zero, we can no longer construct a full-stop equilibrium with zero payoffs. Indeed, the Markovian equilibrium yields the principal's minmax payoff. Intuitively, by successively making the offers associated with the Markovian equilibrium, the principal can secure this payoff. This intuition does not provide a complete argument, because the principal cannot commit to this sequence of offers, and the agent's behavior, given such an offer, depends on his beliefs regarding future offers. Since the argument involves game-theoretic considerations, it must be performed in discrete time, before limits can be taken. Appendix B.10 proves:

Lemma 4.3 *When $\xi > 3$ (fast-learning project) and $q \notin I$ (possible for an intermediate project), there is an equilibrium giving the principal her minmax value of* 0*. When $\xi > 3$ and $q \in I$, the principal's minmax payoff converges (as time periods become short) to the payoff of the full-effort equilibrium presented in Section 4.2.3.*

Having determined the principal's lowest equilibrium payoff (which is also her minmax payoff), we now turn to the agent's lowest equilibrium payoff. In such an equilibrium, it must be the case that the principal is getting her minmax payoff herself (otherwise, we could simply increase delay, and threaten the principal with reversion to the Markovian equilibrium in case she deviates, yielding a new equilibrium, with a lower payoff to the agent). Also, in such an equilibrium, the agent must be indifferent between accepting or rejecting offers (otherwise, by lowering the offer, we could construct an equilibrium with a lower payoff to the agent).

Therefore, we must identify the smallest payoff that the agent can get, subject to the principal getting her minmax payoff, and the agent being indifferent between accepting and rejecting offers. This is an optimization problem, and so we may carry it out directly in the continuous-time framework. This problem turns out to be remarkably tractable, as explained below and summarized in Lemma 4.4. However, readers without any particular penchant for Riccati equations may skip the following derivations without much loss.

Let us denote by v_M, w_M the payoff functions in the Markovian equilibrium, given by (17) and (18), and by s_M the corresponding share (as a function of q). Our purpose, then, is to identify all other solutions (v, w, s, λ) to the differential equations characterizing such an equilibrium, for which $v = v_M$, and in particular, the one giving the lowest possible value of $w(\overline{q})$. Rewriting the differential equations (14) and (15) in terms of beliefs q, (v_M, w, s, λ) solves

$$0 = qps\pi - c - (r\lambda + pq)v_M(q) - pq(1-q)v_M'(q),$$

and

$$0 = qp\pi(1-s) - pq(1-q)w'(q) - (r\lambda + qp)w(q) = c - r\lambda w(q) - pq(1-q)w'(q) + p(1-q)w(q).$$

Since s_M solves the first equation for $\lambda = 1$, any alternative solution (w, s, λ) with $\lambda > 1$ must satisfy (by subtracting the first equation for $(s_M, 1)$ from the first equation for (s, λ))

$$r(\lambda - 1)v_M(q) = qp\pi(s - s_M).$$

Therefore, as is intuitive, $s > s_M$ if and only if $\lambda > 1$: delay allows the principal to increase her share. This allows us to eliminate s from the other two equations, and combining them gives

$$pw(q) = qp\pi - 2c - (r\lambda + pq)v_M(q) - pq(1-q)v_M'(q).$$

Solving for λ in terms of w gives

$$r\lambda = \frac{qp\pi - 2c - pq(1-q)v_M'(q) - pw(q)}{v_M(q)} - pq.$$

Inserting in the second equation for w and rearranging yields

$$pw(q)^2 - (pq\pi - 2c - pq(1-q)v_M'(q) - pv_M(q))w(q) + v_M(q)c = pq(1-q)v_M(q)w'(q).$$

This is a Riccati equation, for which the general solution can be derived from any particular solution. Fortunately, we know one such solution, namely w_M. Define

$$\begin{aligned}\Phi(q) &\equiv \exp\left\{\int [2pw_M(u) - (pu\pi - 2c - pu(1-u)v_M'(u) - pv_M(u))]\frac{du}{pu(1-u)v_M(u)}\right\} \\ &= \frac{(1-q)^{1+\frac{2}{\sigma}}q^{1-\frac{2}{\sigma}}}{(1-\tilde{q})^{1+\frac{2}{\sigma}}\tilde{q}^{1-\frac{2}{\sigma}}}\frac{v_M(\tilde{q})}{v_M(q)}\exp\left\{\frac{c}{\sigma r}\int_{\tilde{q}}^{q}\frac{(\xi+1)u-2}{u(1-u)v_M(u)}du\right\},\end{aligned}$$

for some arbitrary $\tilde{q} \in I$, where the second equality uses our knowledge of v_M. As is well-known, the general solution to the Riccati equation can be written as

$$w(q) = w_M(q) - \Phi(q)\left[C + \int \frac{\Phi(u)}{u(1-u)v_M(u)}du\right]^{-1}.$$

(The particular solution w_M corresponds to $C = \infty$). It remains to determine which solution yields the lowest payoff to the agent. Appendix B.11 proves:

Lemma 4.4 *When $\xi > 3$ (fast-learning) and $q \notin I$ (possible for an intermediate project), there is an equilibrium giving the agent his minmax value of* 0 *(as well as giving the principal payoff* 0*). When $\xi > 3$ and $q \in I$, the infimum over the agent's equilibrium payoffs is given by*

$$w(q) = w_M(q) - \frac{\Phi(q)}{C^* + \int_{\tilde{q}}^{q} \frac{\Phi(u)}{u(1-u)v_M(u)}du}, \tag{20}$$

where

$$C^* \equiv \frac{v_M(\tilde{q})}{(1-\tilde{q})^{1+\frac{2}{\sigma}}\tilde{q}^{1-\frac{2}{\sigma}}} \int_{\underline{q}}^{\tilde{q}} \frac{\left(\frac{1-u}{u}\right)^{\frac{2}{\sigma}}}{v_M(u)^2} \exp\left\{\frac{c}{\sigma r}\int_{\tilde{q}}^{u} \frac{(\xi+1)y-2}{y(1-y)v_M(y)}dy\right\} du. \tag{21}$$

There may or may not be an equilibrium that achieves the infimum over the agent's equilibrium payoffs (see Appendix B.11 for the necessary and sufficient condition). If not, of course, there are equilibria giving the agent payoffs arbitrarily close to this infimum. To simplify subsequent statements (by eliminating straightforward "$\epsilon - \delta$" phrases), we will refer to this infimum as the lowest equilibrium payoff, even if it can only be approximated. Since in this equilibrium the principal also obtains her lowest equilibrium payoff, we will refer to this strategy profile as the worst equilibrium.

The somewhat formidable looking (20)–(21) identify two equilibrium payoffs. One is the agent's payoff in the Markovian equilibrium, given by the function $w_M(q)$ (and corresponding to the solution of (20) we would obtain if we set $C^* = \infty$, eliminating the second term). The other is the lower payoff $w(q)$, corresponding to the value C^* given by (21). Figure 5 shows w and $w_M < w$ for the case $\xi = 4, \sigma = 2$. (In this example, the infimum is not achieved.)

We can check some consistency properties of these results. First, consider an intermediate, fast-learning project. It is straightforward to check that the delay associated with the agent's worst equilibrium payoff grows without bound as $q \nearrow q^*$ (reminiscent of (19)), and so the agent's lowest payoff tends to 0 (as does the principal's, by definition of q^*). The worst equilibrium payoffs for the principal and agent are both 0 for $q > q^*$, and hence these worst payoffs are continuous at q^*.

Second, let us consider a lucrative, fast-learning project, and ask what happens as $q \nearrow 1$, pushing us closer and closer to the case of a development project. If $\xi \geq (1+$

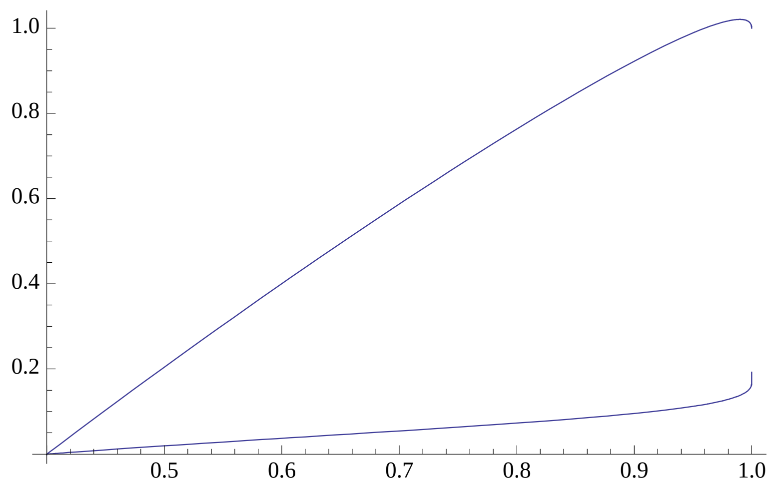

Figure 5: Functions $w_M(q)$ (upper curve) and $w(q)$ (lower curve), giving the agent's payoffs from the Markovian equilibrium (w_M) and the infimum over the agent's equilibrium payoffs (w) as a function of the probability q that the project is good (measured on the horizontal axis) for $\xi = 4$ and $\sigma = 2$ (lucrative, fast-learning project). The horizontal axis is positioned at $\underline{q} = 2/5$. Notice that $w(1) > w_M(1)$, which is consistent with the fact that $\xi < (1+\sigma)^2$, indicating that, as $q \to 1$, this research project converges to the case of a moderately lucrative development project, with $w(1)$ and $w_M(1)$ corresponding to the smallest and largest agent payoff pictured in the third panel of Figure 3.

$\sigma)^2$, the lower payoff $w(q)$ tends to $w_M(1) = c/r$ as $q \to 1$. In this case, there is no equilibrium for $q = 1$ giving the agent a payoff strictly below the Markovian equilibrium payoff, while simultaneously giving the principal a payoff as high as the Markovian payoff. This corresponds to the case of a highly lucrative development project, with its unique equilibrium payoff (cf. Proposition 1 and Figure 3). On the other hand, if $\xi < (1+\sigma)^2$ (but still $\xi \geq 1+\sigma$, so that we have the case of a lucrative research project) the solution $w(q)$ tends to $V^*/\sigma < c/r$ as $q \to 1$. In this case, the limiting values $V^*/\sigma = w(1)$ and $c/r = w_M(1) < w(1)$ are precisely the lowest and highest agent equilibrium payoffs for a development project (Proposition 1 and Figure 3).

The worst equilibrium payoffs for all but a lucrative, slow-learning research project thus converge to the corresponding payoffs for a development project, as $q \nearrow 1$.[21] However, $w_M(\overline{q}) < w(\overline{q})$ for all $\overline{q} < 1$. Hence, there are always non-Markovian equilibria for a research project that yield the agent a payoff below the Markovian equilibrium payoff.

[21] This continuity result is immediate for intermediate research projects, with their zero-payoff equilibria in the case of both development and research projects.

Therefore, there are also always equilibria that yield the principal a payoff above the Markovian equilibrium payoff. The development project, in contrast, is a somewhat special case, since for highly lucrative development projects the Markovian equilibrium payoffs are the only equilibrium payoffs.

Finally, lucrative slow-learning research projects present a discontinuity. For any prior probability less the one, we have an equilibrium giving both agents a payoff of zero, while the lowest equilibrium payoffs for the corresponding development project are bounded away from zero.

4.3.3 The Set of Equilibrium Payoffs: Slow-Learning Projects ($\xi < 3$)

We have shown that the worst equilibrium, in the sense of simultaneously minimizing the equilibrium payoffs of both players, is well defined. To identify the set of equilibrium payoffs, it remains to describe the frontier of the equilibrium payoff set.

As was the case for a development project, efficiency requires that delay (if unavoidable) be postponed as much as possible. Appendix B.12 proves:

Lemma 4.5 *Any equilibrium payoff on the frontier can be achieved by an equilibrium in which there is no delay up to some belief, at which point the equilibrium reverts to the worst equilibrium.*[22]

Hence, as in the development case, a simple class of equilibria spans the entire equilibrium payoff set. They consist of an initial randomization between the worst equilibrium and an equilibrium on the frontier. The latter equilibrium specifies full effort until some time $T < \infty$, after which play reverts to the worst equilibrium from that point on.

We now focus on the equilibrium on the frontier that is best from the point of view of the principal, and briefly describe some of its features. For slow-learning projects, characterizing the principal's favorite equilibrium is straightforward, given Lemma (4.5) and our finding that the agent's lowest equilibrium payoff is zero. Somewhat paradoxically, this means that the principal's maximal equilibrium payoff is rather high, since she can resort to a severe punishment.

Let us fix an arbitrary belief $q_1 \in [\underline{q}, \overline{q}]$ at which point the project is abandoned in favor of a continuation equilibrium with payoffs $(0, 0)$. We can then solve for the corresponding differential equations determining the principal's value v and agent's value w, as a function of the probability q that the project is good, with boundary conditions $v(q_1) = w(q_1) = 0$. To determine the optimal stopping probability q_1, we take the derivative of $v(\overline{q})$ with

[22] More precisely, we must remember that the lowest equilibrium payoff to the agent might not be achieved, so that the "frontier" described here identifies payoffs to which we can come arbitrarily close as period length becomes arbitrarily short.

respect to q_1 to obtain

$$\frac{\partial v(\overline{q})}{\partial q_1} = \left(\frac{1-q}{q}\frac{q_1}{1-q_1}\right)^{\frac{1}{\sigma}} \frac{q(1-q_1)-(1-q)q_1\xi}{q_1^2(1-q_1)^2},$$

whose numerator reduces to $\overline{q}-\underline{q} > 0$ when evaluated at $q_1 = \underline{q}$. Therefore, the principal's favorite equilibrium involves termination at a belief q_1 strictly above $\underline{q}$. In fact, it is not hard to determine the optimal belief at which the project must be abandoned, leading to:

Proposition 3 *For a slow-learning research project, the principal's equilibrium payoff is maximized by an equilibrium in which the agent works at the maximum possible rate until terminating experimentation altogether at some belief $q_1 > \underline{q}$, given by*

$$q_1 = \frac{2\overline{q}}{2\overline{q}-1+\sqrt{1-4\overline{q}(1-\overline{q})\xi}}.$$

The principal's payoff from this equilibrium is increasing in the initial probability $\overline{q}$ that the project is good, and exceeds the payoff available from a Markovian equilibrium.

Figure 6 compares the principal's payoff from this "best" (from her point of view) equilibrium with the corresponding payoff from a Markovian equilibrium. The larger is the initial belief $\overline{q}$ that the project is good, the larger the terminal belief $q_1 > \underline{q}$ at which the project is abandoned.

4.3.4 The Set of Equilibrium Payoffs: Fast-Learning Projects ($\xi > 3$)

The case of a fast-learning project raises similar issues, though is somewhat less tractable. We have:

Proposition 4 *For a fast-learning research project, the principal's equilibrium payoff is maximized by an equilibrium in which the agent works at the maximum possible rate until reaching some belief $q_1 > \underline{q}$, at which point play switches to a continuation equilibrium involving partial effort that minmaxes the agent. The principal's payoff is increasing in the initial probability $\overline{q}$ that the project is good, as is the probability q_1 at which the project is abandoned, and the principal's payoff exceeds that from a Markovian equilibrium.*

The form of the equilibrium again follows from Lemma 4.5. A closed-form solution for q_1 is elusive,[23] but it is not hard to see that this belief increases in $\overline{q}$. In addition, the equilibrium will revert to the continuation giving the agent his lowest payoff before the belief drops to $\underline{q}$. This can be shown by explicitly determining the payoff of the principal

[23]We can solve for q_1 for the case of an intermediate, fast-learning project and sufficiently large $\overline{q}$, in which case the agent's minmax payoff is as in a slow-learning project.

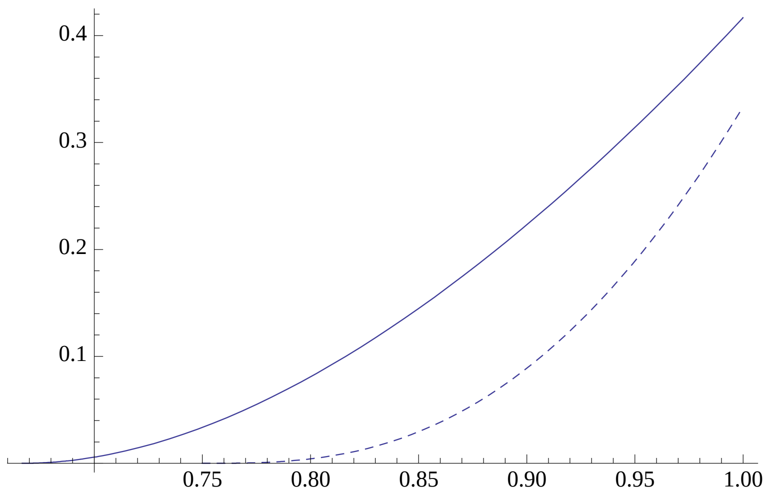

Figure 6: The solid line gives the principal's payoff $v(q)$ from the equilibrium maximizing the principal's payoff, as a function of the initial probability q that the project is good (measured on the horizontal axis). The dotted line gives the principal's payoff from the Markovian equilibrium. In this figure, $\xi = 2$ and $\sigma = 1/2$ (a lucrative, slow-learning research project). We have $q^{**} = 3/4$, so that the principal's payoff in the Markovian equilibrium is zero for $q < 3/4$, and $\underline{q} = 2/3$, so that the Markovian equilibrium abandons the project at $q = 2/3$. If (for example) $\overline{q} = 3/4$, then the principal earns a strictly positive payoff in her best equilibrium, which abandons the project at $q_1 = .7208$.

as a function of q_1, the belief at which switching occurs (the boundary condition at belief q_1 is known, as the principal then gets $v_0(q_1)$ and the agent gets $w(q_1) = \underline{w}(q_1)$). Taking derivatives at $q_1 = \underline{q}$ gives that

$$\frac{\partial v(\overline{q})}{\partial q_1}\Big|_{q_1=\underline{q}} = \frac{4(\xi+1)^3}{\sigma(\xi-1)^2}\left(\frac{\underline{q}}{1-\underline{q}}\frac{1-\overline{q}}{\overline{q}}\right)^{\frac{1}{\sigma}}(\overline{q}(\xi+1)-2) > 0,$$

and so switching before the belief reaches $\underline{q}$ is always better for the principal. Figure 7 compares the payoff from the principal's best equilibrium with the Markovian payoff, for parameters corresponding to a lucrative, fast-learning research project.

In the case of intermediate, fast-learning projects, the principal's payoff-maximizing equilibrium reverses the properties of the Markovian equilibrium. The Markovian equilibrium resorts to delay early in the experimentation process, finishing with a burst of full effort that (in the absence of a success) pushes the project across the abandonment threshold $\underline{q}$. The payoff-maximizing equilibrium, instead, begins with a period of full effort, only to switch (at q_1) to a partial effort continuation. Indeed, any equilibrium on the efficiency frontier has this property, confirming the inefficiency of the Markovian equilibrium.

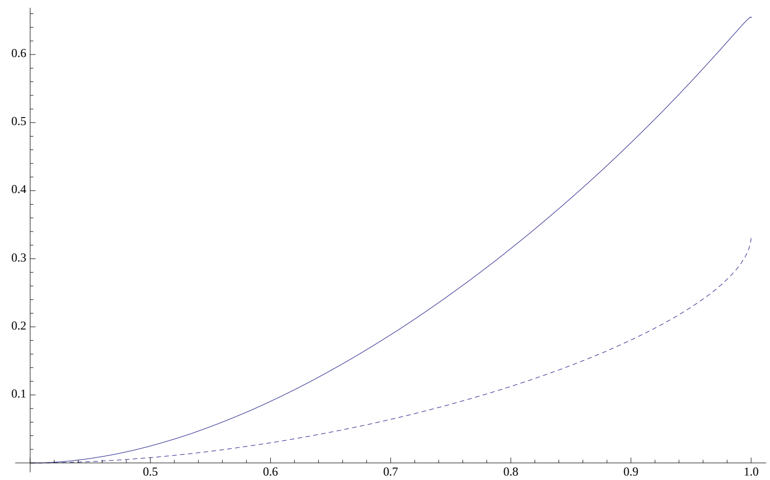

Figure 7: The solid line gives the principal's payoff $v(q)$ from the equilibrium maximizing the principal's payoff, as a function of the initial probability q that the project is good (measured on the horizontal axis). The dotted line gives the principal's payoff from the Markovian equilibrium. In this figure, $\xi = 4$ and $\sigma = 2$ (a lucrative, fast-learning project).

5 Discussion

A fundamental point runs through both Bergemann and Hege [2] and the current paper—if an agent is to have incentives to invest in experimentation, the agent must be compensated not only for the foregone chance to divert the investment funds for personal use, but also for the fact that current experimentation risks giving rise to a success that closes off future experimentation rents. This imposes a dynamic agency cost on top of the expected agency cost. As a result, creating incentives may entail deliberately curtailing the scale of the agent's experimentation, reducing the cost of current incentives by reducing the allure of future experimentation. We see this in the fact that a stationary equilibrium in the case of a development project (known success probability) may require only partial effort from the agent, and in the fact that a Markovian equilibrium in the case of a research project (unknown success probability) may require partial effort for some posteriors.

At this point, our papers part ways. Bergemann and Hege [2] invest the agent with all of the bargaining power by assuming that the agent makes the contract proposal in each period. This in turn ensures that the principal's payoff is zero in every period of every equilibrium. This eliminates many of the intertemporal links that appear in our model, in the process ensuring that the Markovian equilibria are also the only equilibria.

We assume instead that the principal (the venture capitalist) rather than the agent (the experimenting entrepreneur) makes the offers and hence has the bargaining power. We view this as a realistic description of many settings in which venture capital is in short supply (cf. Blass and Yosha [4]). Given the agent's private information (on whether he has actually conducted experiments), however, this significantly complicates the analysis. What new insights have we found?

Once the principal has the bargaining power, both players potentially receive positive payoffs, giving rise to a richer set of intertemporal trade-offs. Our basic result is then that constrained efficiency requires completely front-loading the agent's effort, coupling a phase of full effort with an abrupt, premature termination of the project.[24] Sequential rationality constraints typically preclude attaining this constrained efficient outcome, but the best *equilibrium* outcomes nonetheless feature front-loaded effort and premature termination or attenuation. Our Markovian equilibria are thus joined by a collection of other equilibria, including equilibria with nonstationary outcomes, despite the eminently stationary nature of the project, in the case of a development project; and equilibria that can precisely reverse the pattern of effort found in the Markovian equilibrium (for intermediate, fast-learning research projects) or lead to premature abandonment of the project (for lucrative, slow-learning research projects). The behavioral implications of our model are thus quite different. Indeed, if one finds efficiency compelling, then the various Markovian cases of Bergemann and Hege (in which effort may be front-loaded, or may be back-loaded) give way here to the simple prescription of front-loaded and prematurely reduced effort. Allowing a seemingly more complicated array of equilibrium behavior simplifies the qualitative nature of the results, while bringing on board what appear to be some key features of real venture capital markets.

We have worked throughout with a seemingly restricted set of contracting instruments—the principal can either offer the agent nothing, or can advance the agent the funding c required to conduct an experiment (but no more or less) and then offer the agent a share of the proceeds in the event of a success. Would more general contracts allow us to do more? The key restriction here is that the agent cannot make payments to the principal. If such payments were allowed, the agent could buy the right to receive funding from the principal and retain all of the rewards in the event of a success, eliminating the agency problem. If this were possible, of course, it is not clear why the agent needs the principal in the first place. If we are to retain the essence of the agency problem by imposing a limited liability constraint on the agent, then given the binary (success/failure) nature of the possible experimental outcomes, there is no loss of generality in restricting the principal to creating incentives by offering the agent a (time-dependent) share of the proceeds of a success.

[24] Effort is front-loaded in the Markovian equilibrium of a lucrative, slow-learning project, but for reasons that have nothing to do with efficiency considerations (as shown by the fact that an intermediate, fast-learning project *back-loads* effort).

A Appendix: Formal Description of the Model

The horizon is infinite and time is discrete, indexed by $t = 0, 1, 2, \ldots$. There are two states of the world, $\omega \in \{\omega_0, \omega_1\}$. The project is bad and a success never obtains when the state is ω_0, while if the project is good (state ω_1), success obtains with probability p in any given period in which (full) effort is exerted. There are two players: player 1 is the principal, and player 2 is the agent. Players share a common prior $\overline{q}$ that $\omega = \omega_1$.

In every period t, the players first observe the realization x_t of a random variable X_t that is uniformly drawn from the unit interval, independently across periods, and independently of ω. The principal then chooses an action $a_{1t} \in A_1 = \{NF\} \cup [0, 1]$, where the action NF has the interpretation of no funding for this period, and $s \in [0, 1]$ has the interpretation of the principal providing funding c to the agent and retaining share s of the payoff π in case of success in this period. Conditional on the principal not choosing NF, the agent then chooses an action in $a_{2t} \in A_2 = \{0, 1\}$, with the interpretation that the agent chooses to exert no effort if the action is 0, and exerts effort otherwise. The choice of action by the agent is unobserved, while the actions of the principal are observed. Nature than draws an action a_{0t} that is necessarily 0 (failure) if either $\omega = \omega_0$, the principal chose NF, or the agent chose 0; and is otherwise 0 with probability $1 - p$ and 1 (success) with probability p, independently across periods and of ω and the x_t.

The game ends, if ever, when the first success obtains. Therefore, an outcome is either a finite vector $(x_0, a_{10}, a_{20}, a_{00}, x_1, a_{11}, a_{21}, a_{01}, \ldots, x_t, a_{1t}, a_{2t}, a_{0t})$, with $a_{0\tau} = 0$ if and only if $\tau < t$; or is an infinite vector $(x_0, a_{10}, a_{20}, a_{00}, x_1, a_{11}, a_{21}, a_{01}, \ldots,)$, with $a_{0\tau} = 0$ for all τ. A history of length t for the principal is a vector

$$h_1^t = (x_0, a_{10}, a_{00}, x_1, a_{11}, a_{01}, \ldots, x_{t-1}, a_{1t-1}, a_{0t-1}, x_t) \in H_1^t \equiv ([0, 1] \times A_1 \times A_0)^t \times [0, 1],$$

(set $h_1^0 = \{x_0\} \in H_1^0$), with $a_{0\tau} = 0$ for all τ. A history of length t for the agent is a vector

$$h_2^t = (x_0, a_{10}, a_{20}, a_{00}, x_1, a_{11}, a_{21}, a_{01}, \ldots, x_{t-1}, a_{1t-1}, a_{2t-1}, a_{0t-1}, x_t, a_{1t}),$$

in $H_2^t \equiv ([0, 1] \times A_1 \times A_2 \times A_0)^t \times [0, 1]^2$ (set $h_2^0 = \{x_0, a_{10}\} \in H_2^0$ and take $a_{2t} = \emptyset$ if $a_{1t} = NF$). Notice that we restrict attention to histories in which, in the last period, the principal chose to fund the project, as otherwise the agent has no decision to make in that period. A strategy for the principal is a probability transition $\sigma_1 = (\sigma_{1t})_t$ from $H_1 = \cup_{t \geq 0} H_1^t \to \triangle A_1$ with the interpretation that $s_{1t}(h_1^t)$ is the (possibly random) action chosen by the principal in period t given history h_1^t ($\triangle A_1$ is endowed with the weak*-topology). Similarly, a strategy for the agent is a probability transition $\sigma_2 = (\sigma_{2t})_t$ from $\cup_{t \geq 0} H_2^t \to \triangle A_2$. The random variable ω, along with any strategy profile $\sigma = (\sigma_1, \sigma_2)$ and the collection of random variables $(X_t)_t$ induce a probability distribution denoted by $\mathbb{P}_{(\omega,\sigma)}$ (the random variables X_t are omitted from the notation) over outcomes. This in turn induces a probability distribution over the random time $\tau \in \mathbb{N} \cup \{\infty\}$ at which a success arrives (if ever). The expected payoff of the principal, given a strategy profile σ and a history h_1^t, is then

$$V_1(\sigma|h_1^t) = (1-\delta)\sum\nolimits_{k=0}^{\infty} \delta^k \mathbb{E}_{\omega,\sigma}[\sigma_1[h_1^{t+k}]\pi \cdot \mathbf{1}_{\tau=t+k} - c\mathbf{1}_{\sigma_1[h_1^{t+k}]\neq NF}],$$

where $\mathbf{1}_E$ is the indicator function of the event E. Similarly, the agent's expected payoff, given h_2^t is given by

$$V_2(\sigma|h_2^t) = (1-\delta)\sum\nolimits_{k=0}^{\infty} \delta^k \mathbb{E}_{\omega,\sigma}[(1-\sigma_1[h_1^{t+k}])\pi \cdot \mathbf{1}_{\tau=t+k} + c\cdot \mathbf{1}_{\sigma_2[h_2^{t+k}]=0}\mathbf{1}_{\sigma_1[h_1^{t+k}]\neq NF}].$$

B Appendix: Derivations and Proofs

B.1 Proof of Lemma 3.1

Let $\overline{W}$ be the agent's maximal equilibrium payoff. We can restrict attention to cases in which the principal has offered a contract to the agent, and in which the agent works.[25]

We first note that a lower bound on the principal's payoff is provided by *always* choosing that value s' satisfying (and hence inducing the agent to work, no matter how lucrative a continuation value the agent expects)

$$(1-\delta)p(1-s')\pi + \delta(1-p)\overline{W} = (1-\delta)c + \delta\overline{W},$$

which we can rearrange to give

$$(1-\delta)[ps'\pi - c] = -\delta p\overline{W} + (1-\delta)[p\pi - 2c],$$

and hence a principal payoff of

$$\frac{(1-\delta)[ps'\pi - c]}{1-\delta(1-p)} = \frac{(1-\delta)[p\pi-2c] - \delta p\overline{W}}{1-\delta(1-p)}.$$

We can then characterize $\overline{W}$ as the solution to the maximization problem:

$$\begin{aligned}
\overline{W} \quad &= \quad \max_{s,W,V}(1-\delta)p(1-s)\pi + \delta(1-p)W \\
&s.t. \quad \overline{W} \geq (1-\delta)c + \delta W \\
&\qquad \overline{W} \geq W \\
&\qquad (1-\delta)(ps\pi - c) + \delta(1-p)V \geq \frac{(1-\delta)[p\pi-2c]-\delta p\overline{W}}{1-\delta(1-p)} \\
&\qquad V + W \leq \frac{(1-\delta)(p\pi - c)}{1-\delta(1-p)},
\end{aligned}$$

[25] If c is an upper bound on the agent's payoff conditional on a contract being offered, then it must also be an upper bound on an equilibrium path in which a contract is offered in the current period with probability less than one. If the agent shirks, then we have $\overline{W} = (1-\delta)c + \delta\overline{W}$, giving $\overline{W} = c$.

where W and V are the agent's and principal's continuation values, the first constraint is the agent's incentive constraint, the second establishes $\overline{W}$ as the largest agent payoff, the third imposes the lower bound on the principal's payoff, and the final constraint imposes feasibility. Notice that if the first constraint binds, then (using the second constraint) we immediately have $\overline{W} \leq c$, and so we may drop the first constraint. Next, the final constraint will surely bind (otherwise we can decrease s and increase V so as to preserve the penultimate constraint while increasing the objective), allowing us to write

$$\begin{aligned} \overline{W} &= \max_{s,W}(1-\delta)p(1-s)\pi + \delta(1-p)W \\ & s.t. \quad \overline{W} \geq W \\ & \quad (1-\delta)(ps\pi - c) + \delta(1-p)\left[\frac{(1-\delta)(p\pi - c)}{1-\delta(1-p)} - W\right] = \frac{(1-\delta)[p\pi - 2c] - \delta p\overline{W}}{1-\delta(1-p)}. \end{aligned}$$

Now notice that the objective and the final constraint involve identical linear tradeoffs of s versus W. We can thus assume that $W = \overline{W}$, allowing us to write the problem as

$$\overline{W} = \max_{s}(1-\delta)p(1-s)\pi + \delta(1-p)\overline{W} \tag{22}$$

$$s.t. \quad (1-\delta)(ps\pi - c) + \delta(1-p)\left[\frac{(1-\delta)(p\pi - c)}{1-\delta(1-p)} - \overline{W}\right] = \frac{(1-\delta)[p\pi - 2c] - \delta p\overline{W}}{1-\delta(1-p)}. \tag{23}$$

We now show that this implies $\overline{W} = c$. From (22), we have (subtracting $(1-\delta)c$ from both sides)

$$(1-\delta)(ps\pi - c) = (1-\delta)p\pi + \delta(1-p)\overline{W} - \overline{W} - (1-\delta)c.$$

Now using (23), we can write this as

$$\frac{(1-\delta)[p\pi - 2c] - \delta p\overline{W}}{1-\delta(1-p)} - \delta(1-p)\left[\frac{(1-\delta)(p\pi - c)}{1-\delta(1-p)} - \overline{W}\right] = (1-\delta)p\pi + \delta(1-p)\overline{W} - \overline{W} - (1-\delta)c$$

or, isolating $\overline{W}$,

$$\overline{W}\left[\frac{\delta p}{1-\delta(1-p)} - 1\right] = \frac{(1-\delta)[p\pi - 2c]}{1-\delta(1-p)} - \delta(1-p)\frac{(1-\delta)(p\pi - c)}{1-\delta(1-p)} - (1-\delta)[p\pi - c]$$

or (simplifying the left side, multiplying by -1 and eliminating $(1-\delta)$),

$$\frac{\overline{W}}{1-\delta(1-p)} = (p\pi - c) + \frac{\delta(1-p)(p\pi - c)}{1-\delta(1-p)} - \frac{p\pi - 2c}{1-\delta(1-p)}$$

or

$$\overline{W} = [1-\delta(1-p)](p\pi - c) + \delta(1-p)(p\pi - c) - (p\pi - 2c) = c.$$

■

B.2 Proof of Lemma 3.2

We consider an artificial game in which the principal is free of sequential rationality constraints. Having eliminated such constraints, there is no loss of generality in simplifying the notation by also dispensing with the public randomization device. The principal names, at the beginning of the game a pair of functions $z : H_1 \to [0,1]$ and $s : H_1 \to [0,1]$, giving the probability with which the principal offers a contract and the share offered in that contact, as a function of the principal's history, with the principal's objective being to minimize the agent's payoff subject to the constraint that the principal's payoff in the continuation game starting at each period is at least V^*. We show that the bounds on the agent's payoff given by c (if $z^\dagger > 1$) and $\frac{1-\delta}{\delta p}V^*$ (if $z^\dagger < 1$) apply to this artificial game. The bounds must then also hold in the original game. Since we have equilibria of the original game giving the agent payoff c in the first case and giving the agent a payoff approaching (as $\delta \to 1$) $\frac{1-\delta}{\delta p}V^*$ in the second case, this establishes the result.

First, we note that the agent's incentive constraint can be taken to be binding whenever the agent receives an offer, no matter what the history giving rise to that offer, and must bind after any history that occurs with positive probability. Suppose to the contrary that some history h_1^t has been reached and an offer made, with

$$(1-\delta)(1-s(h_1^t))p\pi + \delta(1-p)W(h_1^t) > (1-\delta)c + \delta W(h_1^t),$$

where $W(h_1^t)$ is the agent's continuation value. Let s^* satisfy this constraint with equality. Then replacing $s(h_1^t)$ with s^* while leaving continuation play unaffected preserves the agent's incentives (since the continuation value of every previous period is decreased, this only strengthens the incentives in previous periods) while increasing the principal's and reducing the agent's payoff (if this history occurs with positive probability), a contradiction.

Let $\underline{W}$ be the agent's minimum equilibrium payoff. Because the agent's incentive constraint always binds, $\underline{W}$ must equal the expected payoff from persistent shirking, and hence is given by

$$\underline{W} = (1-\delta)c \sum\nolimits_{h_1^t \in H_1} \mathbb{P}(h_1^t) z(h_1^t), \tag{24}$$

where $\mathbb{P}(h_1^t)$ is the probability with which history h_1^t appears. The principal's payoff is given by

$$(1-\delta)[p\pi - c] \sum\nolimits_{h_1^t \in H_1} \mathbb{P}(h_1^t) z(h_1^t) - W.$$

Notice that $W(h_1^t) \geq \underline{W}$, for every h_1^t, since otherwise $\underline{W}$ is not the lowest equilibrium payoff possible for the agent. Next, we claim that $\underline{W} = W(h_1^t)$ for any history h_1^t that occurs with positive probability. If not, we could construct an alternative equilibrium that matches the candidate equilibrium for any history that is not a continuation of h_1^t, and that after history h_1^t continues with an equilibrium in the resulting continuation game that gives payoff $\underline{W}$. Because $\underline{W} < W(h_1^t)$, this allows us to reduce the first-period value s_0 while still preserving the (binding) incentive constraint for the agent. The resulting lower first-period payoff and lower continuation value decrease the agent's payoff (and increase the principal's), a contradiction. Using (24), this in turn implies that $z(h_1^t) = z(h_1^0)$ for every h_1^t. However, we have characterized the equilibria that feature a constant value of z, finding that the only such equilibrium gives payoff $W^* = c$ when $z^\dagger > 1$ and that the agent's lowest payoff from such an equilibrium approaches (as $\delta \to 1$) $\frac{1-\delta}{\delta p} V^*$ if $z^\dagger < 1$. ■

B.3 Proof of Lemma 3.3

We consider an equilibrium with payoffs (W_0, V_0). We are interested in an upper bound on the ratio $\frac{V_0 - \underline{V}}{W_0 - \underline{W}}$, which we denote by ζ. It suffices to consider an equilibrium in which a period-0 mixture with probability $(1 - x_0)$ prompts the players to continue with equilibrium payoffs $(\underline{W}, \underline{V})$, and with probability x_0 calls for a current contract s, followed by a period-1 mixture attaching probability $1 - x_1$ between continuation payoffs $(\underline{W}, \underline{V})$ and probability x_1 to continuation play with payoffs (W_1', V_1'), and so on. In addition, we can assume that any contract offered to the agent induces the agent to work.[26] Hence, we have

$$\begin{aligned} V_0 &= x_0 \left[(1-\delta)(ps\pi - c) + \delta(1-p)[x_1 V_1' + (1-x_1)\underline{V}]\right] + (1-x_0)\underline{V} \\ W_0 &= x_0 \left[(1-\delta)p(1-s)\pi + \delta(1-p)[x_1 W_1' + (1-x_1)\underline{W}]\right] + (1-x_0)\underline{W} \\ &\geq x_0 \left[(1-\delta)c + \delta[x_1 W_1' + (1-x_1)\underline{W}]\right] + (1-x_0)\underline{W}, \end{aligned}$$

where the inequality is the agent's incentive constraint. Setting an equality in the incentive constraint, we can solve for

$$(1-\delta)ps\pi = (1-\delta)(p\pi - c) - \delta p[x_1 W_1' + (1-x_1)\underline{W}].$$

[26] Suppose we have a contract that does not induce effort, and hence gives payoffs $(1-\delta)(-c) + \delta V$ and $(1-\delta)(c) + \delta W$ to the principal and agent, respectively, for some continuation payoffs (W, V). There exists an alternative equilibrium with the same continuation payoffs, but in which the principal induces effort by offering a share s satisfying

$$(1-\delta)c + \delta W = (1-\delta)(1-s)p\pi + \delta(1-p)W.$$

Solving this expression gives $(1-\delta)(p\pi - c) - \delta p W = (1-\delta)sp\pi$, and hence a principal payoff of $(1-\delta)(p\pi - 2c) - \delta p W + \delta(1-p)V$. It is then a contradiction to our hypothesis that we are dealing with an extreme equilibrium, hence establishing the result, to show that this latter payoff exceeds $(1-\delta)(-c)+\delta V$, or $(1-\delta)(p\pi - 2c) - \delta p W + \delta(1-p)V > (1-\delta)(-c) + \delta V$, which is $(1-\delta)(p\pi - c) > \delta p(W + V)$, or

$$\frac{(1-\delta)(p\pi - c)}{\delta p} > V + W.$$

The left side is an upper bound on the value of the project without an agency problem, giving the result.

Using this to eliminate the share s from the principal's payoff, and returning to the agent's binding incentive constraint, we obtain

$$\begin{aligned} V_0 - \underline{V} &= x_0\left[(1-\delta)(p\pi - 2c) - \delta p[x_1 W_1' + (1-x_1)\underline{W}] + \delta(1-p)[x_1 V_1' + (1-x_1)\underline{V}] - \underline{V}\right] \\ W_0 - \underline{W} &= x_0\left[(1-\delta)c + \delta[x_1 W_1' + (1-x_1)\underline{W}] - \underline{W}\right] \end{aligned}$$

and hence

$$\zeta \equiv \frac{V_0 - \underline{V}}{W_0 - \underline{W}} = \frac{(1-\delta)(p\pi - 2c) - \delta p[x_1(W_1' - \underline{W}) + \underline{W}] + \delta(1-p)[x_1(V_1' - \underline{V}) + \underline{V}] - \underline{V}}{(1-\delta)c + \delta[x_1(W_1' - \underline{W}) + \underline{W}] - \underline{W}}.$$

We obtain an upper bound on this expression by taking first taking $V_1' - \underline{V} = \zeta(W_1' - \underline{W})$ on the right side and then rearranging to obtain

$$\zeta \leq \frac{(1-\delta)(p\pi - 2c) - \delta p[x_1(W_1' - \underline{W}) + \underline{W}] + (1-\delta)\underline{V}}{(1-\delta)c + \delta[x_1 p(W_1' - \underline{W}) + \underline{W}] - \underline{W}}.$$

We now note that $W_1' - \underline{W}$ appears negatively in the numerator and positively in the denominator, so that an upper bound on ζ is obtained by setting $W_1' - \underline{W} = 0$ on the right side, giving

$$\zeta \leq \frac{(1-\delta)(p\pi - 2c) - \delta p\underline{W} - (1-\delta(1-p))\underline{V}}{(1-\delta)c - (1-\delta)\underline{W}} = \frac{\delta p}{1-\delta}, \tag{25}$$

where the final equality is obtained by using $\underline{W} = \frac{1-\delta}{\delta p}\underline{V}$ to eliminate $\underline{W}$, and then simplifying. ■

B.4 Proof of Lemma 3.4

We now assume that $x_0, x_1 \in (0,1)$ and establish a contradiction. Using the incentive constraint, we can write

$$\begin{aligned} W_0 &= x_0\left[(1-\delta)c + \delta[x_1 W_1 + (1-x_1)\underline{W}]\right] + (1-x_0)\underline{W} \\ V_0 &= x_0\left[(1-\delta)(p\pi - 2c) - \delta p[x_1 W_1' + (1-x_1)\underline{W}] + \delta(1-p)[x_1 V_1' + (1-x_1)\underline{V}] - (1-x_0)\underline{V}\right] \end{aligned}$$

We now identify the rates at which we could decrease x_1 and increase x_0 while preserving the value W_0. Thinking of x_0 as a function of x_1, we can take a derivative of this expression for W_0 to find

$$\frac{dW_0}{dx_1} = \frac{dx_0}{dx_1}\frac{W_0 - \underline{W}}{x_0} + \delta x_0(W_1 - \underline{W}) = 0,$$

and then solve for

$$\frac{dx_0}{dx_1} = \delta x_0^2 \frac{W_1 - \underline{W}}{W_0 - \underline{W}}.$$

Now let us differentiate V_0 to find to find

$$\begin{aligned}\frac{dV_0}{dx_1} &= \frac{dx_0}{dx_1}\frac{V_0-\underline{V}}{x_0} + \delta x_0[(1-p)(V_1-\underline{V}) - p(W_1-\underline{W})] \\ &= -\delta x_0 \frac{W_1-\underline{W}}{W_0-\underline{W}}(V_0-\underline{V}) + \delta x_0[(1-p)(V_1-\underline{V}) - p(W_1-\underline{W})].\end{aligned}$$

It is a contradiction to show that this derivative is negative, since then we could increase the principal's payoff, while preserving the agent's by decreasing x_1. Eliminating the term δx_0 and multiplying by $W_0-\underline{W}>0$, we have

$$[(1-p)(V_1-\underline{V}) - p(W_1-\underline{W})](W_0-\underline{W}) - (V_0-\underline{V})(W_1-\underline{W}) \le 0.$$

We now substitute for $W_0-\underline{W}$ and $V_0-\underline{V}$ to obtain

$$\begin{aligned}&[(1-p)(V_1-\underline{V}) - p(W_1-\underline{W})]x_0\left[(1-\delta)c + \delta[x_1W_1' + (1-x_1)\underline{W}] - \underline{W}\right] \\ -\ & x_0\left[(1-\delta)(p\pi-2c) - \delta p[x_1W_1' + (1-x_1)\underline{W}] + \delta(1-p)[x_1V_1' + (1-x_1)\underline{V}] - \underline{V}\right](W_1-\underline{W}) \\ &\le\ 0.\end{aligned}$$

Deleting the common factor x_0 and canceling terms, this is

$$\begin{aligned}&[(1-p)(V_1-\underline{V}) - p(W_1-\underline{W})]\left[(1-\delta)c - (1-\delta)\underline{W}\right] \\ -\ & \left[(1-\delta)(p\pi-2c) - \delta p\underline{W} + \delta(1-p)\underline{V} - \underline{V}\right](W_1-\underline{W}) \quad \le \quad 0.\end{aligned}$$

Rearranging, we have

$$\frac{(1-p)(V_1-\underline{V}) - p(W_1-\underline{W})}{W_1-\underline{W}} \le \frac{(1-\delta)(p\pi-2c) - \delta p\underline{W} - (1-\delta(1-p))\underline{V}}{(1-\delta)c - (1-\delta)\underline{W}},$$

which follows immediately from the inequality in (25) from the proof of Lemma 3.3. ■

B.5 Proof of Lemma 4.1

We invoke a simple induction argument. In the last period of the game, the agent's value is

$$W(q_1, q_1) = (1-\delta)(1-s_1)q_1p\pi \ge (1-\delta)c,$$

where the inequality is the incentive constraint that the agent want to work, devoid of a continuation value in this case because the next posterior, q_0, is too pessimistic to support further experimentation. Now observe that if the agent holds the private belief $q' > q_1$, then again the agent will be asked to work one period, with a failure ending the game (because the principal's posterior will then drop below $\underline{q}$, even if the agent's does not). Hence,

$$\begin{aligned} W(q_1, q') &= (1-\delta)(1-s_1)q'p\pi \\ &= \frac{q'}{q_1}(1-\delta)(1-s_1)q_1 p\pi \\ &= \frac{q'}{q_1}W(q_1, q') \\ &> (1-\delta)c, \end{aligned}$$

where the final inequality provides the (strict) incentive constraint, ensuring that the agent will indeed work.

Now suppose this relationship holds for all periods $\tau < t$. Then we have

$$\begin{aligned} W(q_t, q') &= (1-\delta)(1-s_t)q'p\pi + \delta(1-q'p)W(q_{t-1}, \varphi(q')) \\ &= \frac{q'}{q_t}\left[(1-\delta)(1-s_t)q_t p\pi + \delta(1-q'p)\frac{q_t}{q'}\frac{\varphi(q')}{q_{t-1}}W(q_{t-1}, q_{t-1})\right] \\ &= \frac{q'}{q_t}\left[(1-\delta)(1-s_t)q_t p\pi + \delta(1-q'p)\frac{1-q_t p}{1-q'p}W(q_{t-1}, q_{t-1})\right] \\ &= \frac{q'}{q_t}\left[(1-\delta)(1-s_t)q_t p\pi + \delta(1-q_t p)W(q_{t-1}, q_{t-1})\right] \\ &= \frac{q'}{q_t}W(q_t, q_t), \end{aligned}$$

where the second equality uses the induction hypothesis, the third uses (2), the fourth rearranges terms, and the remaining two use the definition of W.

This argument builds on the implicit assumption that, given the equilibrium hypothesis that the agent will work in every period, an agent who arrives in period t with posterior $q' > q_t$ will find it optimal to work. To verify this, we need to show

$$(1-\delta)(1-s_t)q'p\pi + \delta(1-q'p)W(q_{t-1}, \varphi(q')) \geq (1-\delta)c + \delta W(q_{t-1}, q').$$

Using (10), this is

$$\begin{aligned} &\frac{q'}{q_t}\left[(1-\delta)c + \delta W(q_{t-1}, q_t) - \delta(1-q_t p)W(q_{t-1}, q_{t-1})\right] \\ &\qquad \geq \quad (1-\delta)c + \delta W(q_{t-1}, q') - \delta(1-q'p)W(q_{t-1}, \varphi(q')). \end{aligned}$$

We can eliminate the terms involving $(1-\delta)c$ from each side, noting that the term eliminated from the left is strictly greater than the term on the right, and then divide by δ, so that it suffices to show

$$\frac{q'}{q_t}\left[W(q_{t-1}, q_t) - (1-q_t p)W(q_{t-1}, q_{t-1})\right] \geq W(q_{t-1}, q') - (1-q'p)W(q_{t-1}, \varphi(q')).$$

This relationship obviously holds (with equality, both sides being zero) when $t = 1$. Using the induction hypothesis that it holds for all $\tau < t$, we can write the period-t version of this inequality as

$$\frac{q'}{q_t}\left[\frac{q_t}{q_{t-1} - (1 - q_t p)}\right] \geq \frac{q'}{q_{t-1}} - (1 - q'p)\frac{\varphi(q')}{q_{t-1}}$$

which (using (2)) is an equality. ■

B.6 Details, Full Effort (No Delay) and Positive Principal Payoff

Because $\lambda_t = 1$, equation (15) reduces to

$$q_t p\pi(1 - s_t) = (r + q_t p)W_t - \dot{W}_t \quad \text{and} \quad rW_t - \dot{W}_t - c = p(1 - q_t)W_t.$$

The second of these equations can be rewritten as

$$rw(q) + pq(1 - q)w'(q) - c = p(1 - q)w(q),$$

where w' is the derivative of w. The solution to this differential equation is

$$w(q) = \frac{pq - r}{p - r}\frac{c}{r} + A(1 - q)^{r/p}q^{1-r/p},$$

for some constant A. Let $\gamma(q) = pq(1 - s)\pi$ (where, with an abuse of notation, s is a function of q) so the first equation writes

$$\gamma(q) = (r + pq)w(q) + pq(1 - q)w'(q) = \frac{p^2 q - r^2}{p - r}\frac{c}{r} + Ap(1 - q)^{r/p}q^{1-r/p},$$

giving us the share s. Finally, using the previous equation to eliminate s, equation (14) simplifies to

$$0 = pq\pi - c - \gamma(q) - (r + pq)v(q) - pq(1 - q)v'(q).$$

The solution to this differential equation is given by

$$v(q) = \frac{pq\pi}{p + r} + \frac{2r^2 - p^2 + pr(1 - 2q)}{r(p^2 - r^2)}c + (B(1 - q) - A)\left(\frac{1 - q}{q}\right)^{r/p}, \tag{26}$$

for some constant B. Note that the function $v(q)$ given by (26) yields

$$v(1) = \frac{\xi - \sigma - 1}{\sigma + 1}\frac{c}{r},$$

which is positive if and only if $\xi \geq 1 + \sigma$.

We have thus solved for the payoffs to both agents, as well as for the share s, over any interval of time in which there is no delay. Note that the function v has at most one inflection point in the unit interval, given, if any, by

$$\frac{(A-B)(p+r)}{2pA-(p+r)B},$$

and so it has at most three zeroes. Note also that, if the interval without delay includes $\underline{q}$, we can solve for the constants of integration A and B using $v(\underline{q}) = w(\underline{q}) = 0$, namely

$$A = \frac{(2\sigma-\xi-1)(\xi-1)^{-1/\sigma}}{2^{(\sigma-1)/\sigma}(1-\sigma)}\frac{c}{r} \text{ and } B = \frac{((\xi+1)^2(1+\sigma)-8\sigma\xi)((\xi-1))^{-1-1/\sigma}}{2^{(\sigma-1)/\sigma}(1-\sigma^2)}\frac{c}{r}.$$

Plugging back into the value for v, we obtain that

$$v'(\underline{q}) = 0, v''(\underline{q}) = \frac{(\xi+1)^3(\xi-3)}{4\sigma(\xi-1)^2}\frac{c}{r}, \tag{27}$$

so that v is positive or negative for q close to $\underline{q}$ according to whether v is convex or concave at $\underline{q}$; it is positive if $\xi > 3$, and negative if $\xi < 3$ (if $\xi = 3$, it is positive if $\xi > 2\sigma - 1$ and negative if $\xi < 2\sigma - 1$, as can be verified from $v'''(\underline{q})$). The closed-form formulas for the case in which the boundary conditions are $v(\underline{q}) = w(\underline{q}) = 0$ are given by (17) and (18). Direct inspection of these formulas gives that $v(1) = (\xi-\sigma-1)/(\sigma+1)$, which is positive if and only if $\xi \geq 1+\sigma$. Therefore, we cannot have full effort for high enough beliefs if $\xi < 1+\sigma$.

B.7 Details, Partial Effort (Delay) and Zero Principal Value

Combining the two equations (15) for the agent gives, using $q_t p s_t \pi = c$,

$$W_t = q_t\pi - 2c/p, \text{ or } w(q) = q\pi - 2c/p.$$

Therefore, differentiating and using Bayes' rule,

$$\dot{W}_t = -pq_t(1-q_t)\pi.$$

Inserting back into the first equality of (15) yields

$$(r\lambda_t + pq_t)W_t = pq_t\pi - c + \dot{W}_t = pq_t^2\pi - c.$$

We obtain our candidate value for the delay

$$\lambda(q) = \frac{2q-1}{pq\pi-2c}\frac{pc}{r} = \frac{(2q-1)\sigma}{q(\xi+1)-2}, \tag{28}$$

which is strictly larger than one if and only if

$$(2\sigma - \xi - 1)q > \sigma - 2. \tag{29}$$

We have thus solved for the values of both players' payoffs ($v(q) = 0$), and for the delay over any interval of time in which there is delay. Note from (29) that the delay λ strictly exceeds 1 at $q = 1$ if and only if $\xi < 1 + \sigma$ and at $\underline{q} = 2/(\xi + 1)$ if and only if $\xi < 3$. In fact, since the left-hand side is linear in q, $\lambda(q) \geq 1$ for all $q \leq 1$ if $\xi \geq 3$ and $\xi \geq 1+\sigma$, in which case this configuration cannot occur. Conversely, if $\xi < 3$ and $\xi < 1 + \sigma$, the value of λ always strictly exceeds one.

B.8 Proof of Proposition 1

The two differential equations (8)–(9) have as solutions, for some $C_1, C_2 \in \mathbb{R}$,

$$W(t) = \frac{c}{r} + C_1 e^{rt}, \text{ and } V(t) = \frac{\xi - \sigma - 1}{\sigma + 1}\frac{c}{r} - C_1 e^{rt} + (C_1 + C_2)e^{r(1+\sigma)t}.$$

If $\xi < 1+\sigma$, then, since the first term of the principal's payoff is strictly negative, it must be that either C_1 or $C_1 + C_2$ is nonzero. Since the solution must be bounded, it implies, as expected, that effort cannot be supported indefinitely. If effort stops at time T, then, since $W(T) = 0$, $C_1 e^{rT} = -c/r$, and C_2 is then obtained from $V(T) = 0$. Eliminating T then yields the following relationship between $V = V(0)$ and $W = W(0)$:

$$V = \frac{\xi}{\sigma + 1}\left[1 - \left(1 - \frac{rW}{c}\right)^{\sigma+1}\right]\frac{c}{r} - W.$$

There exists a strictly positive root of this expression, denoted by $W^\dagger$, if $dV/dW > 0$ when evaluated at $W = 0$. Differentiating, the required condition is $\xi > 1$, providing the lower inequality in the definition of an intermediate project. If $W \in [0, W^\dagger]$, then $V \geq 0$, and these are the values that can be obtained for times T for which the principal's payoff is positive. This yields the result for intermediate projects. For reference, the stationary equilibrium in this region is given by $(V, W) = (0, \frac{\xi-1}{\sigma}\frac{c}{r})$.

Now consider lucrative projects, or $\xi \geq 1 + \sigma$, so that the principal's payoff in the stationary full-effort equilibrium is positive. We need to describe the equilibrium payoffs of potential stationary-outcome partial-effort equilibria. While there are several ways of doing so, we follow the procedure described in more detail in Section 4.2.1, and encompass partial effort in the discount rate. That is, players discount future payoffs at rate $r\lambda$, for $\lambda \geq 1$. The payoffs to the agent and principal, under such a constant rate, are

$$W(t) = \frac{c}{\lambda r}, \qquad V(t) = \frac{\lambda(\xi - 1) - \sigma}{\sigma + \lambda}\frac{c}{\lambda r}.$$

There exists exactly one value of $\lambda \neq 1$ for which the principal's payoff is equal to that obtained for $\lambda = 1$, namely

$$\lambda = \frac{\sigma(1+\sigma)}{\xi - 1 - \sigma},$$

which is larger than one if and only if $\xi \leq (1+\sigma)^2$. As before, if $\xi > (1+\sigma)^2$, then we have the case of a very lucrative project, for which there is no other equilibrium payoff than the stationary payoff (W^*, V^*).

Let us then focus on moderately lucrative projects for which

$$\xi \in \left[(1+\sigma), (1+\sigma)^2\right),$$

in which case $\lambda > 1$, so that there exists an equilibrium in which constant funding is provided, but at a slower rate than possible. The agent's payoff in this equilibrium is

$$\underline{W} = \frac{\xi - 1 - \sigma}{\sigma(\sigma+1)} \frac{c}{r}.$$

We may now solve the differential equations with boundary condition $V(T) = V^*, W(T) = \underline{W}$ for an arbitrary $T \geq 0$. Eliminating T gives the following relationship between $V = V(0)$ and $W = W(0)$:

$$V = \left[\frac{\xi}{\sigma+1} - \left(\frac{(1+\sigma)^2 - \xi - 1}{\sigma(\sigma+1)}\right)^{-\sigma} (1 - rW/c)^{\sigma+1}\right] \frac{c}{r} - W,$$

completing the results for moderately lucrative projects. ■

B.9 Proof of Proposition 2

The proof uses extensively the analysis performed in Appendices B.6 and B.7.

Case 1 ($\xi \geq 3$ and $\xi > 1+\sigma$): To see that this is an equilibrium, note that the value v given by (17) for the initial conditions $v(\underline{q}) = w(\underline{q}) = 0$ is positive everywhere on $[\underline{q}, 1]$. Further, as we observed that $\lambda(q)$ can never exceed one if $\xi \geq 3$ and $\xi > 1+\sigma$, this must be the unique Markovian equilibrium.

Case 2 ($\xi \geq 3$ and $\xi < 1+\sigma$): To see that there is a root in this interval, note that, as we remarked, $\xi \geq 3$ implies that v is positive for q close enough to $\underline{q}$, while $\xi < 1+\sigma$ implies that it is negative for $q = 1$. Since v is continuous in $(\underline{q}, 1)$, the equation $v(q) = 0$ must admit a root in this interval. Since v is convex and positive for q close enough to $\underline{q}$, and negative at $q = 1$, there must be an inflection point in $(\underline{q}, 1)$. As we remarked, such an inflection point must be unique, and so q^* is unique.

To establish sufficiency, we must show that the value of λ exceeds 1 on $(q^*, 1]$. The argument is a little more involved, since q^* admits no closed-form solution. It is easy to check that the coefficient of $((1-q)\underline{q}/(q(1-\underline{q}))^{1/\sigma}$ in (17) is positive given $\xi \geq 3$ and $\xi < 1+\sigma$, so that, by ignoring this term while solving for the root, we obtain a lower bound on q^*. That is, $q^* \geq \tilde{q} = (\sigma-2)(\sigma+1)/[(\xi-\sigma-1)\sigma]$. Since $\lambda(q) > 1$ if and only if $q > (\sigma-2)/(2\sigma-1-\xi)$ (from 29 given that $\xi < 2\sigma+1$ in this case), it suffices now to note that

$$\frac{(\sigma-2)(\sigma+1)}{(\xi-\sigma-1)\sigma} \geq \frac{\sigma-2}{2\sigma-1-\xi}$$

in this case. Note that the inequality is typically strict, so that, in fact, delay does not vary continuously at $q = q^*$.

Uniqueness follows from the fact that there cannot be delay for q close to $\underline{q}$ (as (29) is violated at $q = \underline{q}$ for $\xi \geq 3$ and $\xi < 1+\sigma$), and that the principal's payoff must be continuous in q (λ is bounded for q bounded away from $\underline{q}$), so that $v(q) = 0$ must hold as we move from one configuration to the next. Therefore, there cannot be delay for $q < q^*$. To prove that there cannot be a subinterval (q_1, q_2) of $(q^*, 1]$ in which there is no delay, consider such a maximal interval and note that it would have to be the case that $v(q_1) = 0$ and $w(q_1) = q_1\pi - 2c/p$, by continuity in q of the players' payoff functions. Solving for the differential equations for v, w in such an interval (q_1, q_2), one obtains that, at q_1, $v(q_1) = v'(q_1) = 0$, while

$$v''(q_1) = \frac{\sigma - 2 + q_1(\xi + 1 - 2\sigma)}{q_1^2(1-q_1)^2\sigma^2}.$$

Yet the numerator of this expression is necessarily negative for all $q_1 > \tilde{q}$, and thus, in particular, for $q > q^*$. This contradicts the fact that $v(q)$ must be nonnegative on the interval (q_1, q_2).

Case 3 ($\xi < 3$ and $\xi \geq 1+\sigma$): Recall that q^{**} is defined by the fact that delay is continuous at $q = q^{**}$, i.e. $\lambda(q^{**}) = 1$, or

$$q^{**} = \frac{2-\sigma}{\xi + 1 - 2\sigma},$$

which given that $\xi < 3$ and $\xi \geq 1+\sigma$ is indeed in $(\underline{q}, 1]$. That there must be delay for q close enough to $\underline{q}$ follows from our earlier remarks about the negativity of (17) for such values of q, because $\xi < 3$. Similarly, we already noticed that there cannot be delay for q close enough to 1 if $\xi \geq 1+\sigma$.

For sufficiency, note that λ is decreasing in q over the interval $(\underline{q}, 1)$ in this case, so that $\lambda(q) \geq 1$ over the interval $[\underline{q}, q^{**})$. We can solve for the differential equations giving v and w over the range $[q^{**}, 1]$, with boundary conditions $w(q^{**}) = q^{**}\pi - 2c/p =$

$(q^{**}(\xi+1)-2)c/(\sigma r)$, and $v(q^{**})=0$. It is easy to check that $v''(q^{**})=0$ (compare, for instance, with $v''(q_1)$ above), so the curvature of v is actually zero at q^{**}. However,

$$v'''(q^{**})=\frac{(\xi+1-2\sigma)^5}{\sigma^2(\sigma-2)^2(1+\sigma-\xi)^2},$$

which is strictly positive because $\xi > 2\sigma - 1$ in this case. Since v admits at most one inflection point over the interval $(q^{**}, 1)$, and it is positive at 1, it follows that it is positive over the entire interval.

For necessity, we now rule out equilibria of this type with different values of q^{**}. Because λ is decreasing in q over the interval $(\underline{q}, 1)$, the only other values of the threshold to consider are values of q in $(\underline{q}, q^{**})$. So fix $q_1 \in (\underline{q}, q^{**})$. Since $v(q_1) = 0$ and $w(q_1) = (q_1(\xi+1)-2)c/(\sigma r)$, we can solve for v and w. This gives that $v'(q_1) = 0$ and the same value of $v''(q_1)$ as in the case of a lucrative, low-discount project. However, this value is strictly negative because $\xi > 1+\sigma, \xi \leq 3 \Rightarrow \sigma - 2 + q(\xi+1-2\sigma) < 0$. This implies that v is strictly decreasing at q, and hence strictly negative over some range above q, which is not possible in an equilibrium. More generally, this argument shows that we cannot have an interval without delay that involves values of q lower than q^{**}. Since also we cannot have $\lambda(q) \geq 1$ for values of q strictly above q^{**}, uniqueness follows.

To see that this is an equilibrium, recall that $\lambda(q)$ exceeds one for all values of q if $\xi < 3$ and $\xi < 1+\sigma$.

Case 4 ($\xi < 3$ and $\xi < 1+\sigma$): For necessity, suppose for sake of contradiction that there was an interval (q_1, q_2) in which there was no delay. Again, we can show that, solving the differential equations for v and w, the value of $v''(q_1)$ is as in the case of a lucrative, low-discount project, which is negative in this case. Since $v(q_1) = v'(q_1) = 0$, this implies that the payoff of the principal would be strictly negative for values of q slightly above q_1, a contradiction. ■

B.10 Proof of Lemma 4.3

Let dt be the length of a period, and fix q_1 such that

$$\varphi(q_1)=\frac{(1-pdt)q_1}{1-pdtq_1}<\underline{q}=\frac{2c}{p\pi}.$$

Hence, if an experiment is undertaken at posterior q_1, no further experimentation will occur. Notice that q_1 approaches $\underline{q}$ as dt gets small, though we will suppress the dependence of q_1 on dt. We have

$$\begin{aligned}\underline{W}(q_1) &= (1-\delta)c\\ \overline{W}(q_1) &= (1-\delta)(q_1p\pi-c),\end{aligned}$$

where $W(\cdot)$ is the agent's payoff under the candidate, stationary, always-work equilibrium and $\overline{W}(\cdot)$ is an upper bound on the agent's payoff, where both are a function of the posterior probability of a good project. In particular, $\overline{W}(q_1)$ is simply the entire value of the first-best surplus, at posterior q_1.

Now let us calculate the agent's payoff under the candidate equilibrium. We have

$$\begin{aligned} W(q_t) &= (1-\delta)(1-s_t)q_t p\pi + \delta(1-pq_t)W(q_{t-1}) \\ &= (1-\delta)c + \delta\frac{q_t}{q_{t-1}}W(q_{t-1}), \end{aligned}$$

where the second equality is the incentive constraint. Using this second equality to iterate, we have

$$\begin{aligned} W(q_t) &= (1-\delta)c + \delta\frac{q_t}{q_{t-1}}W_{t-1} \\ &= (1-\delta)c + \delta\frac{q_t}{q_{t-1}}\left((1-\delta)c + \delta\frac{q_{t-1}}{q_{t-2}}W(q_{t-2})\right) \\ &= (1-\delta)c + \delta\frac{q_t}{q_{t-1}}(1-\delta)c + \delta^2\frac{q_t}{q_{t-2}}W(q_{t-2}) \\ &= (1-\delta)c + \delta\frac{q_t}{q_{t-1}}(1-\delta)c + \delta^2\frac{q_t}{q_{t-2}}(1-\delta)c + \delta^3\frac{q_t}{q_{t-3}}W(q_{t-3}) \\ &\vdots \quad \vdots \\ &= (1-\delta)cq_t\left[\frac{1}{q_t} + \frac{\delta}{q_{t-1}} + \frac{d^2}{q_{t-2}} + \frac{\delta^3}{q_{t-3}} + \ldots + \frac{d^{t-2}}{q_2}\right] + \delta^{t-1}\frac{q_t}{q_1}W(q_1). \end{aligned}$$

We now ask what would be the most the principal would have to pay each period, in order to get the agent to work, and what would be the agent's resulting payoff. If this payoff approaches $W^*(q_t)$, as the period length gets small, then the principal's minmax payoff approaches $V^*(q_t)$.

The largest amount the principal has to pay to induce work is obtained by assuming that the agent's incentive constraint is slack, but that should the principal offer anything less, then the equilibrium hypothesis is that the agent shirks, followed by the maximum possible continuation value.[27] The constraint on our desire to make effort as costly as possible for the principal is then that the principal not offer such a large share to the

[27] Alternatively, the largest amount the principal may have to pay in order to induce effort may occur in circumstances in which the agent's incentive constraint binds, so that the agent must be given a value of

$$(1-\delta)c + \delta\frac{q_t}{q_{t-1}}\overline{W}_{t-1}.$$

That effort is more expensive under the hypothesis that even slightly smaller offers induce shirking is then equivalent to the statement that $\overline{W}(q_t) > \frac{q_t}{q_{t-1}}\overline{W}(q_{t-1})$, which is immediate.

agent as to make the agent not prefer to surreptitiously work, or

$$(1-\delta)c + \delta\overline{W}(q_t) = (1-\delta)(1-s_t)pq_t\pi + \delta(1-pq_t)\overline{W}_t(q_t, q_{t-1}). \tag{30}$$

At the same time, we have

$$\overline{W}_t = (1-\delta)(1-s_t)pq_t\pi + \delta(1-pq_t)\overline{W}_{t-1}. \tag{31}$$

We now note that

$$\frac{q_{t-1}}{q_t}\overline{W}(q_t) \geq \overline{W}(q_t, q_{t-1}).$$

In particular, the left side is the value to the agent if the latter continues to work in every subsequent period, which may not be optimal. Hence, we *overestimate* the amount the agent is to be paid if we replace (30) with

$$(1-\delta)c + \delta\overline{W}(q_t) = (1-\delta)(1-s_t)pq_t\pi + \delta(1-pq_t)\frac{q_{t-1}}{q_t}\overline{W}(q_t). \tag{32}$$

Using (32) to eliminate the term involving s_t from (31) and then using (2) for the second and third equalities, we have

$$\begin{aligned}
\overline{W}(q_t) &= (1-\delta)c + \left[\delta - \delta(1-pq_t)\frac{q_{t-1}}{q_t}\right]\overline{W}(q_t) + \delta(1-pq_t)\overline{W}_{t-1} \\
&= \frac{(1-\delta)c}{(1-\delta)p} + \frac{\delta(1-pq_t)}{(1-\delta)p}\overline{W}_{t-1} \\
&= \frac{(1-\delta)c}{(1-\delta)p} + \frac{\delta(1-p)}{(1-\delta)p}\frac{q_t}{q_{t-1}}\overline{W}_{t-1} \\
&\equiv B + A_t\overline{W}_{t-1},
\end{aligned}$$

where

$$\begin{aligned}
B &= \frac{(1-\delta)c}{(1-\delta)p} \\
A_t &= \frac{\delta(1-p)}{(1-\delta)p}\frac{q_t}{q_{t-1}}.
\end{aligned}$$

We now solve for

$$\begin{aligned}
\overline{W}(q_t) &= A_t\overline{W}(q_{t-1}) + B \\
&= A_tA_{t-1}\overline{W}(q_{t-2}) + A_tB + B \\
&= A_tA_{t-1}A_{t-2}\overline{W}(q_{t-3}) + A_tA_{t-1}B + A_tB + B \\
&= A_tA_{t-1}A_{t-2}A_{t-3}\overline{W}(q_{t-4}) + A_tA_{t-1}A_{t-2}B + A_tA_{t-1}B + A_tB + B \\
&\vdots \quad \vdots \\
&= A_t\cdots A_2\overline{W}(q_1) + A_t\cdots A_3B \\
&\qquad\qquad + A_t\cdots A_4B \\
&\qquad\qquad \vdots \\
&\qquad\qquad + A_tA_{t-1}B \\
&\qquad\qquad + A_tB \\
&\qquad\qquad + B.
\end{aligned}$$

Now we compare terms. The first term in the equilibrium payoff $W(q_t)$ is

$$\delta^{t-1}\frac{q_t}{q_1}W(q_1) = \delta^{t-1}\frac{q_t}{q_1}(1-\delta)c,$$

while the bound has as its corresponding term

$$\delta^{t-1}\frac{q_t}{q+1}\frac{(1-p)^{t-1}}{(1-\delta p)^{t-1}}\overline{W}(q_1) = \delta^{t-1}\frac{q_t}{q_1}\frac{(1-p)^{t-1}}{(1-\delta p)^{t-1}}(1-\delta)(q_1p\pi - c).$$

We then note that both terms approach zero as does dt.

Under the equilibrium, the sum of the remaining term comprising $W(q_t)$ is given by $(1-\delta)cq_t$ times

$$\frac{1}{q_t} + \frac{\delta}{q_{t-1}} + \frac{d^2}{q_{t-2}} + \frac{\delta^3}{q_{t-3}} + \ldots + \frac{d^{t-2}}{q_2}.$$

Under our bound, the corresponding term is $(1-\delta)cq_t$ times

$$\delta^{t-2}\frac{1}{q_2}\frac{(1-p)^{t-2}}{(1-\delta p)^{t-1}} + \delta^{t-3}\frac{1}{q_3}\frac{(1-p)^{t-3}}{(1-\delta p)^{t-2}}\ldots + \delta\frac{1}{q_{t-1}}\frac{1-p}{(1-\delta p)^2} + \frac{1}{q_t}\frac{1}{1-\delta p}.$$

We can multiply and divide by $(1-p)$, and then note that $\frac{1-p}{1-\delta p} < 1$, to get an upper bound on our upper bound of

$$\frac{1}{1-p}\left[\frac{1}{q_t} + \frac{\delta}{q_{t-1}} + \frac{d^2}{q_{t-2}} + \frac{\delta^3}{q_{t-3}} + \ldots + \frac{d^{t-1}}{q_1}\right].$$

But this implies that in the limit as dt gets small (and hence we can ignore the first terms)

$$\overline{W}_t \leq \frac{1}{1-p} W_t,$$

which we couple with the observation that $\frac{1}{1-p} \to 1$ as time periods get short to give the result. ■

B.11 Proof of Lemma 4.4

Using the formula for Φ, we get

$$w(q) = w_0(q) - \Phi(q) \left[C + \frac{v_0(\tilde{q})}{(1-\tilde{q})^{1+\frac{2}{\sigma}} q^{*1-\frac{2}{\sigma}}} \int_{\tilde{q}}^{q} \frac{\left(\frac{1-u}{u}\right)^{\frac{2}{\sigma}}}{v_0(u)^2} \exp\left\{ \frac{c}{\sigma r} \int_{\tilde{q}}^{u} \frac{(\xi+1)y-2}{y(1-y)v_0(y)} dy \right\} du \right]^{-1}.$$

The smaller the constant C, the lower the corresponding solution. Let us first state some properties of $\Phi(q)$.

1. $\lim_{q \to 1} \Phi(q) = 0$ if $\xi > (1+\sigma)^2$; if instead $\xi < (1+\sigma)^2$, $\lim_{q\to 1} \Phi(q) = \infty$.
2. $\lim_{q \to \underline{q}} \Phi(q) = 0$ if $\xi > 3$ (which is the case). In fact, $\lim_{q\to \underline{q}} \Phi(q)/v_0(q) = 0$ for $\xi > 3$.

Because we must have $w \leq w_0$, it follows that the only values of C that need be considered are:

$$C \geq C^* = \frac{v_0(\tilde{q})}{(1-\tilde{q})^{1+\frac{2}{\sigma}} q^{*1-\frac{2}{\sigma}}} \int_{\underline{q}}^{\tilde{q}} \frac{\left(\frac{1-u}{u}\right)^{\frac{2}{\sigma}}}{v_0(u)^2} \exp\left\{ \frac{c}{\sigma r} \int_{\tilde{q}}^{u} \frac{(\xi+1)y-2}{y(1-y)v_0(y)} dy \right\} du.$$

Consider the case $C = C^*$, as w is increasing in C. Using l'Hospital's rule, we have that, for $q \to \underline{q}$,

$$\frac{\Phi(q)}{\int_{\underline{q}}^{q} \frac{\Phi(u)}{u(1-u)v_0(u)} du} = \frac{q(1-q)v_0(q)\Phi'(q)}{\Phi(q)} = q(1-q)v_0(q) \ln(\Phi(q))'$$
$$= \frac{2pw_0(q) - (pq\pi - 2c - pq(1-q)v_0'(q) - pv_0(q))}{p},$$

and so

$$w(q) = \frac{pq\pi - 2c - pq(1-q)v_0'(q) - pv_0(q) - pw_0(q)}{p}.$$

Since w_0 satisfies $pw_0(q) = qp\pi - 2c - (r + pq)v_0(q) - pq(1 - q)v_0'(q)$, we obtain

$$w(q) = \frac{(1 - \sigma(1 - q))v_0(q)}{\sigma},$$

which means that non-negativity is satisfied if and only if $1 \geq \sigma(1-\underline{q})$, or $1+\sigma \geq (\sigma-1)\xi$. This condition is not implied by the conditions that define cases 1 or 2, and so it might or might not hold.

At the other end ($q \to 1$), it follows from property 1 above that, if $\xi > (1 + \sigma)^2$, $\lim_{q\to 1} w(q) = w_0(1)$. If instead $\xi < (1 + \sigma)^2$, we may again use l'Hospital's rule to get that $\lim_{q\to 1} w(q) = v_0(1)/\sigma$, and the condition $\xi < (1 + \sigma)^2$ precisely guarantees that $v_0(1)/\sigma \leq w_0(1)$.

On the other hand, for any $C > 0$, and for any $\varepsilon > 0$, there exists $\eta > 0$, for all $q \in (\underline{q}, \underline{q} + \eta)$,

$$\begin{aligned} w(q) &= w_0(q) - \frac{\Phi(q)}{C + \int \frac{\Phi(u)}{u(1-u)v_0(u)}du} \\ &= w_0(q) - \frac{\int \frac{\Phi(u)}{u(1-u)v_0(u)}du}{C + \int \frac{\Phi(u)}{u(1-u)v_0(u)}du} \left(w_0(q) - \frac{(r - p(1 - q))v_0(q)}{p} \right) \\ &> (1 - \varepsilon)w_0(q) > 0, \end{aligned}$$

because

$$v_0(q) = \frac{1}{2} \left(\frac{1}{\xi + 1} + \frac{1}{\xi - 3} \right)^{-1} (q - \underline{q})\, w_0(q) + o(q - \underline{q})^2,$$

so that the term in parenthesis in the penultimate line is positive for $q - \underline{q}$ small enough, and because $\lim_{q\to \underline{q}} \Phi(q)/v_0(q) = 0$, so that the coefficient can be made arbitrarily small by choosing C large enough. Thus, the function w is non-negative everywhere for all $C > 0$. Furthermore, it is continuous in C for fixed q, and so indeed, pointwise, for $q > \underline{q}$, the lowest payoff is

$$w(q) = w_0(q) - \frac{\Phi(q)}{C^* + \int_{\tilde{q}}^{q} \frac{\Phi(u)}{u(1-u)v_0(u)}du},$$

but it might not be achieved.

B.12 Proof of Lemma 4.5

The proof invokes arguments similar to those used to prove Lemmas 3.3 and 3.4. Given W, we consider the value of V that maximizes the principal's payoff among equilibrium payoffs. We can again restrict attention to sequences (x_t, s_t), where, in period t, the worst equilibrium (given the current posterior) is played with probability $1 - x_t$ (determined by the public randomization device); and if not, a share s_t is offered in that period that induces the agent to work. We fix a posterior probability and let W_0 and V_0 be the candidate equilibrium values, with $\underline{W}_0$ and $\underline{V}_0$ being the values of the worst equilibrium given that posterior, and with $W_1, V_1, \underline{W}_1$ and $\underline{V}_1$ being the counterparts for the next period (and the next posterior, under the assumption that the agent worked and generated a failure). We will typically suppress the notation for the posterior probabilities on which these values depend.

Now, let ζ be such that for *any* posterior probability,

$$\frac{V_0 - \underline{V}_0}{W_0 - \underline{W}_0} \leq \zeta.$$

We now note that

$$\begin{aligned} V_0 &= x_0 \left[(1-\delta)(ps\pi - c) + \delta(1-p)[x_1 V_1' + (1-x_1)\underline{V}_1]\right] + (1-x_0)\underline{V}_0 \\ W_0 &= x_0 \left[(1-\delta)p(1-s)\pi + \delta(1-p)[x_1 W_1' + (1-x_1)\underline{W}_1]\right] + (1-x_0)\underline{W}_0 \\ &\geq x_0 \left[(1-\delta)c + \delta[x_1 \theta W_1' + (1-x_1)\theta \underline{W}_1]\right] + (1-x_0)\underline{W}_0, \end{aligned}$$

where the inequality is the agent's incentive constraint and $\theta > 1$ is given by

$$\theta = \frac{q}{\varphi(q)} = \frac{1-pq}{1-p},$$

and hence is the ratio of next period's posterior to the current posterior, given a failure. Setting an equality in the agent's incentive constraint and rearranging gives

$$(1-\delta)ps\pi = (1-\delta)(p\pi - c) + \delta(1-p)[x_1 W_1' + (1-x_1)\underline{W}_1] - \delta[x_1\theta W_1' + (1-x_1)\theta\underline{W}_1].$$

Using this to eliminate the variable s from the value functions gives

$$\begin{aligned} V_0 - \underline{V}_0 &= x_0 \left[(1-\delta)(p\pi - 2c) + \delta(1-p)[x_1 W_1' + (1-x_1)\underline{W}_1] - \delta[x_1\theta W_1' + (1-x_1)\theta\underline{W}_1]\right. \\ &\quad \left. + \delta(1-p)[x_1 V_1' + (1-x_1)\underline{V}_1] - \underline{V}_0\right] && (33) \\ W_0 - \underline{W}_0 &= x_0 \left[(1-\delta)c + \delta[x_1\theta W_1' + (1-x_1)\theta\underline{W}_1] - \underline{W}_0\right]. && (34) \end{aligned}$$

Dividing (33) by (34), we obtain

$$\begin{aligned} \frac{V_0 - \underline{V}_0}{W_0 - \underline{W}_0} &= \frac{(1-\delta)(p\pi - 2c) + [\delta(1-p) - \delta\theta]x_1[W_1' - \underline{W}_1] + [\delta(1-p) - \delta\theta]\underline{W}_1}{(1-\delta)c + \delta\theta[x_1(W_1 - \underline{W}_1) + \underline{W}_1] - \underline{W}_0} \\ &\quad + \frac{\delta(1-p)x_1(V_1' - \underline{V}_1) + \delta(1-p)[\underline{V}_1 - \underline{V}_0]}{(1-\delta)c + \delta\theta[x_1(W_1 - \underline{W}_1) + \underline{W}_1] - \underline{W}_0} \end{aligned}$$

Using the hypotheses that $V_1' - \underline{V}_1 \leq \zeta(W_1' - \underline{W}_1)$, we can substitute and rearrange to obtain an upper bound on ζ, or

$$\zeta \leq \frac{(1-\delta)(p\pi - 2c) + (\delta(1-p) - \delta\theta)(W_1' - \underline{W}_1) + (\delta(1-p) - \delta\theta)\underline{W}_1 + \delta(1-p)\underline{V}_1 - \underline{V}_0}{(1-\delta)c + \delta\theta x_1(W_1' - \underline{W}_1) + \delta\theta\underline{W}_1 - \underline{W}_0 - [\delta(1-p)x_1](W_1 - \underline{W}_1)}.$$

We obtain an upper bound on the right side by setting $W_1' - \underline{W}_1 = 0$, obtaining

$$\zeta \leq \frac{(1-\delta)(p\pi - 2c) + (\delta(1-p) - \delta\theta)\underline{W}_1 + \delta(1-p)\underline{V}_1 - \underline{V}_0}{(1-\delta)c + \delta\theta\underline{W}_1 - \underline{W}_0}.$$

We now differentiate (34) to find

$$\frac{dW_0}{dx_1} = \frac{dx_0}{dx_1}\frac{W_0 - \underline{W}_0}{x_0} + \delta x_0\theta(W_1' - \underline{W}_1)$$

and hence, setting $\frac{dW_0}{dx_1} = 0$,

$$\frac{dx_0}{dx_1} = -\delta x_0^2\theta\frac{W_1' - \underline{W}_1}{W_0 - \underline{W}_0}. \tag{35}$$

Differentiating (33) and using (35), we have

$$\begin{aligned}
\frac{dV_0}{dx_1} &= \frac{dx_0}{dx_1}\frac{V_0 - \underline{V}_0}{x_0} + \delta x_0\left((1-p-\theta)[W_1' - \underline{W}_1] - (1-p)[V_1 - \underline{V}_1]\right) \\
&= -\delta x_0\theta\frac{W_1' - \underline{W}_1}{W_0 - \underline{W}_0}(V_0 - \underline{V}_0) + \delta x_0\left((1-p-\theta)[W_1' - \underline{W}_1] - (1-p)[V_1' - \underline{V}_1]\right).
\end{aligned}$$

It concludes the argument to show that this derivative is negative. Multiplying by $W_0 - \underline{W}_0$, this is

$$(W_0 - \underline{W}_0)\left((1-p-\theta)(W_1' - \underline{W}_1) + (1-p)(V_1' - \underline{V}_1)\right) - (W_1' - \underline{W}_1)(V_0 - \underline{V}_0)\theta < 0.$$

Substituting for $V_0 - \underline{V}_0$ and $W_0 - \underline{W}_0$ from (33)–(34) and dropping the common factor x_0, this is

$$\begin{aligned}
&[(1-p-\theta)(W_1' - \underline{W}_1) + (1-p)(V_1' - \underline{V}_1)]\left((1-\delta)c + \delta(x_1\theta W_1' + (1-x_1)\theta\underline{W}_1) - \underline{W}_0\right) \\
&\quad < \quad (W_1' - \underline{W}_1)\theta\left[(1-\delta)(p\pi - 2c) + \delta(1-p-\theta)[x_1 W_1' + (1-x_1)\underline{W}_1]\right. \\
&\qquad + \left.\delta(1-p)[x_1 V_1' + (1-x_1)\underline{V}_1) - \underline{V}_0\right],
\end{aligned}$$

which simplifies to

$$(1-p-\theta) + (1-p)\frac{V_0' - \underline{V}_0}{W_0' - \underline{W}_0} < \theta\frac{(1-\delta)(p\pi - 2c) + (\delta(1-p) - \delta\theta)\underline{W}_1 + \delta(1-p)\underline{V}_1 - \underline{V}_0}{(1-\delta)c + \delta\theta\underline{W}_1 - \underline{W}_0},$$

which is immediate from the definition of ζ. ■

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