A CONSUMABLE MONEY

An Elementary Discussion of Commodity Money, Fiat Money and Credit: Part I

By

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An Elementary Discussion of Commodity Money, Fiat Money and Credit: Part 1

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Abstract

In this paper we present a series of models, all within the context of a simple two-good economy, which bring out the distinctions among the different types of money and financial institutions. The models emphasize the physical properties of the economic goods, moneys, and trading systems. Part 1 covers models in which the money is a consumable storable; the economies in Part 2 use durable money, fiat money, or credit. Under this framework we are able to successfully contrast the role of private money lenders, banks, bilateral credit systems, and credit clearinghouses. We are also able to model the importance of the bankruptcy or default penalty in supporting the use of fiat.

Keywords: Barley, Gold, Fiat and credit, Evolution of money

JEL Classification: C28, C72, C91, D52, D84, E41, E43, E51, E58, G21, K12, L12, N20, P10

Contents

1 Introduction

In a modern economy there are four "physical" types of goods. They are (1) perishables, whose economic lives terminate within a single time period, and are (to a reasonable approximation) consumed at a point in time; (2) storable consumables (such as a can of beans) which are also consumed at a point in time, but can, at the option of the individual, be carried over for several periods; (3) durables which last for several periods and give off a stream of services;[1] and (4) fiat money which is a fictitious durable with no consumption properties and an imputed value derived from its acceptance and use in transactions (see Kiyotaki and Wright [17] and Bak, Norelykke and Shubik [2]).

Any of the above types of good may be used as money in an economy. Storable consumables such as cigarettes functioned as a medium of exchange in many twentieth century prison camps. The use of durable moneys (such as gold) and fiat moneys (such as our current US dollars) has been much more common.

A difference between a durable good money and a fiat money is that the former may provide both transactions services and a stream of consumption services, while

[1] We abstract away from the important practical aspects of depreciation and decay which apply to both storable consumables and durables.

the latter provides only transactions services. In either case, loan markets provide a means for borrowers to obtain these services earlier in time.

Loan markets can be set up through a variety of financial institutions. Both private (profit-maximizing) and public (central) banks play a large role here. Two other trade arrangements are the bilateral exchange of IOU notes or a clearing-house for such notes, with credit or netting. These arrangements involve high levels of communication and credit evaluation.

In this paper we present a series of models, all within the context of a simple two-good economy, which bring out the distinctions between the different types of money and financial institutions mentioned above. Part 1 covers models in which the money is a consumable storable; the economies in Part 2 use durable money, fiat money, or credit. The models emphasize the physical properties[2] of the economic goods, moneys, and trading systems. Their solutions give us insight into many of the financial systems that have been used throughout history.

2 The Properties of Markets

Modern economies use organized markets. Markets are aggregation, disaggregation devices which facilitate organized, fast, more or less anonymous efficient trade. Their emergence from barter, random encounters or trade at festivals poses many interesting problems in economic anthropology, but are not immediately relevant to the formal analysis here.

The inclusion of a market mechanism together with the specification of price formation lays stress on the importance of the trading technology. This enables us to consider the critical relationship between the real economy and the volume of transactions.

2.1 Three Basic Models

There are three basic structures of increasing complexity which form price in a single period simultaneous move strategic market game. They are:

1. The sell-all model,
2. The buy-sell model, and
3. The double auction model.

For m commodities and a money the dimensions of the strategy set of an individual in Model 1 are m; in Model 2 they are $2m$ and in Model 3 they are $4m$. The strategies are, respectively, of the form $(...b^i_j...)$, $(...b^i_j, q^i_j...)$ and $(...p^i_j, q^i_j, \tilde{p}^i_j, \tilde{q}^i_j...)$ for $j = 1, ..., m$, where b^i_j, q^i_j, p^i_j, $\tilde{p}^i_j$ and $\tilde{q}^i_j$ are respectively bids, offers to sell, personal prices to sell and buy and quantity limits on offers to buy. The last two models are regarded as more general than the first; but for many purposes when dealing with dynamics the first model with just a single consumer good and a money is

[2] Jevons [16] was possibly one of the earliest economists concerned with the physical properties of money.

mathematically the easiest. The double auction market is more complex than the buy-sell model and a sequential version of the double auction offers a good approximation of the functioning of a stock market, but it is not necessary for our purposes here. For many questions involving multistage models, as is indicated here, the sell-all model is the easiest and is sufficient. However for understanding the gains to trade financing aspects of transactions the buy-sell model of Dubey and Shubik [10] is required (see Dubey and Geanakoplos [5], and Shubik [25]). We use the buy-sell model here.

2.2 Micromodeling and Generalization

The approach adopted here is that one has to be painstakingly explicit in the formulation of process details. After one specifies a fully-defined playable game, one then attempts to establish generalizations over classes of mechanisms. Some time ago Hurwicz [15] criticized the possibility that there is too much micromodeling in the specification of a detailed game form, but Dubey, MasColell and Shubik [7] showed the assumption of a continuum of agents removes much of the fine structure of many transactions mechanisms. For truly mass markets many distinctions in structure and information do not matter. But, when numbers are few they do matter. This is why the phrase "mathematical institutional economics" appears to be appropriate when examining markets with a finite number of agents.

2.2.1 The Trading Day

Many economic models chop time into discrete intervals. When a durable (such as gold) is traded or used as a money, we must specify when each of the parties obtains its consumption services. A reasonably good approximation is to split the single time period into three ranges $(0, k_1)$, (k_1, k_2) and $(k_2, 1)$ where k_1 is the percentage of consumption use obtained by the original owner, $k_2 - k_1$ is the percentage of consumption lost in transactions use and $1 - k_2$ is the percentage of consumption use received by the final owner of the asset. A variant of this structure is utilized in Part 2 in order to be explicit about market meetings.

2.2.2 $MV = PQ$ and Velocity

In monetary models involving one or more discrete time periods it is reasonable to ask what is the sequence of moves. Small items such as how many times does a market meet in a single period are critical to determining the velocity of money. In most of the exchange models we can lay out a sequence of: (1) in the morning there is settlement and new financing, followed by(2) bidding and offering in the market at around noon, followed by (3) delivery and consumption of perishables and storables in the evening. In the one-period,[3] one-market models the implicit or explicit construction of the

[3] $M =$ the aggregate means of payment,
$V =$ the velocity or average turns per period,
$P =$ the aggregate price level, and
$Q =$an aggregate measure of trade.

single market meeting often imposes not only $MV = PQ$ but also $PQ \leq M$, in the case where is hoarding.

If we have the market meet many times a day, we may increase the velocity of money (see Dubey, Sahi and Shubik [8]). Although one can investigate the behavior of the system as the time Δt between market meetings approaches zero, in physical fact there is a finite lower bound for the time required to carry out an exchange.

The selection of trading technology makes a difference to the volume of trade. The sell-all model monetizes all assets and thus (although it is a tax collector's delight) it badly over estimates the volume of trade, setting it equal to National Wealth. This is probably not too bad when considering consumers who sell their services and buy almost everything else, but it is not a good model to capture the investment decisions of an economy with independent producers with time lags in production. When we separate production from consumption, we approach a formal understanding of the deep insights of Keynes on the importance of coordination. A decent formal macro-mico-modeling approach requires both a trading model such as the buy-sell model of Dubey and Shubik [10] together with an explicit treatment of independent production. This more realistically illustrates the concerns of the macro-economists for items such the failure of efficient trade. In this paper we cover only some of the central preliminaries and constrain our remarks to trade.

2.2.3 An Aside on Enough Money

Given initial ownership claims and preferences, the upper bound on transactions requirements is given by a sell-all model where the market meets once. The lower bound is somewhat difficult to calculate as it depends on the physical properties of the maximal speed of the markets, the divisibility of the goods and monetary units, and the cost of delivery. It also depends on the thickness of the market. Thus a game with 200 traders meeting once in a period might be deemed to have a thick enough market for competition, but if it met 40 times during the period with an average of only five traders, the market would not necessarily be thick enough to avoid price fluctuations, especially if there were any transactions costs.

3 A Caveat on Modeling

There has been much insightful work on the theory of money over the years The volume of important contributions from Jevons onward has been considerable.[4] The approach offered here is consistent with much of the other work and complementary in several basic ways; but it explicitly utilizes a game theoretic approach. In particular, we stress playable games utilizing the methods of game theory.

General equilibrium theorizing tends to abstract from any institutional structure. As Koopmans described it, it is preinstitutional. Our approach is to seek minimal

[4] No attempt is made here to provide a detailed coverage of the many authors whose work has influenced our thoughts. However the authors have been influenced in particular by Bewley [3], Friedman [11], Grandmont and Younes [12], Gurley and Shaw, [13] Hahn [14], Lucas [18], Magill and Quinzi [19], Samuelson [20], Allais [1], Tobin [30], and many others.

institutional forms of market, banks and other financial institutions which serve as the carriers of process, thereby producing fully defined process models which can be simulated or utilized for experimental gaming. In economic terms the formulation of a strategic market game requires that some transactions technology be specified thus it is necessary to incorporate cash flow or trading conditions. This was abstracted away in the general equilibrium analysis.

These models, with a certain amount of labor (if sufficiently symmetric and of low dimension[5]) can be solved for their noncooperative equilibria. If we assume a continuum of agents, we may prove that the noncooperative equilibria of some strategic market games will coincide with the competitive equilibria of related general equilibrium models. Hence the mathematical techniques developed for general equilibrium can be utilized and modified for strategic market games. Nevertheless the methodology and the stress is different. This is shown when we contemplate a finite number of money lenders and note the importance of relative size of individuals and institutions.

Perhaps it is best to consider the test for our models to be whether they are plausible playable experimental games. The test of trying to nail down all the rules, including the statement of initial and terminal conditions provides a useful exercise in exploring the essential functions of institutions.

In most actual economies, technology, custom and law are constantly in flux, changing the nature of the transactions structure. Thus any formal game with dictated rules for transactions must be interpreted in the context that if the trading structure appears to be too costly, in reality individuals will invent alternative payment systems.

The system appears to tend towards more and more efficient and faster trading technologies with the ideal limit being a moneyless economy with perfect credit supplied with costless credit evaluation. But the ideal economy at the limit can obscure the structure of a system which never attains the limit.

4 Models of Markets with a Storable Consumable Money

4.1 The Markets

Four types of agent are considered: (1) & (2) two types of traders, trading in two goods, (3) a class of monied individuals who may act as consumers, private bankers, and/or lenders and (4) the government central bank. We consider various combinations of these agents with different types of money.

Table 1 shows the different treatments for trade with a storable consumable money. The columns represent the models we cover, while the rows represent types of agents. A cell with a "no" means that the corresponding type of agent is not present in the corresponding economy. Otherwise, the triples show the initial endowments of the two types of goods and the commodity money.

[5] The advantage of closed form solutions is that they make it easier to carry out an extensive sensitivity analysis. As the models discussed here are always fully defined, they are amenable to simulation and gaming methods as well as to computational methods such as those suggested by the work of Scarf [21].

Table 1					
	Cash only	Monied Buyer	Money market	Monopoly Banker	Altruist Banker
Trader 1	$(a, 0, m)$	$(a, 0, m)$	$(a, 0, m_1)$	$(a, 0, m)$	$(a, 0, m)$
Trader 2	$(0, a, m)$	$(0, a, m)$	$(0, a, m_2)$	$(0, a, m)$	$(0, a, m)$
Monied-buyer	no	$(0, 0, M–2m)$	no	no	no
Money-lender	no	no	no	$(0, 0, M–2m)$	no
Central bank	no	no	no	no	$(0, 0, M–2m)$

4.2 Two Trader Types with Trade Using a Consumer Perishable or a Storable Consumable Money

4.2.1 A Simple Symmetric Model

We consider a single trading period with symmetrically placed traders who trade two perishables, utilizing a storable consumable commodity money which enters as a linear separable term in their utility functions. This is a special case with special properties. The more complex situation where the monetary commodity enters in a non-linear manner has been considered elsewhere[6] where the preference structure may influence the selection of an appropriate means of payment.[7] Here, however, in making the comparison with a fiat money the selection of a linear separable term helps to clarify the relationship among monies which are storable consumables, durables and fiat.

In the one-period model, the distinction between a perishable and a storable consumable is obliterated. Both will be consumed before the end of the period. However, the distinction between these and a consumer durable still remains for the one period. The durable would be left over after the end of trade.

Tea, salt and cocoa beans, among others have all been utilized as a money They have the property (which is reflected in the mathematical formulation) that one can use the money in transactions and consume it in the same period.

We consider two types of traders trading in two perishable goods, each with the same utility function $\varphi(x, y)+z$ (φ is assumed to be increasing and concave). There is a continuum of traders of each type. Traders of Type 1 have a total initial endowment of $(a, 0, m)$, while Type 2 traders have $(0, a, m)$. Hence one would expect that Type 1 traders would want to exchange some of their “Good 1” for Type 2 traders' “Good 2” — in the model, the efficient level of consumption is for both trader types to end up consuming $a/2$ of each good. As is shown below even this simple explicit structure is adequate to illustrate many monetary properties.

We work with strategic variables. A strategy for the a trader α of Type 1 is denoted by (q^α, b^α), where $q =$ the total amount of good 1 offered for sale, summed

[6] See Shapley and Shubik [22], Dubey and Shapley [9].

[7] In particular in a multistage economy with a discount factor $\beta > 0$ and a stationary supply of perishables, a storable with a linearly separable utility cannot serve as an efficient money as it would be fully consumed in the first period. In contrast, a durable in sufficient supply would always be efficient. These features are special, pertaining to the selection of the linear separable term [6].

across all Type 1 agents and b = the total amount of money bid for good 2, also summed across all Type 1 agents. The strategy for a trader α of Type 2 is $(\bar{q}^\alpha, \bar{b}^\alpha)$ [8].

We emphasize that the strategies here are integral summations of individual traders' strategies. Since all traders of the same type are identical, one may recover an individual trader's strategy merely by dividing (q, b) or $(\bar{q}, \bar{b})$ by the appropriate measure of the set of corresponding traders. We believe this definition of strategies will help make the overall presentation easier to follow.

Trader 1's utility maximization problem is then as follows:

$$\begin{aligned} &\max_{b,q} \varphi\left(a-q, \frac{b}{\bar{p}}\right) + m + pq - b \\ &\text{s.t. } m - b \geq 0 \qquad\qquad (\lambda) \\ &\qquad 0 \leq q \leq a \text{ and } b \geq 0 \end{aligned} \tag{1}$$

In addition we have the market balance conditions

$$p = \frac{\bar{b}}{q} \text{ and } \bar{p} = \frac{b}{\bar{q}}.$$

The reader will guess that the solution to this model (in terms of the values of the Lagrangian multiplier λ of constraint (λ)) will depend upon the value of m. If m is high i.e., the traders have "enough money," the constraint (λ) will not be tight and the multiplier $\lambda = 0$. Alternatively if the traders have "not enough money," (λ) will be tight and $\lambda > 0$.

In general, the concept of "enough money" means that the value of the money supply at equilibrium is sufficient to cover the volume of competitive trade. In other words, the price of the money is precisely its consumption worth and it carries no extra valuation to account for its shadow price as a binding constraint on transactions. In the simple example below it turns out that "enough money" is the condition $m \geq a/2$.

The full derivation of the solution to this model, for the case where $\varphi(x, y) = 2\sqrt{xy}$, is given in Appendix A. Here we indicate the solution and present Table 2 to provide a sensitivity analysis.

A Sensitivity Analysis We consider model results as m ranges from ∞ down to 0.

Case 1: $m > a/2$. When there is more than enough money, the shadow price of the transactions value of money remains at $\lambda = 0$ and efficient trade is achieved (i.e., each type of trader consumes $a/2$ of each good). Each individual is able to bid using his or her "cans of beans" and keeps any surplus beans, but, in general, will earn back the money spent (i.e., balance the budget in the goods bought or sold in the markets). Thus the consumption of the (linearly separable utility) commodity money equals its initial endowment for each individual.

[8] >From now on, (somwehat sloppily) omit the superscript α, leaving it to the reader to distinguish q^α from q from the context.

Case 2: $m = a/2$. When there is precisely enough money, and it is distributed in a manner that every individual does not encounter her cash constraints, the shadow price of the transactions value of money $\lambda = 0$.

Case 3: $0 < m < a/2$, for example $m = a/10$. When there is a moderate amount of money, but still not enough to avoid a shortage the shadow price of the transactions value of money is positive as is reflected by $\lambda = 3$ Efficient trade is not achieved. The price level is decreasing monotonically as m decreases from $a/2$ to 0.

Case 4: $m \to 0$. When there is little or no money in the economy there is next to no trade in the model.[9] In reality, if this happened, other means of payment would be used and other avenues of trade would spring up.

Table 2

m	p	q	b	λ
$m > a/2$	1	$a/2$	$a/2$	0
$m = a/2$	1	$a/2$	$a/2$	0
$m = a/10$	1/2	$a/5$	$a/10$	3
$m \to 0$	$p \to 0$	$q \to 0$	$b \to 0$	$\lambda \to \infty$

A sensitivity analysis as m varies. The ranges $m > a/2$ (more than enough money), $m = a/2$ (exactly enough), $m < a/2$ (not enough) are considered.

Due to the symmetry in this example there is no need for borrowing or lending. As is noted below when a nonsymmetric initial distribution of the money is made the conditions for a money market appear and the rate of interest equals the marginal value of money in alleviating the trade constraint. We note that because the money is a commodity of value, there is a bound on inflation — individuals would keep the money for consumption if the marginal consumption utility to price did not line up with the other consumption goods.

It is convenient to define the total amount of money in the system as M. Here $M = 2m$. When more agents are considered there will be more components which will sum to M.

Figure 1 shows the price level as M is varied and the level of λ as M is varied. We note that because the money is a commodity of consumption value with a fixed marginal utility of consumption this places an upper bound on the competitive equilibrium prices of the consumable goods. We further note that λ (the shadow price of the cash flow constraint) becomes infinite as m approaches zero.

Figure 2 shows the volume of trade as m is varied. Because of the extreme simplicity of our model the volume of trade has a linear relationship to the amount

[9] For an interesting fictional version of the influence of a shortage of money on economic activity see Sholem Asch [29] *Three Cities*.

of money. Thus for $m_1 = m_2 = m$, total trade will be $2m$ and rises until $m = a/2$, when there is no further incentive for trade.

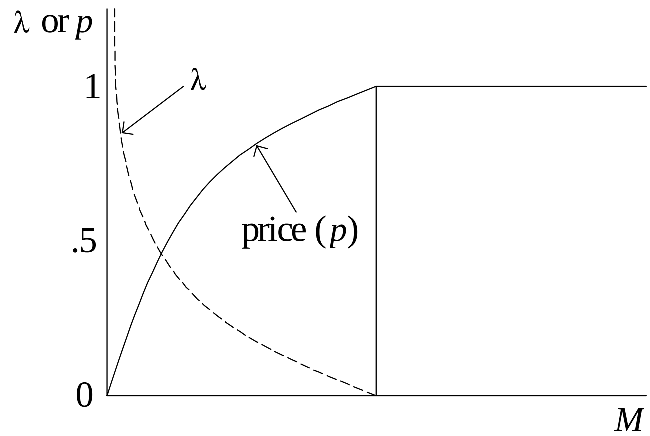

Figure 1. Prices as M increases and λ as M increases

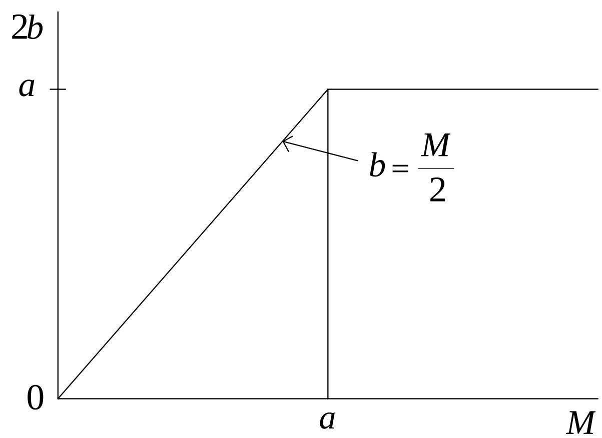

Figure 2. Volume of trade as M increases

4.3 Trade with a Money Market

Suppose we initialize the money holdings so that there is enough money to cover efficient symmetric trade, but it is not distributed equally. In this case we create an inefficiency, as the purchases of one type of trader are constrained by his not having enough money. Trade is impeded by a cash flow constraint. A nonsymmetric efficient outcome can be restored by introducing a money market.

Finally, suppose the types have nonsymmetric endowments of $(a, 0, m_1)$ and $(0, a, m_2)$ where $m_1 > m_2$. A strategy of the Type 1 traders is denoted by (g, q, b) where $g =$ the total amount of money offered in the money market, $q =$ the amount of good 1 offered for sale and $b =$ the amount of money bid for good 2. The notation for Type 2 is $(\bar{q}, \bar{b}, \bar{d})$. Here $\bar{d}$ is the total amount of personal IOU notes bid for the commodity money offered. From considerations of a dimensional analysis it must be considered as a new separate financial instrument, the personal IOU note or promise-to-pay which is monetized by exchanging it for money. This is consistent with the observation in banking practice that a loan to an individual can be interpreted as "monetizing her personal credit."

The players' optimization problems are

$$\max_{b,q,g} \varphi\left(a-q, \frac{b}{\bar{p}}\right) + m_1 - g - b + pq + (1+\rho)g$$

$$\text{s.t. } m_1 - b - g \geq 0 \qquad (\lambda) \tag{2}$$

$$m_2 + \rho g - b + pq \geq 0 \qquad (\mu) \tag{3}$$

$$b, g \geq 0 \text{ and } 0 \leq q \leq a$$

for the lenders and

$$\max_{\bar{b},\bar{q},\bar{d}} \varphi\left(\frac{\bar{b}}{p}, a-\bar{q}\right) + m_2 + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\,\bar{q} - \bar{d} \tag{4}$$

$$\text{s.t. } m_2 - \bar{b} + \frac{\bar{d}}{1+\rho} \geq 0 \qquad (\bar{\lambda})$$

$$m_2 - \frac{\rho\bar{d}}{1+\rho} - \bar{b} + \bar{p}\,\bar{q} \geq 0 \qquad (\bar{\mu})$$

$$\text{and } \bar{b},\ \bar{d} \geq 0,\ 0 \leq \bar{q} \leq a$$

for the borrowers.

The market balance conditions are now:

$$p = \frac{\bar{b}}{q} \text{ and } \bar{p} = \frac{b}{\bar{q}}$$

$$1 + \rho = \frac{d}{g}.$$

Note that if constraint (λ) for the lenders is satisfied then constraint (μ) is satisfied as well; hence (μ) can be omitted. This is not so for the borrowers. If m_2 is small relative to m_1 this constraint may become binding or otherwise if we wish to permit bankruptcy we must introduce an explicit bankruptcy penalty (see Shubik and Wilson [26]).

In all the market balance conditions we remind the reader that the variables b, q, g, $\bar{b}$, $\bar{q}$, and $\bar{d}$ are each integral summations across the continuum of players of a particular type.

Note that in the mechanism the loan is discounted ahead, i.e., the lender gives $\bar{d}/(1+\rho)$ in return for a promise of $\bar{d}$. This provides for the conservation of the commodity money in the system.

4.3.1 Enough Money Badly Distributed

If there is enough money (i.e., $m_1 + m_2 \geq a$), a lender can lend it in the morning, have it used in the market at noon, be repaid in the afternoon and consume it in the evening. We obtain an active money market with $\rho = 0$. The meaning of this is simply that no cash flow constraints are binding. If a can of beans is the unit of money then its price will be equal to its marginal utility in consumption, and there will be no surcharge added to reflect its role in transactions. The owners can lend the beans at their marginal consumption value, get them back "before supper" and consume them on the same day.

In the example above this is easily seen if we consider the initial conditions of $(a, 0, a)$ and $(0, a, 0)$. Traders of Type 1 lend those of Type 2 $a/2$ units of money at an endogenous rate of $\rho = 0$. If there is any more money in the system the rate of interest stays at zero, and trade is optimal but a coordination problem is encountered.[10]

4.3.2 Not Enough Money Badly Distributed

If there is not enough money in the system, trade will be diminished and the price of the monetary commodity goes above its marginal value as a commodity to reflect the shadow price of the constraint to trade. There are two cases which can occur. The first is where the money is not too unequally distributed between the trader types, and the second is where it is distributed in a highly skewed manner. In the latter instance, if bankruptcy of a borrower is to be avoided a new constraint ruling out a negative consumption of the commodity money must be considered. A full appreciation of this phenomenon can be seen by considering a monopolistic banker who owns all or nearly all of an otherwise adequate money supply. This model is considered in Section 4.5.

The full derivation of the solution to our model is given in Appendix B. Three cases are noted: (1) Enough money; (2) not enough money not too unequally distributed; and (3) not enough money highly unequally distributed. Here we indicate the solution and present Table 3 to provide the sensitivity analysis.

A Sensitivity Analysis

[10] Indviduals could borrow randomly at $\rho = 0$, but this could be prevented by charging a rate of $\rho = \varepsilon$ which would prevent frivolous borrowing. The government could then subsidize the traders randomly with the earned revenues. As ε can be made arbitrarily close to zero the coordination problem can almost always be solved easily.

Table 3

m_1+m_2	m_2	$\frac{m_2}{m_1+m_2}$	p	$\bar{p}$	q	$\bar{q}$	b	$\bar{b}$
$m_1+m_2>a$	$<m_1$	any	1	1	$\frac{a}{2}$	$\frac{a}{2}$	$\frac{a}{2}$	$\frac{a}{2}$
$m_1+m_2=a$	$<m_1$	any	1	1	$\frac{a}{2}$	$\frac{a}{2}$	$\frac{a}{2}$	$\frac{a}{2}$
$m_1+m_2=\frac{a}{5}$	$\frac{7a}{80}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{a}{5}$	$\frac{a}{5}$	$\frac{a}{10}$	$\frac{a}{10}$
$m_1+m_2=\frac{a}{5}$	0	0	0.726	0.324	$0.065a$	$0.5a$	$0.162a$	$0.047a$

m_1+m_2	m_2	$\frac{m_2}{m_1+m_2}$	$\bar{d}$	g	ρ	$\bar{\lambda}$	$\bar{\mu}$
$m_1+m_2>a$	$<m_1$	any	?	?	0	0	0
$m_1+m_2=a$	$<m_1$	any	$\frac{m_2}{m_1+m_2}$	$\frac{m_2}{m_1+m_2}$	0	0	0
$m_1+m_2=\frac{a}{5}$	$\frac{7a}{80}$	$\frac{7}{16}$	$\frac{a}{20}$	$\frac{a}{80}$	3	3	3
$m_1+m_2=\frac{a}{5}$	0	0	$0.162a$	$0.038a$	3.251	3.251	3.251

We consider $0 \leq m_2 \leq m_1$ in the four instances where there is more than enough money, enough money, not enough, mildly skewed and not enough, heavily skewed.

Case 1: $m_1 + m_2 > a$. When there is more than enough money, the shadow price of the transactions value of money is $\lambda = 0$, but a problem in coordination appears. The formal mathematics indicates that as an individual is paying a zero interest rate she could demand any amount of the storable consumable store it and return it. The appears to be foolish, but feasible. One could call this a liquidity trap. Basically it loses the coordination aspects of a positive price and $\rho = 0$ reflects the price of the transaction services of the commodity money.

Case 2: $m_1 + m_2 = a$. When there is precisely enough money, and it is distributed nonsymmetrically, traders of one type will encounter their cash constraints, but they will be able to borrow with the shadow price of the transactions value of money $\lambda = 0$.

Case 3: $0 < m_1 + m_2 < a$ and $m_1 \simeq m_2$. *Example,* $m_1 = 9a/80$, $m_2 = 7a/80$. When there is a moderate amount of money, but still not enough to avoid a shortage the shadow price of the transactions value of money is high as is reflected by $\lambda = 3$. The price level is decreasing as m_1 or m_2 decreases. The moderate nonsymmetry of this example avoids the extra budget constraint and the prices of the two consumer goods stay the same.

Case 4: $0 < m_1 + m_2 < a$ and $m_1 \gg m_2$. *Example,* $m_1 = a/5$, $m_2 = 0$. When there is a moderate amount of money in the economy and it is highly nonsymmetrically distributed the borrowers may be constrained by bankruptcy conditions in their ability to borrow. In this formal model where individuals "stick to the rules," we examine what happens. When there is enough money, credit comes free. As the money shortage becomes more and more binding the transactions constraint value of money increases as a function of its total supply. When the borrowers hit the extra bankruptcy constraint, the prices of the goods diverge as is shown in the last line of Table 3.

Figure 3 shows the three zones where there is enough or more than enough money; not enough money for efficient trade, but enough for fully secured lending.[11] We may regard the boundary zones as delineating an economic phase change.

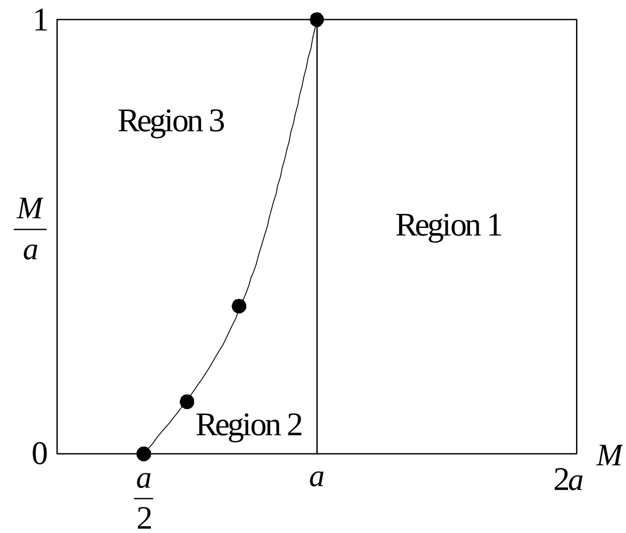

Figure 3. The three trading regions

Table 4 shows some of the points on the boundary between Regions 2 and 3.

Table 4		
M/a	λ	m_2
1	0	1
1/5	3	5/8
1/15	8	5/9
1/505	99	0.505

Table 5 shows some values of the variables in Region 3 where merchants of type 2 have no money $m_2 = 0$. We denote the total money supply of both borrowers and lenders as M $(M = m_1 + m_2)$.

[11] Fully secured lending here implies that at equilibrium the borrower will have enough income to repay the lender without the lender needing to impose a lending constraint on him.

This is not as complex as the securitizing by legally secured durable assets. A somewhat more complex model with durables is needed to illustrate this.

Table 5				
M^2/a^2	M/a	ρ	p	$\overline{p}$
1	1	0	1	1
9/40	0.474	1	0.79	0.632
4/45	0.298	2	0.745	0.447
25/544	0.214	3	0.7276	0.3424

We note that if M/a is very small, a calculation in Appendix B shows that

$$\rho \simeq \frac{a}{\sqrt{2}M} - 1. \tag{5}$$

As long as there is some outside money in the system there are many ways in which one can construct one-period economies which avoid the Hahn paradox of no trade. In particular here it is avoided by the use of a consumable money which thereby maintains its value at the terminal point of the game (which we can call "supper time," or the great bean feast). We may consider the control of the supply of the commodity money contrasting the competitive money market with a private monopolistic and with an altruistic central bank.

The full solution of the Money Market model is given in Appendix B. Even with an example as simple as this the calculations are somewhat messy.[12]

4.4 Trade with a Rich Large Buyer

We introduce a large third agent with the same utility function as the others into the market and study his ability to "live off his money." We consider the economy with the initial endowments as $(a, 0, m)$, $(0, a, m)$ and $(0, 0, M - 2m)$ where $0 \leq m$. There is no loan market in this model.

From the viewpoint of reality the model is bizarre in the extreme, but it merits examination as a link to our study of the merchant-banker who consumes resources. Formally for any value of m the optimizations are of the form

$$\max_{q,b} \varphi\left(a - q, \frac{b}{\overline{p}}\right) + m - b + pq \tag{6}$$

$$\text{s.t. } m - b \geq 0$$

$$q, b \geq 0 \tag{7}$$

$$\max_{\bar{q},\bar{b}} \varphi\left(\frac{\bar{b}}{p}, a - \bar{q}\right) + m - \bar{b} + \bar{p}\bar{q}$$

$$\text{s.t. } m - \bar{b} \geq 0$$

[12]For example, the explicit form of the rate of interest in region 2 is given by:

$$1 + \rho = \sqrt[3]{\left(\frac{1}{27} + \frac{1}{2}K + \frac{1}{18}\sqrt{(12K + 81K^2)}\right)} + \frac{1}{9\sqrt[3]{\left(\frac{1}{27} + \frac{1}{2}K + \frac{1}{18}\sqrt{(12K + 81K^2)}\right)}} - \frac{2}{3}$$

where $K = (2a/(m_1 + m_2))^2$.

for Types 1 and 2 and:

$$\max_{b_1^*, b_2^*} \varphi\left(\frac{b_1^*}{p(b_1^*)}, \frac{b_2^*}{\bar{p}(b_2^*)}\right) + M - 2m - b_1^* - b_2^* \tag{8}$$
$$\text{s.t. } M - 2m - b_1^* - b_2^* \geq 0$$

for the large buyer (where b_1^*, b_2^* are the bids of the large buyer). Individuals of the two agent types being small are, in essence, price takers; but the large buyer takes into account the influence over prices caused by his buying. The price formation in this model must include the bids of the moneyed individual, thus

$$p = \frac{b_1^* + \bar{b}}{q} = \frac{b_1^* + \int_\alpha \bar{b}^\alpha \mu(\alpha)}{\int_\alpha q^\alpha \mu(\alpha)} \tag{9}$$

where m is the measure function defined on the set α of individual traders. A similar expression holds for $\bar{p}$. The notation here emphasizes that b_1^* is the bid of an atomic player and is not an integral, while $\bar{b}$ is the integral sum of bids over a continuum. As the agents of Type 1 and 2 are small, individually they do not influence price. However the large buyer takes into account her influence over price in her optimization.

A way of looking at this model is that we see the power of a single individual whose power is derived from control over transactions. As the distribution of the money is increased to the first two types of trader, for a given total amount of money, the power of the monopsonist decreases. If both trader types together have at least a units of money, the near monopoly power is reduced to zero and any extra commodity money is used only for consumption.

An example illustrates monopoly power. If the initial endowments were $(a, 0, a/12)$, $(0, a, a/12)$, $(0, 0, a/6)$, i.e., the total money supply is $M = a/3$, our calculations show $q = \bar{q} = a/3, \lambda = \rho = 3$ the final endowments would be $(2a/3, a/6, a/6)$, $(a/6, 2a/3, a/6)$, $(a/6, a/6, 0)$ and the utility of the "playboy" has gone from $a/6$ to $a/3$.

If $m = 0$, traders of Type 1 and 2 are unable to trade with each other. However they can still improve their utility by selling their goods to the third agent. The third agent being a monopsonist is in a position to totally exploit the others. Any ε of commodity money[13] he offers them for their goods improves their utility. Thus the outcome from trade is $(0, 0, \varepsilon)$, $(0, 0, \varepsilon)$ and $(a, a, M - 2\varepsilon)$.

For a full solution to this so-called "Playboy Model," see Appendix C.

4.5 Trade with a Consumer Perishable or a Storable Consumable: Two Trader Types with a Monopolistic Money-lender

Suppose we give almost all of the means of payment to one individual. We model the two types of trader as in Section 4.2 (our first model); however now we introduce a third type of agent, the monopolistic money lender. We model the goal of the money

[13] Technically there is a problem at exactly $m = 0$, however if we imagined a minimal unit of payment, such as 0.0001 cents the monopolist would part with this sum and the result would be in equilibrium.

lender in two ways: (1) Her utility function is linear only in the commodity money, or (2) She has exactly the same utility function as the others. This helps us contrast the money lender with the central bank. As is noted below a conceptual difficulty is encountered in treating the central bank's goals.

The two types have symmetric endowments of the goods $(a, 0, m)$ and $(0, a, m)$, but have little money. The money lender has resources $(0, 0, M - 2m)$.

A strategy of traders of Type 1 is denoted by (d, q, b) where $d =$ the amount of personal IOU notes bid for the commodity money offered by the money lender, $q =$ the amount of good 1 offered for sale and $b =$ the amount of money bid for good 2. The strategy of traders of Type 2 is $(\bar{d}, \bar{q}, \bar{b})$ where $\bar{d}$ is the amount of personal IOU notes bid for the commodity money offered by the money lender.

Thus Trader Type 1's problem is[14]

$$\max_{d,q,b} \varphi\left(a - q, \frac{b}{\bar{p}}\right) + m + \frac{d}{1+\rho} - b + pq - d \tag{10}$$

$$\begin{aligned} \text{s.t. } & m + \frac{d}{1+\rho} - b + pq - d \geq 0 \text{ Budget constraint} \\ & m - b + \frac{d}{1+\rho} \geq 0 \text{ Cash flow constraint} \\ & b, d, g \geq 0,\ 0 \leq q \leq a \end{aligned} \tag{11}$$

And Trader Type 2's problem is

$$\max_{\bar{d},\bar{q},\bar{b}} \varphi\left(\frac{\bar{b}}{p}, a - \bar{q}\right) + m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\,\bar{q} - \bar{d} \tag{12}$$

$$\begin{aligned} \text{s.t. } & m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\,\bar{q} - \bar{d} \geq 0 \\ & m - \bar{b} + \frac{\bar{d}}{1+\rho} \geq 0 \\ & \bar{b}, \bar{q}, \bar{d} \geq 0,\ 0 \leq \bar{q} \leq a \end{aligned} \tag{13}$$

The endogenous rate of interest is formed as follows:

$$1 + \rho = \frac{d + \bar{d}}{g} = \frac{\int_\alpha d^\alpha \mu(\alpha) + \int_\gamma \bar{d}^\gamma \mu(\gamma)}{g} \tag{14}$$

where $0 \leq g \leq M - 2m$ is the amount of money offered for loan by the lender.

In the first model the lender's problem is simply to choose g to maximize

$$g\rho(g) \tag{15}$$

Note that we have put ρ as a function of g. This is to emphasize the fact that since the lender is monopolistic, his actions influence the interest rate.

[14] We do not repeat the excess money treatment here for the borrowers, but it is relevant for the single large lender. We consider both when the money supply is less than or equal to "enough" and when there is more than enough. For this example $m = a/2$ is enough.

The second model has the money-lender also act as a consumer. His objective function becomes:

$$\varphi\left(\frac{b_1^*}{p}, \frac{b_2^*}{\overline{p}}\right) + M - 2m + \rho g - b_1^* - b_2^*. \tag{16}$$

Even if the monopolist money lender has far more money than is needed for trade he will not lend it. His monetary power is maximized by making sure that the cash flow constraints are binding. In our section in Part 3 on oligopolistic competition we show how monopoly power is attenuated with many money-lenders or bankers (see also Shubik [23]). However as long as there is a shortage of money, given the transactions structure, the interest rate will be positive.

The price formation in this model must include the bids of the money lender, thus

$$p = \frac{b_1^* + \bar{b}}{q} = \frac{b_1^* + \int_\alpha \bar{b}^\alpha \mu(\alpha)}{\int_\alpha q^\alpha \mu(\alpha)} \tag{17}$$

with a similar expression for $\bar{p}$.

4.6 The Monopolist Who Consumes Only Money

Model 1

We now solve the model of the monopolist money lender whose only concern is monetary maximization. To do this, we need to explore several cases (See Appendix D for the full solution).[15]

Case 1: If $m \geq a/2$ the monopolist is powerless. The traders can obtain all of the gains from trade without borrowing.

Case 2: If $m < a/2$ and is very small in comparison to M the traders cash flow and budget constraints will both be tight, An optimal strategy for the monopolist will be to lend all of his money. His profits will be exactly the amount of money the traders start with, i.e. $2m$. In particular, if the traders start with no money then the monopolist earns zero profits.

Case 3: If $m < a/2$ but is not very small in comparison to M, the traders cash flow is tight but their budget constraint is not tight. The monopolist will lend some amount g as is calculated in Appendix D. The traders cannot obtain all of the gains from trade even if they borrow. The Appendix gives conditions under which it is optimal for the monopolist to lend all of his money, but this will not occur if M is large.

We may obtain an approximate solution by calculation that is illustrative When $m = a/6$, if $M = a$ the bank has $2a/3$, it lends $g = 0.16527a$ at a rate of interest of $\rho = 0.64$ and earns a profit of $0.10577a$. If the bank has more money than $a/3$ it will still lend only 0.16527, it is in the monopolist's self interest to make sure that money is short (competition could weaken this [23]).

[15] A more detailed treatment is presented in Part 3.

A natural question to ask is how much of a profit can the monopolist extract from the traders as a function of the amount of money they have? Is there an amount m such that if each trader type had that amount of money, the profits of the monopolist would be maximized?

4.7 The Paradox of the Powerless Monopolist

Conventional wisdom says "you can't eat money." History and anthropology teaches us that sometimes you can. Societies have used bars of salt, bricks of tea, measures of barley and other storable consumables.

Using a consumable money poses a paradox in monopoly power. Suppose that $m_1 + m_2 = 0$, in words, the traders have no money whatsoever. The monopolist has all of the cans of beans in the world. If her goal is to maximize her monetary profit, she is powerless. The simple physics of conservation indicates that a profit would call for her to be able to extract more money from the traders, but no matter what the prices are the traders will only have the sum of what they borrowed available to pay back hence the only feasible interest rate is $\rho = 0$. But at this rate she makes no profit.

4.7.1 Weak or Strong Pareto Optimality?

The monopolist money-lender is trapped by her own wealth. Does she wish merely to hoard all of her wealth until supper time when she eats it or is she willing to accept a solution showing that she will go along with weak Pareto optimality, in other words is she willing to make interest free loans to the traders, this helps them considerably, gains her nothing, but costs her nothing as she gets her beans back before supper time? If she elects for weak Pareto optimality she is acting as an altruistic bank.

In reality any dominant money-lender who tries to gouge too hard runs the danger of public revolt.

4.8 The Monopolist Who Consumes Everything

The second model has a monopolist with a utility function like that of the traders, hence he is active in all markets. A key difference between this model and the first is that now when the traders have no money the monopolist is no longer impotent but can extract essentially all of the gains from trade for himself.

We omit the complete analysis of this model, because we plan to analyze a similar model in a future paper.

4.9 Trade with a Consumer Perishable or a Storable Consumable: Two Trader Types with an Altruistic Money-lender or Central Bank

We replace the monopolistic money-lender with an altruistic individual or agency who is only interested in facilitating trade rather than maximizing its own consumption or profits. This implies that the goal or payoff function for the central bank involves no consumption of commodities. We may interpret this agency as an outside or government central bank with motivations different from utility maximizing consumers. Giving it the same resources as the monopolistic money-lender yields some extra problems and strikingly different results when there is enough or more than enough money.

The trader optimizations for Type 1 and Type 2 are as before

$$\begin{aligned} &\max_{b,q,d,} \varphi\left(a-q, \frac{b}{\bar{p}}\right) + m + \frac{d}{1+\rho} - b + pq - d \\ &\text{s.t. } m - \frac{\rho d}{1+\rho} - b + pq \geq 0 \qquad (\mu) \\ &\quad m - b + \frac{d}{1+\rho} \geq 0 \qquad (\lambda) \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\max_{\bar{b},\bar{q},\bar{d}} \varphi\left(\frac{\bar{b}}{p}, a-\bar{q}\right) + \left(m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} - \bar{d}\right) \\ &\text{s.t. } m + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\bar{q} - \bar{d} \geq 0 \qquad (\bar{\mu}) \\ &\quad m - \bar{b} + \frac{\bar{d}}{1+\rho} \geq 0 \qquad (\bar{\lambda}) \end{aligned} \tag{19}$$

Also, the endogenous rate of interest is formed as before:

$$1+\rho = \frac{\int_\alpha d^\alpha \mu(\alpha) + \int_\gamma \bar{d}^\gamma \mu(\gamma)}{g}. \tag{20}$$

An informal extension of our basic example serves to illustrate the problems. The formal treatment of the bank's optimization requires considerable formulation and is given in Parts 2 and 3.

Suppose that the central bank has $B \geq a - 2m$ of commodity money. As the traders have $2m$, the total amount of money in the system is a or more and this is enough or more than enough to finance efficient trade. Like the monopolistic money lender the central bank has a choice of the interest rate or specifying the amount of loanable funds it wishes to place on the market. It is easy to check that if the bank announces an interest rate of $\rho = 0$ the traders can achieve efficient trade but a coordination problem is faced.

If the central bank has less than $a - 2m$ then the total amount of money is not sufficient to finance efficient trade. The bank now has a strategic choice which entails deciding to use ρ or g as its strategic variable, in other words in deciding whether to try to control its interest rate or to control the money supply. In this commodity money world all it can do is to use its reserves. It cannot print cans of beans.[16] If it

[16] In a world with international trade a gold shortage would result in an importation of gold from the outside. In this simple model this is not feasible.

offers all of its reserves to the market $g = M - 2m$, where $M < a$ a positive rate of interest appears, trade is not efficient and the central bank makes an undesired profit. If it tries to be more altruistic it sets $\rho = 0$, but then it faces a rationing problem and the well known solution of taking care of "valued customers" and other sociological and informational aspects of distributing a needed rationed resource appear. The lesson here is that although the price of money ρ and the amount of bank money $M - 2m$ appear to be mathematically dual variables. They have different control implications. The bank loses the decentralization and coordination features when ρ is set at zero and it makes an unwanted profit if it uses a quantity strategy.

After the bank has acted and the loans have been repaid what does the central bank do with its stock of storable consumables? Does it store them in a Fort Knox for cans of beans or bars of chocolate? Alternatively does it distribute them via subsidy to the traders who are deemed to be the owners of the government. If this is literally a one-period game, weak Pareto optimality would call for the altruistic government to liquidate the resources it no longer needs. If we view this as an experimental game to be played more than once, a case can be made the referee to keep his reserves for another game.

When we remind ourselves of context, we know that the one-period model is embedded in both time and society. Thus the question of what to do with the inventory of the central bank has an easy interpretation.

4.9.1 A Measure of the Gains from Trade and the Degree of Control

The resources held by the new large agent can be regarded as central bank reserves. If the bank has no desire to make a profit from its services in providing the money supply it can lend the traders its commodity money in a manner to maximize the gains from trade going to the traders,[17] Suppose that $B = a - 2m$. The altruistic bank will issue all of its money, $\rho = 0$ and efficient trade is achieved.

When we compare this amount of commodity money with the amount issued by the monopolistic money lender with only a utility for money the difference between the two indicates what percent of the valuation of resources of the economy can be extracted. This, in essence indicates the money value of the transaction technology, in the sense that this is the upper bound a "currency controller" utilizing one control variable (ρ or g) could extract from a set of traders who obey the transactions rules.[18]

We now may give a strategic interpretation of the relationship between inside and outside money. If $m_1 + m_2 \geq a$ there are enough "trust-substitute pills" that the central bank has no control over transactions. If the "trust-pills" are badly distributed, this can be taken care of by an internal money market. In fact this requires that the lenders trust the new financial instrument, the IOU notes of the

[17] This is mathematically well-defined. The central bank performs a sensitivity analysis over the system as it varies its control variable, utilizing as the optimization criterion a measure of the welfare of the agents. When the bank has enough money the easy criterion is Pareto optimality of the trade.

[18] How to usefully measure the distance that a given allocation is from the Pareto optimal surface, has been a basic problem in the study of trade. See Debreu [4], Smith and Foley [27], Dubey and Geanakoplos [5], Smith and Shubik [28].

borrowers. If a control agent (a central bank or money lender) is introduced such that $B = a - 2m$ as $2m$ varies from 0 to a this agent supplies the "trust pills" for the facilitation of trade.

A somewhat different viewpoint emerges if we consider the money-lender who has a utility function like all the others.[19] The measure of monopolistic resource extraction is in terms of all resources, not just the commodity money.[20] We are now in a position to understand the paradoxical role of the ratio $(m_1 + m_2)/(M - m_1 - m_2) = 2m/B$. The consideration of the case where there is just enough money, or $M + m_1 + m_2 = a$ is instructive. The value of the altruistic bank is at its highest when $M = a$, it decreases monotonically until at $M = 0$ where it has no power and there is no need for it. In contrast, the money maximizing monopolist is impotent at both ends of the spectrum. It optimizes its profit in the interior where it can extract (via an interest rate or a restricted supply of loans) the most commodity money from the traders) in conformity with monopoly theory. When we consider the general utility maximizing monopolist a somewhat different pattern emerges. When $M = a$, the monopolist is at her most powerful point as she can extract essentially all of the gains from trade. At the other extreme when $M = 0$ and $m_1 + m_2 = a$ the monopolist obtains nothing.

5 From Storable Consumables to Fiat

In this part we have been concerned with the properties of an economy utilizing a storable consumable means of payment. Although barley, salt bars or tea bricks and gold bullion and coin have all been used in exchange as history and population have evolved over the last three thousand years, the era of transactions in gold has dominated that of salt or barley or tea. In the last three hundred years gold has been superceded by fiat money and in the last half century credit has grown considerably. We suggest that the physical properties of the means of payment, the density of population and the improvements in communication, information and credit assessment together with the enforcement of contract in a mass anonymous society have made this a natural progression.

In Part 2 our concern is to lay out the path from salt or barley to gold to fiat to credit in a way that is naturally formalized and subject to formal mathematical modeling. Our concern is to present completely well-defined mathematical models which meet the tests that they can be gamed, simulated or have certain solutions computed. We present simple examples but our concern is more with the physical justification of the models and their logic than it is with explicit solution of examples or with static equilibrium existence proofs. The vast literature in general equilibrium theory and in noncooperative game theory with a continuum of agents is such that providing one can establish convexity and compactness of the appropriate functions the proof of existence may be moderately difficult but is reasonably assured. The

[19] This is in keeping with the proposition that "The rich are different from us. They have more money."

[20] We return to this point in Part 2.

main problem lies more in the "economic physics" of producing and justifying the appropriate models which are the carriers of process both in equilibrium and out of equilibrium. Our argument is that fiat money is society's substitute for individual trust. These words almost belong to the popular press and common knowledge. But the appropriate mathematization of these words is the subject of Part 2.

6 Appendix A: Trade with Money

6.1 Two Trader Types

We present the calculations for the model in Section 4.2, with $\varphi(x,y) = 2\sqrt{xy}$. Recall that the initial endowments of Type 1 traders are a units of Type 1 good and m units of money; Type 2 traders begin with a units of Type 2 good and m units of money.

Traders of Type 1 face an optimization described by:

$$\begin{aligned}
&\max_{b,q} 2\sqrt{(a-q)\frac{b}{p}} + m + pq - b \\
&\text{s.t } m - b \geq 0 \text{ (cash flow constraint)} \\
&\qquad \text{and } q, b \geq 0
\end{aligned}$$

Here the decision variables are b (the amount of money bid for the Type 2 good), and q (the amount of Type 1 good that he puts up for sale). We may express the Lagrangian function for the optimization by

$$L = \frac{2}{\sqrt{p}}\sqrt{(a-q)b} + m + pq - b + \lambda(m-b)$$

The first order conditions wrt b and q yield

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-q}{b}} = 1 + \lambda \tag{21}$$

$$\frac{1}{\sqrt{p}}\sqrt{\frac{b}{a-q}} = p \tag{22}$$

$$m - b = 0 \text{ or } \lambda = 0. \tag{23}$$

Similarly, Type 2 traders face:

$$\begin{aligned}
&\max_{\bar{b},\bar{q}} 2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + m + \bar{p}\,\bar{q} - \bar{b} \\
&\text{s.t. } m - \bar{b} \geq 0 \text{ (cash flow constraint)} \\
&\bar{q}, \bar{b} \geq 0
\end{aligned}$$

where decision variables are $\bar{b}$ and $\bar{q}$. The Lagrangian function becomes

$$\bar{L} = \frac{2}{\sqrt{p}}\sqrt{(a-\bar{q})\bar{b}} + m + \bar{p}\,\bar{q} - \bar{b} + \bar{\lambda}(m - \bar{b})$$

The first order conditions wrt $\bar{b}$ and $\bar{q}$ yield

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-\bar{q}}{\bar{b}}} = 1 + \bar{\lambda} \tag{24}$$

$$\frac{1}{\sqrt{p}}\sqrt{\frac{\bar{b}}{a-\bar{q}}} = p \tag{25}$$

$$m - \bar{b} = 0 \text{ or } \bar{\lambda} = 0. \tag{26}$$

Finally, balance conditions are $p = \bar{b}/q$ and $\bar{p} = b/\bar{q}$.

Case 1: If m is large we may assume that $\lambda = 0$ and $\bar{\lambda} = 0$ Equations (21) and (22) give

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-q}{b}} = 1 \Rightarrow \frac{1}{\sqrt{\bar{p}}}\frac{1}{p\sqrt{\bar{p}}} = 1 \text{ or } p\bar{p} = 1. \tag{27}$$

For the symmetric solution $p = \bar{p} = 1$, so $a - q = b$, $a - \bar{q} = \bar{b}$. Hence, from the balance conditions $b = \bar{q}$ and $\bar{b} = q$. This implies that $b = \bar{b} = q = \bar{q} = a/2$.

These results are only valid if the cash flow constraints continue to hold, i.e., $m \geq a/2$. Hence this is what we meant by "m large." Economically speaking, there is enough money to cover efficient trade [24].

Case 2: If m is small the cash flow constraints will be tight, i.e., $\lambda, \bar{\lambda} > 0$ and $b = m$, $\bar{b} = m$. This time equations (21) and (22) imply

$$\frac{1}{\sqrt{\bar{p}}}\frac{1}{(p\sqrt{\bar{p}})} = 1 + \lambda$$

which gives

$$p = \bar{p} = \frac{1}{\sqrt{1+\lambda}}$$

Since $b = m = \bar{b}$, the balance constraints give $q = \bar{q} = m\sqrt{1+\lambda}$. Next, (21) implies that $(1/p)((a-q)/b) = (1+\lambda)^2$, which implies $(1+\lambda)^{1/2}([a-m\sqrt{1+\lambda}]/m) = (1+\lambda)^2$, which in turn gives:

$$(1+\lambda)^{3/2} + (1+\lambda)^{1/2} = \frac{a}{m} \tag{28}$$

Finally, since $p = 1/\sqrt{1+\lambda}$, we have

$$\frac{1}{p^3} + \frac{1}{p} = \frac{a}{m}. \tag{29}$$

This shows how price varies with the money supply.

7 Appendix B: Trade with a Money Market

Here we present the calculations from the model of Section 4.3, where the agents take into account the existence of a money market. We have to introduce the roles of borrowing and lending. We assume initial endowments of money as m_1 and m_2 where $m_1 > m_2$. Hence Type 1 traders become lenders and Type 2 traders borrowers. Again we assume that $\varphi(x,y) = 2\sqrt{xy}$.

Traders of Type 1 face an optimization described by:

$$\begin{aligned} &\max_{b,q,g} 2\sqrt{(a-q)\frac{b}{p}} + m_1 - g - b + pq + (1+\rho)g \\ &\text{s.t. } m - b - g \geq 0 \text{ (cash flow constraint)} \\ &\qquad \text{and } g, q, b \geq 0 \end{aligned}$$

Again, decision variables b and q represent the amounts bid for Type 2 good and the amount offered of Type 1 good, respectively. Another decision variable, g, represents the amount lent to Type 2 traders. This money is lent at an interest rate of ρ, a rate that is determined endogenously to the model.

We may express the Type 1 trader's problem as a Lagrangian function of the form

$$L = \frac{2}{\sqrt{p}}\sqrt{(a-q)b} + m_1 - g - b + pq + (1+\rho)g + \lambda(m - g - b)$$

The first order conditions wrt b, q, and g yield

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-q}{b}} = 1 + \lambda \tag{30}$$

$$\frac{1}{\sqrt{p}}\sqrt{\frac{b}{a-q}} = p \tag{31}$$

$$\lambda = \rho \tag{32}$$

$$m - b - g \geq 0 \text{ or } \lambda = 0 \tag{33}$$

Type 2 traders face

$$\begin{aligned} &\max_{\bar{b},\bar{q},\bar{d}} 2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + \left(m_2 + \frac{\bar{d}}{1+\rho} + \bar{p}\,\bar{q} - \bar{b} - \bar{d}\right) \\ &\text{s.t. } m_2 + \frac{\bar{d}}{1+\rho} - \bar{b} \geq 0 \text{ (cash flow constraint } (\bar{\lambda})) \\ &\qquad \bar{b}, \bar{q}, \bar{d} \geq 0 \end{aligned}$$

Here $\bar{d}/(1+\rho)$ represents the amount that the Type 2 traders borrow from the Type 1 traders (and so they must pay back $\bar{d}$). We also require a budget constraint (or "no bankruptcy" constraint) that

$$m_2 + \frac{\bar{d}}{1+\rho} + \bar{p}\,\bar{q} - \bar{b} - \bar{d} \geq 0 \text{ (budget constraint } (\bar{\mu}))$$

thus the Lagrangian expression becomes:

$$\bar{L} = \frac{2}{\sqrt{\bar{p}}}\sqrt{(a-\bar{q})\bar{b}} + \left(m_2 + \frac{\bar{d}}{1+\rho} + \bar{p}\,\bar{q} - \bar{b} - \bar{d}\right) + \bar{\lambda}\left(m + \frac{\bar{d}}{1+\rho} - \bar{b}\right) + \bar{\mu}\left(m_2 + \frac{\bar{d}}{1+\rho} + \bar{p}\,\bar{q} - \bar{b} - \bar{d}\right)$$

The first order conditions with respect to b, q and $\bar{d}$ yield

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-\bar{q}}{\bar{b}}} = 1 + \bar{\lambda} + \bar{\mu} \tag{34}$$

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{\bar{b}}{a-\bar{q}}} = \bar{p} + \bar{p}\bar{\mu} \tag{35}$$

$$\bar{\lambda} = \rho(1+\bar{\mu}) \tag{36}$$

$$m_2 + \frac{\bar{d}}{1+\rho} - \bar{b} = 0 \text{ or } \bar{\lambda} = 0 \tag{37}$$

$$m_2 + \frac{\bar{d}}{1+\rho} - \bar{b} + \bar{p}\,\bar{q} - \bar{d} = 0 \text{ or } \bar{\mu} = 0 \tag{38}$$

and balance conditions are $p = \bar{b}/q$, $\bar{p} = b/\bar{q}$, and $1+\rho = \bar{d}/g$.

Case 1: If $m_1 + m_2 \geq a$, we may assume that $\lambda = 0$, $\bar{\lambda} = 0$ and $\bar{\mu} = 0$. Equations (30) and (31) give

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-q}{b}} = 1 \Rightarrow \frac{1}{\sqrt{\bar{p}}}\frac{1}{p\sqrt{\bar{p}}} = 1 \text{ or } p\bar{p} = 1$$

For the symmetric solution $p = \bar{p} = 1$. This gives $a - q = b$, $a - \bar{q} = \bar{b}$, which together with the balance conditions $b = \bar{q}$ and $\bar{b} = q$ give one solution as $q = b = \bar{q} = \bar{b} = a/2$. In addition, condition (32) implies $\rho = 0$. The quantities $\bar{d}$ and g will be any quantities such that $\bar{d} = g$ and the cash flow constraints hold; in particular any value of $\bar{d} = g$ in the interval $[a/2 - m_2, m_1 - a/2]$ is valid.

To summarize, the prices are 1, the interest rate is 0, and the final consumption bundles are $(a/2, a/2, m_1)$ and $(a/2, a/2, m_2)$.

Case 2: If $m_1 + m_2 < a$, we see that the cash flow constraint will become tight, i.e., $\lambda > 0$. We now have an extra case distinction to make. We must consider whether the bankruptcy constraint becomes binding, i.e., whether $\bar{\mu} > 0$. This will depend on the rate of interest and the relative sizes of $m_1$and m_2. The case $\bar{\mu} = 0$ is considered as Case 2, while $\bar{\mu} > 0$ is Case 3.

In Case 2, constraints (30) and (31) imply that

$$\frac{1}{\sqrt{\bar{p}}}\frac{1}{(p\sqrt{\bar{p}})} = 1 + \lambda$$

Hence, assuming equal prices (we may do this in Case 2 because Trader Type 2's extra constraint – his budget constraint – is not binding) we have $p = \bar{p} = 1/\sqrt{1+\lambda}$. In addition condition (32), (36) and the assumption that $\bar{\mu} = 0$ imply that $\lambda = \bar{\lambda} = \rho$; and from $b + g = m_1$, $\bar{b} - \bar{d}/(1+\rho) = m_2$ and $1 + \rho = \bar{d}.g$ we obtain

$$b + \bar{b} = m_1 + m_2. \tag{39}$$

Since $\bar{b}/q = b/\bar{q} = p$ we have

$$\frac{\bar{b}+b}{\bar{q}+q} = p \Rightarrow \frac{m_1+m_2}{\bar{q}+q} = \frac{1}{\sqrt{1+\lambda}} \Rightarrow \bar{q}+q = \sqrt{1+\lambda}(m_1+m_2)$$

Now by condition (30) we have $(1/\bar{p})((a-q)/b) = (1+\lambda)^2$, which implies $(a-q)/b = (1+\lambda)^{3/2}$; similarly $(a-\bar{q})/\bar{b} = (1+\bar{\lambda})^{3/2} = (1+\lambda)^{3/2}$. Hence $(1+\lambda)^{3/2} = (2a-q-\bar{q})/(b+\bar{b}) = [2a-\sqrt{1+\lambda}(m_1+m_2)]/(m_1+m_2)$; this gives

$$(1+\lambda)^{3/2} + (1+\lambda)^{1/2} = \frac{2a}{m_1+m_2} \tag{40}$$

Next, $(1/\bar{p})((a-q)/b) = (1+\lambda)^2$ implies $[a - \bar{b}\sqrt{1+\lambda}]/b = (1+\lambda)^{3/2}$, which in turn gives $a = \bar{b}(1+\lambda)^{1/2} + b(1+\lambda)^{3/2}$. Similarly, $(1/p)((a-\bar{q})/\bar{b}) = (1+\lambda)^2$ gives $a = b(1+\lambda)^{\frac{1}{2}} + \bar{b}(1+\lambda)^{3/2}$. These imply $b = \bar{b}$. But then (39) gives $b = \bar{b} = (m_1+m_2)/2$.

Next, starting from $(a-q)/b = (1+\lambda)^{3/2}$ gives $q = a - (m_1+m_2)/2 \cdot (1+\lambda)^{3/2}$. Substituting in for $m_1 + m_2$ using (40) gives

$$q = a - \frac{a(1+\lambda)^{3/2}}{(1+\lambda)^{3/2} + (1+\lambda)^{1/2}} = \frac{a}{2+\lambda} \tag{41}$$

Finally,

$$g = m_1 - b = m_1 - \frac{m_1+m_2}{2} = \frac{m_1-m_2}{2} \tag{42}$$

$$\bar{d} = (1+\rho)(b-m_2) = (1+\lambda)\left(\frac{m_1-m_2}{2}\right) \tag{43}$$

We remark that computationally, given m_1, m_2, and a, one would first find λ using (40) and then use the value of λ to find the value of all the other variables.

We also remark that all of these results are valid only so long as the budget constraint ($\bar{\mu}$) is satisfied. This is equivalent to $\bar{p}\,\bar{q} \geq \bar{d}$. But

$$\bar{p}\,\bar{q} \geq \bar{d} \Rightarrow \frac{1}{\sqrt{1+\lambda}}\frac{a}{2+\lambda} \geq (1+\lambda)\left(\frac{m_1-m_2}{2}\right) \tag{44}$$

$$\Rightarrow \frac{m_1+m_2}{m_1-m_2} \geq 1+\lambda \text{ or } m_2 \geq \frac{\lambda m_1}{2+\lambda} \text{ or } \frac{m_2}{m_1+m_2} \geq \frac{\lambda}{2(1+\lambda)} \tag{45}$$

Case 3: We now consider the extra case where $m_1 + m_2 < a$ (so the interest rate is positive) and the ratio $m_2/(m_1 + m_2)$ is small (so that (45) doesn't hold). This is the case where constraint $(\bar{\mu})$ holds tightly. It is then easy to see that borrowers could not have an optimum at a point where $(\bar{\lambda})$ holds loosely. Hence we may assume that all three constraints (λ), $(\bar{\lambda})$, and $(\bar{\mu})$ hold tightly. The fact that the borrowers have an extra "meaningful" constraint here will imply that there is a basic asymmetry to the game, causing p to be different from $\bar{p}$.

We again start with constraints (30) and (31) implying that $1/\sqrt{\bar{p}} \cdot 1/(p\sqrt{\bar{p}}) = 1 + \lambda$, which gives

$$p\,\bar{p} = \frac{1}{1+\lambda} \tag{46}$$

Next, (35) implies that $(1/p)(\bar{b}/(a - \bar{q})) = (\bar{p} + \bar{p}\bar{\mu})^2$, which gives $p\bar{p}^2 = 1/(1 + \bar{\mu})^2(\bar{b}/(a - \bar{q}))$. Now (34) tells us that $\bar{b}/(a - \bar{q}) = 1/p(1 + \bar{\lambda} + \bar{\mu})^2$; hence we have $p^2\bar{p}^2 = 1/(1 + \bar{\mu})^2(1 + \bar{\lambda} + \bar{\mu})^2$. This gives $p\bar{p} = 1/(1 + \bar{\mu})(1 + \bar{\lambda} + \bar{\mu})$, and so

$$1 + \lambda = (1 + \bar{\mu})(1 + \bar{\lambda} + \bar{\mu}) \tag{47}$$

Now we know from (32) that $\rho = \lambda$, and so (36) gives $\bar{\lambda} = \lambda(1 + \bar{\mu})$. Hence (47) implies that $\bar{\mu} = 0$ and $\bar{\lambda} = \lambda$.

Next, because constraints (λ) and $(\bar{\lambda})$ are tight, $b + g = m_1$and $\bar{b} - \bar{d}/(1+\rho) = m_2$. Together with the balancing condition $1 + \rho = \bar{d}/g$ we obtain

$$b + \bar{b} = m_1 + m_2. \tag{48}$$

Next, since $(\bar{\lambda})$ and $(\bar{\mu})$ are tight, we have $\bar{p}\,\bar{q} = \bar{d}$. But also $\bar{p}\,\bar{q} = b$ because of the balancing condition. Hence $\bar{d} = b$. This implies $\bar{b} - b/(1 + \rho) = m_2$ Substituting in for $\bar{b}$ using (48) gives

$$\begin{aligned} b &= (\frac{1+\lambda}{2+\lambda})m_1 = \bar{d} && (49)\\ \bar{b} &= m_1 + m_2 - b = m_2 + (\frac{1}{2+\lambda})m_1 && (50)\\ g &= m_1 - b = \left(\frac{1}{2+\lambda}\right) m_1. && (51) \end{aligned}$$

We now are in a position to show how to calculate λ. We start with (31), which implies that $p^2\bar{p} = b/(a - q)$. Using (46), we have $p = (1 + \lambda)b/(a - \bar{p}/p)$, which is $ap - \bar{b} = (1 + \lambda)b$, or

$$p = \frac{(1+\lambda)b + \bar{b}}{a} = \frac{m_1 + m_2 + \lambda b}{a}. \tag{52}$$

Similarly, starting with equation (35) and again using (46), we may arrive at

$$\bar{p} = \frac{(1+\lambda)\bar{b} + b}{a} = \frac{m_1 + m_2 + \lambda\bar{b}}{a} \tag{53}$$

Now we can substitute into the $p\,\bar{p} = 1/(1+\lambda)$ equation, obtaining

$$\frac{m_1 + m_2 + \lambda b}{a} \cdot \frac{m_1 + m_2 + \lambda \bar{b}}{a} = \frac{1}{1+\lambda}. \tag{54}$$

Finally, we can use our expressions for b and $\bar{b}$, obtaining a fifth-degree equation for λ in terms of the input parameters m_1, m_2, and a. This can be solved computationally.

An interesting case is when $m_2 = 0$. Here $b = ((1+\lambda)/(2+\lambda))m_1$ and $\bar{b} = (1/(2+\lambda))m_1$. Hence the equation for λ is

$$\frac{m_1 + \lambda(\frac{1+\lambda}{2+\lambda})m_1}{a} \cdot \frac{m_1 + \lambda(\frac{1}{2+\lambda})m_1}{a} = \frac{1}{1+\lambda}, \tag{55}$$

which is

$$\frac{2m_1^2}{a^2}(\lambda^2 + 2\lambda + 2)\left(\frac{1+\lambda}{2+\lambda}\right)^2 = 1. \tag{56}$$

The table below gives λ for certain values of m_1/a:

Table 5

$\frac{M^2}{a^2}$	$\frac{M}{a}$	$\lambda = \rho$	p	$\bar{p}$
1	1	0	1	1
$\frac{9}{40}$	0.474	1	$\frac{5}{3}0.474 = 0.79$	$\frac{4}{3}0.474 = 0.632$
$\frac{4}{45}$	0.298	2	$\frac{5}{2}0.298 = 0.745$	$\frac{3}{2}0.298 = 0.447$
$\frac{25}{544}$	0.214	3	$\frac{17}{5}0.214 = 0.7276$	$\frac{8}{5}0.214 = 0.3424$

Finally, suppose that m_1/a is very small, so that $\lambda = \rho$ is very large. Then $\lambda^2 + 2\lambda + 2$ is approximated by $(1+\lambda)^2$ and $(1+\lambda)/(2+\lambda)$ is approximately 1. Equation (56) becomes $(2m_1^2/a^2)(\lambda+1)^2 \cong 1$, which implies $(\sqrt{2}m_1/a)(\lambda+1) \cong 1$, or $\lambda = \rho \cong a/\sqrt{2}m_1 - 1$.

8 Appendix C: The Rich Large Buyer (Playboy) Model

In this Appendix we go back to the perfectly symmetric model of Appendix A, except that we add a single large trader, the "playboy," who is endowed with a large amount of money but no goods of Type 1 or 2. The playboy has a utility function identical to all of the other traders.

To formalize things, we put the initial endowment of Type 1 trader at a units of Type 1 good and m units of money, while the Type 2 trader has a units of Type 2 good and m units of money. The playboy has only $M - 2m$ units of money, where M represents the total amount of money in the game. Again we assume that all of the players' utility functions are $\varphi(x, y) = 2\sqrt{xy}$.

The Type 1 and Type 2 traders' optimization problems and Lagrangian conditions are identical to those from Appendix A, and we reproduce them here without comment. First, for Trader 1,

$$\max_{b.q} 2\sqrt{(a-q)\frac{b}{p}} + m + pq - b$$
$$\text{s.t } m - b \geq 0 \text{ (cash flow constraint) } (\lambda)$$
$$\text{and } q, b \geq 0$$

The associated first-order conditions for this problem are

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-q}{b}} = 1 + \lambda \tag{57}$$

$$\frac{1}{\sqrt{\overline{p}}}\sqrt{\frac{b}{a-q}} = p \tag{58}$$

$$m - b = 0 \text{ or } \lambda = 0. \tag{59}$$

The Type 2 traders face the following problem:

$$\max_{\bar{b},\bar{q}} 2\sqrt{(a-\overline{q})\frac{\bar{b}}{p}} + m + \overline{p}\,\overline{q} - \bar{b}$$
$$\text{s.t. } m - \bar{b} \geq 0 \text{ (cash flow constraint) } (\bar{\lambda})$$
$$\bar{q}, \bar{b} \geq 0$$

The first-order equations here are

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-\bar{q}}{\bar{b}}} = 1 + \bar{\lambda} \tag{60}$$

$$\frac{1}{\sqrt{p}}\sqrt{\frac{\bar{b}}{a-\bar{q}}} = p \tag{61}$$

$$m - \bar{b} = 0 \text{ or } \bar{\lambda} = 0 \tag{62}$$

The playboy's problem is to decide how much of his money to bid for Type 1 goods, to bid for Type 2 goods, or to hoard. Hence his decision variables will be b_1^* and b_2^*, where b_1^* is the amount he bids for Type 1 goods and b_2^* is the amount he bids for Type 2 goods. His payoff is simply the amount of utility he gets from consuming Type 1 and Type 2 goods, plus that from left-over commodity money, i.e.,

$$2\sqrt{\frac{b_1^* b_2^*}{p\overline{p}}} + M - 2m - b_1^* - b_2^* \tag{63}$$

However, the playboy's bids, unlike the traders', can influence price. This is because the playboy is a single large trader, not a continuum of infinitesimal price-taking individuals like our Type 1 and Type 2 traders Hence p and $\overline{p}$ must be modeled as dependent on b_1^* and b_2^*. In fact, since $p = (b_1^* + \int_\alpha \bar{b}^\alpha)/\int_\alpha q^\alpha = (b_1^* + \bar{b})/q$ and $\overline{p} = (b_2^* + \int_\alpha b^\alpha)/\int_\alpha q^\alpha = (b_2^* + b)/\bar{q}$, his optimization problem becomes

$$\max_{b_1^*, b_2^*} 2\sqrt{\frac{qb_1^*}{(b_1^*+\bar{b})}\frac{\overline{q}b_2^*}{(b_2^*+b)}} + M - 2m - b_1^* - b_2^* \tag{64}$$

$$\text{s.t. } M - 2m - b_1^* - b_2^* \geq 0 \ (\lambda_P) \tag{65}$$

$$b_1^*,\ b_2^* \geq 0 \tag{66}$$

The first-order Lagrangian conditions, with respect to the decision variables $b_1^*$and b_2^* are

$$\sqrt{\frac{\overline{q}b_2^*}{(b_2^*+b)}\frac{(b_1^*+\bar{b})}{qb_1^*}}\ \frac{\bar{b}q}{(b_1^*+\bar{b})^2} = 1 + \lambda_P \tag{67}$$

$$\sqrt{\frac{qb_1^*}{(b_1^*+\bar{b})}\frac{(b_2^*+b)}{\overline{q}b_2^*}}\ \frac{b\overline{q}}{(b_2^*+b)^2} = 1 + \lambda_P \tag{68}$$

$$M - 2m - b_1^* - b_2^* = 0 \text{ or } \lambda_P = 0 \tag{69}$$

We remark that in all cases the markets for Good 1 and Good 2 are isomorphic; hence in an optimal solution we may assume "symmetry", i.e., that $p = \overline{p}$, $q = \overline{q}$, $b = \bar{b}$, and $b_1^* = b_2^*$. One consequence of this is that conditions (67) and (68) simplify to

$$\frac{\bar{b}q}{(b_1^*+\bar{b})^2} = 1 + \lambda_P \tag{70}$$

$$\frac{b\overline{q}}{(b_2^*+b)^2} = 1 + \lambda_P \tag{71}$$

Finally, the balancing constraints are as follows.

$$p = \frac{b_1^*+\bar{b}}{q} \text{ and } \overline{p} = \frac{b_2^*+b}{\overline{q}} \tag{72}$$

Again we remind ourselves that while the quantities b and $\bar{b}$ are really integrals, the quantities b_1^* and b_2^* are the bids of atomic players and so are not integrals.

Case 1: We first consider the case where m is large. Here the traders' cash flow constraints are nonbinding, so $\lambda = \bar{\lambda} = 0$. Hence (58) implies $\sqrt{(a=q)/b} = 1/p\sqrt{\overline{p}}$ and so (57) implies $p = \overline{p} = 1$ (and so $q \neq 0$). Re-combining with (57) gives $(a = q)/b = 1$. But then (70), symmetry, and the balancing constraint (with $p = 1$) give $\bar{b}q = (b_1^* + \bar{b})^2(1 + \lambda_P) = q^2(1 + \lambda_P)$. Hence $b = q(1 + \lambda_P) = \bar{b}$. But then $1 = p = (b_1^* + \bar{b})/q = (b_1^* + q(1 + \lambda_P))/q \Rightarrow b_1^* + q\lambda_P = 0$. But q is positive and both b_1^* and λ_P are nonnegative; hence $b_1^* = \lambda_P = 0$. So $b = q(1 + \lambda_P) = q$, and $(a = q)/b = 1$ implies $b = q = \bar{b} = \bar{q} = a/2$.. Also, b_1^* and b_2^* are both equal to 0. Hence in this case the playboy keeps his money and consumes nothing, while both traders consume $(a/2, a/2)$.

We remark that in order for the above results to hold, the cash flow constraints must be satisfied. In this case this means that $m \geq a/2$. So this is the precise definition of "m is large."

Case 2: The next case is where m is small and M is large. So the traders have little money, but the rich monopolist has a lot of money. Hence we expect the traders' cash flow constraints to be tight and the playboy's to be loose. In mathematical terms, this means λ and $\bar{\lambda}$ are positive, while $\lambda_P = 0$. The immediate consequence of this is that $b = \bar{b} = m$.

As in case 1 we first use (58) and (57) to calculate an expression for price – but here we get $p = \overline{p} = 1/\sqrt{1+\lambda}$. But then (57) implies $(1/p)((a-q)/b) = (1+\lambda)^2$, which gives $q = a - pb(1+\lambda)^2 = a - m(1+\lambda)^{3/2} = \bar{q}$.

Next, since $p = \overline{p} = 1/\sqrt{1+\lambda}$, the balancing conditions give $b + b_2^* = q/\sqrt{1+\lambda}$. This implies $b + b_2^* = a/\sqrt{1+\lambda} - m(1+\lambda)$. Since $b = m$, this gives $b_2^* = a/\sqrt{1+\lambda} - m(2+\lambda) = b_1^*$.

We also have $bq = (b_1^* + b)^2$. This gives $m(a - m(1+\lambda)^{3/2}) = (a/\sqrt{1+\lambda} - m(1+\lambda))^2 = (1/(1+\lambda))(a - m(1+\lambda)^{3/2})^2$, which is

$$(1+\lambda)^{3/2} + (1+\lambda) = \frac{a}{m}. \tag{73}$$

We remark that since $p = 1/\sqrt{1+\lambda}$, this can be written as (compared with (29) from Appendix A):

$$\frac{1}{p^3} + \frac{1}{p^2} = \frac{a}{m}. \tag{74}$$

We also remark that in order for the above analysis to be valid, it must be that condition (λ_P) must hold, i.e., $M - 2m - b_1^* - b_2^* \geq 0$. Substituting in the solutions obtained above, this condition (which is what we mean by "M is large") becomes

$$\frac{M}{2} \geq \frac{a}{\sqrt{1+\lambda}} - m(1+\lambda). \tag{75}$$

Finally, we provide an example. Suppose $m = a/12$ and $M - 2m = a/3$ (so $M = a/2$). Then equation (73) gives $\lambda = 3$, and then we may compute $b = \bar{b} = a/12$, $p = \overline{p} = 1/2$, $q = \bar{q} = a/3$, and $b_1^* = b_2^* = a/12$. Note also that condition (75) holds

Case 3: The final case is where both M and m are small. This means that constraints (λ), $(\bar{\lambda})$, and (λ_P) are tight, i.e., $\lambda = \bar{\lambda} > 0$ and $\lambda_P > 0$. Hence $b = \bar{b} = m$ and $b_1^* = b_2^* = M/2 - m$. So in order to solve the model all we have to do is find p and q.

As above, we may use the Lagrangian conditions on the traders' problems to arrive at $p = \overline{p} = 1/\sqrt{1+\lambda}$ and $q = \bar{q} = a - m(1+\lambda)^{3/2}$. To find λ, note that $p = (b_1^* + \bar{b})/q \Rightarrow 1/\sqrt{1+\lambda} = (M/2)/(a - m(1+\lambda)^{3/2})$. Crossmultiplying gives $M\sqrt{1+\lambda}/2 = a - m(1+\lambda)^{\frac{3}{2}}$, or $1/\sqrt{1+\lambda} - (m/a)(1+\lambda) = M/2a$.

Summarizing, our solution is: $b = \bar{b} = m$, $b_1^* = b_2^* = M/2 - m$, $p = \overline{p} = 1/\sqrt{1+\lambda}$, and $q = \bar{q} = a - m(1+\lambda)^{3/2}$, where λ satisfies $1/\sqrt{1+\lambda} - (m/a)(1+\lambda) = M/2a$.

9 Appendix D: Trading with a Money-lender

9.1 The Money Maximizing Lender

All traders are endowed with m units of commodity money. The monopolistic money-lender is endowed with $M - 2m$ units of money (where $0 \leq m \leq a/2$) The lender's objective is solely to end up with the most possible money.

The Type 1 traders face an optimization described by:

$$\max_{b,q,d} 2\sqrt{(a-q)\frac{b}{\bar{p}}} + m + \frac{d}{1+\rho} - b + pq - d$$

$$\text{s.t. } m - b + \frac{d}{1+\rho} \geq 0 \text{ (cash flow constraint)}, \qquad (\lambda)$$

$$m - b + \frac{d}{1+\rho} + pq - d \geq 0 \text{ (budget constraint)} \qquad (\mu) \tag{76}$$

$$q, b, d \geq 0 \tag{77}$$

The first order conditions wrt b, q, and d yield

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{a-q}{b}} = 1 + \lambda + \mu \tag{78}$$

$$\frac{1}{\sqrt{\bar{p}}}\sqrt{\frac{b}{a-q}} = p + \mu p \tag{79}$$

$$\frac{1}{1+\rho} - 1 + \frac{\lambda}{1+\rho} + \frac{\mu}{1+\rho} - \mu = 0 \tag{80}$$

$$m - b + \frac{d}{1+\rho} = 0 \text{ or } \lambda = 0 \tag{81}$$

$$m - b + \frac{d}{1+\rho} + pq - d = 0 \text{ or } \mu = 0 \tag{82}$$

Similarly, the Type 2 traders face the optimization below:

$$\max_{\bar{b},\bar{q},\bar{d}} 2\sqrt{(a-\bar{q})\frac{\bar{b}}{p}} + m + \frac{\bar{d}}{1+\rho} - b + \bar{p}\bar{q} - \bar{d}$$

$$\text{s.t. } m - \bar{b} + \frac{\bar{d}}{1+\rho} \geq 0 \text{ (cash flow constraint)}, \qquad (\bar{\lambda})$$

$$m - \bar{b} + \frac{\bar{d}}{1+\rho} + \bar{p}\bar{q} - \bar{d} \geq 0 \text{ (budget constraint)}, \qquad (\bar{\mu}) \tag{83}$$

$$\bar{q}, \bar{b}, \bar{d} \geq 0 \tag{84}$$

The optimization conditions here are

$$\frac{1}{\sqrt{p}}\sqrt{\frac{a-\bar{q}}{\bar{b}}} = 1 + \bar{\lambda} + \bar{\mu} \tag{85}$$

$$\frac{1}{\sqrt{p}}\sqrt{\frac{\bar{b}}{a-\bar{q}}} = \bar{p} + \bar{\mu}\bar{p} \tag{86}$$

$$\frac{1}{1+\rho} - 1 + \frac{\bar{\lambda}}{1+\rho} + \frac{\bar{\mu}}{1+\rho} - \bar{\mu} = 0 \tag{87}$$

$$m - \bar{b} + \frac{\bar{d}}{1+\rho} = 0 \text{ or } \bar{\lambda} = 0 \tag{88}$$

$$m - \bar{b} + \frac{\bar{d}}{1+\rho} + \bar{p}\bar{q} - \bar{d} = 0 \text{ or } \bar{\mu} = 0 \tag{89}$$

The banker's optimization is expressed as:

$$\max_{g \text{ or } \rho} M - 2m + \rho g(\rho) \tag{90}$$

$$\text{s.t. } M - 2m + \rho g(\rho) \geq 0 \ (\lambda_B) \tag{91}$$

$$\text{and } g \text{ or } \rho \geq 0 \tag{92}$$

Finally, the balance conditions are $p = \bar{b}/q$, $\bar{p} = b/\bar{q}$ and $1 + \rho = (d + \bar{d})/g$.

Our general approach here is to first solve the traders' problems, and then solve the banker's problem once we know what the traders do as a function of ρ. In essence we solve the trading problem parametrically for ρ then consider $g(\rho)$ and optimize.

We remark that here again the problems for Types 1 and 2 are isomorphic and so we may assume a symmetric solution where $b = \bar{b}$, $d = \bar{d}$, $p = \bar{p}$, and $q = \bar{q}$.

Case 1: First we consider the case where m is large. In this instance neither the cash flow or budget constraint are binding for the traders, i.e., $\lambda = \mu = \bar{\lambda} = \bar{\mu} = 0$. Hence (78) implies $(1/\sqrt{\bar{p}})\sqrt{(a-q)/b} = 1$, which gives $p = \bar{p} = (a-q)/b$. Also (79) implies $p\sqrt{\bar{p}} = \sqrt{b/(a-q)}$, which is $p^3 = b/(a-q)$. Hence $p = 1/p^3$, which gives $p = \bar{p} = 1$.

Next, the balancing conditions give $1 = p = \bar{b}/q = b/q$, so $b = q$ Since also $1 = p = (a-q)/b$, we have $b = \bar{b} = q = \bar{q} = a/2$.

Next, we note that equation (80) (with $\lambda = \mu = 0$) implies $\rho = 0$. Hence the bank gains zero profits. But if this is to be an equilibrium we must then have $d = \bar{d} = 0$; otherwise the bank could generate a profit by putting ρ and g at small positive values.

The above results are valid only so long as constraints (λ), (μ), $(\bar{\lambda})$, $(\bar{\mu})$, and (λ_B) hold, i.e.,

$$m - \frac{a}{2} \geq 0 \ (\lambda) \text{ and } (\bar{\lambda}) \tag{93}$$

$$m - \frac{a}{2} + 0 + \frac{a}{2} \geq 0 \ (\mu) \text{ and } (\bar{\mu}) \tag{94}$$

$$M - 2m \geq 0 \ (\lambda_B) \tag{95}$$

i.e., $m \geq a/2$. So this is what is meant by "m is large."

Case 2: The second case is where m is very small: In this instance both the cash flow and budget constraints will hold with equality, so $m - b + d/(1+\rho) = 0$ and $pq = d$. In addition, λ, μ, $\bar{\lambda}$, and $\bar{\mu}$ are all positive.

First, we note that $pq = d$ (see above) and $pq = \bar{b} = b$ (balancing condition plus symmetry). Hence $b = d$ $(= \bar{b} = \bar{d})$. Hence, $m - b + d/(1+\rho) = 0$ implies $m - b + b/(1+\rho) = 0$. This gives $b = ((1+\rho)/\rho)m = d = \bar{b} = \bar{d}$. Furthermore,

$$g = g(\rho) = \frac{d + \bar{d}}{1+\rho} = \frac{\frac{2(1+\rho)}{\rho}m}{1+\rho} = \frac{2m}{\rho}. \tag{96}$$

Hence the banker maximizes $\rho g(\rho) = \rho(2m/\rho) = 2m$. In other words, the bank's profits are $2m$ no matter what he does. Formally he can set g anywhere in $(0, M - 2m]$, with $\rho = 2m/g$, and attain profits of $2m$ Another way to say this is that the function $g(\rho)$ has unit elasticity.

Since the bank is indifferent among its feasible strategies, it may wish to choose a policy by which it would benefit the traders most. If so, it will set g as high as possible and ρ as low as possible, i.e., $g = M - 2m$ and $\rho = 2m/(M - 2m)$.

Given the values of g and ρ the bank sets, we may now calculate the optimal values of the traders' decision variables. Note that we've already calculated $b = d = \bar{b} = \bar{d} = [(1+\rho)/\rho]m$.

First, we have (79) implies $\sqrt{(a-q)/b} = 1/p\sqrt{\bar{p}}(1+\mu)$ Together with (78) this implies $1/p\bar{p} = (1+\lambda+\mu)(1+\mu)$, for which symmetry implies $p = \bar{p} = 1/\sqrt{(1+\lambda+\mu)(1+\mu)}$ Now (80) implies $\lambda = \rho(1+\mu)$, so $p = \bar{p} = 1/\sqrt{1+\rho}(1+\mu)$. Thus $p(1+\mu) = 1/\sqrt{1+\rho}$. But then (79) $\Rightarrow$ $\sqrt{(a-q)/b} = 1/p\sqrt{\bar{p}}(1+\mu) = \sqrt{1+\rho}/\sqrt{\bar{p}}$, i.e., $(a-q)/b = (1+\rho)/\bar{p}$. Hence $(a-q)/\bar{p}\,\bar{q} = (1+\rho)/\bar{p}$. Rearranging, we have $q = a/(2+\rho) = \bar{q}$. Also, $p = \bar{b}/q = [((1+\rho)/\rho)m]/a/(2+\rho) = (1+\rho)(2+\rho)m/\rho a = \bar{p}$

Finally, for the multipliers, we have

$$1 + \mu = \frac{1}{p\sqrt{1+\rho}} = \frac{\rho a}{(1+\rho)^{3/2}(2+\rho)m} \tag{97}$$

$$\lambda = \rho(1+\mu) = \frac{\rho^2 a}{(1+\rho)^{3/2}(2+\rho)m} \tag{98}$$

We remark that the quantities above are valid so long as the multiplier μ is nonnegative. This gives a condition of

$$\frac{\rho a}{(1+\rho)^{3/2}(2+\rho)m} \geq 1, \text{ or } \frac{m}{a} \leq \frac{\rho}{(1+\rho)^{3/2}(2+\rho)} \tag{99}$$

Note that the maximum value of $\rho/(1+\rho)^{3/2}(2+\rho)$ (on the interval $\rho \in [0,\infty)$) is about .12, which is much less than one half. Hence "Case 1" and "Case 2" do not cover all possibilities, i.ee, we must have at least one "intermediate value for m" case.

Case 3: Now suppose that neither Case 1 nor Case 2 holds. Hence exactly one of the constraints (λ) and (μ) holds tightly. But with $m < a/2$ (and so $\rho, d > 0$) it is impossible for (μ) to hold tightly but not (λ); if so, then Type 1 traders could improve by simultaneously lowering d and raising b, while preserving equality in (μ). Hence the only case to consider here is for $\lambda > 0$ (so $m - b + d/(1+\rho) = 0$) and $\mu = 0$. Also, $\bar{\lambda} = \lambda > 0$ and $\bar{\mu} = 0$.

First, note that (80) implies $\lambda = \rho$.

Next, we see that (78) is $(1/\sqrt{\bar{p}})\sqrt{(a-q)/b} = 1+\lambda$ and (79) implies $\sqrt{(a-q)/b} = 1/p\sqrt{\bar{p}}$, hence $1/p\bar{p} = 1+\lambda$. Using symmetry, we have $p = \bar{p} = 1/\sqrt{1+\lambda} = 1/\sqrt{1+\rho}$.

Next, since $m - b + d/(1+\rho) = 0$ and $m - \bar{b} + \bar{d}/(1+\rho) = 0$ we have $b + \bar{b} = (d+\bar{d})/(1+\rho) + 2m = g + 2m$. But now, since $\bar{b}/q = p = \bar{p} = b/\bar{q}$, we have $(b+\bar{b})/(q+\bar{q}) = 1/\sqrt{1+\rho}$, which is $q + \bar{q} = \sqrt{1+\rho}(b+\bar{b}) = \sqrt{1+\rho}(g+2m)$.

Next, we see that (78) implies $(1/\bar{p})((a-q)/b) = (1+\lambda)^2 = (1+\rho)^2$, so $(a = q)/b = (1+\rho)^{3/2} = (a = \bar{q})/\bar{b}$. This implies $(1+\rho)^{3/2} = (2a - q - \bar{q})/(b+\bar{b}) = [2a - \sqrt{1+\rho}(g+2m)]/(g+2m) = 2a/(g+2m) - \sqrt{1+\rho}$. Rearranging gives

$$g = g(\rho) = \frac{2a}{(1+\rho)^{3/2} + (1+\rho)^{1/2}} - 2m. \tag{100}$$

This implies

$$b = \bar{b} = \frac{g+2m}{2} = \frac{a}{(1+\rho)^{3/2} + (1+\rho)^{1/2}}. \tag{101}$$

Also,

$$q = \bar{q} = \frac{\sqrt{1+\rho}(g+2m)}{2} = \frac{a}{2+\rho}. \tag{102}$$

The banker will choose ρ so as to maximize $\rho g(\rho) = 2a\rho/[(1+\rho)^{3/2} + (1+\rho)^{1/2}] - 2m\rho$. This can be done computationally. We remark that the maximization is valid only so long as M is large enough so that the ρ so obtained does not cause $g(\rho)$ to be more than $M - 2m$. This is certainly true if $M \geq a$. Otherwise, $g(\rho)$ will stay at the bound of $M - 2m$ and ρ will satisfy $2a/[(1+\rho)^{3/2} + (1+\rho)^{1/2}] = M$. (The formulas for the other variables follow, using this "modified" value of ρ.)

References

[1] Allais, M., 1947. *Economie et interet*. Paris: Imprimerie Nationale, pp. 238–241.

[2] Bak, P., S.F. Norrelykke and M. Shubik, 1999. "Dynamics of Money," *Phys. Rev.,* E60: 2528–32.

[3] Bewley, T., 1980. "The Optimum Quantity of Money." In J.H. Kareken and N. Wallace (eds.), *Models of Monetary Economies* Minneapolis: Federal Reserve Bank of Minneapolis, pp. 169–210.

[4] Debreu, G., 1951. "The Coefficient of Resource Utilization," *Econometrica*, 19.

[5] Dubey, P., and J. Geanakoplos. 2003 "IS-LM. Inside and Outside Fiat Money, Gains to Trade, and IS-LM," *Economic Theory*, 21: 347–397.

[6] Dubey, P., J. Geanakoplos, and M. Shubik.. 2003. "Is Gold an Efficient Store of Value?", *Economic Theory*, 21(4): 767–782.

[7] Dubey, P., A. Mas Colell, and M. Shubik, 1980. "Efficiency Properties of Strategic Market Games: An Axiomatic Approach," *Journal of Economic Theory*, 22: 339–362

[8] Dubey, P., S. Sahi and M. Shubik. 1993. "A Repeated Trade and the Velocity of Money," *Journal of Mathematical Economics*, 22(2): 125–137.

[9] Dubey, P. and L.S. Shapley, 1977. "Noncooperative Exchange with a Continuum of Traders," Rand p-5964, Santa Monica, CA.

[10] Dubey, P., and M. Shubik, 1978. "The Noncooperative Equilibria of a Closed Trading Economy with Market Supply and Bidding Strategies," *Journal of Economic Theory*, 17(1): 1–20.

[11] Friedman, M., 1969. *The Optimum Quantity of Money and Other Essays*. Chicago: Aldine.

[12] Grandmont, J.M., and Y. Younes, 1972. "On the Role of Money and the Existence of a Monetary Equilibrium," *Review of Economic Studies*, 39: 355–372.

[13] Gurley, J.G., and E.S. Shaw, 1960. *Money in a Theory of Finance*. Washington, DC: Brookings.

[14] Hahn, F.H., 1965. "On Some Problems of Proving the Existence of an Equilibrium in a Monetary Economy. In F H Hahn and F. Brechling (eds.), *The Theory of Interest Rates*. New York: MacMillan

[15] Hurwicz, L., 1980. "Explaining the Existence and Value of Money: Comment." In J.H. Karaken and N. Wallace (eds.), *Models of Monetary Economies*. Minneapolis: Federal Reserve Bank of Minneapolis.

[16] Jevons, W.S., 1875. *Money and the Mechanism of Exchange*. London: MacMillan.

[17] Kiyotaki, N., and R. Wright, 1989. "On Money as a Medium of Exchange," *Journal of Political Economy*, 97: 927–993.

[18] Lucas, R.E., 1980. "Equilibrium in a Pure Currency Economy," *Economic Enquiry*, 18: 203–220.

[19] Magill, M. and M. Quinzii, 1992. "Real Effects of Money in General Equilibrium," *Journal of Mathematical Economics*, 302–342.

[20] Samuelson, P.A., 1958. "An Exact Consumption-loan Model of Interest with or without the Social Contrivance of Money," *Journal of Political Economy*, 66: 467–480.

[21] Scarf, H.S. with T. Hansen, 1973. *The Computation of Economic Equilibria*. New Haven, CT: Yale University Press

[22] Shapley, L.S., and M. Shubik, 1977. "Trade Using One Commodity as a Means of Payment," *Journal of Political Economy*, 85(5): 937–968.

[23] Shubik, M., 1976. "A Noncooperative Model of a Closed Trading Economy with Many Traders and Two Bankers," *Zeitschrift fur Nationalokonomie*, 36: 49–60.

[24] Shubik, M. 1999. *The Theory of Money and Financial Institutions*, Vol. 1. Cambridge, MA: MIT Press.

[25] Shubik, M. 1999. *The Theory of Money and Financial Institutions*, Vol. 2. Cambridge, MA: MIT Press.

[26] Shubik, M., and C. Wilson, 1977."The Optimal Bankruptcy Rule in a Trading Economy Using Fiat Money," *Zeitschrift fur Nationalokonomie*, 37: 333–354.

[27] Smith, E., and D.K. Foley, 2002. "Thermodynamics and Economic General Equilibrium Theory," submitted to *Econometrica* 12/2002

[28] Smith, E., and M. Shubik, 2003. "Strategic Freedom, Constraint and Symmetry in One-period Markets with Cash and Credit Payment," CFDP 1420, New Haven CT.

[29] Sholem, Asch, 1937. *Three Cities*. London: Victor Gollancz Ltd.

[30] Tobin, J., 1982. "Money and Finance in the Macroeconomic Process," *Journal of Money, Credit and Banking*, 14.