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FRANK PLUMPTON RAMSEY♠

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> ‘If I was to write a *Weltanschaung* I should call it not “What I believe” but “What I feel.” This is connected with Wittgenstein’s view that philosophy does not give us beliefs, but merely relieves feelings of intellectual discomfort.
>
> …..
>
> I don’t feel the least humble before the vastness of the heavens. The stars may be large, but they cannot think or love; and these are qualities which impress me far more than size does.’

♠ A slightly shorter version was prepared for the **Handbook of the History of Economic Analysis**, edited by *Gilbert Faccarello* & *Heinz Kurz*, Edward Elgar Publishing, 2011. References with only dates in parenthesis refer to the contents of the ‘Mellor Edited’ collection of 1978 (Ramsey, 1931; 1978)..

A Tragically Truncated Life

> "[Ramsey's] death at the height of his powers deprives Cambridge of one of its intellectual glories and contemporary philosophy of one of its profoundest thinkers."
>
> Braithwaite[1] (19310, p. ix.

In one of those extraordinary serendipities with which greatness is tinged, and *Frank Ramsey* – despite the cruelty of a life cut short at its prime – was blessed with an abundance of it, he was born the year two of the defining works in *Ethics* and the *Foundations of Mathematics* of the 20th Century were published: *Principia Ethica* by G. E. Moore and the *Principles of Mathematics* by Bertrand Russell, the latter presaging Russell's monumental *Principia Mathematica*, written jointly with Whitehead, after ten more years of struggle. Then, with equal irony, Ramsey died the year his remarkable *On a Problem of Formal Logic* (Ramsey, 1930) was published and Kurt Gödel announced his famous *Incompleteness Theorem*(s) in Königsberg.

With the celebrated Paris-Harrington results (Paris & Harrington, 1977), the connection between Ramsey's posthumously published classic and Gödel's pioneering results have been shown to be woven from the same foundational fabric of the *Entscheidungsproblem* that Hilbert had formulated, to settle, hopefully decisively, the *grundlagenkrise* of the 1920s, precipitated by Brouwer's intuitionistic and constructive challenges. It is poignant to reflect that Ramsey himself veered towards *Intuitionism*, in mathematical philosophy, inspired by Hermann Weyl's work, at the end of his short life[2]

[1] The senior author – Velupillai, himself into the proverbial 'three score years and tenth' decade of his life, was young enough to have had Ramsey *père*'s two books, one on **Statics** and the other on **Dynamics**, as his high school textbooks in *applied mathematics*. Moreover, in his very first year as a graduate student at King's College, Cambridge, the College of both Ramsey and Braithwaite, he had the pleasure of a long conversation with the latter on Ramsey's *Truth and Probability*.

[2] This is suggested by Maria Carla Galavotti in her superb edition of Ramey's Notes on Philosophy, Probability and Mathematics (Ramsey, 1991), p. 22:

> "Ramsey's conversion to intuitionism by the end of his life is mentioned by Braithwaite [1931]."

Our own reading of Braithwaite's *Introduction* does not convince us of this 'conversion', but this may well be due to nuances of interpretation. In any case, see Ramsey (1991), chapters 53 & 54;. Brian McGuinness points out that Wittgenstein's important acquaintance with Brouwer's work was through Ramsey (McGuinness, 2002, p. 10). On the other hand, Ramsey (1925, p. 219) did refer to 'the Bolshevik menace of Brouwer and Weyl'! But, then, it was, after all, in Brouwer's Journal, *Composito Mathematica*, that the first results by Erdös & Szekeres (1935), on what came to eventually to be called Ramsey Theory, was published!

The bare bones of Frank Ramsey's family background are simple enough to document, although behind this simplistic account there is a richness that deserves to be explained and expanded. He was the eldest of Agnes Mary – née Wilson - and Arthur Stanley Ramsey's four children. His brother, Michael, later to become the Archbishop of Canterbury, was born one year later and his sisters, Bridget and Margaret were, respectively, 4 and 14 years younger. Frank's father, Arthur Stanley, was also a mathematician, Fellow, Tutor, Bursar and President of Magdalene College, Cambridge. Frank himself, at the time of his death, was a Fellow of King's College, Cambridge. He had been a scholar at Winchester, from 1915 and went up to Trinity College, Cambridge, in 1920, graduating as a Wrangler in 1923. He was elected to a Fellowship at King's College in 1924, to a University Lectureship in Mathematics in 1926 and took over as the Director of Studies in Mathematics at King's. He married Lettice Barker (d. 12/7/1985), five years his senior in age, in September, 1925 and they had two daughters, Jane and Sarah; only the latter survived the deaths of her Parents, separated by 55 years, to produce three grandchildren, Stephen, Belinda and Matthew Burch.

The Eponymous F. P. Ramsey

> "In his paper, Hugh [Mellor] called Ramsey 'eponymous.' I must admit that this word was not already in my vocabulary and I looked it up in Webster's Dictionary which said:
>
> Eponymous, adj:
>
> of, relating to, or being the person for whom something is or is believed to be named.
>
> Of course! The study of ramsey theory has become so important that his name has become an adjective, along with other immortal mathematical lower case adjectives, including abelian, boolean, cartesian,…. ."
>
> Harary (1983), p. 2

Moore's *Principia Ethica* (1903), Russell's (& Whitehead's) *Principia Mathematica* (1913), Hardy's *Pure Mathematics* (1913), Keynes' *Treatise on Probability* (1921) and Wittgenstein's *Tractatus Logico Philosophicus*[3] (1921/1922) redefined the intellectual landscape of their respective fields, as Ramsey himself reached his precociously youthful maturity in the approaching post-war period. To these themes, in which he excelled and made what can only be described as outstandingly original contributions, he added, under the influence of Maynard Keynes (and A. C. Pigou, also a Fellow at King's), also a mastery of economics that few, then or since, matched for analytical brilliance and originality.

[3] The manuscript version of which was placed at Frank Ramsey's disposal by C.K.Ogden, with whom he eventually translated it while still in his 'teens'.

The *Eponymous F. P. Ramsey* (Mellor, 1983), has given birth to *Ramsey Economics* (Samuelson, 1969, p. vii), *Ramsey Pricing* and the *Ramsey Taxation Formula* (Tirole, 1988, Mas-Colell, et. al., 1995) *The Ramsey Sentence* (Carnap, 1966, chapter 26), *Ramsey Theorem* (Wang, 1981), *Ramsey Problem* (Joshi, 1997), *Ramsey Theory* and *Ramsey Number* (Bollobás, 2001, chapter 12). Beyond these, there are the contributions to a *simplified theory of types* (Ramsey, 1925 & 1926)[4], to resolve the perplexities of the logical and semantic paradoxes that plagued the *Principia Mathematica* and the discussions and debates with Wittgenstein on *finitism* (Marion, 1998) the role of *tautologies* and the definition of *identity* in mathematics (Monk, 1990, pp. 245-6). Finally, there is the acknowledged influence of Ramsey on *Production of Commodities by Means of Commodities* (Sraffa, 1960).

It is, of course, impossible to discuss all of the above in any kind of justifiable detail within the severely restricted space we have at our disposal. We hope the suggested references give some pathway into the depth and richness of the *Eponymous F. P. Ramsey* world. We confine ourselves to five core contributions, with economic relevance, past, present – and, we are sure, future.

The Three Classic Contributions to Economics

> "When he did descend from his accustomed stony heights, he still lived without effort in a rarer atmosphere than most economists care to breathe, and handled the technical apparatus of our science with the easy grace of one accustomed to something far more difficult."
> Keynes (1930 [1933]), p. 295.

'Officially', in the past and the present, Ramsey's contributions to economics have been confined to his three articles on *Truth and Probability* (1926), *A Contribution to the Theory of Taxation* (1927a) and *A Mathematical Theory of Saving* (1928). With hindsight at our convenient disposal, it is easy to see that *Ramsey's Theorem* (1928) has become highly relevant for various frontiers in economic theory. In addition, there is the explicit acknowledgement by Sraffa, in the preface to his *magnum opus*, of Ramsey's 'mathematical help'.

[4] Leon Chwistek, whose work Ramsey was acquainted with, claims priority on this point (cf, Chwistek, 1949, chapter VI).

Truth and Probability, written as a critical review of *A Treatise on Probability*[5] (Keynes, *op.cit*) is justly celebrated as the original contribution to the axiomatization of subjective probability – predating the equally important work of Bruno De Finetti by many years and anticipating the more orthodox von Neumann-Morgenstern work by almost two decades – and providing foundations for expected utility maximization underpinning for rational behaviour. However, it is rarely – if ever – remembered that Ramsey added an important caveat[6] to this classic contribution (ibid, p. 85):

> "[N]othing has been said [in my paper] about degrees of belief when the number of alternatives is *infinite*. About this I have nothing useful to say, except that I doubt if the mind is capable of contemplating more than a *finite* number of alternatives."

et, with princely unconcern, economists (of many persuasions) continue to invoke Ramsey as one of the forefathers of their rationality frameworks.

Similarly, Ramsey's[7] injunctions *against* discounting the future, as 'a practice which is ethically indefensible and arises merely from the weakness of the imagination' (ibid, p. 261)[8], are no longer even added as a footnote qualification when deriving the *Keynes-Ramsey Rule* within the framework of the '*Ramsey-Cass-Koopmans*' model of optimal aggregate economic growth in neoclassical economics[9]. Moreover, the published version of this modern classic omitted two additional sections, where it can be seen that Ramsey thought hard and deeply about the problem of discounting the future[10]

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[5] Incidentally, many – even well-meaning scholars – seem to insist that Keynes accepted Ramsey's criticisms and gave up on 'partial orders'. This simply cannot be true, given the many ways in which Keynes insisted on the unbridgeable formal gap between 'risk' and 'uncertainty', particularly in the *General Theory*.

[6] De Finetti, too, was equally explicit in his rejection of σ-algebras and the embracing of *finite-additivity* when axiomatising subjective probability.

[7] Perhaps 'Pigou-inspired'?

[8] It is our firm *belief* that Keynes' *Economic Possibilities for our Grandchildren* (Keynes, 1930), given not only in 1928, the year that he published Ramsey's '*Saving*' paper in the **EJ**, but also at Winchester College, Ramsey's old school, was fully consistent with the way the problem of discounting the future was solved by the notion of 'bliss' by Ramsey.

[9] The early classics of optimal economic growth, for example by Koopmans (1965) and Samuelson (1965), mentioned, even with approval, the Ramsey injunctions and tried to accommodate it in some form. This is no longer the case in the current crop of advance macroeconomic textbooks (eg., Romer, 2006).

[10] They can be found as document # 006-07-01 in the 'Ramsey Collection' held in the Hillman Library of the University of Pittsburgh.

Ramsey (1927a) has spawned two strands of optimal pricing research in economics: one, a kind of classic optimal taxation literature, links up with the kind of work that originates with Mirrlees and Diamond. The other, more along the line of public utility and public goods pricing has resulted in the Ramsey-Boiteux *inverse elasticity formula* (which, though famously also derived in Robinson (1933), has no reference to Ramsey!). However, it remains a puzzle to us that the link with *Lindahl pricing* has never been developed. Perhaps it is a line of research for the future.

Beyond the 'Three Classics'

> "How frequently does an intriguing problem come up over lunch time[11], only to have it solved the next morning? How many mathematical problems are seemingly intractable? Decades go by without a hint of progress. What a delight when problem is worked on over many many years with progress occurring incrementally until it finally succumbs. … in discrete mathematics, my vote is for the asymptotics of the Ramsey number $R(3,k)$[12], The story begins in 1931, is resolved in 1995, with a coda in 2008, and *the final story perhaps not yet told*."
> Spencer (2011), p. 27; italics added

As economists who try to extol the virtues of an algorithmic approach to the subject, Ramsey's lasting contributions to economic theory are not those that have traditionally been considered 'classic': his contributions to the problem of optimal savings, pricing and taxation and the formalisation of one version of subjective probability and subjective expected value theory. We think his interaction with Sraffa, particularly in Sraffa's insistence on remaining 'faithful' to a formalism in terms of equations and *constructing* solutions[13], knowing that general algorithms do not exist, in view of the Ruffini-Abel result, for higher order equations, is of fundamental importance for anyone interested in computable economics.

i. Sraffa's Mathematical Economics

> "I am also indebted for [invaluable mathematical help at different periods to the late Mr Frank Ramsey …. ."
> Sraffa (1960), pp. vi-vii.

The Ramsey-Sraffa interaction is clearly and, so far as we can tell, exhaustively, presented in Kurz and Salvadori (2001). The only comment we would like to add here is the following.

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[11] This is a reference to the 'party problem' version of Ramsey theorem: how many guests are necessary, as a minimum, to make sure that a certain (finite) number of them, all of whom either know each other or do not know anyone of them.

[12] The Ramsey number $R(3,k)$, is defined as the smallest integer n, such that any undirected graph, $G=(V,E)$ with n vertices must contain either a *clique* of size k or an independent set of size l.

[13] Velupillai has been making this point for over 35 years! His most recent foray into this issue can be found in Velupillai (2008).

The first of the three comments by Ramsey, summarised by Sraffa on 26 June, 1928 (ibid, p. 262), refers essentially to the Ruffini-Abel result on *the unsolvability of the quintic*, which Ramsey would, as a student of mathematics (particularly of Hardy) have known very well.

The interesting point here is the affinity of Abel's approach to his *unsolvability proof* with Gödel's later unsolvability proof strategies and the present understanding of the relevance of such things to variations of Ramsey's Theorem (cf., Paris-Harrington, op.cit., Pesic, 2003 and Shanker, 1988).

The unwritten chapter in this story is the role of *mathematical proof* - particularly the place of finitism and constructivism in the philosophy and methodology of proof in Sraffa (1960) - in Sraffa's economics and how influential Ramsey might have been, had he lived – just a little longer!

ii. Ramsey Theory and Ramsey's Theorem

> "Let us summarize in one paragraph the fetal development preceding Ramsey theory's birth. As we believe now, David Hilbert's cube lemma was the first Ramseyan result but it did not influence anyone at the time and thus did not give birth to Ramsey theory. Issai Schur's 1916 theorem could have remained unnoticed too, but Schur was first to realize that he had run into something new and striking. And so Schur continued by conjecturing the result on monochromatic arithmetic progressions. .. Then came Frank Plumpton Ramsey who delivered the Two Commandments, the principles of the theory later named in his honour.
> Soifer (2011), p. 2

We come now to what we think will be Ramsey's lasting contribution to economic theory, in the future: the growing influence of *Ramsey's Theorem*[14] (Ramsey (1930)) on graph theory, combinatorics, recursion theory and computational complexity theory. We, as adherents and practitioners of computable economics are convinced that Ramsey Theory will come to play an increasingly important computational role in this field. To substantiate this conjecture we have to refer to what are called the *two principles of Ramsey theory*. However, to sate them we have, first, to state, in some minimally formal way, the Ramsey Theorem. We opt for the lucid, yet concise formulation in Jockush (1972):

[14]As Soifer clarified, elegantly, by Soifer (2011), p. 10 (italics in the original):

> "The Ramsey theorem occupies a unique place in Ramsey theory. It is powerful tool. It is also a philosophical principle stating …. that a 'complete disorder is an impossibility. Any structure will necessarily contain an orderly substructure.' It is, therefore, imperative to call the Ramsey theorem by a much better fitting name: the *Ramsey principle*."

These are stated above.

Ramsey's Theorem (ibid, p. 233, Theorem A):

For $A \subseteq N$, where N: the set of natural numbers, $[A]^n$ the class of all n-element subsets of A. Partition $[N]^n$ into finitely many sub-classes, C_i, i=1,2,…., p and call the partition P.

$H(P)$: the class of those infinite sets $A \subseteq N$, s.t., $[A]^n \subseteq C_i$ for some i.

Then, $H(P)$ is nonempty for every such partition.

Ramsey Theory's First Principle:

Structure is preserved under finite partitions.

Ramsey Theory's Second Principle:

There is always an appropriate notion of size, s.t., any sufficiently large structure always contains the desired well-organized sub-structure

Ramsey's Theorem in 'Colouring' From:

For given $r, k \in N$, assign r colours to the k-element subsets of the initial set. Then, one can find a 'large' subset, all of whose k-tuples are of the same colour.

Now, suppose we endow – as faithful *computable economists* – P with a *recursive structure*. Then we ask what we can say about $H(P)$, from many points of view: computational complexity theory (*how fast do functions grow*?), for example. Recent research shows, for example, that this kind of question can be answered using, for example, proof strategies developed in *Kolomogorov Complexity Theory* (cf. Li & Vitanyi, 1990).

Ramsey's remarkable Theorem retains its fertile life, spawning research in many directions and at the frontiers of many fields, from the foundations of mathematics to graph theory; from combinatorics to computational complexity theory; from ergodic theory to topological dynamics.

Above all, in being a source for the derivation of the celebrated Paris-Harrington theorem[15], it also links up with *Goodstein's theorem* and the *Goodstein's sequence*[16] and the recent

[15] The notion of 'large', as used in the 'colouring form' of the Ramsey theorem, used by 'Paris-Harrington' definition of a 'large set of integers' is, for example, as follows: *A set that has at least as*

development in rational-valued nonlinear dynamics (Goodstein, 1944 and Paris & Tavakol, 1993). This latter class of dynamical systems go beyond the exoticness of sensitively dependent dynamics and a new class, labelled super sensitive dynamical systems has been defined. The trajectories of such systems are almost always (in the strict mathematical sense) in transition and their attractors are invariably trivial.

In conclusion, the *Ramseyan* precept for intellectual adventures that we have culled out of our research on the remarkable work of this prodigal genius can by summarised by his own paraphrasing of Wittgenstein's well known aphorism:

"What we can't say, we can't say and we can't whistle it either."

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many elements as its smallest integer; eg: {3,15,25,27} is 'large' because it contains at least three elements – but {102, 103, 105, 107,109,111,113,115} is not 'large'.

[16] This with the perennial problem of the finite-infinite divide that preoccupied Ramsey in almost all his mathematical and philosophical works – and even in his classic contribution to Truth and Probability, as we have emphasised above. Clearly, this is the divide that he was reluctant to bridge by 'discounting' an infinite horizon optimization problem to determine an optimum savings rate. The philosophical and ethical doubts he expressed in this connection were underpinned by his lifelong unease with non-finitist proofs – but not, for that reason, going all the way towards the ultra-finitist constructivist (which Brouwer was not; nor was Bishop).

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