MPRA
Munich Personal RePEc Archive

Rational choice by two sequential criteria

García-Sanz, María D. and Alcantud, José Carlos R.
Universidad de Salamanca

16. March 2010

Rational Choice by two Sequential Criteria

María D. García-Sanz [1,2], José Carlos R. Alcantud [1]

Facultad de Economía y Empresa, Universidad de Salamanca, E 37008 Salamanca, Spain

Abstract

This paper contributes to the theory of rational choice under multiple criteria. We perform a preliminary study of the properties of decision made by the sequential application of rational choices. This is then used to obtain a characterization of set-valued choice functions that are rational by two sequential criteria, which follows the approach initiated by Manzini and Mariotti [16] for single-valued choice functions. Uniqueness is not guaranteed but our proof is constructive and an explicit solution is provided in terms of approximation choice functions.

Key words: choice function, rational choice, compound function.
JEL Classification: D0.

* Corresponding author
Email addresses: `dgarcia@usal.es` (María D. García-Sanz), `jcr@usal.es` (José Carlos R. Alcantud).

[1] Both authors acknowledge financial support by the Spanish Ministerio de Ciencia e Innovación under Project ECO2009-07682, and by Junta de Castilla y León under Project SA024A08 and GR-99 funding. We thank Paola Manzini and Marco Mariotti for valuable comments.

[2] This paper is part of García-Sanz's doctoral dissertation under the supervision of the second author and Ignacio García-Jurado. Financial support by Ministerio de Educación y Ciencia under Project SEJ2007-67068 is gratefully acknowledged.

1 Introduction

The literature on abstract choice theory abounds with analysis of the possible *rationality* of a choice function via satisfaction of consistency properties, although the precise meaning of the term "rationality" is subject to interpretations. The identification of rational choice as "optimizing behavior" has a long tradition in the literature. A standard position identifies a rational choice function with the result of the optimization of a generating binary relation irrespective of its properties (cf., Arrow [6], Richter [21,22], Wilson [31], Sen [24], Blair et al. [9], Suzumura [28], etc). In addition, the literature vastly deals with enhanced generating relations, and the implications of acyclicity, transitivity, quasi-transitivity, ... are extensively studied.

Another aspect to bear in mind is the possibility of incorporating different criteria into the process. Previous contributions have approached this topic in several forms. Kalai et al. [15] study the rationality of a choice function by multiple binary relations, when the choice is a single element and all the relations are simultaneously applied to each set of alternatives. Houy [13] analyses whether the order of the criteria affects the final choice or not. Various procedures of choice for lexicographic applications of multiple criteria have been considered by Houy and Tadenuma [14]. In this paper we focus on the case where a number of criteria are applied in a sequential way, a process that we call *sequential choice*. Under this general perspective, the bahavior of a decision maker (DM) is conceived of as rational not only when his choice function derives from a single binary relation, but also when it is the sequential application of such type of choices.

Our contribution adds to a branch of the literature from which we pinpoint two focal references. In the first place, Manzini and Mariotti [16] are concerned with singled-valued choice functions that are the result of a sequential application of binary relations. They provide a complete analysis of the case of two and three relations. In the second place, Apesteguía and Ballester [5] complement this achievement and characterize singled-valued choice functions that can be rationalized by the sequential application of any number of rational choice functions. By contrast with these works, we approach the problem in terms of set-valued choice functions although we restrict ourselves to the sequential application of two relations. We also emphasize that we do not require the asymmetry restriction imposed by them to the binary relations in their model. In addition to those references, Masatlioglu et al. [18] study a related model under single-valued choice functions. They admit the possibility of selecting an alternative x in the presence of y when the DM prefers y, because he simply does not realize that y is also available. This model concerns a DM that only pays attention to a subset from each set of alternatives, which can be considered as a set-valued choice function. From this subset a final unique selection is made.

Since we aim at identifying choices that arise from a sequential application of rational choice functions, a crucial previous step is the analysis of the behavior of rationality with respect to composition (technically: of the rationality properties of the *compound* choice function). To this purpose, in Section 2 we set our notation and recall rationality properties of choice that are the key to analyse rational choice functions. Then in Section 3 we study some axioms for choice functions that permit to infer relevant properties of their compound function. Aizerman and Aleskerov [3] made a partial study of this issue, to which we add with further conclusions. Besides, the usual characterizations of rationalizability of choice functions permit to state direct corollaries in terms of rationality of a choice function obtained by the composition of two rational choice functions.

Finally we approach the problem of identifying choice functions that arise as the composition of two rational choice functions, that is, choice functions that are *rational by two sequential criteria*. We give a complete characterization via two testable necessary and sufficient conditions, and an explicit expression for a solution (uniqueness can not be guaranteed) is provided. This constitutes Section 4. We conclude with some final remarks in Section 5.

2 Definitions and properties of rationality

In this section we set the notation and introduce properties of rationality for a choice function that are common in related literature.

Along this paper X denotes a general set of alternatives and $\mathcal{P}(X)$ is the set of subsets of X. A *binary relation* on X is a subset $R \subseteq X \times X$, and we interpret xRy –a shorthand for $(x, y) \in R$– as "x is weakly preferred to y". Besides, R produces a strict relation P_R (the asymmetric part of R) and an indifference relation I_R (the symmetric part of R) on X according to:

$$xP_Ry \Leftrightarrow \{xRy \text{ and not } yRx\}, \text{ and } xI_Ry \Leftrightarrow \{xRy \text{ and } yRx\}.$$

The following properties of a binary relation are relevant in our study.

Definition 1 *Let R be a binary relation on X.*

- *R is* reflexive *if xRx for all $x \in X$.*
- *R is* transitive *if whenever xRy and yRz it is true that xRz.*
- *R is* complete *if for any $x, y \in X$ either xRy or yRx is true.*
- *R is an* ordering *if it is transitive and complete.*

- R *is* quasi-transitive *if its asymmetric part* P_R *is transitive.*
- R *is* acyclic *if for any finite sequence of alternatives* $\{x_1, \ldots, x_t\}$ *such that*

$$(x_1, x_2) \in P_R, (x_2, x_3) \in P_R, \ldots, (x_{t-1}, x_t) \in P_R$$

one has $(x_t, x_1) \notin P_R$.

Definitions 2 and 3 below formalize the concepts of *decisive choice function* and *rational choice function.*

Definition 2 *Let $\mathcal{D}$ be a nonempty domain of nonempty subsets of X, that is, $\varnothing \neq \mathcal{D} \subseteq \mathcal{P}(X)$ and $S \neq \varnothing$ for all $S \in \mathcal{D}$. A decisive choice function on $\mathcal{D}$ is a map $\mathcal{C} : \mathcal{D} \to \mathcal{P}(X)$ such that $\varnothing \neq \mathcal{C}(S) \subseteq S$ for all $S \in \mathcal{D}$.*

Unless otherwise stated, along this paper we are bound by two technical restrictions. Firstly, all choice functions are decisive thus *choice function* holds for *decisive choice function.* Secondly, $\mathcal{D}$ is the domain consisting of all finite and nonempty subsets of X.[3]

Definition 3 *A choice function $\mathcal{C}$ on $\mathcal{D}$ is rational if there exists a binary relation R on X such that $\mathcal{C}(S) = \mathcal{C}_R(S) = \{x \in S : \forall y \in S,\ (x, y) \in R\}$, for any set of alternatives $S \in \mathcal{D}$.*

Definition 3 captures the idea that the choice is made by optimization of a preference relation R, and we also say that R rationalizes the choice function $\mathcal{C}$. When the preference relation R is an ordering, it renders *complete rationality* and we say that $\mathcal{C}$ *is full rational*; if R is quasi-transitive then we say that the choice function $\mathcal{C}$ is *quasi-transitive rational*; and if R is acyclic then the choice function $\mathcal{C}$ is *acyclic rational.* Of course, full rationality implies quasi-transitive rationality, which in turn implies acyclic rationality. As $\mathcal{D}$ contains all singletons and pairs of alternatives, any binary relation that rationalizes a decisive choice function on $\mathcal{D}$ is reflexive and complete.

Definitions 4 and 5 below concern a choice function $\mathcal{C}$ on a domain $\mathcal{D}$. They respectively deal with contraction and expansion properties that are crucial for our analysis.

Definition 4 *The choice function $\mathcal{C}$ satisfies*

- *The **Chernoff condition**, also **CH**, if for any $S, T \in \mathcal{D}$ such that $S \subseteq T$ we have $\mathcal{C}(T) \cap S \subseteq \mathcal{C}(S)$.*

[3] The results would not be affected if the domain includes all the infinite subsets as well, and even some of them are true for domains containing all pairs and all triples from X only. We set this framework in order to avoid unnecessary technicalities and concentrate on the sequential application of criteria.

- ***Arrow's axiom**, also **A**, if for any $S, T \in \mathcal{D}$ such that $S \subseteq T$ it is true that $\mathcal{C}(T) \cap S = \mathcal{C}(S)$.*
- *The **Superset property**, also **SUP**, if for all $S, T \in \mathcal{D}$ such that $S \subseteq T$ and $\mathcal{C}(T) \subseteq \mathcal{C}(S)$ we have $\mathcal{C}(S) = \mathcal{C}(T)$.*

It is obvious that Arrow's axiom is stronger than the Chernoff condition.

Definition 5 *We say that $\mathcal{C}$ satisfies **Property $\boldsymbol{\gamma}$** (cf., Sen [24]), also $\boldsymbol{\gamma}$, if for any collection $\{M_i\}_{i \in I}$ of subsets of $\mathcal{D}$ the following holds true:*

$$x \in \mathcal{C}(M_i) \text{ for all } i \in I \text{ entails } x \in \mathcal{C}\left(\cup_{i \in I} M_i\right)$$

This property is stronger than the *Concordance property* which establishes that $\mathcal{C}(S) \cap \mathcal{C}(T) \subseteq \mathcal{C}(S \cup T)$ throughout. We are especially interested in a generalization of Property γ that we call the *binariness property* [4].

Definition 6 *The choice function $\mathcal{C}$ satisfies the **Binariness property**, also **B**, if for any $S \in \mathcal{D}$ we have: $x \in \mathcal{C}(\{x, y\})$ for all $y \in S$ implies $x \in \mathcal{C}(S)$.*

Properties **CH**, **SUP**, and $\boldsymbol{\gamma}$ are independent, but Arrow's axiom is stronger than any of them. The next diagram summarizes the relationships among the properties above.

$$\text{Arrow's axiom} \Rightarrow \begin{cases} \text{Chernoff condition} \\ \text{Superset property} \\ \text{Property } \gamma \Rightarrow \text{Binariness} \end{cases}$$

3 Rationality of a compound choice function

As has been mentioned, we are interested in identifying choice functions that arise from a sequential application of rational choice functions. Therefore a crucial previous step is the analysis of the behavior of rationality with respect to composition. From a technical perspective, we proceed to study the preservation of certain *rationality* properties of choice functions under the operation of composition, because such properties characterize rationalizability.

We first formalize the idea of a sequential application of two criteria of decision making.

[4] This postulate is also named the *Direct Condorcet Property* or *Condorcet consistency* in the literature.

Definition 7 *Let X be a set of alternatives and $\mathcal{C}_1 : \mathcal{D}_1 \to \mathcal{P}(X)$ and $\mathcal{C}_2 : \mathcal{D}_2 \to \mathcal{P}(X)$ two choice functions with respective domains $\mathcal{D}_1$ and $\mathcal{D}_2$, in such way that $\mathcal{C}_1(\mathcal{D}_1) \subseteq \mathcal{D}_2$. We define the composition of $\mathcal{C}_1$ and $\mathcal{C}_2$, also the compound function of $\mathcal{C}_1$ and $\mathcal{C}_2$, as the map $\mathcal{C}_2 \circ \mathcal{C}_1 : \mathcal{D}_1 \to \mathcal{P}(X)$ given by*

$$(C_2 \circ C_1)(A) = C_2(C_1(A)) \text{ for all } A \in \mathcal{D}_1$$

Following our convention we assume $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}$, the domain of all finite and nonempty subsets of alternatives.

Now we recall a characterization theorem for full rational choice functions.

Theorem 1 (Arrow [6]) *A choice function $\mathcal{C}$ over $\mathcal{D}$ is full rational if and only if it satisfies Arrow's axiom.*

Aizerman and Aleskerov [3] have established that Arrow's axiom is preserved under the composition of choice functions. Thus we can state:

Corollary 1 *Whenever we have full rational choice functions $\mathcal{C}_1, \ldots, \mathcal{C}_n$ on $\mathcal{D}$, the choice function $\mathcal{C}_n \circ \ldots \circ \mathcal{C}_1$ is full rational too.*

We continue our inspection with the case of quasi-transitive rational choice functions. We begin recalling the following characterization theorem.

Theorem 2 (Blair et al. [9], p. 367) *A choice function $\mathcal{C}$ on $\mathcal{D}$ is quasi-transitive rational if and only if it satisfies **CH**, **SUP** and **B**.*

Although the composition of two choice functions that satisfy **CH** does not necessarily preserve such property, if $\mathcal{C}_1$ satisfies **A** and $\mathcal{C}_2$ satisfies **CH** then $\mathcal{C}_2 \circ \ \mathcal{C}_1$ satisfies **CH** too (cf., Aizerman and Aleskerov [3]). We proceed to study to what extent properties **SUP** and **B** are preserved by composition. Propositions 1 and 2 below provide insights in this respect.

Proposition 1 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be choice functions on $\mathcal{D}$. If $\mathcal{C}_1$ satisfies **A** and $\mathcal{C}_2$ satisfies **SUP**, then $\mathcal{C}_2 \circ \mathcal{C}_1$ satisfies **SUP**.*

Proof. Let us select $S, T \in \mathcal{D}$ such that $S \subseteq T$ and $(\mathcal{C}_2 \circ \mathcal{C}_1)(T) \subseteq (\mathcal{C}_2 \circ \mathcal{C}_1)(S)$. We must prove that $(\mathcal{C}_2 \circ \mathcal{C}_1)(S) = (\mathcal{C}_2 \circ \mathcal{C}_1)(T)$.

As $\mathcal{C}_1$ satisfies **A** we have $\mathcal{C}_1(T) \cap S = \mathcal{C}_1(S)$, thus $\mathcal{C}_1(S) \subseteq \mathcal{C}_1(T)$.

Because $\mathcal{C}_2\left(\mathcal{C}_1(T)\right) \subseteq \mathcal{C}_2\left(\mathcal{C}_1(S)\right)$ and $\mathcal{C}_2$ satisfies **SUP**, we can conclude $(\mathcal{C}_2 \circ \mathcal{C}_1)(S) = (\mathcal{C}_2 \circ \mathcal{C}_1)(T)$. □

Proposition 2 *Let $\mathcal{C}_1$ be a choice function on $\mathcal{D}$ that satisfies **B** and **CH**. If $\mathcal{C}_2$ is a choice function on $\mathcal{D}$ that satisfies **B** then $\mathcal{C}_2 \circ \mathcal{C}_1$ satisfies **B** too.*

Proof.

We have to prove that for any $S \in \mathcal{D}$

$$x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(\{x, y\}), \forall y \in S \Rightarrow x \in (\mathcal{C}_2 \circ \mathcal{C}_1)(S)$$

Since $\mathcal{C}_2$ satisfies **B** we have

$$x \in \mathcal{C}_2(\{x, z\}),\ \forall z \in \mathcal{C}_1(S) \Rightarrow x \in \mathcal{C}_2(\mathcal{C}_1(S))$$

Then we are done if we prove that $x \in \mathcal{C}_2(\{x, z\})$ holds true for any $z \in \mathcal{C}_1(S)$.

We first observe that $z \in \mathcal{C}_1(S)$ implies $z \in \mathcal{C}_1(\{x, z\})$ for all $x \in S$ because $\mathcal{C}_1$ satisfies **CH**.

Moreover from $x \in \mathcal{C}_2(\mathcal{C}_1(\{x, y\}))$ for all $y \in S$, we deduce $x \in \mathcal{C}_1(\{x, y\})$ for all $y \in S$. Therefore $\mathcal{C}_1(\{x, z\}) = \{x, z\}$ for all $z \in \mathcal{C}_1(S)$.

Also from $x \in \mathcal{C}_2(\mathcal{C}_1(\{x, y\}))\ \forall y \in S$ we obtain

$$x \in \mathcal{C}_2(\mathcal{C}_1(\{x, z\})) = \mathcal{C}_2(\{x, z\})\ \forall z \in \mathcal{C}_1(S)$$

which concludes the proof. □

Corollary 2 below establishes the quasi-transitive rationality of the compound choice function of a full rational choice function with a quasi-transitive rational. Afterwards a short analysis of the structure of the composition of quasi-transitive rational choice functions complements such result.

Corollary 2 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be choice functions defined on $\mathcal{D}$. If $\mathcal{C}_1$ is full rational and $\mathcal{C}_2$ is quasi-transitive rational, then $\mathcal{C}_2 \circ \mathcal{C}_1$ is quasi-transitive rational.*

If we relax the assumptions of Corollary 2 to quasi-transitive rationality to $\mathcal{C}_1$, then the compound choice function may not satisfy the superset property as the next example proves.

Example 1 *Let $X = \{x, y, z, t\}$ and let us define the next choice functions*

$\mathcal{C}_1$ *and* $\mathcal{C}_2$ *on its nonempty subsets according to:* [5]

$$
\begin{array}{lll}
\mathcal{C}_1(\{x,y\}) = \{x,y\} & & \\
\mathcal{C}_1(\{x,z\}) = \{x,z\} & \mathcal{C}_1(\{x,y,z\}) = \{x,y,z\} & \\
\mathcal{C}_1(\{x,t\}) = \{t\} & \mathcal{C}_1(\{x,y,t\}) = \{y,t\} & \mathcal{C}_1(\{x,y,z,t\}) = \{y,z,t\} \\
\mathcal{C}_1(\{y,z\}) = \{y,z\} & \mathcal{C}_1(\{x,z,t\}) = \{z,t\} & \\
\mathcal{C}_1(\{y,t\}) = \{y,t\} & \mathcal{C}_1(\{y,z,t\}) = \{y,z,t\} & \\
\mathcal{C}_1(\{z,t\}) = \{z,t\} & &
\end{array}
$$

$$
\begin{array}{lll}
\mathcal{C}_2(\{x,y\}) = \{x,y\} & & \\
\mathcal{C}_2(\{x,z\}) = \{x,z\} & \mathcal{C}_2(\{x,y,z\}) = \{x,y,z\} & \\
\mathcal{C}_2(\{x,t\}) = \{x,t\} & \mathcal{C}_2(\{x,y,t\}) = \{x,y,t\} & \mathcal{C}_2(\{x,y,z,t\} = \{x,y,z\} \\
\mathcal{C}_2(\{y,z\}) = \{y,z\} & \mathcal{C}_2(\{x,z,t\}) = \{x,z\} & \\
\mathcal{C}_2(\{y,t\}) = \{y,t\} & \mathcal{C}_2(\{y,z,t\}) = \{y,z\} & \\
\mathcal{C}_2(\{z,t\}) = \{z\} & &
\end{array}
$$

Both choice functions satisfy property $\boldsymbol{\gamma}$ *(therefore* ***B****),* ***CH*** *and* ***SUP****. Their composition* $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ *is given by*

$$
\begin{array}{lll}
\mathcal{C}(\{x,y\}) = \{x,y\} & & \\
\mathcal{C}(\{x,z\}) = \{x,z\} & \mathcal{C}(\{x,y,z\}) = \{x,y,z\} & \\
\mathcal{C}(\{x,t\}) = \{t\} & \mathcal{C}(\{x,y,t\}) = \{y,t\} & \mathcal{C}(\{x,y,z,t\}) = \{y,z\} \\
\mathcal{C}(\{y,z\}) = \{y,z\} & \mathcal{C}(\{x,z,t\}) = \{z\} & \\
\mathcal{C}(\{y,t\}) = \{y,t\} & \mathcal{C}(\{y,z,t\}) = \{y,z\} & \\
\mathcal{C}(\{z,t\}) = \{z\} & &
\end{array}
$$

which does not satisfy ***SUP*** *because* $\mathcal{C}(\{x,z,t\}) = \{z\} \subsetneqq \mathcal{C}(\{x,z\})$.

From Proposition 2 we obtain the following consequence.

Corollary 3 *Let* $\mathcal{C}$ *be a choice function on* $\mathcal{D}$*. If* $\mathcal{C}$ *is a compound function of quasi-transitive rational choice functions, then* $\mathcal{C}$ *satisfies* ***B****.*

[5] Because all choice functions in the paper are decisive we avoid the redundant assertion '$\mathcal{C}(\{a\}) = \{a\}$ for every $a \in X$' throughout.

Finally we recall the conditions for a choice function to be acyclic rational or, equivalently, rational. [6]

Theorem 3 (Blair et al. [9]) *A choice function on $\mathcal{D}$ is acyclic rational if and only if it satisfies the Chernoff condition and the binariness property.*

An appeal to Proposition 2 and Theorem 3 produces the next immediate consequence.

Corollary 4 *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be choice functions defined on $\mathcal{D}$. If $\mathcal{C}_1$ is full rational and $\mathcal{C}_2$ is acyclic rational then $\mathcal{C}_2 \circ \mathcal{C}_1$ is acyclic rational.*

Nevertheless if we compound two quasi-transitive rational choice functions we obtain a choice function that is not necessarily acyclic rational, because it may not satisfy the Chernoff condition as the next example proves.

Example 2 *Let us consider the choice functions*

$$\begin{array}{lll}
\mathcal{C}_1(\{x,y\}) = \{x,y\} & & \\
\mathcal{C}_1(\{x,z\}) = \{x,z\} & \mathcal{C}_1(\{x,y,z\}) = \{x,y\} & \\
\mathcal{C}_1(\{x,t\}) = \{x,t\} & \mathcal{C}_1(\{x,y,t\}) = \{x,y,t\} & \mathcal{C}_1(\{x,y,z,t\}) = \{x,y,t\} \\
\mathcal{C}_1(\{y,z\}) = \{y\} & \mathcal{C}_1(\{x,z,t\}) = \{x,z,t\} & \\
\mathcal{C}_1(\{y,t\}) = \{y,t\} & \mathcal{C}_1(\{y,z,t\}) = \{y,t\} & \\
\mathcal{C}_1(\{z,t\}) = \{z,t\} & &
\end{array}$$

and

$$\begin{array}{lll}
\mathcal{C}_2(\{x,y\}) = \{x,y\} & & \\
\mathcal{C}_2(\{x,z\}) = \{x,z\} & \mathcal{C}_2(\{x,y,z\}) = \{x,y,z\} & \\
\mathcal{C}_2(\{x,t\}) = \{x,t\} & \mathcal{C}_2(\{x,y,t\}) = \{x,y,t\} & \mathcal{C}_2(\{x,y,z,t\} = \{x,y,z\} \\
\mathcal{C}_2(\{y,z\}) = \{y,z\} & \mathcal{C}_2(\{x,z,t\}) = \{x,z\} & \\
\mathcal{C}_2(\{y,t\}) = \{y,t\} & \mathcal{C}_2(\{y,z,t\}) = \{y,z\} & \\
\mathcal{C}_2(\{z,t\}) = \{z\} & &
\end{array}$$

*Both $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy γ (therefore **B**), **CH** and **SUP**.*

[6] Suzumura [29, page 35] establishes that a choice function on $\mathcal{D}$ is acyclic rational if and only if it is rational, and that this equivalence does not hold for general domains.

The compound function $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$ is given by

$$\begin{array}{lll} \mathcal{C}(\{x,y\}) = \{x,y\} & & \\ \mathcal{C}(\{x,z\}) = \{x,z\} & \mathcal{C}(\{x,y,z\}) = \{x,y\} & \\ \mathcal{C}(\{x,t\}) = \{x,t\} & \mathcal{C}(\{x,y,t\}) = \{x,y,t\} & \mathcal{C}(\{x,y,z,t\}) = \{x,y,t\} \\ \mathcal{C}(\{y,z\}) = \{y\} & \mathcal{C}(\{x,z,t\}) = \{x,z\} & \\ \mathcal{C}(\{y,t\}) = \{y,t\} & \mathcal{C}(\{y,z,t\}) = \{y,t\} & \\ \mathcal{C}(\{z,t\}) = \{z\} & & \end{array}$$

and it does not satisfy ***CH*** *because $\{x,z,t\} \subseteq \{x,y,z,t\}$ but*

$$\mathcal{C}(\{x,y,z,t\}) \cap \{x,z,t\} = \{x,t\} \nsubseteq \{x,z\} = \mathcal{C}(\{x,z,t\})$$

Our next Proposition gathers some conclusions from the analysis above.

Proposition 3 *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are choice functions on $\mathcal{D}$ that satisfy* ***CH*** *and* ***B*** *then $\mathcal{C}_2 \circ \mathcal{C}_1$ may not satisfy* ***CH*** *(Example 2), thus it may not be acyclic rational (⇔ rational). Therefore the compound choice function of two acyclic rational (⇔ rational) choice functions is not necessarily rational. Even if $\mathcal{C}_1$ and $\mathcal{C}_2$ are quasi-transitive rational (⇔ both satisfy* ***CH****,* ***B*** *and* ***SUP****), then $\mathcal{C}_2 \circ \mathcal{C}_1$ is not necessarily rational.*

We conclude this Section with the following immediate consequence of Proposition 2.

Corollary 5 *Let $\mathcal{C}$ be a choice function on $\mathcal{D}$. If $\mathcal{C}$ is a compound function of acyclic rational (or equivalently, rational) choice functions on $\mathcal{D}$, then $\mathcal{C}$ satisfies* ***B****.*

4 Choice functions rational by two sequential criteria

Along this Section our primitive concept is the choice made by a decision-maker for some sets of alternatives. We investigate when this behavior can be explained by the sequential application of two rational choice functions. Theorem 5 below completely characterizes such class of choice functions. Some examples help to clarify its implications.

To this porpose we first introduce the concepts of *upper and lower approximations* of a choice function in a class (Aizerman and Aleskerov [3]). They try to approximate a non-rational choice function by a rational one and in the end, yield a solution to the problem we have posed ourselves. We emphasize that not only we characterize the choice functions that can be written as the composition of two rational ones but also we give an explicit solution to that problem.

Let $\mathcal{Q}$ an arbitrary class of choice functions on $\mathcal{D}$. A common interpretation is that $\mathcal{Q}$ contains all choice functions that verify certain relevant properties.

Definition 8 *The upper approximation in $\mathcal{Q}$ of a choice function $\mathcal{C}$ is a choice function $\mathcal{C}^u \in \mathcal{Q}$ such that $\mathcal{C}(S) \subseteq \mathcal{C}^u(S)$ for any $S \in \mathcal{D}$, with the property that if $\bar{\mathcal{C}}$ is another choice function in $\mathcal{Q}$ satisfying $\mathcal{C}(S) \subseteq \bar{\mathcal{C}}(S)$ for all $S \in \mathcal{D}$, it must be the case that $\mathcal{C}^u(S) \subseteq \bar{\mathcal{C}}(S)$ for all $S \in \mathcal{D}$. We stress the fact that $\mathcal{C}^u$ is forcefully decisive because so is $\mathcal{C}$ and $\mathcal{C}(S) \subseteq \mathcal{C}^u(S)$ throughout.*

Definition 9 *The lower approximation in $\mathcal{Q}$ of a choice function $\mathcal{C}$ is a (possibly indecisive) choice function $\mathcal{C}^l \in \mathcal{Q}$ such that $\mathcal{C}^l(S) \subseteq \mathcal{C}(S)$ for any $S \in \mathcal{D}$, with the property that if $\bar{\mathcal{C}}$ is another choice function in $\mathcal{Q}$ satisfying $\bar{\mathcal{C}}(S) \subseteq \mathcal{C}(S)$ for all $S \in \mathcal{D}$, it must be the case that $\bar{\mathcal{C}}(S) \subseteq \mathcal{C}^l(S)$ for all $S \in \mathcal{D}$.*

When a choice function is the sequential application of two rational choice functions, the latter functions verify **CH** and **B**. Focusing on such class for analysis is meaningful and in fact, to our purposes the relevant class is the one containing the choice functions that verify **CH** and $\boldsymbol{\gamma}$ that we denote by $\mathcal{Q}_0$. The next result (Theorem 5.15 in Aizerman and Aleskerov [3]) assures that upper approximations in $\mathcal{Q}_0$ exist and also that lower approximations in $\mathcal{Q}_0$ exist under **B**.

Theorem 4 *For any choice function $\mathcal{C}$ on $\mathcal{D}$ the upper approximation in $\mathcal{Q}_0$ exists, and it is given by the expression*

$$\mathcal{C}^u(S) = \{x \in S : \forall y \in S \text{ there exists } S' \in \mathcal{D} \text{ with } x, y \in S' \text{ and } x \in \mathcal{C}(S')\}$$

for any $S \in \mathcal{D}$.

*Moreover if $\mathcal{C}$ satisfies **B** then the lower approximation of $\mathcal{C}$ in $\mathcal{Q}_0$ exists, and it is given by the expression*

$$\mathcal{C}^l(S) = \{x \in S : x \in \mathcal{C}(\{x, y\}),\ \forall y \in S\} \text{ for any } S \in \mathcal{D}.$$

Remark 1 *By virtue of Theorem 3 we conclude that $\mathcal{C}^u$ in Theorem 4 is rational. Alternatively it is easy to check that*

$$R^u \text{ defined as } xR^uy \Leftrightarrow \exists S \in \mathcal{D} : x, y \in S \text{ and } x \in \mathcal{C}(S) \text{ rationalizes } \mathcal{C}^u$$

The rationality of $\mathcal{C}^l$ in Theorem 4 can not be directly derived from Theorem 3 *because $\mathcal{C}^l(S)$ may be empty for some $S \in \mathcal{D}$. Nevertheless the complete binary relation R^l given by xR^ly if and only if $x \in \mathcal{C}(\{x,y\})$ rationalizes $\mathcal{C}^l$.*

We now introduce a new property for a choice function $\mathcal{C}$ on $\mathcal{D}$. It provides the solution to the main problem of the paper.

Definition 10 *A choice function $\mathcal{C}$ satisfies **Property** $\mathbf{P}$ if for any $x, y \in X$ and $S, T \in \mathcal{D}$ such that $\{x,y\} \subseteq S \subseteq T$, the following holds true:*

$$\textit{if } \mathcal{C}(\{x,y\}) = \{x\} \textit{ and } x \in \mathcal{C}(T), \textit{ then it must be the case that } y \notin \mathcal{C}(S)$$

Property $\mathbf{P}$ is weaker than the Chernoff condition thus any rational choice function satisfies it. Moreover this property allows for cyclical patterns.

The next example shows that if a choice function on $\mathcal{D}$ satisfies property $\mathbf{P}$ then it must satisfy neither binariness nor the Chernoff condition, the two necessary and sufficient conditions for such choice function to be rational.

Example 3 *Let $\mathcal{C}$ be the choice function defined on the domain of nonempty subsets of $X = \{x,y,z,t\}$ given by*

$$\begin{array}{lll}
\mathcal{C}(\{x,y\}) = \{x,y\} & & \\
\mathcal{C}(\{x,z\}) = \{x\} & \mathcal{C}(\{x,y,z\}) = \{y\} & \\
\mathcal{C}(\{x,t\}) = \{x,t\} & \mathcal{C}(\{x,y,t\}) = \{x,y,t\} & \mathcal{C}(\{x,y,z,t\}) = \{x,y,t\} \\
\mathcal{C}(\{y,z\}) = \{y\} & \mathcal{C}(\{x,z,t\}) = \{x,t\} & \\
\mathcal{C}(\{y,t\}) = \{y,t\} & \mathcal{C}(\{y,z,t\}) = \{y,t\} & \\
\mathcal{C}(\{z,t\}) = \{z,t\} & &
\end{array}$$

This choice function does not satisfy the binariness property because $x \in \mathcal{C}(\{x,y\})$ and $x \in \mathcal{C}(\{x,z\})$ but $x \notin \mathcal{C}(\{x,y,z\})$. Moreover it does not satisfy the Chernoff condition because $\mathcal{C}(\{x,y,z,t\}) \cap \{x,y,z\} = \{x,y\} \nsubseteq \mathcal{C}(\{x,y,z\})$. Nonetheless $\mathcal{C}$ satisfies property $\mathbf{P}$.

In addition there exist choice functions satisfying Property $\boldsymbol{\gamma}$ which do not satisfy property $\mathbf{P}$ as we can observe in Example 4 below.

Example 4 *Let $\mathcal{C}$ be a choice function defined on the domain of nonempty* *subsets of* $X = \{x, y, z, t\}$ *given by*

$$\begin{array}{lll} \mathcal{C}(\{x,y\}) = \{y\} & & \\ \mathcal{C}(\{x,z\}) = \{z\} & \mathcal{C}(\{x,y,z\}) = \{y,x\} & \\ \mathcal{C}(\{x,t\}) = \{x,t\} & \mathcal{C}(\{x,y,t\}) = \{y,t\} & \mathcal{C}(\{x,y,z,t\}) = \{y,t\} \\ \mathcal{C}(\{y,z\}) = \{y\} & \mathcal{C}(\{x,z,t\}) = \{z\} & \\ \mathcal{C}(\{y,t\}) = \{y,t\} & \mathcal{C}(\{y,z,t\}) = \{y,t\} & \\ \mathcal{C}(\{z,t\}) = \{z\} & & \end{array}$$

It is simple to check that $\mathcal{C}$ *satisfies Property* $\boldsymbol{\gamma}$. *However it does not satisfy* **P** *since* $\{x, y\} \subseteq \{x, y, z\} \subseteq \{x, y, z, t\}$ *and* $\mathcal{C}(\{x, y\}) = \{y\}$, $y \in \mathcal{C}(\{x, y, z, t\})$, *but* $x \in \mathcal{C}(\{x, y, z\})$.

Despite this performance we proceed to prove that a choice function satisfying properties **P** and $\boldsymbol{\gamma}$ can be written as the composition of two rational choice functions. We formalize this concept in the next definition.

Definition 11 *A choice function* $\mathcal{C}$ *on* $\mathcal{D}$ *is* rational by two sequential criteria *if there exist two rational choice functions* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *on* $\mathcal{D}$ *(*$\mathcal{C}_2$ *being possibly indecisive) such that* $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$, *i.e., such that the composition of* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *yields* $\mathcal{C}$. [7]

Rational choice functions are obviously rational by two sequential criteria, but the converse is not true as shown by Example 5 below. In turn, Lemma 1 shows that the composition of rational choice functions satisfies $\boldsymbol{\gamma}$.

Lemma 1 *If* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are rational choice functions on* $\mathcal{D}$, *then the compound choice function* $\mathcal{C}_2 \circ \mathcal{C}_1$ *satisfies Property* $\boldsymbol{\gamma}$.

Proof. Let us denote the binary relations that rationalize $\mathcal{C}_1$ and $\mathcal{C}_2$ by R_1 and R_2 respectively, i.e., for all $S \in \mathcal{D}$,

$$\mathcal{C}_1(S) = \{x \in S : (x, y) \in R_1,\ \forall y \in S\} = \mathcal{C}_{R_1}(S)$$

$$\mathcal{C}_2(S) = \{x \in S : (x, y) \in R_2,\ \forall y \in S\} = \mathcal{C}_{R_2}(S)$$

We denote $\mathcal{C} = \mathcal{C}_2 \circ \mathcal{C}_1$, i.e., $\mathcal{C}(S) = \mathcal{C}_{R_2}(\mathcal{C}_{R_1}(S))$ throughout. We proceed to check that $\mathcal{C}$ satisfies Property $\boldsymbol{\gamma}$.

Let $\{S_i\}_{i \in I}$ be a collection of sets in $\mathcal{D}$ such that $x \in \mathcal{C}(S_i)$ for all $i \in I$. We must prove that $x \in \mathcal{C}(\bigcup_{i \in I} S_i)$. Since $x \in \mathcal{C}_{R_2}(\mathcal{C}_{R_1}(S_i))$ for all $i \in I$ thus

[7] Observe that $\mathcal{C}_2$ must be nonempty-valued on $\{\mathcal{C}_1(S) : S \in \mathcal{D}\}$.

$x \in \mathcal{C}_{R_1}(S_i)$ for all $i \in I$, this leads to

$$xR_1y \text{ for all } y \in S_i \ \ \forall i \in I \ \Rightarrow \ xR_1y \ \ \forall y \in \bigcup_{i\in I} S_i \ \Rightarrow \ x \in \mathcal{C}_{R_1}(\bigcup_{i\in I} S_i)$$

If $x \notin \mathcal{C}_{R_2}(\mathcal{C}_{R_1}(\bigcup_{i\in I} S_i))$, then there exists $z \in \mathcal{C}_{R_1}(\bigcup_{i\in I} S_i)$ such that $\neg(xR_2z)$.

But on the other hand $z \in \mathcal{C}_{R_1}(\bigcup_{i\in I} S_i)$ implies that $z \in \mathcal{C}_{R_1}(S_i)$ for some $i \in I$. Because $x \in \mathcal{C}_{R_2}(\mathcal{C}_{R_1}(S_i))$ for all $i \in I$ we obtain that xR_2z, a contradiction that proves $x \in \mathcal{C}(\bigcup_{i\in I} S_i)$. □

Lemma 1 assures that $\boldsymbol{\gamma}$ is a necessary condition for a choice function to be rational by two sequential criteria. By contrast, Example 5 below shows that this is not the case for **CH**: a choice function satisfying $\boldsymbol{\gamma}$ and not **CH** can be written as the composition of its rational approximation functions. Moreover it makes explicit the fact that we can not assure uniqueness of the solution to our problem.

Example 5 *Let $X = \{x, y, z, t\}$. We define the choice function $\mathcal{C}$ on the domain of nonempty subsets of X as:*

$$\begin{array}{lll}
\mathcal{C}(\{x,y\}) = \{x,y\} & & \\
\mathcal{C}(\{x,z\}) = \{x,z\} & \mathcal{C}(\{x,y,z\}) = \{x,y\} & \\
\mathcal{C}(\{x,t\}) = \{x,t\} & \mathcal{C}(\{x,y,t\}) = \{x,y,t\} & \mathcal{C}(\{x,y,z,t\}) = \{x,y,t\} \\
\mathcal{C}(\{y,z\}) = \{y\} & \mathcal{C}(\{x,z,t\}) = \{x,z\} & \\
\mathcal{C}(\{y,t\}) = \{y,t\} & \mathcal{C}(\{y,z,t\}) = \{y,t\} & \\
\mathcal{C}(\{z,t\}) = \{z\} & &
\end{array}$$

This choice function satisfies Property $\boldsymbol{\gamma}$ and it does not satisfy the Chernoff condition because $\{x,z,t\} \subseteq \{x,y,z,t\}$ but $\mathcal{C}(\{x,y,z,t\}) \cap \{x,z,t\} = \{x,t\} \nsubseteq \{x,z\} = \mathcal{C}(\{x,z,t\})$. Nevertheless it satisfies the weaker property **P**.

Theorem 4 provides the explicit expressions for the upper and lower rational *approximations of $\mathcal{C}$ in $\mathcal{Q}_0$.*

$$\begin{array}{lll}
\mathcal{C}^u(\{x,y\}) = \{x,y\} & & \\
\mathcal{C}^u(\{x,z\}) = \{x,z\} & \mathcal{C}^u(\{x,y,z\}) = \{x,y\} & \\
\mathcal{C}^u(\{x,t\}) = \{x,t\} & \mathcal{C}^u(\{x,y,t\}) = \{x,y,t\} & \mathcal{C}^u(\{x,y,z,t\}) = \{x,y,t\} \\
\mathcal{C}^u(\{y,z\}) = \{y\} & \mathcal{C}^u(\{x,z,t\}) = \{x,z,t\} & \\
\mathcal{C}^u(\{y,t\}) = \{y,t\} & \mathcal{C}^u(\{y,z,t\}) = \{y,t\} & \\
\mathcal{C}^u(\{z,t\}) = \{z,t\} & &
\end{array}$$

$$\begin{array}{lll}
\mathcal{C}^l(\{x,y\}) = \{x,y\} & & \\
\mathcal{C}^l(\{x,z\}) = \{x,z\} & \mathcal{C}^l(\{x,y,z\}) = \{x,y\} & \\
\mathcal{C}^l(\{x,t\}) = \{x,t\} & \mathcal{C}^l(\{x,y,t\}) = \{x,y,t\} & \mathcal{C}^l(\{x,y,z,t\}) = \{x,y\} \\
\mathcal{C}^l(\{y,z\}) = \{y\} & \mathcal{C}^l(\{x,z,t\}) = \{x,z\} & \\
\mathcal{C}^l(\{y,t\}) = \{y,t\} & \mathcal{C}^l(\{y,z,t\}) = \{y\} & \\
\mathcal{C}^l(\{z,t\}) = \{z\} & &
\end{array}$$

The equality $\mathcal{C} = \mathcal{C}^l \circ \mathcal{C}^u$ can be checked directly.

Nevertheless these choice functions $\mathcal{C}^u$ and $\mathcal{C}^l$ do not provide a unique solution to our problem in this particular case as we can see by replacing $\mathcal{C}^l$ with the choice function $\bar{\mathcal{C}}$ defined as follows.

$$\begin{array}{lll}
\bar{\mathcal{C}}(\{x,y\}) = \{x,y\} & & \\
\bar{\mathcal{C}}(\{x,z\}) = \{x,z\} & \bar{\mathcal{C}}(\{x,y,z\}) = \{x,y,z\} & \\
\bar{\mathcal{C}}(\{x,t\}) = \{x,t\} & \bar{\mathcal{C}}(\{x,y,t\}) = \{x,y,t\} & \bar{\mathcal{C}}(\{x,y,z,t\}) = \{x,y,z\} \\
\bar{\mathcal{C}}(\{y,z\}) = \{y,z\} & \bar{\mathcal{C}}(\{x,z,t\}) = \{x,z\} & \\
\bar{\mathcal{C}}(\{y,t\}) = \{y,t\} & \bar{\mathcal{C}}(\{y,z,t\}) = \{y,z\} & \\
\bar{\mathcal{C}}(\{z,t\}) = \{z\} & &
\end{array}$$

$\bar{\mathcal{C}}$ satisfies property $\boldsymbol{\gamma}$ and the Chernoff condition too. Some simple computations show that $\mathcal{C} = \bar{\mathcal{C}} \circ \mathcal{C}^u$.

Our main Theorem identifies the class of choice functions that are rational by two sequential criteria.

Theorem 5 *A choice function $\mathcal{C}$ on $\mathcal{D}$ is rational by two sequential criteria if and only if it satisfies properties $\boldsymbol{\gamma}$ and $\mathbf{P}$. In this case an explicit –but not* *unique– decomposition is* $\mathcal{C} = \mathcal{C}^l \circ \mathcal{C}^u$.

Proof.

To prove sufficiency, recall that Theorem 4 yields the existence of the upper and lower approximations of $\mathcal{C}$ in $\mathcal{Q}_0$, both being rational choice functions by Remark 1. The same theorem gives their respective expressions:

$$\mathcal{C}^u(S) = \{x \in S : \forall y \in S \text{ there exists } S' \in \mathcal{D} \text{ such that } x, y \in S' \text{ and } x \in \mathcal{C}(S')\}$$

and

$$\mathcal{C}^l(S) = \{x \in S : x \in \mathcal{C}(\{x, y\}),\ \forall y \in S\}.$$

Let us now prove that $\mathcal{C} = \mathcal{C}^l \circ \mathcal{C}^u$.

i) $\mathcal{C}(S) \subseteq (\mathcal{C}^l \circ \mathcal{C}^u)(S)$.
If $x \in \mathcal{C}(S)$ then $x \in \mathcal{C}^u(S)$ because we can select $S' = S$ for all $y \in S$.
Let us now suppose that $x \notin \mathcal{C}^l(\mathcal{C}^u(S))$. In this case there exists $y \in \mathcal{C}^u(S)$ such that $\{y\} = \mathcal{C}(\{x, y\})$ because of the definition of $\mathcal{C}^l$.
From $y \in \mathcal{C}^u(S)$ we obtain that for all $s \in S$ there exists $S_{ys} \in \mathcal{D}$ such that $y, s \in S_{ys}$ and $y \in \mathcal{C}(S_{ys})$.
As $\mathcal{C}$ satisfies Property $\boldsymbol{\gamma}$ we conclude $y \in \mathcal{C}(\cup_{s\in S} S_{ys})$.
Then we have $\{x, y\} \subseteq S \subseteq \underset{s\in S}{\cup} S_{ys}$ with $\{y\} = \mathcal{C}(\{x, y\})$ and $y \in \mathcal{C}(\cup_{s\in S} S_{ys})$.
Applying now that $\mathbf{P}$ is verified by $\mathcal{C}$ we conclude $x \notin \mathcal{C}(S)$, against the hypothesis. Therefore $x \in \mathcal{C}^l(\mathcal{C}^u(S))$.

ii) $\mathcal{C}(S) \supseteq (\mathcal{C}^l \circ \mathcal{C}^u)(S)$:
Select $x \in \mathcal{C}^l(\mathcal{C}^u(S))$, thus by the definition of $\mathcal{C}^l$ we have that $x \in \mathcal{C}(\{x, y\})$ for all $y \in \mathcal{C}^u(S)$. In particular $x \in \mathcal{C}(\{x, y\})$ for all $y \in \mathcal{C}(S)$ (because $\mathcal{C}(S) \subseteq \mathcal{C}^u(S)$). As we have that $\mathcal{C}$ satisfies Property $\boldsymbol{\gamma}$ we conclude $x \in \mathcal{C}(\mathcal{C}(S)) \subseteq \mathcal{C}(S)$.

Example 5 accounts for lack of uniqueness.

Conversely, Lemma 1 ensures that if $\mathcal{C}$ is the compound choice function of two rational choice functions $\mathcal{C}_{R_1}$ and $\mathcal{C}_{R_2}$ then $\mathcal{C}$ satisfies property $\boldsymbol{\gamma}$, thus it remains to prove that it satisfies property $\mathbf{P}$ too.

Let us select $\{x, y\} \subseteq S \subseteq T$ such that $\{x\} = (\mathcal{C}_{R_2} \circ \mathcal{C}_{R_1})(\{x, y\})$ and $x \in (\mathcal{C}_{R_2} \circ \mathcal{C}_{R_1})(T)$. We prove that $y \notin \mathcal{C}(S)$.

Indeed let us suppose that $y \in \mathcal{C}(S)$. Then we have

$(y, s) \in R_1$ for all $s \in S$ and $(y, s') \in R_2$ for all $s' \in S$ such that $s' R_1 s$ for all $s \in S$

As we also have that $x \in (\mathcal{C}_{R_2} \circ \mathcal{C}_{R_1})(T)$ we obtain $(x, t) \in R_1$ for all $t \in T$. In particular:

i) $(y, x) \in R_1$ because $x \in S$, and $(x, y) \in R_1$ because $y \in T$, which implies that $\mathcal{C}_{R_1}(\{x, y\}) = \{x, y\}$.
ii) $(y, x) \in R_2$ because xR_1t for all $t \in T$ and $S \subseteq T$, which implies that $y \in \mathcal{C}_{R_2}(\{x, y\})$.

Thus $y \in \mathcal{C}_{R_2}(\{x, y\}) = \mathcal{C}_{R_2}(\mathcal{C}_{R_1}(\{x, y\})) = \mathcal{C}(\{x, y\})$ which contradicts the hypothesis and concludes the proof. $\square$

Examples with three alternatives can be designed where $\mathcal{C}^l$ in Theorem 5 is indecisive even if $\mathcal{C}$ is single-valued. That is the case of e.g., the choice function in Masatlioglu et al. [18, page 10] that illustrates their definition of *an attention filter*. Nonetheless as has been explained, $\mathcal{C}^l$ must be nonempty-valued on the subset $\{\mathcal{C}^u(S) : S \in \mathcal{D}\}$ due to the fact that $\mathcal{C} = \mathcal{C}^l \circ \mathcal{C}^u$ and $\mathcal{C}$ is decisive.

5 Concluding Remarks

We have studied the class of set-valued choice functions that results when we compound rational choice functions. As a previous step we have performed an analysis of relevant properties of the compound function that stem from axioms for choice functions. The following tables gather these results.

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_2 \circ \mathcal{C}_1$
A	A	A $\star$
A	CH	CH $\star$
A	C	C $\star$
A	SUP	SUP $\dagger$
B+CH	B	B $\ddagger$

Table 1

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_2 \circ \mathcal{C}_1$
FR	FR	FR
FR	QTR	QTR
FR	AR	AR
QTR	QTR	B
AR	AR	B

Table 2

In Table 1, cases $\star$ are proved in Aizerman and Aleskerov [3]. Example 2 proves that even if **SUP** and $\boldsymbol{\gamma}$ are imposed then composition does not preserve the Chernoff condition. Nonetheless this property is transmitted to the compound function when $\mathcal{C}_2$ satisfies it and $\mathcal{C}_1$ satisfies the stronger Arrow's axiom. We complement this study with assertions † and ‡ that are proved in Propositions 1 and 2 respectively.

By using the classical axiomatizations of salient specifications of rationality these assertions produce Table 2, where FR, QTR and AR hold for full rational, quasi-transitive rational and acyclic rational ($\Leftrightarrow$ rational) respectively. Corollaries 1 to 5 make the assertions in this table explicit.

We have then tackled an inverse problem: we study when a DM's behavior can be explained as the sequential application of two rational choice functions, that is, when her choice function is rational by two sequential criteria. We obtain a characterization of this type of set-valued choice functions. Uniqueness is not guaranteed but our proof is constructive and an explicit solution is provided in terms of approximation choice functions.

Our results refer to the domain $\mathcal{D}$ consisting of all the finite and nonempty subsets of the grand set. They would not be affected if the domain includes all the infinite subsets as well. Some of them remain true for domains that contain all pairs and all triples from the set of alternatives X. We have focused on this case in order to avoid unnecessary technicalities and concentrate on the sequential choice.

Manzini and Mariotti [16] first studied the particular case of single-valued choice functions. By considering that a choice function $\mathcal{C}$ is rational when there is an *asymmetric* relation P such that $\mathcal{C}(S) = \{x \in S | \nexists y \in S \text{ for which } (y, x) \in P\}$, they characterize single-valued choice functions that are rational by two and three sequential criteria. Apesteguía and Ballester [5] extend the Manzini and Mariotti's result to the sequential application of an arbitrary number of rational choice functions. Our article refers to the wider class of set-valued choice functions, nevertheless it does not compare to their results because asymmetry is not an issue. The model in Masatlioglu et al. [18] is especially close to ours when the filter of attention (the selection of the subset of alternatives that the DM actually has in mind when he makes the choice) satisfies the Chernoff condition, but their approach is bound by uniqueness of the selections and their axiomatics is stated accordingly.

References

[1] M.A. Aizerman, New Problems in the general choice theory, Review of a research trend, Soc. Choice Welfare 2 (1985), 235-282.

[2] M.A. Aizerman, A.V. Malishevski, General theory of best variants choice: some aspects, IEEE Trasactions on Automatic Control 26 (1981), No.5. 1030-1040.

[3] M.A. Aizerman, F. Aleskerov, Theory of Choice, North-Holland, 1995.

[4] J. Apesteguía, M.A. Ballester, Minimal books of rationales, Documento de Trabajo N^o. 1 (2005), (Universidad Pública de Navarra. Departamento de Economía).

[5] J. Apesteguía, M.A. Ballester, Choice by sequential procedures, EconPaper, 2009.

[6] K.J. Arrow, Rational choice functions and orderings, Economica 26 (1959), 121-127.

[7] T. Bandyopadhyay, K. Sengupta, Revealed preference axioms for rational choice, Econ. J. 101 (1991), 202-213.

[8] T. Bandyopadhyay, K. Sengupta, Intransitive indifference and rationalizability of choice functions on general domains, Math. Soc. Sci. 46 (2003), 311-326.

[9] D.H. Blair, G. Bordes, J.S. Kelly, K. Suzumura, Impossibility theorems without collective rationality, J. Econ. Theory 13 (1976), 361-379.

[10] W. Bossert, Y. Sprumont, K. Suzumura, Consistent rationalizability, Economica 72 (2005), 185-200.

[11] W. Bossert, Y. Sprumont, K. Suzumura, Rationalizability of choice functions on general domains without full transitivity, Soc. Choice Welfare 27 (2006), issue 3, 435-458.

[12] W. Bossert, K. Suzumura, Social norms and rationality of choice, Cahiers de recherche (2007), Universite de Montreal, Departement de sciences economiques.

[13] N. Houy, Rationality and order-dependent sequential rationality, Theory Dec. 62(2) (2007), 119-134.

[14] N. Houy, K. Tadenuma, Lexicographic compositions of multiple criteria for decision making, J. Econ. Theory 144 (2009), 1770-1782.

[15] G. Kalai, A. Rubinstein, R. Spiegler, Rationalizing choice functions by multiple rationales, Econometrica 70 (2002), No. 6, 2481-2488.

[16] P. Manzini, M. Mariotti, Sequentially rationalizable choice, Amer. Econ. Rev. 97 (2007), 1824-1839.

[17] P. Manzini, M. Mariotti, Choice by lexicographic semiorders, IZA Discussion Paper No. 4046 (2009).

[18] Y. Masatlioglu, D. Nakajima, E.Y. Ozbay, Revealed attention, NajEcon Working Paper Reviews number 814577000000000409 (2009).

[19] H. Moulin, Choice functions over a finite set: a summary, Soc. Choice Welfare 2 (1985), 147-160.

[20] K. Nehring, Maximal elements of non-binary choice-functions on compact sets. Econ. Letters 50 (1996), 337-340.

[21] M.K. Richter, Revealed preference theory, Econometrica 34 (1966), No.3. 635-645.

[22] M.K. Richter, Rational choice, in: J.S. Chipman, L. Hurwicz, M.K. Richter, H.F. Sonnenchein (Eds.), Preferences, Utility, and Demand, Harcourt Brace Jovanovich (1971), New York.

[23] C. Rodríguez Palmero, J.L. García Lapresta, Maximal elements for irreflexive binary relations on compact sets, Math. Soc. Sci. 43 (2002), 55-60.

[24] A. Sen, Choice functions and revealed preference, Rev. Econ. Stud. 38 (1971), 307-312.

[25] A. Sen, Maximization and the act of choice, Econometrica 65 (1997), No. 4, 745-779.

[26] M.R. Sertel, A.V.D. Bellem, Synopses in the theory of choice, Econometrica 47 (1979), No.6, 1367-1389.

[27] M.R. Sertel, A.V.D. Bellem, Comparison and choice, Theory Dec. 14 (1982), 35-50.

[28] K. Suzumura, Rational choice and revealed preference, Rev. Econ. Stud. 43 (1976), 149-158.

[29] K. Suzumura, Rational Choice, Collective Decisions, and Social Welfare, Cambridge University Press, 1983.

[30] V.I. Vol'skiy, Characteristic conditions for a class of choice functions, Systems and Control Letters 2 (1982), No.3. 189-193.

[31] R.B. Wilson, The finer structure of revealed preference, J. Econ. Theory 2 (1970), 348-353.