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# Robust Stochastic Stability*

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**Abstract**

A strategy profile of a game is called robustly stochastically stable if it is stochastically stable for a given behavioral model independently of the specification of revision opportunities and tie-breaking assumptions in the dynamics. We provide a simple radius-coradius result for robust stochastic stability and examine several applications. For the logit-response dynamics, the selection of potential maximizers is robust for the subclass of supermodular $N$-player binary-action games. For the mistakes model, robust selection results obtain for best-reply dynamics in the same class of games under the weaker condition of strategic complementarity. Further, both the selection of risk-dominant strategies in coordination games under best-reply and the selection of "Walrasian" strategies in aggregative games under imitation are robust.

**Keywords:** Learning in games, stochastic stability, radius-coradius theorems, logit-response dynamics, mutations, imitation.
**JEL Classification Numbers:** C72, D83.

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# 1 Introduction

The concept of stochastic stability was introduced in Game Theory in a series of seminal papers by Blume (1993), Kandori, Mailath, and Rob (1993), Young (1993), and Ellison (1993). In the meantime, a large number of applications have been developed, a series of theoretical improvements have ensued, and several striking results have been proven relying on this concept. Among the best known results, we single out three which have had a lasting impact in the literature. First, the selection of risk-dominant equilibria (even in the presence of alternative, Pareto-efficient ones) in coordination games (Kandori, Mailath, and Rob, 1993; Kandori and Rob, 1995; Sandholm, 1998) under best-reply or imitation dynamics. Second, the selection of potential maximizers in exact potential games in logit-response dynamics (Blume, 1993, 1997). Third, the selection of Walrasian equilibria in oligopolies with imitating firms (Vega-Redondo, 1997), which has been shown to generalize to the class of aggregative games by Alós-Ferrer and Ania (2005). These are all important insights which have shaped our understanding of equilibrium (and non-equilibrium) selection and stability.

The literature has also made a number of weaknesses apparent, some of which have the status of unwritten "folk wisdom". The main weakness of the stochastic stability literature as a whole is probably the fact that many results might depend, or might be perceived to depend on modeling details, thereby casting doubt on the main insights gained from this approach. A number of failed robustness checks have demonstrated this issue. We would like to argue that, while some of these checks are substantial and have further sharpened our intuition, other have arisen due to a fundamental lack of robustness in the very concept of stochastic stability.

Among the substantial results we count the analysis in Robson and Vega-Redondo (1996), which showed that the selection of risk-dominant equilibria under the imitation dynamics of Kandori, Mailath, and Rob (1993) depends on the postulated interaction structure, with "round-robin" interaction leading to risk-dominant equilibria but true random matching favoring Pareto-efficient ones (this distinction would not exist if myopic best-reply is assumed). In our opinion, this result does not correspond to a weakness in the general approach. On the contrary, it is a substantial contribution that points at the importance of the interaction structure and should not be considered a robustness check. Indeed, the importance of both the interaction structure and the behavioral rule for equilibrium selection has been made apparent in the related literature on games in networks (see Weidenholzer, 2010 for a review). For instance, Morris (2000) shows that best-reply dynamics lead to risk-dominant equilibria in quite general networks, while Alós-Ferrer and Weidenholzer (2008) show that imitation favors Pareto-efficient outcomes under comparatively mild conditions on the network.

Among the more worrying failed robustness tests we count the fact that changing the specification of either revision opportunities or tie-breaking assumptions might sometimes influence the long-run outcomes in a given dynamic specification. This affects, for instance, the well-known result that the original logit dynamics of Blume (1993, 1997) selects potential maximizers in exact potential games. Alós-Ferrer and Netzer (2010) have shown that this result depends crucially on the assumption of asynchronous learning, that is, a dynamic specification in which every period one and only one agent is selected and allowed to revise his or her strategy, while all other players are required to stay put. If this assumption is dispensed with and more general revision processes are allowed for, the result vanishes away.

Tie-breaking assumptions are also not always harmless. Suppose that a behavioral rule specifies the set of strategies that a player might choose from, e.g. the set of payoff maximizers given other players' strategies (as in the case of a myopic best-reply dynamics) or the set of strategies leading to currently maximal, observed payoffs (as in the case of an imitation dynamics). Even abstracting from revision opportunities, this still does not fully specify the dynamics. One might for instance require that all maximizing strategies be chosen with positive (maybe equal) probability; it might, however, be equally reasonable to postulate that players who are already employing one of the optimal strategies do not switch away. These are all reasonable choices, which sometimes have consequences for the dynamic analysis (contrast e.g. Oechssler, 1997 and Alós-Ferrer, 2003; see also Sandholm, 1998).

Both the specification of revision opportunities and tie-breaking assumptions might be argued to be orthogonal to the analysis of the long-run predictions associated to a given behavioral rule and interaction structure. It is precisely for this reason that the possible dependence of long-run predictions on them is an important consideration. A result which depends on such modeling details should not be considered to be on equal grounds with a result which is immune to the specification thereof. In the present research, we aim to provide and apply a simple result which helps establish when a long-run prediction is robust to the specification of revision opportunities and/or tie-breaking assumptions.

The remainder of the paper is structured as follows. Section 2 introduces the general framework for the analysis. Section 3 introduces the robustness concept and presents the main result. Section 4 presents a first application to evolutionary stability and aggregate-taking behavior for perturbed imitation dynamics. Section 5 analyzes $N$-player binary-action games both for the logit-response dynamics and the popular best-reply mistakes model. Section 6 discusses the limits of the approach and Section 7 concludes.

# 2 Learning in Games: A General Framework

## 2.1 Stage Model

Consider a finite population of $N$ agents who repeatedly interact in discrete time $t = 1, 2, \ldots$ according to a pre-specified stage model, formalized as a finite, normal-form game $\Gamma = (I, (S_i, u_i)_{i \in I})$, where $I = \{1, 2, ..., N\}$ is the set of players, $S_i$ are the strategy sets, and $u_i : S \to \mathbb{R}$ are payoff functions, where $S = \prod_{i \in I} S_i$. We let $S_{-i} = \prod_{j \neq i} S_j$ be the set of pure strategy profiles of all players except $i$, and we also write $s = (s_i, s_{-i})$ and $u_i(s_i, s_{-i})$.

The strategies chosen and the stage model determine the payoffs agents receive at the end of the period $t$. The stage model can simply be taken to be an arbitrary, asymmetric $N$-player game, as in Blume (1993) or Alós-Ferrer and Netzer (2010), or it can incorporate additional structure. For example, it might specify that agents play a bilateral finite game sequentially against each other agent in the population (*round robin tournament*), as in Kandori, Mailath, and Rob (1993) (hereafter KMR), where the bilateral game is a symmetric $2 \times 2$ coordination game.

## 2.2 Behavioral Rules and Correspondences

The game is played by boundedly rational players, whose behavior is summarized by *behavioral rules*. At the beginning of each period, a certain subset of agents is chosen to update their actions (we will further specify revision opportunities below). Each player chooses a pure strategy according to a pre-specified behavioral rule $B_i : S \mapsto \Delta S_i$. That is, $B_i(s)(s'_i)$ is the probability with which player $i$ will choose strategy $s'_i \in S_i$ after the profile $s \in S$ has been played.

A simple behavioral rule which has been extensively studied in the literature of learning in games is the *myopic best-reply* rule, where players are assumed to be able to compute best-replies to the current profile of strategies of their opponents, and choose one of them. In games with alternative best-replies, the need for tie-breaking gives rise to a family of rules. That is, a rule $B_i^{BR}$ is a best-reply rule if

$$B_i^{BR}(s)(s'_i) > 0 \quad \Longrightarrow \quad u_i(s'_i, s_{-i}) \geq u_i(s''_i, s_{-i}) \; \forall \; s''_i \in S_i. \tag{1}$$

Let us call $\Gamma$ a symmetric game if $S_i = S_j = S_0$ for all $i, j \in I$ and payoffs are given by a symmetric mapping, i.e. the payoff of a player choosing strategy $s_i$ against the profile of strategies $s_{-i}$ is $u_i(s_i, s_{-i}) = u(s_i|s_{-i})$, where the latter is invariant to permutations of $s_{-i}$. For symmetric games, a second prominent example of behavioral rule (or, rather, family thereof), is given by *imitate-the-best* rules as in KMR, Vega-Redondo, 1997, or Alós-Ferrer and Ania, 2005, where players just adopt one of the strategies leading to the highest, currently observed payoff. That is, again taking into account the need for tie-breaking assumptions, a rule $B_i^{IB}$ is an imitate-the-best rule if

$$B_i^{IB}(s)(s'_i) > 0 \quad \Longrightarrow \quad \begin{array}{c} s'_i = s_j \text{ for some } j \in I \text{ with} \\ u_j(s) \geq u_k(s) \; \forall \; k \in I. \end{array} \tag{2}$$

Note that, formally, a rule might rely on the payoff functions in order to specify the strategies to be chosen, but the interpretation on the actual knowledge of the game that players have might be very different. In a best-reply rule, the use of the payoff function amounts to assuming that players do know the payoff function and can use it to (myopically) optimize their behavior. In the case of an imitation rule, the use of the payoff function is just a modeling device capturing the informational assumption that players observe realized payoffs, but do not necessarily know the game or are able to perform optimizing computations.

The description of both the best-reply and the imitation rule allows for different tie-breaking assumptions. We will now provide a formal approach to their specification. A *behavioral correspondence* for player $i$ is a correspondence $\widehat{B}_i : S \twoheadrightarrow S_i$. That is, $\widehat{B}_i(s)$ is the set of strategies $s'_i \in S_i$ which player $i$ *might* choose after the profile $s \in S$ has been played. A behavioral rule $B_i$ is said to *agree* with a behavioral correspondence $\widehat{B}_i$ if

$$B_i(s)(s'_i) > 0 \quad \Longrightarrow \quad s'_i \in \widehat{B}_i(s) \tag{3}$$

for all $s'_i \in S_i$ and all $s \in S$. For instance, myopic best-reply rules as in (1) are those agreeing with the best-reply correspondence

$$\widehat{B}_i^{BR}(s) = \{s'_i \in S_i \mid u_i(s'_i, s_{-i}) \geq u_i(s''_i, s_{-i}) \ \forall s''_i \in S_i\}.$$

Imitate-the-best rules as in (2) are those agreeing with the imitation correspondence

$$\widehat{B}_i^{IB}(s) = \{s'_i \in S_i \mid s'_i = s_j \text{ for some } j \in I \text{ with } u_j(s) \geq u_k(s) \ \forall k \in I\}.$$

Given a behavioral correspondence $\widehat{B}_i$, we say that a behavioral rule $B_i$ agreeing with $\widehat{B}_i$ is *reasonable* if $B_i(s)(s_i) > 0$ whenever $s_i \in \widehat{B}_i(s)$, where $s_i$ is player $i$'s strategy in the profile $s$, and $B_i(s)(s'_i) > 0$ for all $s'_i \in \widehat{B}_i(s)$ whenever $s_i \notin \widehat{B}_i(s)$. With a reasonable behavioral rule, players who find their current behavior to be optimal according to the behavioral correspondence will not abandon it for sure (although they might also not stick to it for sure). Also, the rule respects anonymity of the strategies in the sense that only the consequences of their use matter, as evaluated by the correspondence.[1] Let $\mathscr{T}_i$ denote the set of all reasonable behavioral rules that agree with a given correspondence $\widehat{B}_i$ (we suppress dependency of $\mathscr{T}_i$ on $\widehat{B}_i$ for notational convenience).

[1]Lexicographic conditions as e.g. choosing the most popular action in case of ties can be built into the behavioral *correspondence*.

Now consider two reasonable behavioral rules $B_i^1$ and $B_i^2$ from $\mathscr{T}_i$. We say that $B_i^1$ is (weakly) *more sluggish* than $B_i^2$, written $B_i^1 \preceq B_i^2$, if $B_i^1(s)(s_i') > 0$ implies $B_i^2(s)(s_i') > 0$, for all $s_i' \in S_i$ and all $s \in S$. That is, the support of $B_i^2$ is always weakly larger than the support of $B_i^1$. We say that the two rules are *equally sluggish*, written $B_i^1 \simeq B_i^2$, if $B_i^1 \preceq B_i^2$ and $B_i^2 \preceq B_i^1$, so that the sets $\{s_i' \in S_i \mid B_i^1(s)(s_i') > 0\}$ and $\{s_i' \in S_i \mid B_i^2(s)(s_i') > 0\}$ always coincide. By construction, the relation $\simeq$ is a binary equivalence relation on $\mathscr{T}_i$. In the following, we will informally identify two behavioral rules if they are equally sluggish, i.e. if they differ in specific (strictly positive) probabilities assigned to strategies, but not in their support. Formally, we work in the quotient set $\mathscr{T}_i/\simeq$, on which the sluggishness-relation $\preceq$ becomes a partial order.

Among all rules in $\mathscr{T}_i$, we consider two distinguished rules (modulo equal sluggishness). The *cautious rule* $B_i^0$ is the rule which specifies $B_i^0(s)(s_i) = 1$ whenever $s_i \in \widehat{B}_i(s)$. That is, under the cautious rule a player will always stick to his or her current action if this is one of the optimal ones according to the behavioral correspondence $\widehat{B}_i$. The *random tie-breaking rule* $B_i^X$ is the rule given by $B_i^X(s)(s_i') > 0$ for all $s_i' \in \widehat{B}_i(s)$, that is, all strategies that are optimal according to $\widehat{B}_i$ are always chosen with strictly positive probability. The following observation is now immediate.

**Lemma 1.** *Any reasonable behavioral rule $B_i$ satisfies $B_i^0 \preceq B_i \preceq B_i^X$.*

That is, the poset $\mathscr{T}_i/\simeq$ has a top and a bottom element. It is straightforward to show that it is actually a complete lattice.

Finally, we denote profiles of reasonable behavioral rules for all players by $B = (B_i)_{i\in I} \in \mathscr{T} := \prod_{i\in I} \mathscr{T}_i$. Consider the product order on $\mathscr{T}$, i.e. $B^1 \preceq B^2$ if and only if $B_i^1$ is weakly more sluggish than $B_i^2$ for all $i \in I$. Then we also obtain $B^0 \preceq B \preceq B^X$ for any reasonable profile $B \in \mathscr{T}$ and the two extreme profiles $B^0 = (B_i^0)_{i\in I}$ and $B^X = (B_i^X)_{i\in I}$.

## 2.3 Revision opportunities

A learning dynamics for a game $\Gamma$ is made of a behavioral rule for each player, which includes tie-breaking assumptions, and a specification of revision opportunities, i.e. a way of determining which players receive the opportunity to update their actions in a given period. Intuitively, revision opportunities are closely related to the speed of the dynamics. A dynamics where only one agent is allowed to revise per period is more gradual than one where the whole population might switch away simultaneously, enabling abrupt transition phenomena. We consider a general class of revision processes which encompasses a wide range of possibilities.

**Definition 1.** A *revision process* is a probability measure $q$ on the set of subsets of $I$, $\mathcal{P}(I)$, such that

$$\forall\, i \in I,\ \exists J \subseteq I \text{ such that } i \in J \text{ and } q_J > 0 \tag{4}$$

where, for each $J \subseteq I$, $q_J = q(J)$ is interpreted as the probability that *exactly* players in $J$ receive revision opportunities (independently across periods).

This definition is taken from Alós-Ferrer and Netzer (2010). Condition (4) implies that each player gets the opportunity to revise with strictly positive probability. A revision process is called *regular* if $q_i = q_{\{i\}} > 0$ for all $i \in I$, so that for each player there is a strictly positive probability of being the *only* player who is allowed to revise. Let $\mathscr{Q}$ denote the set of all regular revision processes.

Analogously to the previous section, we can define a binary relation $\preceq$ on $\mathscr{Q}$ as follows.[2] For any $q, q' \in \mathscr{Q}$ we say that $q'$ is (weakly) *quicker* than $q$, written $q \preceq q'$, if $q_J > 0$ implies $q'_J > 0$ for any $J \subseteq I$. That is, the revision process $q'$ includes more possibilities than $q$. Say that $q$ and $q'$ have *the same speed*, written $q \simeq q'$, if $q \preceq q'$ and $q' \preceq q$. By construction, the relation $\simeq$ is a binary equivalence relation. Consider again the quotient set $\mathscr{Q}/\simeq$, where two revision processes belong to the same class if and only if they have the same speed, i.e. they differ in specific probabilities assigned to player subsets $J \subseteq I$, but not in their support. We will again identify two processes that have the same speed and treat $\preceq$ as a partial order.

Among all processes in $\mathscr{Q}$, we again consider two distinguished elements (modulo equal speed). The *asynchronous learning* process $q^{AL}$ satisfies $q^{AL}_J = 0$ whenever $|J| \geq 2$. The *independent learning* process $q^{IL}$, on the

[2]We use the same symbols for the binary relations on $\mathscr{T}$ and on $\mathscr{Q}$ for convenience.

other hand, satisfies $q^{IL}_J > 0$ for all $J \subseteq I$.[3] The following observation is now again immediate.

**Lemma 2.** *Any regular revision process $q$ satisfies $q^{AL} \preceq q \preceq q^{IL}$.*

Therefore the poset $\mathcal{Q}/\simeq$ has a top and a bottom element as well. It is again a simple exercise to show that it is actually a complete lattice.

### 2.4 Stochastic Stability

Starting from a profile of behavioral rules $B$, we can apply a *noise process* to derive associated profiles of behavioral rules with noise $B^\varepsilon = (B^\varepsilon_i)_{i\in I}$, where $\varepsilon \in (0,1)$ measures how strongly players' behavior is perturbed from $B$.[4]

For the first noise process that we consider, the *mistakes model*, we fix a noise rule $E_i : S \mapsto \Delta S_i$ for every player $i \in I$, where $E_i(s)(s'_i)$ is independent of $s$ and satisfies $E_i(s)(s'_i) > 0$ for all $s'_i \in S_i$. Then each player's behavioral rule $B_i$ is perturbed to $B^{M,\varepsilon}_i$ by

$$B^{M,\varepsilon}_i(s)(s'_i) = (1-\varepsilon)B_i(s)(s'_i) + \varepsilon E_i(s)(s'_i). \tag{5}$$

For instance, the best-reply version of the well-known KMR model, first studied in Kandori and Rob (1995), proceeds exactly like this to derive the best-reply with mistakes $B^{BR,M,\varepsilon}_i$ from an unperturbed best-reply rule $B^{BR}_i$. As $\varepsilon \to 0$, behavior converges to the best-reply rule. The noisy version $B^{IB,M,\varepsilon}_i$ of an imitate-the-best rule $B^{IB}_i$ can be constructed analogously.[5] Importantly, the tie-breaking assumptions implicit in $B^{BR}_i$ or $B^{IB}_i$ carry over to the noisy rules when the mistakes approach is used. When we start from

[3]These concepts are again taken from Alós-Ferrer and Netzer (2010). The model of Blume (1993) postulates $q_i = 1/N$ and is therefore an instance of asynchronous learning. Independent inertia as in Sandholm (1998), where $q_J = p^{|J|}(1-p)^{N-|J|}$ for some $p > 0$, is an instance of independent learning. The simultaneous learning process, where $q_I = 1$, is the simplest example of a process which is not regular.

[4]See Bergin and Lipman (1996) for a general treatment of (state-dependent) noise processes and their selection properties.

[5]The original KMR model can be readily interpreted as a model of imitation (see KMR p.31, Rhode and Stegeman, 1996, and Sandholm, 1998) where agents mimic the actions which led to highest payoffs in the last period. In a celebrated result, KMR show that their dynamics select risk-dominant equilibria, rather than payoff-dominant ones, in $2 \times 2$ symmetric coordination games.

a behavioral correspondence such as $\widehat{B}_i^{BR}$ or $\widehat{B}_i^{IB}$, for instance, the mistakes model associates to every behavioral rule $B_i \in \mathscr{T}_i$ a distinct behavioral rule with noise $B_i^\varepsilon$, which converges to $B_i$ as $\varepsilon \to 0$. We say that noise processes with this property *respect tie-breaking.*

The second noise process that we will consider is the *logit choice function*, which has been used in the literature to obtain noisy versions of the best-reply dynamics (see e.g. Blume, 1993 or Alós-Ferrer and Netzer, 2010). Formally, the probability of player $i$ choosing $s_i'$ given $s$ only depends on the profile $s_{-i}$ of actions of the opponents and is given by

$$B_i^{BR,L,\varepsilon}(s)(s_i') = \frac{e^{(1/\varepsilon)u_i(s_i',s_{-i})}}{\sum_{s_i''\in S_i} e^{(1/\varepsilon)u_i(s_i'',s_{-i})}}. \tag{6}$$

Again, all actions are chosen with strictly positive probability whenever $\varepsilon > 0$, and choice concentrates on myopic best-replies as $\varepsilon \to 0$. The logit perturbation, however, leaves no freedom in tie-breaking assumptions. As $\varepsilon \to 0$, the behavioral rule $B_i^{BR,L,\varepsilon}$ converges to the specific best-reply rule that breaks ties with equal probabilities. The logit approach is therefore not suited to associate a distinct noisy rule to every $B_i \in \mathscr{T}_i$ for a given behavioral correspondence $\widehat{B}_i$. By using logit choice, we rather select a specific $B_i \in \mathscr{T}_i$ (the one with equal tie-breaking).[6] We say that noise processes with this property *impose tie-breaking.*

Other particular examples of noise processes could also be considered. For instance, Myatt and Wallace (2003) and Dokumaci and Sandholm (2008) consider dynamics based on a probit choice function, which, as in the case of logit, impose tie-breaking. Sandholm (2010) considers general "noisy revision protocols" (where the word revision is used in a different sense as in this paper) including the mistakes model and logit and probit choice. The two prominent examples presented above are those for which we develop specific applications later in this paper.

Now consider any perturbed learning dynamics $(B^\varepsilon, q)$ derived from an unperturbed dynamics $(B, q)$ according to some noise process. Suppose that all $B_i^\varepsilon$ have full support whenever $0 < \varepsilon < 1$, as in the examples above. Then,

[6] One could also study the logit perturbation $B_i^{IB,L,\varepsilon}$ of an imitate-the-best dynamics $B_i^{IB}$, which would converge to the equal tie-breaking imitation rule as noise vanishes.

the perturbed dynamics becomes an irreducible and aperiodic Markov chain on the state space $S$ with transition probabilities

$$P^{\varepsilon}_{s,s'} = \sum_{J \subseteq I} q_J \prod_{i \in J} B^{\varepsilon}_i(s)(s'_i), \tag{7}$$

and it has a unique invariant distribution, denoted $\mu^{\varepsilon}$. A strategy profile $s \in S$ is *stochastically stable* if $\lim_{\varepsilon \to 0} \mu^{\varepsilon}(s) > 0$. Stochastic stability for the mistakes model can be characterized along the lines introduced in KMR or Young (1993), with a useful "radius-coradius" sufficient condition developed by Ellison (2000). Alós-Ferrer and Netzer (2010) provide an analogous, general characterization for the logit-response dynamics, and a similar radius-coradius result.[7]

## 3 Robustness

We are interested in the following two concepts of robustness. First, suppose we consider a given profile of behavioral rules with noise $B^{\varepsilon}$, based on some underlying profile of unperturbed behavioral rules $B$. Hence we treat as fixed a specification of tie-breaking assumptions. This is always the case when the noise process imposes tie-breaking, as with the logit-response dynamics, but it can be done for any behavioral rule and noise process as detailed above. Robustness now refers to the specification of revision opportunities alone.

**Definition 2.** Fix a profile of behavioral rules with noise $B^{\varepsilon}$. A state $s \in S$ is *robustly stochastically stable for* $B^{\varepsilon}$ if it is stochastically stable for *any* regular revision process $q \in \mathscr{Q}$.

Second, suppose we consider a profile of behavioral correspondences $\widehat{B} = (\widehat{B}_i)_{i \in I}$, with $\mathscr{T}$ being the set of profiles of reasonable behavioral rules that agree player-wise with $\widehat{B}$. For each $B \in \mathscr{T}$ we then apply a noise process that respects tie-breaking to associate a distinct profile with noise $B^{\varepsilon}$. The prime example is the mistakes model. Robustness then refers to the specification of both tie-breaking assumptions and revision opportunities.

[7] Several earlier contributions have studied logit behavior for special classes of games or dynamics (e.g. Blume, 1993, 1997; Maruta, 2002; Myatt and Wallace, 2008a,b).

**Definition 3.** Fix a profile of behavioral correspondences $\widehat{B}$ and a noise process that respects tie-breaking. A state $s \in S$ is *robustly stochastically stable for* $\widehat{B}$ *and the noise process* if it is robustly stochastically stable for all $B^\varepsilon$ that the noise process associates to the elements of $\mathscr{T}$.

We will now provide a method that allows us to identify robustly stochastically stable states. We first introduce the following auxiliary concepts.

**Definition 4.** A *monotone operator on revision processes* is a mapping

$$\begin{array}{rccc} T: & \mathscr{Q} \times S & \mapsto & \mathbb{R} \\ & (q, s) & \mapsto & T^q(s) \end{array}$$

such that $T^q(s) \geq T^{q'}(s)$ for all $s \in S$ whenever $q \preceq q'$.

Analogously, we can define monotone operators with respect to both revision opportunities and tie-breaking rules as follows. Fix a profile of behavioral correspondences $\widehat{B}$, which induces the set $\mathscr{T}$. Consider the product order on $\mathscr{T} \times \mathscr{Q}$, i.e. $(B, q) \preceq (B', q')$ if and only if $B \preceq B'$ and $q \preceq q'$.

**Definition 5.** A *monotone operator on revision processes and tie-breaking rules* is a mapping

$$\begin{array}{rccc} T: & \mathscr{T} \times \mathscr{Q} \times S & \mapsto & \mathbb{R} \\ & (B, q, s) & \mapsto & T^{B,q}(s) \end{array}$$

such that $T^{B,q}(s) \geq T^{B',q'}(s)$ for all $s \in S$ whenever $(B, q) \preceq (B', q')$.

In contrast to the usual approach, we will not define radius and coradius from a primitive such as cost (Ellison, 2000) or waste (Alós-Ferrer and Netzer, 2010). The only property of the different radius and coradius concepts that we need in the following is that they are monotone operators that yield sufficient conditions for stochastic stability.

**Definition 6.** Fix a profile of behavioral rules with noise $B^\varepsilon$. A *radius-coradius pair* $(R, CR)$ *for* $B^\varepsilon$ is a pair of monotone operators on revision processes such that, whenever $R^q(s) > CR^q(s)$ for some $s \in S$, it follows that $s$ is the only stochastically stable state for revision process $q$ and $B^\varepsilon$.

It is easy to see that the existing radius and coradius concepts are in fact radius-coradius pairs in the sense of this definition. For instance, the radius of Ellison (2000) for the mistakes model is essentially the minimal number of mistakes needed to leave the basin of attraction (under the unperturbed dynamics) of a state. If one considers a revision process that enables more transitions, the transitions which realize the minimum under the previous dynamics are still feasible. Hence the minimum can only become weakly smaller. Similarly, the coradius of Ellison (2000) is the maximum across states $s'$ outside the basin of attraction of $s$, of all the minimum number of mistakes required for transitions from $s'$ to the basin of attraction of $s$. Again, if a revision process allows for more transitions, the minima can only weakly decrease and the maximum among all the minima can only be weakly smaller than before. The reasoning for logit-response is analogous, with the number of mistakes replaced by the utility differences between the chosen actions and the myopically optimal ones. Once monotonicity is established, the fact that the property embodied in the definition above is fulfilled follows from the radius-coradius theorems in Ellison (2000) and Alós-Ferrer and Netzer (2010).

When we start from behavioral correspondences $\widehat{B}$ and a noise process that respects tie-breaking, we can define analogous concepts.

**Definition 7.** Fix a profile of behavioral correspondences $\widehat{B}$ and a noise process that respects tie-breaking. A *radius-coradius pair* $(R, CR)$ *for* $\widehat{B}$ *and the noise process* is a pair of monotone operators on revision processes and tie-breaking rules such that, whenever $R^{B,q}(s) > CR^{B,q}(s)$ for some $s \in S$, it follows that $s$ is the only stochastically stable state for revision process $q$ and the profile $B^{\varepsilon}$ associated to $B$ by the noise process.

Again, the radius and coradius due to Ellison (2000) satisfy these requirements when the noise process is the mistakes model. The intuition for monotonicity provided for Definition 6 also applies here, because we enable more transitions when we make tie-breaking less sluggish.

The following proposition embodies the main idea behind our results.

**Proposition 1.** *(i) Fix a profile of behavioral rules with noise $B^\varepsilon$. Let $(R, CR)$ be a radius-coradius pair for $B^\varepsilon$. Let $q^1, q^2 \in \mathscr{Q}$ with $q^1 \preceq q^2$. If there exists $s \in S$ such that*

$$R^{q^2}(s) > CR^{q^1}(s),$$

*then for any $q$ with $q^1 \preceq q \preceq q^2$, $s$ is the unique stochastically stable state.*

*(ii) Fix a profile of behavioral correspondences $\widehat{B}$ and a noise process that respects tie-breaking. Let $(R, CR)$ be a radius-coradius pair for $\widehat{B}$ and the noise process. Let $(B^1, q^1), (B^2, q^2) \in \mathscr{T} \times \mathscr{Q}$ with $(B^1, q^1) \preceq (B^2, q^2)$. If there exists $s \in S$ such that*

$$R^{B^2,q^2}(s) > CR^{B^1,q^1}(s),$$

*then for any $(B, q) \in \mathscr{T} \times \mathscr{Q}$ with $(B^1, q^1) \preceq (B, q) \preceq (B^2, q^2)$, $s$ is the unique stochastically stable state (for the profile $B^\varepsilon$ associated to $B$).*

*Proof.* We prove statement $(ii)$. Statement $(i)$ is proven analogously. Consider an arbitrary $(B, q) \in \mathscr{T} \times \mathscr{Q}$ with $(B^1, q^1) \preceq (B, q) \preceq (B^2, q^2)$. It suffices to notice that, by monotonicity,

$$R^{B,q}(s) \geq R^{B^2,q^2}(s) > CR^{B^1,q^1}(s) \geq CR^{B,q}(s),$$

which implies the statement by definition of radius-coradius pair. □

Using part $(i)$ of this result and Lemma 2, we obtain an immediate corollary which delivers a simple condition for robust stochastic stability given a behavioral rule with noise.

**Corollary 3.** *Fix a profile of behavioral rules with noise $B^\varepsilon$. Let $(R, CR)$ be a radius-coradius pair for $B^\varepsilon$. If there exists $s \in S$ such that*

$$R^{q^{IL}}(s) > CR^{q^{AL}}(s),$$

*then $s$ is the unique robustly stochastically stable state for $B^\varepsilon$.*

This corollary applies directly to the logit-response dynamics or to any mistakes model for pre-specified tie-breaking assumptions. The result states that establishing robust stochastic stability is just as simple (or just as complex) as establishing stochastic stability with the help of a radius-coradius result. The only difference is that one must focus on asynchronous learning for computing the coradius and on independent learning for computing the radius.

Using part (*ii*) of Proposition 1 and Lemmas 1 and 2, we also obtain an immediate corollary about robustness with respect to tie-breaking rules in addition to revision processes.

**Corollary 4.** *Fix a profile of behavioral correspondences* $\widehat{B}$ *and a noise process that respects tie-breaking. Let* $(R, CR)$ *be a radius-coradius pair for* $\widehat{B}$ *and the noise process. If there exists* $s \in S$ *such that*

$$R^{B^X, q^{IL}}(s) > CR^{B^0, q^{AL}}(s),$$

*then* $s$ *is the unique robustly stochastically stable state for* $\widehat{B}$ *and the noise process.*

Hence, even when we require robustness to cover both revision processes and tie-breaking assumptions, a radius-coradius result applies. Again we need to focus on two different, focal dynamic specifications only.

# 4 Symmetric Games and ESS

Consider a symmetric game $\Gamma$ as defined above. Relevant examples include Cournot oligopolies, rent-seeking games, and other classes of games (see Alós-Ferrer and Ania, 2005 for further details). Following Schaffer (1988, 1989), a strategy $s^* \in S_0$ is a strict, globally stable ESS (where ESS stands for "Evolutionarily Stable Strategy") if for all $s' \in S_0$, $s' \neq s^*$,

$$u(s^*|s', \overset{m}{\ldots}, s', s^*, \ldots, s^*) > u(s'|s', \overset{m-1}{\ldots}, s', s^*, \ldots, s^*) \qquad (8)$$

for all $m \in \{1, \ldots, N-1\}$ . In words, a globally stable strategy earns larger payoffs than any alternative strategy in any profile where only those two strategies are present. In the case of a Cournot oligopoly, the Walrasian quantity has been shown by Vega-Redondo (1997) to be strictly, globally stable, and stochastically stable in imitation-based dynamics with mistakes. This result has been extended by Alós-Ferrer and Ania (2005) as follows. First, any strict, globally stable ESS in a symmetric game is stochastically stable in a mistakes dynamics with imitate-the-best and independent inertia; Alós-Ferrer and Schlag (2009) observe that this result holds for an even broader class of imitation rules. Second, strict global stability, which might seem a restrictive concept, includes a family of outcomes of special interest. Informally, an aggregative game is a symmetric game such that players' payoffs depend only on the own strategy and an aggregate of all strategies. It is quasi-submodular if the own strategy and the aggregate exhibit an ordinal substitutability (see Alós-Ferrer and Ania, 2005 for details). In every quasi-submodular, aggregative game (which again includes Cournot oligopolies, rent-seeking games, and other examples), strict global stability follows from a more economic concept, Aggregate-Taking-Strategy, i.e. the generalization of Walrasian equilibrium where each player maximizes payoffs taking the aggregate of all strategies as given.

Here we show that strict, globally stable ESS (and hence aggregate-taking strategies in quasi-submodular, aggregative games) are actually robustly stochastically stable for imitation dynamics and the mistakes model. In other words, the associated selection result is robust to the specification of both revision opportunities and tie-breaking rules.

**Proposition 2.** *Let $s^*$ be a strict, globally stable ESS of a symmetric game. Consider the profile of imitation correspondences $\widehat{B}^{IB} = (\widehat{B}_i^{IB})_{i \in I}$. Then $(s^*, \ldots, s^*)$ is robustly stochastically stable for $\widehat{B}^{IB}$ and the mistakes model.*

*Proof.* We will rely on Corollary 4 and radius-coradius from Ellison (2000). Consider the dynamics with independent learning and random tie-breaking. If one mutant appears at the state $(s^*, \ldots, s^*)$, inequality (8) with $m = 1$ indicates that $s^*$-players earn strictly more than the mutant. Hence, tie-breaking is irrelevant and one mutation is not enough to leave the basin of attraction of $(s^*, \ldots, s^*)$. We conclude that $R^{B^X, q^{IL}}(s^*, \ldots, s^*) > 1$.

Consider now the dynamics with asynchronous learning and cautious tie-breaking. It is easy to see that it suffices to consider monomorphic states $(s, \ldots, s)$ for the calculation of coradius. For any such state with $s \neq s^*$, if a single mutation to $s^*$ occurs, by inequality (8) with $m = N - 1$, we obtain that the mutant earns strictly more than the incumbents. In the asynchronous learning dynamics, if the mutant is selected to revise, the state will remain unchanged. Eventually, an incumbent will be selected and switch to $s^*$. By (8) with $m = N-2$, the next incumbent to be selected will also switch to $s^*$. Iterating, the dynamics will reach the state $(s^*, \ldots, s^*)$ and we conclude that $CR^{B^0, q^{AL}}(s^*, \ldots, s^*) = 1$. The conclusion follows from Corollary 4. □

This results strengthens the ones in Vega-Redondo (1997) and Alós-Ferrer and Ania (2005) and shows that the relevance of the concept of strictly globally stable ESS due to Schaffer (1988, 1989) for imitation models goes beyond particular modeling assumptions.

# 5 Symmetric Binary Action Games

## 5.1 Notation and Definitions

Now let $\Gamma$ be a symmetric binary action game (see e.g. Kim, 1996; Maruta, 2002; Blume, 2003), where the players' strategy sets are given by $S_i = \{A, B\}$. Symmetry implies that each player's payoff depends only on the own action and on the *number* of opponents choosing each action.[8] Thus, given a strategy profile $s \in S$, denote by $m(s) \in \{0, ..., N\}$ the number of players choosing action $A$ in $s$. Let $\pi^A(n)$ be the payoff of an $A$-player given that $n$ players choose action $A$ altogether (including the respective player herself), and let $\pi^B(n)$ be the payoff of a $B$-player if $n$ players are choosing $A$. We can then write the payoff functions as

$$u_i(A, s_{-i}) = \pi^A(m(A, s_{-i})) \tag{9}$$

and

$$u_i(B, s_{-i}) = \pi^B(m(B, s_{-i})). \tag{10}$$

Furthermore, we define $\Delta(n) := \pi^A(n) - \pi^B(n-1)$ for $1 \leq n \leq N$ as the payoff change of a player who switches from action $B$ to action $A$, given that $n-1$ of the opponents choose action $A$, so that the overall number of $A$-players is $n$ after the switch. Throughout, we assume that $\Delta(n) \neq 0$ for some

[8]Sandholm (2010) also considers symmetric binary action games, concentrating on the asymptotics as noise vanishes *and* population size goes to infinity. Staudigl (forthcoming) follows the same approach for asymmetric binary action games.

$n$, i.e. we exclude the trivial case where the players' payoffs are completely unaffected by their own choice of action. We consider two examples.

*Example* 1. Consider a unanimity game (e.g. Young, 1998a, Section 9) where $\pi^B(0) > 0$ and $\pi^A(N) > 0$, but $\pi^A(n) = 0$ if $n < N$ and $\pi^B(n) = 0$ if $n > 0$. The game has two strict Nash equilibria, $\vec{A} = (A, \ldots, A)$ and $\vec{B} = (B, \ldots, B)$. In addition, every state $s \in S$ with $2 \leq m(s) \leq N-2$ is a non-strict Nash equilibrium. The difference function $\Delta(n)$ of the unanimity game is given by $\Delta(1) = -\pi^B(0)$, $\Delta(n) = 0$ for all $2 \leq n \leq N-1$, and $\Delta(N) = \pi^A(N)$.

*Example* 2. The unanimity game can be generalized in different ways. As a particularly interesting example for our purpose, consider a team project game with two projects, $A$ and $B$, where each of the $N$ players must participate in exactly one of the projects. Participation is costless, but the success of project $A$ requires the participation of at least $n_A \leq N$ players, while project $B$ is successful if and only if at least $n_B \leq N$ players participate. Assume further that $n_A + n_B > N + 1$, which implies that the two projects cannot be realized jointly, and that there is the possibility that none of them is successful. If project $A$ ($B$) is successful, it generates an overall benefit of size $a > 0$ ($b > 0$), which is distributed equally among all participating players. Players who do not participate in a successful project obtain a payoff of zero. Hence the payoffs are given by

$$\pi^A(n) = \begin{cases} a/n & \text{if} \quad n \geq n_A, \\ 0 & \text{if} \quad n < n_A, \end{cases} \qquad \pi^B(n) = \begin{cases} 0 & \text{if} \quad n > N - n_B, \\ b/(N-n) & \text{if} \quad n \leq N - n_B. \end{cases}$$

The two profiles $\vec{A} = (A, \ldots, A)$ and $\vec{B} = (B, \ldots, B)$ are again strict Nash equilibria, and states $s \in S$ with $N - n_B + 2 \leq m(s) \leq n_A - 2$ are non-strict Nash equilibria. We obtain the difference function

$$\Delta(n) = \begin{cases} a/n & \text{if} \quad n_A \leq n \leq N, \\ 0 & \text{if} \quad N - n_B + 2 \leq n \leq n_A - 1, \\ -b/(N-n+1) & \text{if} \quad 1 \leq n \leq N - n_B + 1. \end{cases}$$

The team project game becomes the unanimity game if $n_A = n_B = N$.[9]

[9] Maruta (2002) and Maruta and Okada (2009) generalize unanimity games to the

We believe it to be commonly known that *every* symmetric binary action game is an exact potential game in the sense of Monderer and Shapley (1996). However, we are not aware of a formal statement of this fact in the literature, so we present the result in the following lemma, together with a straightforward potential function.[10]

**Lemma 5.** *Any symmetric binary action game is a potential game with potential function*

$$\rho(s) = \begin{cases} \sum_{j=1}^{m(s)} \Delta(j) & \text{if} \quad m(s) \geq 1, \\ 0 & \text{if} \quad m(s) = 0. \end{cases} \tag{11}$$

*Proof.* See Appendix. □

Exact potential games are relevant for the logit-response dynamics, which, as mentioned in the introduction, selects the potential-maximizing states as stochastically stable under certain assumptions. We will now introduce two additional properties that will become crucial for the analysis of robust stochastic stability: *supermodularity* and *strategic complementarity* (Topkis, 1998). To define supermodularity, we first impose an order $\leq$ on the strategy set $\{A, B\}$ by defining the convention $B \leq A$. We then obtain a partial order (also denoted $\leq$) on each of the sets $S_{-i}$, by using the product order derived from $\leq$.

**Definition 8.** $\Gamma$ is supermodular if for each $i \in I$ and all $s_{-i}, s'_{-i} \in S_{-i}$ with $s_{-i} \leq s'_{-i}$, it holds that

$$u_i(A, s_{-i}) - u_i(B, s_{-i}) \leq u_i(A, s'_{-i}) - u_i(B, s'_{-i}). \tag{12}$$

different class of "binary coordination games" (see our discussion of Definition 8 below). Our team project game is also related to the collective-action games studied by Myatt and Wallace (2008a,b) for general quantal response dynamics under asynchronous learning (and also simultaneous learning in Myatt and Wallace, 2008b). The games in Myatt and Wallace (2008a,b) are not necessarily symmetric, they exhibit a single project only, and *all* players obtain a positive payoff if the project is successful.

[10] Maruta (2002) shows that symmetric binary coordination games are exact potential games, with a potential function as given in (11). Myatt and Wallace (2008b) show that their collective-action games are potential games under a symmetry condition, again with a potential function similar to (11).

Definition 8 is the standard definition that requires the individuals' payoff functions to have increasing differences in $(s_i, s_{-i})$. For symmetric binary action games, it is easy to show that property (12) is equivalent to the difference function $\Delta(n)$ being weakly increasing in $n$ (see Lemma 7 in the Appendix). Such games are called binary coordination games by Maruta (2002) and Maruta and Okada (2009).[11] The unanimity game (Example 1) is supermodular. Another example would be a population game where $N$ players are matched pairwise in a round-robin tournament to play a symmetric $2 \times 2$ coordination game (see Section 5.3). Figure 1 depicts an exemplary difference function $\Delta$ of a supermodular game. As indicated in the figure, we will denote by $\underline{n}$ the smallest value such that $\Delta(n) \geq 0$ for all $n > \underline{n}$ (and hence $\Delta(n) < 0$ for all $n \leq \underline{n}$). Analogously, we denote by $\overline{n}$ the largest value such that $\Delta(n) \leq 0$ for all $n \leq \overline{n}$ (and hence $\Delta(n) > 0$ for all $n > \overline{n}$).

If $\Gamma$ is supermodular, only the monomorphic states $\vec{A} = (A, ..., A)$ or $\vec{B} = (B, ..., B)$ can maximize the potential $\rho$. This is because the potential of a state $s$ is the sum of weakly increasing elements $\Delta(n)$ up to $m(s)$, so that only the states with $m(s) = 0$ or $m(s) = N$ can be maximizers of $\rho$.[12]

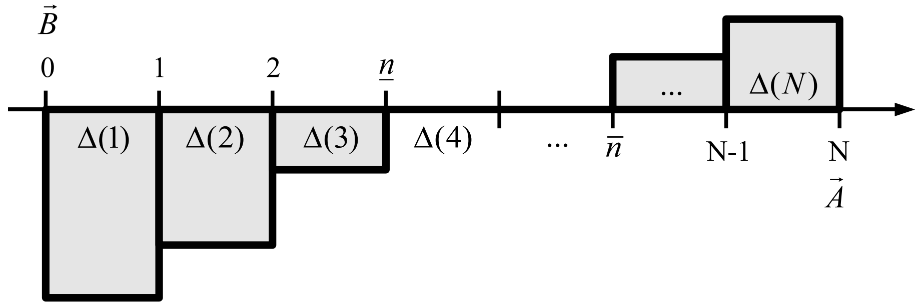


Figure 1: Supermodularity

[11]Technically speaking, Maruta (2002) requires the difference function $\Delta$ to be strictly increasing. Maruta and Okada (2009) allow for games that are not necessarily symmetric.

[12]The case where $\Delta(n) = 0$ for all $0 \leq n \leq N$ has been excluded by assumption.

The team project game from Example 2 is not supermodular (except if $n_A = n_B = N$ so that it becomes the unanimity game), because $\Delta(n)$ is decreasing from 1 to $N - n_B + 1$, and from $n_A$ to $N$. Still, the game satisfies the weaker condition of strategic complementarity (see Vives, 2005).

**Definition 9.** $\Gamma$ exhibits strategic complementarity if for each $i \in I$ and $s_{-i} \in S_{-i}$,

$$u_i(A, s_{-i}) \geq u_i(B, s_{-i}) \text{ implies } u_i(A, s'_{-i}) \geq u_i(B, s'_{-i}) \tag{13}$$

for all $s'_{-i} \in S_{-i}$ with $s_{-i} \leq s'_{-i}$.

The definition is again standard, requiring best-responses to be weakly increasing: if $A$ is a best-response against $s_{-i}$, then the same holds for any $s'_{-i}$ with $s_{-i} \leq s'_{-i}$. Conversely, condition (13) also implies that if $B$ is a best-response against $s_{-i}$, it remains a best-response against any $s'_{-i} \leq s_{-i}$. We can again give a characterization of strategic complementarity in terms of the difference function: a symmetric binary action game exhibits strategic complementarity if and only if two values $\underline{n}$ and $\overline{n}$ as described above do exist, i.e. $\Delta(n) < 0$ for all $n \leq \underline{n}$, $\Delta(n) \geq 0$ for all $n > \underline{n}$, $\Delta(n) \leq 0$ for all $n \leq \overline{n}$ and $\Delta(n) > 0$ for all $n > \overline{n}$. Thus any supermodular game exhibits strategic complementarity, but the converse is not true. Figure 2 illustrates the case of a game that exhibits strategic complementarity but is not supermodular.

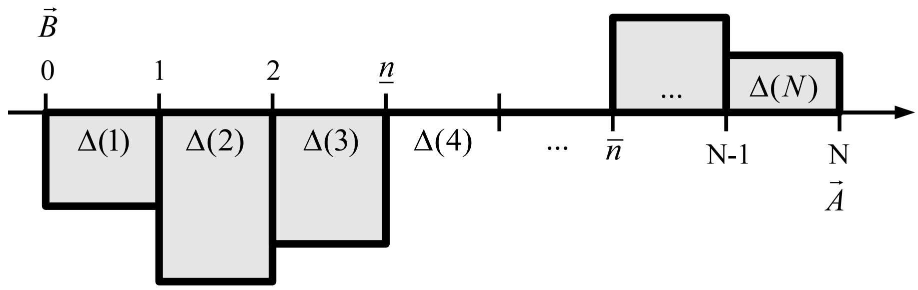


Figure 2: Strategic Complementarity

As before, observe that only the states $\vec{A} = (A, ..., A)$ or $\vec{B} = (B, ..., B)$ can maximize the potential of a symmetric binary action game with strategic complementarities.

### 5.2 Logit-Response

We now turn to the (best-reply based) logit-response dynamics for symmetric binary action games. An earlier result by Blume (1993, 1997) for the class of exact potential games implies that the potential maximizing strategy profile will be stochastically stable under asynchronous learning. With the potential function from Lemma 5, the difference in potential between two states $s$ and $s'$ corresponds to the accumulated utility changes of moving asynchronously from $s$ to $s'$. Moving towards a profile with larger potential is thus always easier under logit response, if only one player can update at a time.

Consider the unanimity game in Example 1. We only need to compare the value of the potential for $\vec{A}$ and $\vec{B}$. Straightforward calculations reveal that $\rho(\vec{B}) = 0$ and $\rho(\vec{A}) = (a - b)/N$, so that a project is stochastically stable with asynchronous logit response if and only if it is Pareto efficient.

We now want to examine under which conditions the selection of the potential maximizer is robust. We apply Corollary 3 to obtain the following result.

**Theorem 1.** *Let $\Gamma$ be a supermodular symmetric binary action game. Then, the potential maximizing states are robustly stochastically stable for the logit-response dynamics.*

*Proof.* See Appendix. □

This result is interesting in itself. The selection of potential maximizers in exact potential games (Blume, 1993, 1997) has been shown to be knife-edge by Alós-Ferrer and Netzer (2010), in the sense that it neither holds for general revision processes beyond asynchronous learning even for exact potential games, nor for generalized potential games even for asynchronous learning.[13] For the particular class of $N$-player binary action games, The-

[13]Interestingly, however, Okada and Tercieux (2008) show that, under supermodularity,

orem 1 shows that, if one additionally assumes supermodularity, potential maximizers do become a robust prediction. Hence, the relevance of potential maximizers does extend beyond asynchronous learning, and the result becomes a generalization of the original result by Blume (1993, 1997), at the price of considering a smaller class of games.

Theorem 1 has a straightforward intuition. In supermodular games, the waste of a non-best-reply, i.e. the absolute value of the difference between its associated payoff and that of the best-reply (see Alós-Ferrer and Netzer, 2010), is decreasing in the number of opponents already choosing that non-optimal action. Hence, with a logit choice rule, a player's mistake becomes more likely the more players have already made that mistake before. Minimal waste paths in and out of the basin of attraction of an absorbing state are therefore constructed by letting players switch sequentially, as under asynchronous learning, so that stochastically stable states under asynchronous learning are stochastically stable for any regular revision process.

The above mentioned selection of the Pareto efficient equilibrium in the unanimity game is therefore robust by its supermodularity property. The result that the risk-dominant equilibrium of a symmetric $2 \times 2$ coordination game played in a round-robin tournament or on a (weighted) network (Young, 1998b) will be selected by the logit dynamics is also robust due to supermodularity. The same is true for the results that Maruta (2002) obtains for binary coordination and hence supermodular games under asynchronous logit response.

To investigate the case where supermodularity is not satisfied, we will now analyze the team project game. Theorem 1 relies on the fact that, in supermodular games, the basins of attraction become shallower with distance to the absorbing state, as illustrated in Figure 1. The opposite holds in the team project game, where the basins do become deeper: leaving a successful project is more damaging if there are fewer people active in the project, and the benefit has to be shared among a smaller number of people. The property of deepening basins is illustrated in Figure 3, for a case with $N = 7, n_A = 6, n_B = 5$ and $a > b$.

the asynchronous version of the logit-response dynamics selects local potential maximizers, a generalization of potential maximizers.

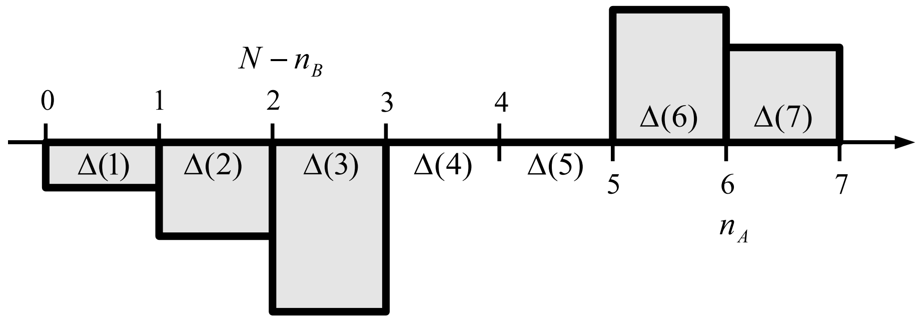


Figure 3: Team Project Game

First, the size of the basins does not depend on the specific (regular) revision process. We can construct a zero waste path from state $s$ to $\vec{B}$ (to $\vec{A}$) if and only if $m(s) \leq n_A - 1$ (respectively $m(s) \geq N - n_B + 1$), so that

$$B(\vec{A}) = \{s \in S | m(s) \geq N - n_B + 1\}$$

and

$$B(\vec{B}) = \{s \in S | m(s) \leq n_A - 1\}.$$

Consider asynchronous learning first. We immediately obtain $R^{q^{AL}}(\vec{A}) = CR^{q^{AL}}(\vec{B}) = a \sum_{j=n_A}^{N} 1/j$ and $R^{q^{AL}}(\vec{B}) = CR^{q^{AL}}(\vec{A}) = b \sum_{j=n_B}^{N} 1/j$, by adding the utility losses of moving through the basins one step at a time. With independent learning, the waste of a transition that involves several players changing their action simultaneously is the sum of individual myopic utility losses. Thus whenever a basin of attraction becomes deeper as we move away from the absorbing state, jumping directly out of the basin by letting a sufficient number of players mutate simultaneously will cause a smaller waste than a path of sequential mutations, where each step makes the next one less likely. Leaving $B(\vec{A})$ like this requires $N - n_A - 1$ players to change actions simultaneously, each at a waste of $a/N$, so that we obtain $R^{q^{IL}}(\vec{A}) = CR^{q^{IL}}(\vec{B}) = (a/N)(N - n_A + 1)$, and $R^{q^{IL}}(\vec{B}) = CR^{q^{IL}}(\vec{A}) = (b/N)(N - n_B + 1)$ analogously. Based on these calculations, we can now provide the following result.

**Proposition 3.** *Consider the team project game and the logit-response dynamics. Assume w.l.o.g. that $n_B \leq n_A$. Then there exist critical values*

$$\Lambda^{R,B} \leq \Lambda^{IL} \leq \Lambda^{AL} \leq \Lambda^{R,A} \tag{14}$$

*such that state $\vec{A}$ ($\vec{B}$, respectively) is*
*(i) stochastically stable with asynchronous learning iff $a/b \geq (\leq)\, \Lambda^{AL}$,*
*(ii) stochastically stable with independent learning iff $a/b \geq (\leq)\, \Lambda^{IL}$,*
*(iii) robustly stochastically stable if $a/b > \Lambda^{R,A}$ ($a/b < \Lambda^{R,B}$).*
*The first and third inequalities in (14) are strict if and only if $n_A < N$.*
*The second inequality in (14) is strict if and only if $n_B < n_A$.*

*Proof.* See Appendix. □

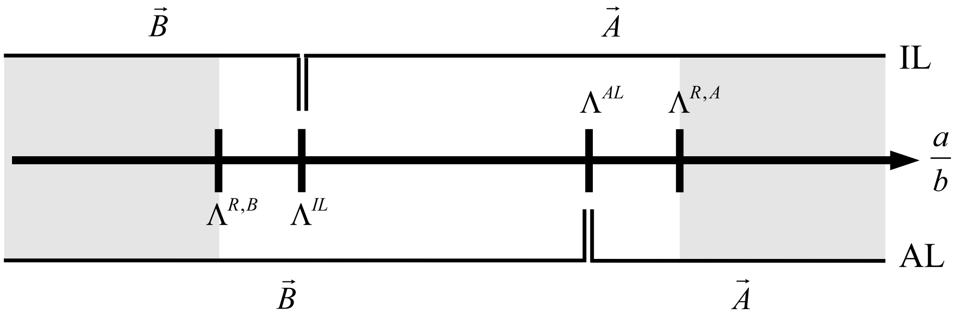


Figure 4: Illustration of Proposition 3

The proposition is illustrated in Figure 4, for the case where $n_B < n_A < N$, so that all inequalities in (14) are strict.

To make statements about stochastic stability, we need to compare the ratio of project payoffs $a/b$ to the respective critical values, which in turn are the ratio of the appropriate radius and coradius terms. For instance, the critical values $\Lambda^{IL}$ and $\Lambda^{AL}$ are given by

$$\Lambda^{IL} = \frac{N - n_B + 1}{N - n_A + 1}, \quad \Lambda^{AL} = \frac{\sum_{j=n_B}^{N} 1/j}{\sum_{j=n_A}^{N} 1/j},$$

as shown in the proof of the proposition. Hence the conditions for stochastic stability under independent and asynchronous learning both reflect a trade-off between a project's payoff and its participation requirement: larger payoffs favor stochastic stability, and a large minimum number of participants works against it. If, for instance, the less resource intensive project $B$ also exhibits the larger payoff ($b > a$), then we immediately obtain that $\vec{B}$ is stochastically stable for both $q^{IL}$ and $q^{AL}$ (because $a/b < 1 \leq \Lambda^{IL} \leq \Lambda^{AL}$). The more interesting case arises if project $A$ has a strict payoff advantage ($a > b$) while project $B$ requires strictly fewer participants for success ($n_A > n_B$). Then we have $\Lambda^{IL} < \Lambda^{AL}$, which implies that independent learning is more likely to select the payoff dominant project: whenever $\Lambda^{IL} < a/b < \Lambda^{AL}$, the payoff dominant state $\vec{A}$ is already stochastically stable with independent learning, but still $\vec{B}$ under asynchronicity.[14] Most importantly, we do *not* have a robustly stochastically stable state in this situation. Hence the team project game shows that Theorem 1 cannot be generalized from supermodular games to the broader class of games with strategic complements.

Proposition 3 still delivers sufficient conditions for robustness: the advantage of one project over the other has to be sufficiently pronounced. State $\vec{A}$, for example, is robustly stable if $a/b > \Lambda^{R,A}$, which is illustrated by a shaded gray area in Figure 4. The critical values for robustness are given by

$$\Lambda^{R,B} = \frac{N - n_B + 1}{N \sum_{j=n_A}^{N} 1/j}, \quad \Lambda^{R,A} = \frac{\sum_{j=n_B}^{N} 1/j}{N - n_A + 1},$$

which follows immediately from applying our robust radius-coradius result from Corollary 3.

### 5.3 Mistakes Model

In this section, we examine robust stochastic stability for the mistakes model based on a myopic best-reply dynamics. Hence we proceed in parallel to the previous section for the logit-response dynamics, but, since the mistakes model is a noise process that respects tie-breaking, we will also investigate robustness with respect to tie-breaking assumptions, using Corollary 4.

Analyzing the team project game under asynchronous learning and cautious tie-breaking is in fact straightforward. First, the size of the basins

[14] The reason is that the radius is linearly decreasing in the participation requirement if learning is independent, but convex if learning is asynchronous. The relative advantage of a smaller participation requirement is thus greater under asynchronous learning.

of attraction remains exactly as before. To move out of $B(\vec{B})$ we need $N - n_B + 1$ consecutive costly mutations towards any state $s$ with $m(s) = N - n_B + 1$, which can then be connected to $\vec{A}$ at zero cost. Hence we have $R^{B^0,q^{AL}}(\vec{B}) = CR^{B^0,q^{AL}}(\vec{A}) = N - n_B + 1$. Analogously, $R^{B^0,q^{AL}}(\vec{A}) = CR^{B^0,q^{AL}}(\vec{B}) = N - n_A + 1$. This implies that the project with smaller participation requirement will be stochastically stable. Specifically, both $A$ and $B$ are stochastically stable in the unanimity game, as already pointed out by Young (1998a), so that the asynchronous mistakes model cannot distinguish between the two projects.[15] As it turns out, these findings for the team project game are robust due to the fact that the game exhibits strategic complementarity.

**Theorem 2.** *Let $\Gamma$ be a symmetric binary action game with strategic complementarity. Consider the profile of best-reply correspondences $\widehat{B}^{BR} = (\widehat{B}_i^{BR})_{i \in I}$. Then, $\vec{A}$ is robustly stochastically stable for $\widehat{B}^{BR}$ and the mistakes model if and only if $\underline{n} + \overline{n} \leq N$, and $\vec{B}$ is robustly stochastically stable if and only if $\underline{n} + \overline{n} \geq N$.*

*Proof.* See Appendix. □

Compared to the logit-response dynamics, the mistakes model requires only the weaker property of strategic complementarity for robustness of its selection result in symmetric binary action games. Strategic complementarity implies that the basin of attraction of each monomorphic state contains in its interior no area where the unperturbed dynamics would lead away from the monomorphic state, and thus it suffices to compare the size of the basins, irrespective of the specific regular revision process and tie-breaking assumptions. In this sense, the mistakes model is more robust than the logit-response dynamics, delivering robust selection results for a larger class of games. The reason is, of course, that it makes use of the payoff structure of the game to a lesser extent.

As an immediate application, we obtain the robustness of a classical selection result.

[15] See Maruta and Okada (2009) for a treatment of generalized, asymmetric unanimity games under perturbed adaptive play as in Young (1993).

**Corollary 6.** *Consider a symmetric $2 \times 2$ coordination game played in a round-robin tournament. For myopic best-reply and the mistakes model, coordination on the risk-dominant equilibrium is robustly stochastically stable.*

Similarly, the comparable results of Maruta and Okada (2009) for the symmetric case are robust due to strategic complementarity of their binary coordination games.

We want to conclude this section by presenting an example without strategic complementarity, in which the mistakes model's selection is in fact not robust. Consider a game with difference function as displayed in Figure 5. Clearly, strategic complementarity fails, because we have $u_i(A, s_{-i}) < u_i(B, s_{-i})$ for $s_{-i}$ such that $m(A, s_{-i}) = 1$, but $u_i(A, s'_{-i}) = u_i(B, s'_{-i})$ for any $s'_{-i}$ with $m(A, s_{-i}) = 2$ and hence $s_{-i} \leq s'_{-i}$.

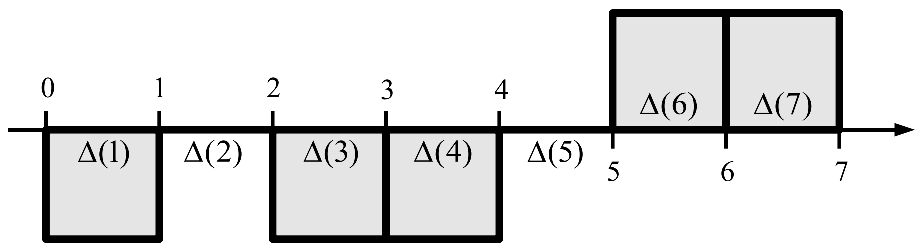


Figure 5: Non-Complementarity

With asynchronous learning, leaving $\vec{B}$ requires 3 mutations. After an initial mutation, a second player can switch to $A$ at zero cost, but two additional mutations become necessary afterwards. $\vec{A}$ can be left with only two mutations, so that $\vec{B}$ is stochastically stable. With independent learning, however, $\vec{B}$ can be left with only one mutation. After reaching a state $s$ with $m(s) = 1$, all remaining $B$-players are (myopically) indifferent between $A$ and $B$, and the basin of attraction of $\vec{B}$ can be left without additional cost whenever at least 3 of them switch actions simultaneously. Hence $\vec{A}$ is stochastically stable. The difference between the two revision processes arises because the basin of attraction of $\vec{B}$ contains states in its interior where the unperturbed dynamics no longer gravitates back to $\vec{B}$. Simultaneous mutations then allow for direct and costless jumps out of the basin.

## 6 Limitations of the Approach

Radius-coradius results are sufficient conditions. Our approach shares this limitation with previous results. In some cases, this is not an issue. For instance, in the team project game, standard radius-coradius approaches are always able to identify the stochastically stable state (except for the non-generic case where both $\vec{A}$ and $\vec{B}$ are stable). However, in general there might be cases where robustly stochastically states cannot be identified by applying Corollaries 3 or 4.

Corollary 3 asserts that a state is robustly stochastically stable for some noisy behavioral rule profile $B^{\varepsilon}$ if its radius under independent learning is larger than its coradius under asynchronous learning, $R^{q^{IL}}(s) > CR^{q^{AL}}(s)$. This is not the same as being stochastically stable for both independent and asynchronous learning, or $R^{q^{IL}}(s) > CR^{q^{IL}}(s)$ and $R^{q^{AL}}(s) > CR^{q^{AL}}(s)$. Robustness is stronger as it requires stochastic stability also for all intermediate regular revision processes. In the following example we want to illustrate that a state can be stochastically stable for both independent and asynchronous learning, but fail to be stochastically stable for *all* regular revision processes. For this purpose, we again use the logit-response dynamics.

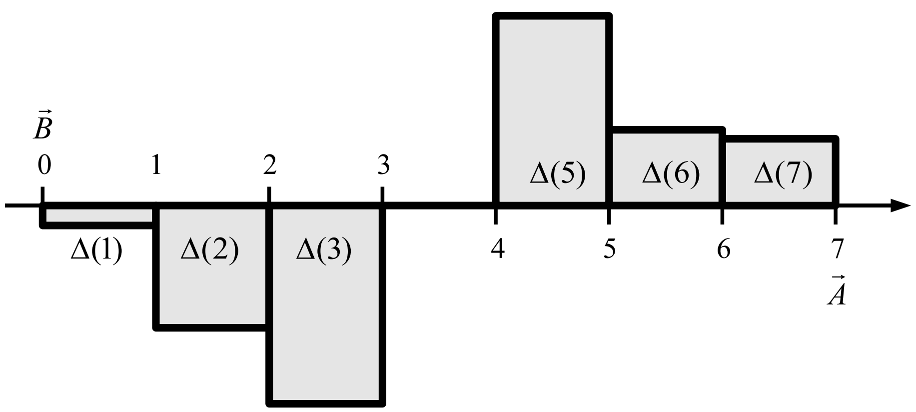


Figure 6: Example 3

*Example* 3. Consider a symmetric binary action game with $N = 7$ and difference function $\Delta(n)$ as given in Figure 6. If learning is asynchronous and $-[\Delta(1)+\Delta(2)+\Delta(3)] < [\Delta(5)+\Delta(6)+\Delta(7)]$, then $\vec{A}$ is stochastically stable because the utility losses of moving out of $B(\vec{B})$ are strictly smaller than those of leaving $B(\vec{A})$. If learning is independent and $-\Delta(1) < \Delta(7)$, the same is true because leaving $B(\vec{A})$ causes a waste of $3\Delta(7)$ while leaving $B(\vec{B})$ causes only $-3\Delta(1)$.

But now consider a regular revision process $q \in \mathscr{Q}$ where $q_{\{1,2\}} > 0$ and $q_J = 0$ for all other $J \subseteq I$ with $|J| \geq 2$. Under the assumption that $-\Delta(2) < -[\Delta(1) + \Delta(3)]/2$, a minimal waste path out of $B(\vec{B})$ is constructed by letting a single player, say player 3, switch to $A$ first, followed by a simultaneous switch of players 1 and 2.[16] The waste associated to this path is $R^q(\vec{B}) = CR^q(\vec{A}) = -[\Delta(1) + 2\Delta(2)]$. If $\Delta(6) < [\Delta(5) + \Delta(7)]/2$, the minimal waste path out of $B(\vec{A})$ has the analogous updating structure, leading to $R^q(\vec{A}) = CR^q(\vec{B}) = [\Delta(7) + 2\Delta(6)]$. Now consider a game where

$$\Delta(1) = -1, \Delta(2) = -10, \Delta(3) = -20, \Delta(5) = 20, \Delta(6) = 7, \Delta(7) = 5,$$

which satisfies all above assumptions. Then $R^q(\vec{B}) = 21 > 19 = CR^q(\vec{B})$. Hence $\vec{B}$ is stochastically stable under $q$, so that $\vec{A}$ is not robustly stochastically stable, despite being stochastically stable under both asynchronous and independent learning.

In view of the last example, we want to conclude this section by discussing the tightness of our sufficient conditions for robustness. Recall the analysis of team project games in Proposition 3. We are not allowed to conclude that, for instance, the bound $\Lambda^{R,A}$ is not tight because $\Lambda^{AL} < \Lambda^{R,A}$ when $n_A < N$. In fact, conditions that are both sufficient *and* necessary for robust stochastic stability are out of reach already in this rather simple class of games. Instead, we will show that our sufficient radius-coradius condition

[16] Since the basin of attraction is deepening with distance to $\vec{B}$, the simultaneous switch of players 1 and 2 must clearly be used in a minimal waste path that leaves $B(\vec{B})$. The only question is whether the two simultaneous strategy changes should occur right in the beginning, leading to overall waste $-[2\Delta(1) + \Delta(3)]$, or in step 2, leading to waste $-[\Delta(1) + 2\Delta(2)]$. The latter expression is smaller whenever $-\Delta(2) < -[\Delta(1) + \Delta(3)]/2$.

is tight in some but not in other cases. First, when $n_A = N$, then the sufficient condition is tight, because $\Lambda^{R,B} = \Lambda^{IL}$ and $\Lambda^{AL} = \Lambda^{R,A}$ according to Proposition 3. State $\vec{A}$, for instance, is then robustly stable if $a/b > \Lambda^{R,A}$, but not if $a/b < \Lambda^{R,A}$. On the other hand, our condition will not be tight if $n_B = n_A < N$ and $a > b$. For this specific parameter constellation, it is straightforward to verify (see Lemma 8 in the Appendix) that $\vec{A}$ is not only stochastically stable under asynchronous and independent learning but in fact robustly stochastically stable. However, we have $\Lambda^{AL} = 1 < \Lambda^{R,A}$ according to Proposition 3. Then, when $\Lambda^{AL} \leq a/b \leq \Lambda^{R,A}$, we are in a situation where our approach is not able to identify the robustly stochastically stable state, although it exists.

# 7 Conclusion

Stochastic stability is and remains an important concept in Game Theory. Unfortunately, it is sometimes too flexible a concept, and different assumptions might lead to different results. In our view, stochastic stability is well suited to analyze questions of outcome selection in noisy environments, as long as the different ingredients of the model are clearly differentiated. Ideally, a strong, clearcut result is one linking a particular behavioral assumption (captured by a behavioral rule or correspondence) under a particular interaction structure (as a proxy for the socioeconomic setting, e.g. the network structure) to the selection of a particular outcome.

Failing that, of course more subtle elements of the model might have an influence on long-run outcomes, and it is still important to understand the reasons behind this influence. However, we should be careful with results which depend crucially on modeling assumptions such as revision opportunities or tie-breaking assumptions, unless there is a clear interpretation thereof in the problem at hand. Our concept of robust stochastic stability aims to differentiate clearcut predictions from more subtle ones. For noisy behavioral rules where specific tie-breaking assumptions are built into the rule, as in the case of logit choice, robust stochastic stability requires robustness with respect to the speed of the dynamics, as captured by the specification of revision opportunities. For noisy behavioral rules which remain silent (or are less vocal) on the issue of tie-breaking, as e.g. those based on the mistakes model, robustness should also include the latter.

We have provided an easy-to-use sufficient condition for robust stochastic stability, and have illustrated its application for different games and dynamics. The condition makes use of an order structure of the space of dynamics, by observing that the radius and coradius concepts introduced in the literature are monotone operators in this space. Our result itself reduces to a radius-coradius approach, with the difference that the radius is taken with respect to the "quickest" dynamics (independent learning and random tie-breaking) and the coradius is taken with respect to the "slowest" one (asynchronous learning and cautious tie-breaking). It is interesting to observe that, in the quest to obtain results which are independent of certain parts of the specification of the dynamics, we are led to concentrate on these two particular, extreme dynamics.

In our applications, we have found that both the celebrated selection of risk-dominant strategies in coordination games under noisy best-reply (Kandori and Rob, 1995; Sandholm, 1998) and the selection of "Walrasian" strategies in aggregative games under noisy imitation (Vega-Redondo, 1997; Alós-Ferrer and Ania, 2005) turn out to be robust. The selection of potential maximizers in exact potential games under the logit-response dynamics (Blume, 1993, 1997), shown to be generally non-robust by Alós-Ferrer and Netzer (2010), turns out to be robust for the subclass of supermodular, $N$-player binary-action games. Best-reply with mistakes delivers robust selection results in the same class of games under the weaker condition of strategic complementarity. These results, which illustrate the usefulness of our main result, are also of independent interest for the literature of learning in games.

# Appendix

*Proof of Lemma 5.* We verify that $\rho$ is a potential for $\Gamma$. From (9) and (10) we obtain that for each $i \in I$ and $s_{-i} \in S_{-i}$,

$$\pi_i(A, s_{-i}) - \pi_i(B, s_{-i}) = \pi^A(m(A, s_{-i})) - \pi^B(m(B, s_{-i})) = \Delta(m(A, s_{-i})),$$

because $m(B, s_{-i}) = m(A, s_{-i}) - 1$. For the same reason, it follows from (11) that

$$\rho(A, s_{-i}) - \rho(B, s_{-i}) = \Delta(m(A, s_{-i})),$$

which verifies that $\rho$ is a potential. □

**Lemma 7.** *For symmetric binary action games, property (12) holds if and only if the difference function $\Delta(n)$ is weakly increasing in $n$.*

*Proof.* Assume first that $\Delta$ is weakly increasing. Consider any player $i \in I$ and profiles $s_{-i}, s'_{-i} \in S_{-i}$ with $s_{-i} \leq s'_{-i}$. From the definition of the product order on $S_{-i}$ it follows that $m(A, s_{-i}) \leq m(A, s'_{-i})$. Thus $\Delta(m(A, s_{-i})) \leq \Delta(m(A, s'_{-i}))$, which is equivalent to condition (12), implying that $\Gamma$ is supermodular.

Next assume that $\Delta$ is not weakly increasing, i.e. there exist values $1 \leq n < n' \leq N$ such that $\Delta(n) > \Delta(n')$. Fix any player $i \in I$ and consider the profiles $s_{-i} = (A, \overset{n-1}{\ldots}, A, B, \overset{N-n}{\ldots}, B)$ and $s'_{-i} = (A, \overset{n'-1}{\ldots}, A, B, \overset{N-n'}{\ldots}, B)$. By construction, $s_{-i} \leq s'_{-i}$ and $m(A, s_{-i}) = n < n' = m(A, s'_{-i})$. Then, $\Delta(m(A, s_{-i})) > \Delta(m(A, s'_{-i}))$, which contradicts (12) and implies that $\Gamma$ is not supermodular. □

*Proof of Theorem 1.* We will rely on Corollary 3 and radius-coradius from Alós-Ferrer and Netzer (2010). Let $\underline{n}$ and $\overline{n}$ be as given in the text. Since $\Delta(n)$ is weakly increasing in $n$ (supermodularity), $\underline{n}$ and $\overline{n}$ do exist and $\underline{n} \leq \overline{n}$ holds.

Fix an arbitrary regular revision process $q \in \mathcal{Q}$. The waste caused by a single player switching from $B$ to $A$ in the presence of $n$ other $A$-players is $\max\{-\Delta(n+1), 0\}$ and hence zero if and only if $n \geq \underline{n}$. Analogously, the waste that is generated if one of $n$ $A$-players switches to $B$ is $\max\{\Delta(n), 0\}$

and zero if and only if $n \leq \overline{n}$. We can thus construct zero waste paths from any $s$ to $\vec{B}$ (to $\vec{A}$) if $m(s) \leq \overline{n}$ (or $m(s) \geq \underline{n}$, respectively), letting one player switch at a time. On the other hand, for any $s$ with $m(s) > \overline{n}$ (with $m(s) < \underline{n}$), any switch to $B$ (to $A$) causes strictly positive waste, irrespective of the specific revising set chosen. The basins of attraction (of the unperturbed best-reply dynamics) are thus $B(\vec{B}) = \{s \in S | m(s) \leq \overline{n}\}$ and $B(\vec{A}) = \{s \in S | m(s) \geq \underline{n}\}$, for any regular process $q \in \mathscr{Q}$.

Now consider asynchronous learning and fix some $s' \notin B(\vec{B})$. Construct a minimal waste path $P = (s', ..., \vec{B})$ by letting $A$-players switch to $B$ sequentially. We obtain the waste $W(P) = \sum_{j=\overline{n}+1}^{m(s')} \Delta(j)$. Since $\Delta(n)$ is positive for all $n \geq \overline{n} + 1$, this expression is maximal if $s' = \vec{A}$ so that $m(s') = N$, which yields $CR^{q^{AL}}(\vec{B}) = \sum_{j=\overline{n}+1}^{N} \Delta(j)$. From analogous arguments we obtain $CR^{q^{AL}}(\vec{A}) = -\sum_{j=1}^{\underline{n}} \Delta(j)$.

Now consider independent learning. Since $\Delta(n)$ is increasing in $n$ due to supermodularity, the waste caused by a $B$-player switching to $A$ in the presence of $n$ $A$-players ($\max\{-\Delta(n+1), 0\}$) is decreasing in $n$. Analogously, the waste of an $A$-player switching to $B$ ($\max\{\Delta(n), 0\}$) is increasing. Hence the waste caused by several players switching simultaneously is weakly larger than the waste caused by sequential switching, so that among minimal waste paths out of $B(\vec{B})$ and $B(\vec{A})$, there are always paths that make use of sequential revisions only. This immediately implies

$$R^{q^{IL}}(\vec{B}) = \sum_{j=1}^{\overline{n}+1} \max\{-\Delta(j), 0\} = -\sum_{j=1}^{\underline{n}} \Delta(j)$$

and

$$R^{q^{IL}}(\vec{A}) = \sum_{j=\underline{n}}^{N} \max\{\Delta(j), 0\} = \sum_{j=\overline{n}+1}^{N} \Delta(j).$$

Now suppose $\vec{B}$ is the unique potential maximizer, i.e. $\sum_{j=1}^{N} \Delta(j) < 0$. This can be rearranged to $-\sum_{j=1}^{\underline{n}} \Delta(j) > \sum_{j=\overline{n}+1}^{N} \Delta(j)$, because $\Delta(n) = 0$ for all $\underline{n} < n \leq \overline{n}$. This is equivalent to $R^{q^{IL}}(\vec{B}) > CR^{q^{AL}}(\vec{B})$ and implies that $\vec{B}$ is the unique robustly stochastically stable state by Corollary 3. The argument for $\vec{A}$ is analogous.

If both $\vec{A}$ and $\vec{B}$ maximize the potential ($\sum_{j=1}^{N} \Delta(j) = 0$), our robust radius-coradius result is not applicable. From the previous arguments about supermodularity it is still true that minimal waste revision trees can be constructed using singleton revising sets only. Hence the stochastically stable states for any regular revision process must be the same as for asynchronous learning, and thus the potential maximizers. □

*Proof of Proposition 3.* (*i*) *Asynchronous Learning.* $R^{q^{AL}}(\vec{A}) > CR_L^{q^{AL}}(\vec{A})$ can be rearranged to $a/b > (\sum_{j=n_B}^{N} 1/j)/(\sum_{j=n_A}^{N} 1/j) =: \Lambda^{AL}$, so that $\vec{A}$ is stochastically stable under asynchronous learning if $a/b > \Lambda^{AL}$. Analogously, $\vec{B}$ is stochastically stable if $a/b < \Lambda^{AL}$. If $R^{q^{AL}}(\vec{A}) = CR_L^{q^{AL}}(\vec{A})$, so $a/b = \Lambda^{AL}$, it follows immediately that the stochastic potential (again, see Alós-Ferrer and Netzer, 2010) of both $\vec{A}$ and $\vec{B}$ is identical and both are stochastically stable.

(*ii*) *Independent Learning.* The proof is analogous, using $R^{q^{IL}}$ and $CR^{q^{IL}}$ instead, which yields the critical value $\Lambda^{IL} := (N - n_B + 1)/(N - n_A + 1)$.

(*iii*) *Robust Stochastic Stability.* Corollary 3 implies that $\vec{A}$ is robustly stochastically stable if $R^{q^{IL}}(\vec{A}) > CR^{q^{AL}}(\vec{A})$, which can be rearranged to $a/b > (N \sum_{j=n_B}^{N} 1/j)/(N - n_A + 1) =: \Lambda^{R,A}$. Analogously, $\vec{B}$ is robustly stochastically stable if $a/b < (N - n_B + 1)/(N \sum_{j=n_A}^{N} 1/j) =: \Lambda^{R,B}$.

*Ranking of* $\Lambda^{IL}$ *and* $\Lambda^{AL}$. If $n_A = n_B$, then $\Lambda^{AL} = \Lambda^{IL} = 1$ holds. Suppose then that $n_B < n_A$. Condition $\Lambda^{AL} > \Lambda^{IL}$ can be rearranged to

$$\gamma(n_A, n_B) := (N - n_A + 1) \sum_{j=n_B}^{N} \frac{1}{j} - (N - n_B + 1) \sum_{j=n_A}^{N} \frac{1}{j} > 0. \qquad \text{(A1)}$$

Define $\delta(n_A, n_B) := \gamma(n_A + 1, n_B) - \gamma(n_A, n_B)$ as the change of $\gamma$ if $n_A$ is increased by 1. We obtain

$$\delta(n_A, n_B) = \frac{1}{n_A}(N - n_B + 1) - \sum_{j=n_B}^{N} \frac{1}{j} = \frac{1}{n_A} \sum_{j=n_B}^{N} 1 - \sum_{j=n_B}^{N} \frac{1}{j} = \sum_{j=n_B}^{N} \left( \frac{1}{n_A} - \frac{1}{j} \right). \qquad \text{(A2)}$$

Since $\gamma(n, n) = 0$ holds, and $n_A > n_B$ by assumption, we can write $n_A = n_B + x$, $x > 0$, and

$$\gamma(n_A, n_B) = \gamma(n_B + x, n_B) = \sum_{i=0}^{x-1} \delta(n_B + i, n_B) = \sum_{i=0}^{x-1} \sum_{j=n_B}^{N} \left( \frac{1}{n_B + i} - \frac{1}{j} \right),$$

where the second equality follows from iterating the differences $\delta$, and the third from substitution of (A2). This expression can be transformed to

$$\begin{aligned}\gamma(n_A,n_B) &= \sum_{i=0}^{x-1}\sum_{j=n_B}^{N}\frac{1}{n_B+i}-\sum_{i=0}^{x-1}\sum_{j=n_B}^{N}\frac{1}{j}\\ &= \sum_{i=0}^{x-1}\sum_{j=n_B}^{n_B+x-1}\frac{1}{n_B+i}+\sum_{i=0}^{x-1}\sum_{j=n_B+x}^{N}\frac{1}{n_B+i}-\sum_{i=0}^{x-1}\sum_{j=n_B}^{n_B+x-1}\frac{1}{j}-\sum_{i=0}^{x-1}\sum_{j=n_B+x}^{N}\frac{1}{j}\\ &= \sum_{i=0}^{x-1}\sum_{j=0}^{x-1}\frac{1}{n_B+i}+\sum_{i=0}^{x-1}\sum_{j=n_B}^{N-x}\frac{1}{n_B+i}-\sum_{i=0}^{x-1}\sum_{j=0}^{x-1}\frac{1}{j+n_B}-\sum_{i=0}^{x-1}\sum_{j=n_B}^{N-x}\frac{1}{j+x},\end{aligned}$$

where the first equality follows after separating the summands, the second equality follows by breaking the sums with index $j$ into two partial sums each, and the third equality follows from redefining indices. Now observe that, in the resulting expression, the first and the third double-sum are identical and cancel out. This leaves us with

$$\gamma(n_A,n_B)=\sum_{i=0}^{x-1}\sum_{j=n_B}^{N-x}\left(\frac{1}{n_B+i}-\frac{1}{j+x}\right). \tag{A3}$$

First, the sums in (A3) are not empty, because $x \geq 1$ and $n_B+x=n_A \leq N$. But then (A3) is strictly positive, because in each summand $i < x$ and $n_B \leq j$ holds, so that each summand is strictly positive. Hence $\Lambda^{AL} > \Lambda^{IL}$.

*Ranking of* $\Lambda^{R,B}$ *and* $\Lambda^{R,A}$. Consider $\Lambda^{R,A}$ first. Observe that $N\sum_{j=n_A}^{N} 1/j = \sum_{j=n_A}^{N} N/j \geq \sum_{j=n_A}^{N} N/N = N-n_A+1$, with strict inequality if and only if $n_A < N$. This implies

$$\Lambda^{R,A}=\frac{N\sum_{j=n_B}^{N}1/j}{N-n_A+1}\geq\frac{N\sum_{j=n_B}^{N}1/j}{N\sum_{j=n_A}^{N}1/j}=\Lambda^{AL},$$

with strict inequality if and only if $n_A < N$. By the same argument, $\Lambda^{R,B} \leq \Lambda^{IL}$, again with strict inequality if and only if $n_A < N$. □

*Proof of Theorem 2.* We will rely on Corollary 4 and radius-coradius from Ellison (2000). Arguing as in the proof of Theorem 1, the basins of attraction for a symmetric binary action game with strategic complementarity and hence well-defined values $\underline{n}$ and $\overline{n}$, are given by $B(\vec{B}) = \{s \in S | m(s) \leq \overline{n}\}$ and $B(\vec{A}) = \{s \in S | m(s) \geq \underline{n}\}$, for any regular process $q \in \mathcal{Q}$.

Consider asynchronous learning, fix any $s' \notin B(\vec{B})$, and construct a minimal cost path $P = (s', ..., \vec{B})$ by letting $A$-players switch to $B$ sequentially. Note that this transition path is unaffected by tie-breaking considerations. The cost of this path is $C(P) = m(s') - \overline{n}$, which is maximal if $s' = \vec{A}$ so that $m(s') = N$, and we thus have $CR^{B^0,q^{AL}}(\vec{B}) = N - \overline{n}$. We analogously obtain $CR^{B^0,q^{AL}}(\vec{A}) = \underline{n}$.

Now consider independent learning. The fact that $\underline{n}$ players have to mutate to eventually leave $B(\vec{B})$ is unaffected by the possibility that some mutations could occur simultaneously (and analogously for $B(\vec{A})$, where $N-\overline{n}$ mutations need to occur). Again, note that tie-breaking considerations are not relevant for this argument. Hence we obtain $R^{B^X,q^{IL}}(\vec{B}) = \underline{n}$ and $R^{B^X,q^{IL}}(\vec{A}) = N - \overline{n}$.

Now suppose $\underline{n}+\overline{n} > N$. This is identical to $R^{B^X,q^{IL}}(\vec{B}) > CR^{B^0,q^{AL}}(\vec{B})$ and implies that $\vec{B}$ is the unique robustly stochastically stable state by Corollary 4. The same holds for $\vec{A}$ if $\underline{n} + \overline{n} < N$.

If $\underline{n} + \overline{n} = N$, our approach is again not applicable. The previous arguments, however, imply that both $\vec{A}$ and $\vec{B}$ have minimal stochastic potential in this case (see Kandori and Rob, 1995), under any regular revision process und tie-breaking assumption, so that both are robustly stable. □

**Lemma 8.** *Consider the team project game and the logit-response dynamics. If $n_B = n_A < N$ and $a > b$, $\vec{A}$ is robustly stochastically stable.*

*Proof.* Following the general approach from Alós-Ferrer and Netzer (2010) for finding stochastically stable states of the logit-response dynamics, let $q \in \mathcal{Q}$ be an arbitrary regular revision process and $(T, \gamma)$ a minimum waste $\vec{B}$ revision-tree under $q$. Then the tree $T$ contains a path $P = (s_1, ..., s_n)$ where $s_1 = \vec{A}$ and $s_n = \vec{B}$, and which satisfies that $W(P, \gamma|_P) = W(T, \gamma)$. Here, $\gamma|_P$ denotes the restriction of $\gamma$ to $P$. Clearly $W(P, \gamma|_P) \leq W(T, \gamma)$ must hold, since the revision path $(P, \gamma|_P)$ is a part of $(T, \gamma)$. Since any state $s$ that is not on $P$ can be connected to either $\vec{A}$ or $\vec{B}$ at zero waste, with the help of singleton revising sets, and $(T, \gamma)$ is a minimal waste revision-tree by assumption, we must have $W(P, \gamma|_P) = W(T, \gamma)$.

We can now construct an inverted path $P' = (s'_1, ..., s'_n)$ where $s'_1 = \vec{B}$

and $s'_n = \vec{A}$, together with a revision selection $\gamma'|_{P'}$ by using the same revising sets in the same order as before, i.e. $\gamma'|_{P'}(s'_j, s'_{j+1}) = \gamma|_P(s_j, s_{j+1})$ for all $j = 1, ..., n-1$, and by letting the same players switch to the opposite action. Then, whenever $W((s_j, s_{j+1}), \gamma|_P(s_j, s_{j+1})) > 0$ we obtain $W((s'_j, s'_{j+1}), \gamma'|_{P'}(s'_j, s'_{j+1})) < W((s_j, s_{j+1}), \gamma|_P(s_j, s_{j+1}))$ because $b < a$, and $W((s'_j, s'_{j+1}), \gamma'|_{P'}(s'_j, s'_{j+1})) = W((s_j, s_{j+1}), \gamma|_P(s_j, s_{j+1})) = 0$ otherwise, because $n_A = n_B$. Hence $W(P', \gamma'|_{P'}) < W(P, \gamma|_P)$. Connecting all states which are not on $P'$ to either $\vec{A}$ or $\vec{B}$ within singleton revising sets at zero waste yields a revision $\vec{A}$-tree $(T', \gamma')$ with $W(T', \gamma') < W(T, \gamma)$, which implies that $\vec{A}$ is stochastically stable under (any) $q \in \mathscr{Q}$. □

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