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Heterogeneous Discount Factors in an Assignment Model with Search Frictions

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Heterogeneous Discount Factors in an Assignment Model with Search Frictions*

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Abstract

We look at a simple market with two-sided heterogeneity and pairwise meetings. On the supply side are two landlord types who differ in the quality of their apartments. On the demand side is a continuum of tenant types who differ in their valuations for apartment types and in their patience. For infinitesimal search frictions and an atomless tenant type distribution, we fully characterize all possible steady state equilibria in a typical region of the parameter space. Our main finding is that the heterogeneous discount factors can cause strong deviations from the Walrasian outcome even when we asymptotically remove all the search frictions. All conventional frictional models with non-Walrasian limits are qualitatively different from the model in this paper.

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1 Introduction

In this paper, we discuss how discount factor heterogeneities can influence the equilibrium outcomes in a frictional assignment model. Conventional assignment models belong to the perfect matching literature which dates back to the early sixties. The focus in this literature are games with two groups of heterogeneous players in which each member of one group wants to be matched with one member of the other group. In the original paper of Gale and Shapley (1962), players have pure ordinal preferences over all potential partners. Assignment models, however, are characterized by cardinal preferences and transferable utility among players. The first paper which explicitly studies an assignment game is the one of Shapley and Shubik (1972).

An important implicit assumption on which all equilibrium concepts in the perfect matching literature rely is that meetings between players can be costlessly arranged, i.e., that search frictions are zero. Matching models with positive search frictions, i.e., imperfect matching models had long not been considered, but recently a small and fast growing literature on this topic has begun to emerge. An already published survey of this literature is Burdett and Coles (1999). Notice that imperfect matching models must be sharply distinguished from these conventional search models in which match qualities realize randomly after the match creations.

Transferable as well as nontransferable utility models do exist in the imperfect matching literature. A more or less complete list of all transferable utility models is Albrecht and Vroman (1998), Lu and McAfee (1996), Shimer and Smith (1996) and Sattinger (1995). Typical nontransferable utility models are Bloch and Ryder (1994), Burdett and Coles (1997), Eeckhout (1996), Morgan (1996) and Smith (1997). The above mentioned papers differ significantly in the way in which they model the trader flows in a steady state. Burdett and Coles (1997), Lu and McAfee (1996) and Morgan (1996) assume that there is an exogenously given inflow of traders which must be balanced by an equivalent outflow in a steady state. Bloch and Ryder (1994) and Eeckhout (1996) do not explicitly model the trader flows in a steady state because they just assume that all exiting pairs are automatically replaced by identical entering pairs. The remaining papers assume infinitely lived agents whose matches become randomly dissolved by nature so that the destruction of matches in a steady state must be balanced by an equivalent creation.

Most authors in the imperfect matching literature build on Becker's (1973) perfect matching model in which there is a unique ranking of players on both sides of the market. In the nontransferable utility version of this frictionless model, positively assortative matching, i.e., a strictly positive correlation between partner qualities does always arise. In the transferable utility version, on the other hand, positively assortative matching is only possible with supermodular productive interactions, i.e., with payoff matrices which are complementary in qualities.

Under a single discount factor as the major friction source, it turns out that Becker's (1973) results need not carry over to a search setting. In a joint paper, Shimer and Smith (1996) show for a model with transferable utility that supermodular productive interactions do not guarantee a correlation between partner qualities which is everywhere positive. Similarly, Smith (1997) shows for a nontransferable utility model that a unique ranking of players on both sides of the market does not guarantee a matching pattern which is approximately positively assortative. The intuition behind that is simply that with a common discount factor, the highest ranked players need not be the choosiest ones because their high match value makes them suffer most from a match delay. For a discount factor which approaches 1, however, this intuition breaks down because even the highest ranked players face very low search costs in this case.

In this paper, we deviate from the widespread assumption of a common discount factor because we assume that some of the lower ranked players on one side of the market are much more patient than all other players. An advantage of this nature is obviously irrelevant in a frictionless model, and whether this advantage can yield higher partner qualities in a frictional model is the main question which we want to address in this paper.

Under the modeling choices of this paper which includes transferable utility and which we discuss in a separate section, we get a clear-cut answer to our main question. Even for discount factors which approach 1, we do not only obtain an equilibrium with a positive correlation between partner qualities, but also equilibria in which some or all of the more patient but low ranked players can match with partners of the highest rank. Such deviations from positively assortative matching always turn out to be driven by a high steady state population share of the patient players which makes it hard for the impatient players to meet one another. The fact that such a crowding out can even take place when all players become infinitely patient is due to a crowding intensity which goes to infinity at the same time. For homogenous discount factors and small frictions, we never obtain deviations from positively assortative matching because the crowding intensity must always remain finite in such cases.

The rest of the paper is organized as follows. Section 2 presents a frictionless benchmark model which is essentially a flow version of Becker's (1973) transferable utility model. Section 3 specifies the way in which we introduce search frictions into the perfect matching model in Section 2. Section 4 derives necessary and sufficient equilibrium conditions. Section 5 shows that our results do not stem from an equilibrium multiplicity problem. Section 6 establishes equilibrium existence. Section 7 discusses our modeling choices, and Section 8 shows that all conventional frictional models with a non-Walrasian limit are qualitatively different from ours.

2 The Walrasian Benchmark Model

On the supply side of a housing market are two types of landlords who differ in the quality of their apartment. On the demand side is a continuum of tenant types who differ in their valuations for apartment types. A single apartment is demanded by each tenant and supplied by each landlord. The utility of a landlord or tenant who does not trade is normalized to zero. Type i tenants are willing to pay a price of $\beta_{ij} > 0$ for a type j apartment where $i \in [0, \nu^T]$ and $j \in \{1, 2\}$. The supply prices of all landlords are zero and the demand prices of all tenants are monotonic and complementary in types, i.e., $\beta_{i2} > \beta_{i1}$, $\partial\beta_{i2}/\partial i > 0$, $\partial\beta_{i1}/\partial i > 0$ and $\partial\beta_{i2}/\partial i > \partial\beta_{i1}/\partial i$ when $i \in [0, \nu^T]$.[1]

An infinite number of periods follow after an initial period. In each period, apartments become available and prospective tenants become interested in trading. The measure of all type j apartments which become available in each period is $\nu_j^L > 0$ for $j = 1$ and $j = 2$, the measure of all prospective tenants who arrive at the market in each period is $\nu^T > 0$, and the tenant type distribution is uniform on the interval $[0, \nu^T]$, i.e., its density is equal to 1 for all $i \in [0, \nu^T]$.

A single trader is of measure zero, and each trader maximizes his utility for a given set of apartment prices. A landlord has to decide on whether and when to hire out. A tenant has to decide on whether, when and what type of

[1]For simplicity, we assume that β_{i2} and β_{i1} are continuous and differentiable with respect to i.

apartment to rent. The discount factor of both landlord types is $\delta > 0$, and the discount factor of a type i tenant is $\delta_i = \delta < \delta_P < 1$ for all $i \in [\nu_P^T, \nu^T]$ with $0 < \nu_P^T < \nu^T$, and $\delta_i = \delta_P$ for all $i \in [0, \nu_P^T)$.[2]

In the following, we limit our attention to cases where the initial stocks of all market participants are zero, and where the sequences of the market clearing prices remain constant over time. Such steady states have a simple structure because the markets for both apartment types must clear on a period by period basis. Trades between different periods do not take place because the trading opportunities remain the same as time passes by. For a complete description of the trader flows in a steady state, we denote the density of the type distribution of all tenants who receive a type j apartment in each period by ν_{ij} for $j = 1$, $j = 2$ and all $i \in [0, \nu^T]$. With a strictly positive correlation between partner types, the tenants with the highest demand strength must obtain the best apartments. Hence, we can describe a steady state of this nature by

$$\nu_{i2}^{PA} = \begin{cases} 1 & \text{if } i \in [i^{**}, \nu^T], \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{i1}^{PA} = \begin{cases} 1 & \text{if } i \in [i^{*}, i^{**}), \\ 0 & \text{otherwise,} \end{cases}$$

where $i^{**} = \max(\nu^T - \nu_2^L, 0)$ and $i^* = \max(\nu^T - \nu^L, 0)$.

As Becker (1973), we find that supermodular productive interactions lead to positively assortative matching in a market clearing equilibrium. A Walrasian apartment assignment must maximize total surplus, and with type complementarity, surplus improvements are always possible when significantly many tenants receive better apartments than some other tenants with a higher demand strength. We omit the proof of the following proposition because it just sharpens this simple argument.

Proposition 1 *The unique competitive equilibrium which does exist is one with* $\nu = \nu^{PA}$.[3]

[2] Unless explicitly mentioned, we assume throughout the paper that $\delta_P \neq \delta$, i.e., that discount factors are heterogeneous.

[3] By $\nu = \nu^{PA}$ we mean that $\nu_{ij} = \nu_{ij}^{PA}$ for $j = 1$, $j = 2$ and almost all $i \in [0, \nu^T]$. For notational simplicity, we use analogous abbreviations repeatedly throughout the paper.

3 The Search Model

We use the same framework as in the previous section except for the assumption that a Walrasian auctioneer calls market clearing prices. Instead, we assume that the formation of prices takes place through decentralized bargaining in a market with pairwise meetings. Each trader can only enter the market at the time when she arrives, and each tenant who enters the market must decide on a desired level of quality. All traders who dwell in the market meet at most one partner per period. Landlords with an apartment of type $j \in \{1, 2\}$ meet a tenant of a type not greater than $\bar{i} \in [0, \nu^T]$ with probability $\int_0^{\bar{i}} \pi_{ij}^L di$. Tenants who choose quality level $j \in \{1, 2\}$ meet only landlords with an apartment of type j, and they meet such a landlord with probability $\bar{\pi}_j^T$. The probability densities π_{ij}^L and the matching probabilities $\bar{\pi}_j^T$ are determined by the stocks of traders and the tenant's quality decision:

$$\pi_{ij}^L = \begin{cases} \dfrac{\mu_{ij}^T}{\mu_j^T} & \text{if} \quad \mu_j^L \leq \mu_j^T \cap \mu_j^T > 0, \\ \dfrac{\mu_{ij}^T}{\mu_j^L} & \text{if} \quad \mu_j^L > \mu_j^T > 0, \\ 0 & \text{if} \quad \mu_j^T = 0, \end{cases}$$

$$\bar{\pi}_j^T = \begin{cases} 1 & \text{if} \quad \mu_j^T \leq \mu_j^L \cap \mu_j^L > 0, \\ \dfrac{\mu_j^L}{\mu_j^T} & \text{if} \quad \mu_j^T > \mu_j^L > 0, \\ 0 & \text{if} \quad \mu_j^L = 0, \end{cases}$$

where μ_{ij}^T is the density of the type distribution of all tenants who chose quality level j, $\mu_j^T = \int_0^{\nu^T} \mu_{ij}^T di$ is the measure of all tenants who chose quality level j, and μ_j^L is the measure of all available apartments of type j.

A single take-it-or-leave-it offer must be made when a meeting takes place, and a fair chance move determines who can make this offer. All traders who reach an agreement leave the market, and all traders who reject an offer or whose offer gets rejected terminate the meeting and remain unmatched until they face a new chance in the next period. The personal history of each trader is private information, but the type of partner in a match is observable. The discount factor of the patient trader types δ_P, the discount factor of the impatient trader types δ, and a small entry fee $e > 0$ are the only frictions sources.[4] All traders must choose a strategy which specifies the behavior in all possible decision nodes. The common risk aversion is zero.

4 Search Equilibrium

We analyze symmetric subgame perfect Nash equilibria in which all traders of the same type choose the same strategy, and we focus on steady state cases in which the stocks of all landlord types and the stocks of all tenant types who chose a given quality level remain constant over time. We also assume that the behavior in a match is independent of what happened in previous matches. This stationarity assumption is not restrictive because the probability of getting matched twice with the same partner is zero.

For a given set of stationary strategies, we can compute an unmatched market participant's expected utility at the beginning of a period discounted to that period: v_j^L is the expected utility of an unmatched landlord with an apartment of type $j \in \{1, 2\}$, v_{ij}^T is the expected utility of an unmatched type $i \in [0, \nu^T]$ tenant who chose quality level j, and $v_i^T = \max(v_{i1}^T, v_{i2}^T)$ is the expected utility of a type i tenant who enters the market. These value functions are time independent because all matching probabilities remain the same as time progresses. The expected utility of a landlord who enters the market with a type j apartment is v_j^L.

The unique subgame perfect equilibrium in a bargaining situation between a type i tenant and a landlord with an apartment of type j is an outcome in which the trader who makes the offer receives all the gain from trade. A disagreement yields a total utility of $v_j^L\delta + v_{ij}^T\delta_i$ because both traders must wait one period before the next chance arrives. An agreement, on the other hand, yields a total utility of β_{ij}, and, consequently, the gain from trade is $\beta_{ij} - v_j^L\delta - v_{ij}^T\delta_i$. No exchange of goods can take place if $\beta_{ij} < v_j^L\delta + v_{ij}^T\delta_i$. For the knife edge case when $\beta_{ij} = v_j^L\delta + v_{ij}^T\delta_i$, we assume that trade does always take place. The expected utility of an unmatched type i tenant who chose quality level j is

$$(1)\quad v_{ij}^T = (1 - \bar{\pi}_j^T)v_{ij}^T\delta_i + \frac{\bar{\pi}_j^T}{2}v_{ij}^T\delta_i + \frac{\bar{\pi}_j^T}{2}(v_{ij}^T\delta_i + \max(0, \beta_{ij} - v_j^L\delta - v_{ij}^T\delta_i)).$$

[4] In order to establish the existence of an equilibrium for cases when $\nu^T < \nu^L$, we need an entry fee or a similar friction source which can be arbitrarily small.

She does not meet a partner with probability $(1-\bar{\pi}_j^T)$, she receives an offer with probability $\bar{\pi}_j^T/2$, and she can make an offer with probability $\bar{\pi}_j^T/2$. Her expected utility is $v_{ij}^T\delta_i$ if she does not meet a partner, $v_{ij}^T\delta_i$ if she receives an offer, and $v_{ij}^T\delta_i + \max(0, \beta_{ij} - v_j^L\delta - v_{ij}^T\delta_i) = \max(v_{ij}^T\delta_i, \beta_{ij} - v_j^L\delta)$ if she can make an offer. Similarly, we obtain that

$$(2) \qquad \begin{aligned} v_j^L &= (1-\int \pi_{ij}^L di)v_j^L\delta + (\int \frac{\pi_{ij}^L}{2}di)v_j^L\delta \\ &\quad + \int \frac{\pi_{ij}^L}{2}(v_j^L\delta + \max(0, \beta_{ij} - v_j^L\delta - v_{ij}^T\delta_i))di.^5 \end{aligned}$$

The traders who enter the market in a steady state must always replace identical traders who leave the market. For a complete description of these trader flows, we use again the per-period type distribution of all tenants who rent type j apartments. As in the frictionless benchmark model, we denote by ν_{ij} the per-period density of all type i tenants who rent an apartment of type j. A steady state requires that the per-period density of all type i tenants who choose quality level j when they enter the market must also be ν_{ij}, and that the per-period measure of all landlords who enter the market with a type j apartment must be $\int \nu_{ij}di$. The density resp. measure of traders who arrive at the market in a given period is an upper bound for the density resp. measure of traders who can enter the market in this period. Hence,

$$(3) \qquad \nu_{i1} + \nu_{i2} \leq 1$$

and

$$(4) \qquad \int \nu_{ij}di \leq \nu_j^L.$$

The per-period density of all type i tenants who meet a landlord with an apartment of type j is $\bar{\pi}_j^T\mu_{ij}^T$, and

$$(5) \qquad \bar{\pi}_j^T\mu_{ij}^T = \nu_{ij} \text{ if } \nu_{ij} > 0$$

because all these meetings must end up with trade if some of them do. Market participants who do not trade cannot be ignored in our model because they

[5] At an abuse of notation and throughout the paper, we use $\int$ as a shorthand for $\int_0^{\nu^T}$.

influence the matching probabilities. In the following, however, we will focus on equilibria without such superfluous market participants. We assume that,

$$\mu_{ij}^T = 0 \text{ if } \nu_{ij} = 0, \tag{6}$$

and that

$$\mu_j^L = 0 \text{ if } \int \nu_{ij} di = 0. \tag{7}$$

All stock and flow variables which satisfy (3),..,(6) and (7) represent a steady state, and a given steady state is possible in a subgame perfect equilibrium iff there are value functions which support the necessary entry, quality, and exit decisions, and at the same time satisfy the Bellman equations (1) and (2). The remaining equilibrium conditions must, therefore, establish that the value functions do support the decisions which induce the right trader flows. A trader must be willing to enter the market if some members of his type enter the market, and he must be willing to stay out of the market if some members of his type stay out of the market. Hence,

$$v_j^L \geq e \text{ if } \int \nu_{ij} di > 0, \tag{8}$$

$$v_i^T \geq e \text{ if } \nu_{i1} + \nu_{i2} > 0, \tag{9}$$

$$v_j^L \leq e \text{ if } \int \nu_{ij} di < \nu_j^L, \tag{10}$$

and

$$v_i^T \leq e \text{ if } \nu_{i1} + \nu_{i2} < 1. \tag{11}$$ [6]

Type i tenants must be willing to choose quality level j and must be able to trade with landlords who possess an apartment of type j if $\nu_{ij} > 0$. Hence,

$$v_{ij}^T = \max(v_{i1}^T, v_{i2}^T) = v_i^T \text{ if } \nu_{ij} > 0 \tag{12}$$

and

$$\beta_{ij} \geq v_j^L \delta + v_{ij}^T \delta_i \text{ if } \nu_{ij} > 0. \tag{13}$$

[6] For simplicity, we ignore incentive compatibility violations which could occur for some traders with an overall measure of zero. Strictly speaking, (8) should require that $v_j^L \geq e$ if $\nu_{\hat{i},j} > 0$ for some $\hat{i} \in [0, \nu^T]$, and (10) should require that $v_j^L \leq e$ if $\nu_{\tilde{i},j} < 1$ for some $\tilde{i}$ $\in [0, \nu^T]$.

Notice that our simplifying assumption of a once-in-a-lifetime quality choice does not matter in a steady state because optimal quality decisions remain optimal over time.

Equations (1) to (13) are necessary and sufficient conditions for a search equilibrium if they hold for all $i \in [0, \nu^T]$ and $j \in \{1, 2\}$. The exogenous variables are β_{ij}, ν_j^L, ν^T, ν_P^T, δ, δ_P and e. The endogenous variables are ν_{ij}, μ_{ij}^T, μ_j^L, v_j^L, v_i^T and v_{ij}^T.

5 Equilibrium Characterization for Infinitesimal Search Frictions

The complete equilibrium characterization for arbitrarily small search frictions depends on the structure of the exogenously given trader flows. In the following, we omit a full discussion of all possible cases because similar equilibrium configurations appear everywhere. In order to avoid redundancies, we only provide a detailed treatment for the parameter region where $\nu_I^T = \nu^T - \nu_P^T < \nu_1^L$ and $\nu_I^T < \nu_2^L < \nu_P^T$, but for the sake of completeness, we describe the remaining regions without proofs.

The special assumptions of our model permit implausible "shut-down" equilibria in which one or both apartment types are not traded so that $\int \nu_{i1} di = 0$ or $\int \nu_{i2} di = 0$. In the following, we ignore such equilibria because they are not robust with respect to infinitesimal stochastic perturbations of the quality or entry decisions. Their existence is driven by a coordination failure in which high resp. low quality landlords do not enter the market because there are no tenants who choose a high resp. low quality level, and in which the tenants do not choose a high resp. low quality level because there are no high resp. low quality landlords who enter the market.

Even for infinitesimal search frictions, we find substantial deviations from the Walrasian outcome in our model, and the critical feature which makes this possible is a high or low quality market in which the number of tenants per apartment goes to infinity resp. the matching probabilities of all tenants go to zero as the frictions become asymptotically removed. Lemma 1(i) reveals that such an ever growing tightness in the high resp. low quality market can only arise if this market becomes exclusively populated with patient tenant types, i.e. , if $\int_I \nu_{i2} di$ resp. $\int_I \nu_{i1} di$ approaches 0.[7] Lemma 1(ii) establishes that the tightness in a market without impatient tenants must always grow at a lower rate than the patience of the patient tenant types.

Lemma 1 *Let $\{(\delta^m, \delta_P^m, e^m)\}_{m=1}^{m=\infty}$ be a sequence of frictional parameter which converges to* $(1,\ 1, 0)$ *and assume that an equilibrium does exist for each element of this sequence.*

(i) If $\xi < \int_I \nu_{ij}^m di$ holds for a $j \in \{1,2\}$, a $\xi > 0$ and all $m \in$ N, then $\{\bar{\pi}_j^{T\,m}\}_{m=0}^{m=\infty}$ must be bounded below by an $\eta > 0$.

(ii) If $0 = \int_I \nu_{ij}^m di \leq \xi < \int_P \nu_{ij}^m di$ holds for a $j \in \{1,2\}$, a $\xi > 0$ and all $m \in$ N, then $\lim_{m\to\infty} (1-\delta_P^m)/\ \bar{\pi}_j^{T\,m} = 0$.

In our proof of this preparatory lemma, we now argue that the equilibrium conditions (1), (2) and (13) could never hold at once for sufficiently small search frictions if Lemma 1 were false. For a given matching matrix ν_{ij} and by using (5) and (6), we, thereby, need to figure out all possible steady state stock configurations, i.e., the so called Berveridge curve.

Exploiting (1), we can show that

$$\frac{1}{2}\max(0, \beta_{ij} - v_j^L \delta - v_{ij}^T \delta_i) = \frac{v_{ij}^T (1-\delta_i)}{\bar{\pi}_j^T}.$$

Using this expression and (2), note that

$$v_j^L = \int \frac{\pi_{ij}^L}{\bar{\pi}_j^T} \frac{(1-\delta_i)}{(1-\delta)} v_{ij}^T di. \tag{14}$$

When we manipulate (1) and (13), we can verify that

$$v_{ij}^T = \frac{\bar{\pi}_j^T \frac{1}{2}(\beta_{ij} - v_j^L \delta)}{1-\delta_i + \bar{\pi}_j^T \frac{1}{2}\delta_i} \text{ if } \nu_{ij} > 0. \tag{15}$$

Our equilibrium condition (6) demands that $\mu_{ij}^T = 0$ if $\nu_{ij} = 0$ and, therefore, that $\pi_{ij}^L = 0$ if $\nu_{ij} = 0$. With (14), we, thus, find that

$$v_j^L = \int \frac{\pi_{ij}^L (1-\delta_i)\frac{1}{2}(\beta_{ij} - v_j^L \delta)}{(1-\delta)(1-\delta_i + \bar{\pi}_j^T \frac{1}{2}\delta_i)} di. \tag{16}$$

[7] Throughout the paper, $\int_I$ resp. $\int_P$ is short for $\int_{\nu_P^T}^{\nu^T}$ resp. $\int_0^{\nu_P^T}$.

Before we can continue, we have to figure out all possible steady-state stock configurations for the case when $\int \nu_{ij}di > 0$. From (5), we know that ν_{ij}=$\bar{\pi}_j^T \mu_{ij}^T$ if $\nu_{ij} > 0$. From (6), we know that $\mu_{ij}^T = 0$ if $\nu_{ij} = 0$. Hence, we observe that

$$\int \nu_{ij} di = \int \bar{\pi}_j^T \mu_{ij}^T di = \bar{\pi}_j^T \mu_j^T > 0$$

which implies that $\mu_j^L \neq 0$ and $\mu_j^T \neq 0$. With $0 < \mu_j^L < \mu_j^T$, we obtain that

$$\bar{\pi}_j^T \mu_j^T = (\mu_j^L/\mu_j^T)\mu_j^T = \mu_j^L = \int \nu_{ij} di < \mu_j^T,$$

$$\bar{\pi}_j^T = \int \nu_{ij} di / \mu_j^T \text{ and}$$

$$\pi_{ij}^L = \mu_{ij}^T/\mu_j^T = \nu_{ij}/(\bar{\pi}_j^T \mu_j^T) = \nu_{ij}/\int \nu_{ij} di.$$

With $0 < \mu_j^T \leq \mu_j^L$, on the other hand, we obtain that

$$\bar{\pi}_j^T \mu_j^T = \mu_j^T = \int \nu_{ij} di \leq \mu_j^L,$$

$$\bar{\pi}_j^T = 1 \text{ and}$$

$$\pi_{ij}^L = \mu_{ij}^T/\mu_j^L = \nu_{ij}/(\bar{\pi}_j^T \mu_j^L) = \nu_{ij}/\mu_j^L.$$

The discussion in the previous paragraph adds up to the conclusion that for $\int \nu_{ij} di > 0$, either

(a) $\mu_j^L = \int \nu_{ij} di < \mu_j^T, \quad \bar{\pi}_j^T = \int \nu_{ij} di/\mu_j^T \quad \text{and} \quad \pi_{ij}^L = \nu_{ij}/\int \nu_{ij} di$

or

(b) $\mu_j^T = \int \nu_{ij} di \leq \mu_j^L, \quad \bar{\pi}_j^T = 1 \quad \text{and} \quad \pi_{ij}^L = \nu_{ij}/\mu_j^L.$

Under (a), (16) requires that

$$v_j^L \int \nu_{ij} \geq \int_I \frac{\nu_{ij}\frac{1}{2}(\beta_{ij} - v_j^L \delta)}{(1 - \delta + \bar{\pi}_j^T \frac{1}{2}\delta)} di. \tag{17}$$

[8]

If $\{(\delta^m, \delta_P^m, e^m)\}_{m=1}^{m=\infty}$ is a parameter sequence which converges to $(1, 1, 0)$ and if $0 < \xi < \int_I \nu_{ij}^m di$, then $0 < \xi/(2\nu_I^T) < \nu_{ij}^m$ must hold for some $i \in [\nu_P^T, \nu^T]$

[8] Using (1) and (13), notice that $v_{ij}^T \geq 0$ for all i and that $\beta_{ij} - v_j^L \delta \geq 0$ if $\nu_{ij} > 0$.

with measure greater than some $\xi' > 0$.[9] Since $0 \leq \beta_{ij} - {v_j^L}^m \delta^m$ must hold for the smallest such i, there must also be some $i \in [\nu_P^T, \nu^T]$ with measure greater than $\frac{2}{3}\xi' > 0$ such that $0 < \xi/(2\nu_I^T) < \nu_{ij}^m$ and

$$0 < \frac{\xi'}{3} \min_{i'}\{\frac{\partial \beta_{ij}}{\partial i}(i')\} < \beta_{ij} - {v_j^L}^m \delta^m .^{10}$$

Hence, under (a), (17) would certainly be violated for some large enough m, if $\{{\bar{\pi}_j^T}{}^m\}_{m=0}^{m=\infty}$ were *not* bounded below by an $\eta > 0$.[11] Under (b), on the other hand, ${\bar{\pi}_j^T}{}^m = 1$ would certainly be violated for some large enough m, if $\{{\bar{\pi}_j^T}{}^m\}_{m=0}^{m=\infty}$ were *not* bounded below by an $\eta > 0$. To summarize, Lemma 1(i) must be true.

Under (a), (16) requires that

$$v_j^L \textstyle\int \nu_{ij} \geq \displaystyle\int_P \frac{\nu_{ij}(1-\delta_P)\frac{1}{2}(\beta_{ij} - v_j^L \delta)}{(1-\delta)(1-\delta_P + \bar{\pi}_j^T \frac{1}{2}\delta_P)} di. \tag{18}$$

If $\{(\delta^m, \delta_P^m, e^m)\}_{m=1}^{m=\infty}$ is a parameter sequence which converges to $(1,1,0)$ and if $0 < \xi < \int_P \nu_{ij}^m di$, then $0 < \xi/(2\nu_P^T) < \nu_{ij}^m$ must hold for some $i \in [0, \nu_P^T)$ with measure greater than some $\xi' > 0$. Since $0 \leq \beta_{ij} - {v_j^L}^m \delta^m$ must hold for the smallest such i, there must also be some $i \in [0, \nu_P^T)$ with measure greater than $\frac{2}{3}\xi' > 0$ such that $0 < \xi/(2\nu_P^T) < \nu_{ij}^m$ and

$$0 < \frac{\xi'}{3} \min_{i'}\{\frac{\partial \beta_{ij}}{\partial i}(i')\} < \beta_{ij} - {v_j^L}^m \delta^m .$$

Hence, under (a), (18) would certainly be violated for some large enough m, if $\{(1-\delta_P^m)/\ {\bar{\pi}_j^T}{}^m\}_{m=0}^{m=\infty}$ were bounded below by an $\eta > 0$. Under (b), on the other hand, ${\bar{\pi}_j^T}{}^m = 1$ would certainly be violated for some large enough m, if $\{(1-\delta_P^m)/\ {\bar{\pi}_j^T}{}^m\}_{m=0}^{m=\infty}$ were bounded below by an $\eta > 0$. To summarize, Lemma 1(ii) must be true. □

[9] Among all impatient types, some i with $\xi/(2\nu_I^T) < \nu_{ij}^m \leq 1$ and a sufficiently small measure $\xi' > 0$, will certainly not be enough to raise the average density above ξ/ν_I^T given that the density of all other i is less than $\xi/(2\nu_I^T)$.

[10] The lowest $\xi'/3$ of all $i \in [\nu_P^T, \nu^T]$ with $0 < \xi/(2\nu_I^T) < \nu_{ij}^m$ cannot be in an interval smaller than $\xi'/3$ since according to (3), $\nu_{ij}^m \leq 1$ for all i.

[11] Exploiting (4), (13), ${v_{ij}^T}^m \geq 0$ and the fact that $\nu_{ij}^m > 0$ for some i if $0 < \xi < \int_I \nu_{ij}^m di$, observe that ${v_j^L}^m \int \nu_{ij}^m di$ must remain finite as m goes to ∞.

Our preliminary Lemma 1 is very helpful for the purpose of singling out the possible equilibrium configuration for arbitrarily small search frictions. In the parameter region where $\nu_I^T < \nu_2^L < \nu_P^T$ and $\nu_I^T < \nu_1^L$, it turns out that only equilibria with positively assortative matching, equilibria with crowding out in the high quality market and equilibria with crowding out in both markets do exist when the search frictions become asymptotically removed. In an equilibrium with crowding out in the high quality market, the patient tenant types with the highest demand strength completely take over the high quality market, and the tenant types with the highest demand strength among the remaining tenants share the low quality market. For $\nu_I^T < \nu_1^L$, we can describe the matching pattern in an equilibrium of this nature by

$$\nu_{i2}^{CH} = \begin{cases} 1 & \text{if } i \in [i^{\times\times}, \nu_P^T), \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{i1}^{CH} = \begin{cases} 1 & \text{if } i \in [i^{\times}, i^{\times\times}) \cup [\nu_P^T, \nu^T], \\ 0 & \text{otherwise,} \end{cases}$$

where $i^{\times\times} = \max(0, \nu_P^T - \nu_2^L)$ and $i^{\times} = \max(0, \nu^T - \nu^L)$. An equilibrium with crowding out in both markets is characterized by patient tenants who completely take over both markets in such a way that the ones with the highest demand strength receive the best apartments. No such equilibrium is possible when $\nu_P^T < \nu_2^L$, but for $\nu_2^L < \nu_P^T$, we can describe such an equilibrium by

$$\nu_{i2}^{CB} = \begin{cases} 1 & \text{if } i \in [i^{++}, \nu_P^T), \\ 0 & \text{otherwise,} \end{cases}$$

$$\nu_{i1}^{CB} = \begin{cases} 1 & \text{if } i \in [i^{+}, i^{++}), \\ 0 & \text{otherwise,} \end{cases}$$

where $i^{++} = \max(0, \nu_P^T - \nu_2^L)$ and $i^{+} = \max(0, \nu_P^T - \nu^L)$.

To convince ourselves that no other than the above mentioned equilibrium configurations do exist for arbitrarily small frictions, we need to examine all matching matrices which obey (3) and (4). For this purpose, we start out with those in which only the high quality market is exclusively populated with patient tenant types.

Lemma 2 *Equilibria with* $\nu \neq \nu^{CH}$, $0 = \int_I \nu_{i2} di \leq \xi < \int_P \nu_{i2} di$ *and* $\xi < \int_I \nu_{i1} di$ *do not exist in an* ϵ *neighborhood of* $(\delta, \delta_P, e) = (1, 1, 0)$ *if* $\nu_I^T < \nu_1^L$ *and* $\xi > 0$.

In our proof of Lemma 2, we first argue that under sufficiently small search frictions, our incentive constraints (8) to (12) can never be compatible with a matching pattern in which, on the one hand, only the high quality market is exclusively populated with patient tenant types, and which, on the other hand, deviates from ν_{ij}^{CH} in the high quality market. In Step1 and Step 2, we, thereby, need to distinguish the case when all high quality landlords enter the market from the case when some of them stay out of the market. In Step 3 and Step 4, we repeat this exercise for a matching pattern which coincides with ν_{ij}^{CH} in the high quality market, but deviates from ν_{ij}^{CH} in the low quality market. Throughout the proof, we frequently use the fact that

$$(19) \qquad \upsilon_{ij}^{T} = \frac{\bar{\pi}_j^T \frac{1}{2} \max(0, \beta_{ij} - \upsilon_j^L \delta)}{1 - \delta_i + \bar{\pi}_j^T \frac{1}{2} \delta_i}$$

which we obtain when we somewhat manipulate the equilibrium condition (1).

Step 1. If $0 < \xi < \int_I \nu_{i1} di$, $0 = \int_I \nu_{i2} di < \xi < \int_P \nu_{i2} di = \nu_2^L$ and $\nu_{i2} = 1$ for almost all $i \in [i^{\times\times}, \nu_P^T)$, then $\nu_P^T = \int_P di \geq \int_P \nu_{i2} di = \nu_2^L$ must be the case since (3) demands for all $i \in [0, \nu_P^T)$ that $1 \geq \nu_{i2}$. Hence, note that $i^{\times\times} = \nu_P^T - \nu_2^L$ must also be the case. Since $\int_{i^{\times\times}}^{\nu_P^T} \nu_{i2} di = \nu_P^T - \nu_P^T + \nu_2^L = \nu_2^L$ and since (4) demands that $\int \nu_{i2} di \leq \nu_2^L$, we, consequently, observe that $\nu_{i2} = 0$ must hold for almost all $i \in [0, i^{\times\times}) \cup [\nu_P^T, \nu^T]$, i.e., that $\nu_{i2} = \nu_{i2}^{CH}$ must hold for almost all $i \in [0, \nu^T]$ under these circumstances.

The argument above reveals that there must be some i'' with positive measure such that $\nu_{i''2} < 1$ and $i'' \in [i^{\times\times}, \nu_P^T)$ if we assume that $0 < \xi < \int_I \nu_{i1} di$, that $0 = \int_I \nu_{i2} di < \xi < \int_P \nu_{i2} di = \nu_2^L$ and that $\nu_{i2} \neq \nu_{i2}^{CH}$ for some nonnegligible i.[12] At the same time, there must be some i' with positive measure such that $\nu_{i'2} > 0$ and $i' < i^{\times\times}$ because $\int_{i^{\times\times}}^{\nu_P^T} \nu_{i2} di < \nu_2^L$ and since $\int_P \nu_{i2} di = \nu_2^L$.[13] Since $\nu_{i'2} > 0$, (15), (12), and (19) require that

$$\upsilon_{i'2}^{T} = \frac{\bar{\pi}_2^T \frac{1}{2} (\beta_{i'2} - \upsilon_2^L \delta)}{1 - \delta_P + \bar{\pi}_2^T \frac{1}{2} \delta_P} \geq \upsilon_{i'1}^{T} = \frac{\bar{\pi}_1^T \frac{1}{2} \max(0, \beta_{i'1} - \upsilon_1^L \delta)}{1 - \delta_P + \bar{\pi}_1^T \frac{1}{2} \delta_P}$$

[12] Using(3), note that $\nu_{i''2}$ cannot be greater than 1.

[13] Since (3) requires for all $i \in [i^{\times\times}, \nu_P^T)$ that $\nu_{i2} \leq 1$ and since by assumption, $\nu_{i2} < 1$ for some nonnegligible $i \in [i^{\times\times}, \nu_P^T)$, $\int_{i^{\times\times}}^{\nu_P^T} \nu_{i2} di < \int_{i^{\times\times}}^{\nu_P^T} di = \nu_2^L$ must hold. Note that $i^{\times\times} = \nu_P^T - \nu_2^L$ is again valid since $\nu_P^T = \int_P di \geq \int_P \nu_{i2} di = \nu_2^L$ must be the case.

and since $\beta_{i''2}-\beta_{i'2} > \beta_{i''1}-\beta_{i'1} > 0$ by type complementarity and monotonicity, Lemma 1 implies that $\beta_{i'2} - v_2^L\delta \geq \max(0, \beta_{i'1} - v_1^L\delta)$ and that $\beta_{i''2} - v_2^L\delta > \max(0, \beta_{i''1} - v_1^L\delta)$ must hold for (δ, δ_P, e) sufficiently close to (1,1,0). For $i = i''$, however, this leads us into a contradiction with (12) and (11) since by using (19), Lemma 1 and again (19) resp. by using (19), $\beta_{i''2} > \beta_{i'2}$, $\nu_{i'2} > 0$, (15), (12) and (9), we can conclude that

$$v_{i''2}^T \;\geq\; \frac{\bar{\pi}_2^T \frac{1}{2}(\beta_{i''2} - v_2^L\delta)}{1-\delta_P+\bar{\pi}_2^T\frac{1}{2}\delta_P} \;>\; \frac{\bar{\pi}_1^T \frac{1}{2}\max(0, \beta_{i''1} - v_1^L\delta)}{1-\delta_P+\bar{\pi}_1^T\frac{1}{2}\delta_P} \;=\; v_{i''1}^T$$

resp. that

$$v_{i''2}^T \;\geq\; \frac{\bar{\pi}_2^T \frac{1}{2}(\beta_{i''2} - v_2^L\delta)}{1-\delta_P+\bar{\pi}_2^T\frac{1}{2}\delta_P} \;>\; \frac{\bar{\pi}_2^T \frac{1}{2}(\beta_{i'2} - v_2^L\delta)}{1-\delta_P+\bar{\pi}_2^T\frac{1}{2}\delta_P} \;=\; v_{i'2}^T \;=\; v_{i'}^T \;\geq e$$

for (δ, δ_P, e) sufficiently close to (1,1,0).[14] To summarize, $\nu_{i2} = \nu_{i2}^{CH}$ must hold for almost all i if $0{=}\int_I \nu_{i2}di{<}\xi{<}\int_P \nu_{i2}di = \nu_2^L$, if $0 < \xi{<}\int_I \nu_{i1}di$ and if (δ, δ_P, e) sufficiently close to (1,1,0).

Step 2. If $0 < \xi{<}\int_I \nu_{i1}di$, $0{=}\int_I \nu_{i2}di{<}\xi{<}\int_P \nu_{i2}di < \nu_2^L$ and $\nu_{i2} = 1$ for almost all $i \in [i^{\times\times}, \nu_P^T)$, then $\nu_P^T \geq \nu_2^L$ cannot be the case since it would imply $i^{\times\times}{=}\nu_P^T - \nu_2^L$ and $\int_{i^{\times\times}}^{\nu_P^T} \nu_{i2}di{=}\nu_2^L{\leq} \int_P \nu_{i2}di$. Hence, notice that $\nu_P^T < \nu_2^L$ and $i^{\times\times}{=}0$ must be the case. Since $\nu_{i2} = 1$ for almost all $i \in [i^{\times\times}, \nu_P^T)$ and since $\int_I \nu_{i2}di = 0$ implies that $\nu_{i2} = 0$ for almost all $i \in [\nu_P^T, \nu^T]$, we, consequently observe that $\nu_{i2} = \nu_{i2}^{CH}$ for almost all $i{\in}[0, \nu^T]$ under these circumstances.

The comment above discloses that there must be some i'' with positive measure such that $\nu_{i''2} < 1$ and $i'' \in [i^{\times\times}, \nu_P^T)$ if we assume that $0 < \xi{<}\int_I \nu_{i1}di$, that $0{=}\int_I \nu_{i2}di{<}\xi{<}\int_P \nu_{i2}di < \nu_2^L$ and that $\nu_{i2} \neq \nu_{i2}^{CH}$ for some nonnegligible i. Further, we can deduce from (8) and (10) that $v_2^L = e$ and that $v_1^L \geq e$. For $i = i''$, however, this leads us immediately into a contradiction with (12) and (11) since by using (19), $\beta_{i''2} > \beta_{i''1}$, Lemma 1(ii), $\beta_{i''1} > 0$, $1-\delta_P+\bar{\pi}_1^T\frac{1}{2}\delta_P{>}\ \bar{\pi}_1^T\frac{1}{2}$ and again (19) resp. by using (19), $\beta_{i''2} > 0$, and Lemma 1(ii), we find that

$$v_{i''2}^T \;=\; \frac{\bar{\pi}_2^T \frac{1}{2}\max(0, \beta_{i''2} - e\delta)}{1-\delta_P+\bar{\pi}_2^T\frac{1}{2}\delta_P} \;>\; \beta_{i''1} - e\delta$$

[14] Since $\nu_{i''2} < 1$ and since (3) demands that $\nu_{i''1} + \nu_{i''2} \leq 1$, note that $\nu_{i''1} + \nu_{i''2} < 1$ or $\nu_{i''1} > 0$. The incentive constraints (12) and (11), thus, demand that $v_{i''1}^T \geq v_{i''2}^T$ or that $e \geq v_{i''}^T \geq v_{i''2}^T$.

$$\geq \max(0, \beta_{i''1} - v_1^L\delta) > \frac{\bar{\pi}_1^T \frac{1}{2} \max(0, \beta_{i''1} - v_1^L\delta)}{1 - \delta_P + \bar{\pi}_1^T \frac{1}{2}\delta_P} = v_{i''1}^T$$

resp. that

$$v_{i''2}^T = \frac{\bar{\pi}_2^T \frac{1}{2} \max(0, \beta_{i''2} - e\delta)}{1 - \delta_P + \bar{\pi}_2^T \frac{1}{2}\delta_P} > e$$

for (δ, δ_P, e) sufficiently close to (1,1,0).[15] To summarize, $\nu_{i2} = \nu_{i2}^{CH}$ must hold for almost all i if $0{=}\int_I \nu_{i2}di{<}\xi{<}\int_P \nu_{i2}di < \nu_2^L$, if $0 < \xi{<}\int_I \nu_{i1}di$ and if (δ, δ_P, e) sufficiently close to (1,1,0).

Step 3. If $\nu_{i2} = \nu_{i2}^{CH}$ for almost all i, $0{<}\xi{<} \int_I \nu_{i1}di{\leq} \int \nu_{i1}di = \nu_1^L$ and $\nu_{i1} = 1$ for almost all $i \in [i^\times, i^{\times\times}){\cup}[\nu_P^T, \nu^T]$, then

$$\begin{aligned} \max(0, \nu_P^T - \nu_2^L) + \nu_I^T &= \textstyle\int_0^{i^{\times\times}} di + \int_I di \\ &\geq \textstyle\int_0^{i^{\times\times}} \nu_{i1}di + \int_I \nu_{i1}di = \int \nu_{i1}di = \nu_1^L \end{aligned}$$

must be the case since (3) demands for all $i \in [0, i^{\times\times}){\cup}[\nu_P^T, \nu^T]$ that $1 \geq \nu_{i1}$ and for almost all $i \in [i^{\times\times}, \nu_P^T)$ that $\nu_{i1} = 0$. With $\nu_1^L > \nu_I^T$ as a qualification in Lemma 2, hence, note that $\nu_P^T - \nu_2^L{\geq} \nu_1^L - \nu_I^T{>} 0$, $i^{\times\times} = \nu_P^T - \nu_2^L$, $i^\times = \nu^T - \nu^L$ and $i^{\times\times} - i^\times = \nu_1^L - \nu_I^T > 0$ must also be the case. Since $\int_{i^\times}^{i^{\times\times}} \nu_{i1}di{+}\int_I \nu_{i1}di{=}\nu_1^L - \nu_I^T + \nu_I^T{=}\nu_1^L$ and since (4) requires that $\int \nu_{i1}di{\leq} \nu_1^L$, we, consequently, observe that $\nu_{i1} = 0$ must hold for almost all $i \in [0, i^\times){\cup}[i^{\times\times}, \nu_P^T)$, i.e., that $\nu_{i1} = \nu_{i1}^{CH}$ must hold for almost all $i{\in}[0, \nu^T]$ under these circumstances.

The exposition in the previous paragraph amounts to the conclusion that there must be some i'' with positive measure such that $\nu_{i''1}{<}1$, $\nu_{i''2}{=}0$ and $i'' \in [i^\times, i^{\times\times}){\cup}[\nu_P^T, \nu^T]$ if we assume that $\nu_{i2} = \nu_{i2}^{CH}$ for almost all i, that $0{<}\xi{<} \int_I \nu_{i1}di{\leq} \int \nu_{i1}di = \nu_1^L$ and that $\nu_{i1} \neq \nu_{i1}^{CH}$ for some nonnegligible i. At the same time, there must be some i' with positive measure such that $\nu_{i'1} > 0$ and $i' < i^\times$ since $\int_{i^\times}^{i^{\times\times}} \nu_{i1}di{+}\int_I \nu_{i1}di{<} \nu_1^L$, since $\int \nu_{i1}di = \nu_1^L$ and since (3) requires for almost all $i \in [i^{\times\times}, \nu_P^T)$ that $\nu_{i1} = 0$.[16] For $i = i''$, however,

[15] Here, the same comment as in footnote 14 applies.

[16] Since (3) requires that $\nu_{i1} \leq 1$ for all $i \in [i^\times, i^{\times\times}){\cup}[\nu_P^T, \nu^T]$ and since by assumption, $\nu_{i1} < 1$ for some nonnegligible $i \in [i^\times, i^{\times\times}){\cup}[\nu_P^T, \nu^T]$, $\int_{i^\times}^{i^{\times\times}} \nu_{i1}di + \int_I \nu_{i1}di < \int_{i^\times}^{i^{\times\times}} di + \int_I di = \nu_1^L$ must hold. Note that $i^{\times\times} - i^\times{=}\nu_1^L - \nu_I^T{>}0$ is again valid since $\max(0,\nu_P^T{-}\nu_2^L){+}\nu_I^T{=} \int_0^{i^{\times\times}} di + \int_I di \geq \int_0^{i^{\times\times}} \nu_{i1}di + \int_I \nu_{i1}di = \nu_1^L$ must be the case.

this leads us into a contradiction with (11) since by using (19), $\beta_{i''1} > \beta_{i'1}$, Lemma 1(i), $\nu_{i'1} > 0$, (15), (12) and (9), we can conclude that

$$v^T_{i''1} \;\geq\; \frac{\bar{\pi}^T_1 \frac{1}{2}(\beta_{i''1} - v^L_1 \delta)}{1 - \delta_{i''} + \bar{\pi}^T_1 \frac{1}{2}\delta_{i''}} \;>\; \frac{\bar{\pi}^T_1 \frac{1}{2}(\beta_{i'1} - v^L_1 \delta)}{1 - \delta_{i'} + \bar{\pi}^T_1 \frac{1}{2}\delta_{i'}} \;\geq\; v^T_{i'1} \;=\; v^T_{i'} \;\geq e$$

for (δ, δ_P, e) sufficiently close to $(1, 1, 0)$.[17] To summarize, $\nu_{i1} = \nu^{CH}_{i1}$ must hold for almost all i if $0<\xi< \int_I \nu_{i1} di \leq \int \nu_{i1} di = \nu^L_1$, if $\nu_{i2} = \nu^{CH}_{i2}$ holds for almost all i and if (δ, δ_P, e) is sufficiently close to $(1, 1, 0)$.

Step 4. If $\nu_{i2} = \nu^{CH}_{i2}$ for almost all i, $0<\xi< \int_I \nu_{i1} di \leq \int \nu_{i1} di < \nu^L_1$ and $\nu_{i1} = 1$ for almost all $i \in [i^\times, i^{\times\times}) \cup [\nu^T_P, \nu^T]$, then $\nu^L \leq \nu^T$ cannot be the case since with $\nu^T_I < \nu^L_1$ as a qualification in Lemma 2, it would imply that $0< \nu^L_1 - \nu^T_I \leq \nu^T_P - \nu^L_2$, $i^{\times\times} = \nu^T_P - \nu^L_2$, $i^\times = \nu^T - \nu^L$, $i^{\times\times} - i^\times = \nu^L_1 - \nu^T_I > 0$ and that $\nu^L_1 = \int_{i^\times}^{i^{\times\times}} \nu_{i1} di + \int_I \nu_{i1} di \leq \int \nu_{i1} di$. Hence, note that $\nu^L > \nu^T$ and $i^\times = 0$ must be the case. Since $\nu_{i1} = 1$ for almost all $i \in [i^\times, i^{\times\times}) \cup [\nu^T_P, \nu^T]$ and since (3) demands that $\nu_{i1} = 0$ for almost all $i \in [i^{\times\times}, \nu^T_P)$, we, consequently observe that $\nu_{i1} = \nu^{CH}_{i1}$ must hold for almost all $i \in [0, \nu^T]$ under these circumstances.

The discussion in the previous paragraph shows that there must be some i'' with positive measure such that $\nu_{i''1} < 1$, $\nu_{i''2} = 0$ and $i'' \in [i^\times, i^{\times\times}) \cup [\nu^T_P, \nu^T]$ if we assume that $\nu_{i2} = \nu^{CH}_{i2}$ for almost all i, that $0<\xi< \int_I \nu_{i1} di \leq \int \nu_{i1} di < \nu^L_1$ and that $\nu_{i1} \neq \nu^{CH}_{i1}$ for some nonnegligible i. For $i = i''$, however, this leads us immediately into a contradiction with (11), since by using (19), (8), (10), $\beta_{i''1} > 0$ and Lemma 1(i), we find that

$$v^T_{i''1} \;=\; \frac{\bar{\pi}^T_1 \frac{1}{2} \max(0, \beta_{i''1} - e\delta)}{1 - \delta_{i''} + \bar{\pi}^T_1 \frac{1}{2}\delta_{i''}} \;>\; e$$

for (δ, δ_P, e) sufficiently close to (1,1,0).[18] To summarize, $\nu_{i1} = \nu^{CH}_{i1}$ must hold for almost all i if $0<\xi< \int_I \nu_{i1} di \leq \int \nu_{i1} di < \nu^L_1$, if $\nu_{i2} = \nu^{CH}_{i2}$ holds for almost all i and if (δ, δ_P, e) is sufficiently close to $(1, 1, 0)$. This finishes our proof of Lemma 2. □

To complete our quest for possible equilibrium matching configurations, we need to examine all matching matrices in which only the low quality market is exclusively populated with patient tenant types, all matrices in which both markets are exclusively populated with patient tenant types,

[17] Since $\nu_{i''2} = 0$ and $\nu_{i''1} < 1$, (11) demands that $e \geq v^T_{i''} \geq v^T_{i''1}$.

[18] Here, the same comment as in footnote 17 applies.

and all matching matrices in which some impatient tenant types appear in both markets.

Lemma 3 *Equilibria with* $\nu{\neq}\nu^{PA}$*,* $\xi{<}\int_I \nu_{i2}di$ *and* $0{=}\int_I \nu_{i1}di{\leq}\xi{<}\int_P \nu_{i1}di$ *do not exist in an* ϵ *neighborhood of* $(\delta, \delta_P, e){=}(1, 1, 0)$ *if* $\nu_I^T < \nu_2^L$ *and* $\xi > 0$*.*

Lemma 4 *Equilibria with* $\nu{\neq}\nu^{CB}$*,* $0{=}\int_I \nu_{i2}di{\leq}\xi{<}\int_P \nu_{i2}di$ *and* $0{=}\int_I \nu_{i1}di{\leq}\xi$ ${<}\int_P \nu_{i1}di$ *do not exist in an* ϵ *neighborhood of* $(\delta, \delta_P, e){=}(1, 1, 0)$ *if* $\xi > 0$*.*

Lemma 5 *Equilibria with* $\nu{\neq}\nu^{PA}$*,* $\xi{<}\int_I \nu_{i2}di$ *and* $\xi{<}\int_I \nu_{i1}di$ *do not exist in an* ϵ *neighborhood of* $(\delta, \delta_P, e){=}(1, 1, 0)$ *if* $\xi > 0$*.*

We omit the proofs of Lemma 3, Lemma 4 and Lemma 5 because they require exactly the same steps as the proof of Lemma 2. The only basic difference is that these proofs apply Lemma 1(i) and Lemma 1(ii) in different markets. The proof of Lemma 4 applies Lemma 1(ii) in both markets, the proof of Lemma 5 applies Lemma 1(i) in both markets, and the proof of Lemma 3 applies Lemma 1(i) in the high and Lemma 1(ii) in the low quality market.

6 Existence

In this section, we show that the various equilibrium configurations which we singled out in the previous section do indeed exist for arbitrarily small search frictions. We summarize most of the discussion in section 5 and 6 as follows:

Proposition 2 *For* $\nu_I^T{<}\nu_2^L{<}\nu_P^T$ *and* $\nu_I^T{<}\nu_1^L$*, the three possible equilibria which do exist in an* ϵ *neighborhood of* $(\delta,\delta_P,e){=}(1,1,0)$ *are as follows:*

(a) an equilibrium with crowding out in the high quality market;

(b) an equilibrium with crowding out in both markets;

(c) an equilibrium with positively assortative matching.[19]

[19]Lemma 2, 3, 4 and 5 do not rule out equilibria with $\nu{\neq}\nu^{CH}$, $\nu{\neq}\nu^{CB}$ and $\nu{\neq}\nu^{PA}$ in which the measure of all impatient resp. patient tenant types in a given market is greater than zero but approaches zero as the frictions become removed, i.e., in which $\xi > 0$ is violated. Such equilibria, probably, do indeed exist, but similar arguments as in the proof of Lemma 2 show that they must converge to an equilibrium with $\nu{=}\nu^{CH}$, an equilibrium with $\nu{=}\nu^{CB}$, an equilibrium with $\nu{=}\nu^{PA}$ or a "shut-down" equilibrium as the frictions become small.

We could establish similar results for all possible parameter regions: Except for equilibria with crowding out in the low quality market, no other equilibrium configurations than the one mentioned in Proposition 2 do exist for infinitesimal frictions in general. An equilibrium with positively assortative matching and an equilibrium with crowding out in the high quality market does always exist for infinitesimal frictions. An equilibrium with crowding out in the low quality market does exist in an ϵ neighborhood of $(\delta,\delta_P,e)=(1,1,0)$ iff $\nu_I^T>\nu_2^L$, and an equilibrium with crowding out in both markets does exist in such an ϵ neighborhood iff $\nu_2^L<\nu_P^T$. $\nu_I^T<\nu_1^L$ is an assumption which is irrelevant from the viewpoint of equilibrium existence. In Proposition 2, we just impose this assumption because it simplifies the description of the matching pattern in a search equilibrium with crowding out in the high quality market.

The patient tenant types with the highest demand strength completely take over the low quality market, and the tenant types with the highest demand strength among the remaining tenants share the high quality market in an equilibrium with crowding out in the low quality market. For $\nu_I^T<\nu_2^L$ and arbitrarily small search frictions, no such equilibrium turns out to be possible because the impatient tenant types must always leave some high quality apartments for the patient tenant types with the highest demand strength. For $\nu_P^T<\nu_2^L$ and arbitrarily small search frictions, no equilibrium with crowding out in both markets turns out to be possible simply because there are not even enough patient tenants for a complete take over of all high quality apartments.

6.1 Crowding Out in the High Quality Market

Lemma 6 *For $\nu_2^L<\nu_P^T$, $\nu_I^T<\nu_1^L$, and any given (δ,e) sufficiently close $(1,0)$, one and at most one equilibrium with $\nu=\nu^{CH}$ does exist if δ_P is sufficiently close to 1.*

In an equilibrium with $\nu=\nu^{CH}$, not impatient tenant types get the high quality apartments, but patient tenant types whose demand strength is strictly lower. Lemma 6, consequently, shows that strong deviations from the Walrasian outcome even remain possible in our model if we asymptotically remove all search frictions. The critical feature which is responsible for this surprising result is a high quality market which becomes infinitely crowded with patient tenant types during the removal of the search frictions: All impatient tenant types would like to meet a high quality landlord in a search equilibrium with $\nu = \nu^{CH}$, but all of them choose a low quality level because the tenant per apartment ratio in the high quality market is very high. δ_P is an important parameter in such an equilibrium because $\lim_{\delta_P \to 1} \mu_2^T/\mu_2^L = \infty$ for any given (δ, e). The discount factor of the patient tenant types must be close enough to 1 in order to make sure that a sufficiently crowded high quality market induces a low quality choice for all impatient tenant types.

In our argument on the existence of an equilibrium with $\nu = \nu^{CH}$, we first verify that *some* of our equilibrium conditions uniquely determine all endogenous variables in the low quality market. As a second step, we repeat this exercise for the high quality market and finally, we show that all these endogenous variables do indeed satisfy *all* equilibrium conditions if δ_P is sufficiently close to 1. Throughout the proof, we focus on the case when $\nu^L > \nu^T$, and omit the case when $\nu^T > \nu^L$ because it would not add much. For the purpose of demonstration, $\nu^L > \nu^T$ is more rewarding because all equilibrium conditions apply in some way, and because it shows why we need an entry fee in our model. Notice also that $\nu_1^L > \nu_I^T$ is a direct consequence of $\nu^L > \nu^T$ and $\nu_P^T > \nu_2^L$.

Step 1. With $\nu = \nu^{CH}$, $\nu^L > \nu^T$ and $\nu_P^T > \nu_2^L$, the equilibrium condition (10) must apply because some low quality landlords must remain without a tenant. Formally, we obtain $i^{\times} = 0$, $i^{\times\times} = \nu_P^T - \nu_2^L$, $\int \nu_{i2} di = \nu_2^L$ and $0 < \int \nu_{i1} di = \nu_I^T + \nu_P^T - \nu_2^L < \nu_1^L$ which according to (10) and (8) requires that $v_1^L = e$, i.e., that low quality landlords must be indifferent between entering the market and staying out of the market. With (16), we, thus, find that

$$e = \int \frac{\pi_{i1}^L (1-\delta_i) \frac{1}{2} (\beta_{i1} - e\delta)}{(1-\delta)(1-\delta_i + \bar{\pi}_1^T \frac{1}{2} \delta_i)} di. \tag{20}$$

Since $\int \nu_{i1} di = \nu^T - \nu_2^L$, we know from our discussion in the proof of Lemma 1 that either

(a) $\mu_1^L = \nu^T - \nu_2^L < \mu_1^T, \quad \bar{\pi}_1^T = \dfrac{\nu^T - \nu_2^L}{\mu_1^T} \quad$ and $\quad \pi_{i1}^L = \dfrac{\nu_{i1}}{\nu^T - \nu_2^L}$

or

(b) $\mu_1^T = \nu^T - \nu_2^L \leq \mu_1^L, \quad \bar{\pi}_1^T = 1 \quad$ and $\quad \pi_{i1}^L = \dfrac{\nu_{i1}}{\mu_1^L}.$

Under (b) and for small e, the right hand side of equation (20) is monotonically and continuously decreasing in μ_1^L and approaches 0 as μ_1^L goes to infinity. Under (a) and for small e, the right hand side of equation (20) is monotonically and continuously increasing in μ_1^T and approaches

$$\int \frac{\frac{1}{2}(\beta_{i1} - e\delta)}{(1-\delta)} \frac{\nu_{i1}}{\nu^T - \nu_2^L} di$$

as μ_1^T goes to infinity. Thus, there must be a unique $\mu_1^L > 0$ and $\mu_1^T > 0$ which solve (20) since for (δ, e) sufficiently close to $(1, 0)$, $e > 0$ is certainly in between these extremes. For given $\mu_1^L > 0$ and $\mu_1^T > 0$, on the other hand, we observe that (5), (6), and (19) uniquely determine μ_{i1}^T and v_{i1}^T :

$$\mu_{i1}^T = \frac{\nu_{i1}}{\bar{\pi}_1^T} \text{ if } \nu_{i1} > 0, \tag{21}$$

$$\mu_{i1}^T = 0 \text{ if } \nu_{i1} = 0, \text{ and} \tag{22}$$

$$v_{i1}^T = \frac{\bar{\pi}_1^T \frac{1}{2} \max(0, \beta_{i1} - v_1^L \delta)}{1 - \delta_i + \bar{\pi}_1^T \frac{1}{2} \delta_i} \text{ with} \tag{23}$$

$$v_1^L = e. \tag{24}$$

Notice, thereby, that we need a positive entry fee in order to establish existence.

Step 2. Now, we proof that our equilibrium conditions also pin down the tightness in the high quality market. Since (10) does not apply there, this is a quite different task than before. As a first step, we show that the market conditions in the low quality market determine $v_{i^{\times\times}2}^T$. Then, we use this result for the derivation of an equation which uniquely determines all endogenous variables in the high quality market.

Our incentive compatibility condition (12) requires that $v_{i2}^T - v_{i1}^T \geq 0$ for all $i \in [i^{\times\times}, \nu_P^T)$ and that $v_{i2}^T - v_{i1}^T \leq 0$ for all $i \in [i^\times, i^{\times\times})$. Using (19), note that $v_{i2}^T - v_{i1}^T$ must be continuous at $i = i^{\times\times}$ since with respect to i and at $i = i^{\times\times}$, the functions β_{i1}, β_{i2} and δ_i are continuous by assumption. Hence, we observe $v_{i^{\times\times}2}^T = v_{i^{\times\times}1}^T$, i.e., that type $i^{\times\times}$ tenants must be indifferent between a high and a low quality choice.

Employing $v_{i^{\times\times}2}^T = v_{i^{\times\times}1}^T$, $\nu_{i^{\times\times}2} = 1$ and (15), we can verify that

$$(1 - \delta_{i^{\times\times}} + \bar{\pi}_2^T \tfrac{1}{2} \delta_{i^{\times\times}}) v_{i^{\times\times}1}^T = \bar{\pi}_2^T \tfrac{1}{2} (\beta_{i^{\times\times}2} - v_2^L \delta)$$

and that

$$v_{i2}^T = \frac{\bar{\pi}_2^T \frac{1}{2}(\beta_{i2} - v_2^L \delta)}{1 - \delta_i + \bar{\pi}_2^T \frac{1}{2}\delta_i} \text{ if } \nu_{i2} > 0.$$

When we subtract the former from the latter equation, we find that

$$v_{i2}^T - \frac{1 - \delta_{i^{\times\times}} + \bar{\pi}_2^T \frac{1}{2}\delta_{i^{\times\times}}}{1 - \delta_i + \bar{\pi}_2^T \frac{1}{2}\delta_i} v_{i^{\times\times}1}^T = \frac{\bar{\pi}_2^T \frac{1}{2}(\beta_{i2} - \beta_{i^{\times\times}2})}{1 - \delta_i + \bar{\pi}_2^T \frac{1}{2}\delta_i} \text{ if } \nu_{i2} > 0.$$

Our equilibrium condition (6) requires that $\mu_{i2}^T = 0$ if $\nu_{i2} = 0$ and, hence, that $\pi_{i2}^L = 0$ if $\nu_{i2} = 0$. Exploiting (14), $v_{i^{\times\times}2}^T = v_{i^{\times\times}1}^T$, $\nu_{i^{\times\times}2} = 1$ and (15), we can further show that

$$(1 - \delta)\left(\frac{\beta_{i^{\times\times}2}}{\delta} - \frac{1 - \delta_{i^{\times\times}} + \bar{\pi}_2^T \frac{1}{2}\delta_{i^{\times\times}}}{\bar{\pi}_2^T \frac{1}{2}\delta} v_{i^{\times\times}1}^T\right) = \int \frac{\pi_{i2}^L}{\bar{\pi}_2^T}(1 - \delta_i) v_{i2}^T di.$$

Notice finally that these comments amount to

$$\begin{aligned}(25) \qquad 0 \;=\; & \frac{\beta_{i^{\times\times}2}}{\delta} - \frac{1 - \delta_{i^{\times\times}} + \bar{\pi}_2^T \frac{1}{2}\delta_{i^{\times\times}}}{\bar{\pi}_2^T \frac{1}{2}\delta} v_{i^{\times\times}1}^T \\ & - \int \pi_{i2}^L \frac{(1 - \delta_i)}{(1 - \delta)} \frac{\frac{1}{2}(\beta_{i2} - \beta_{i^{\times\times}2})}{(1 - \delta_i + \bar{\pi}_2^T \frac{1}{2}\delta_i)} di \\ & - \int \frac{\pi_{i2}^L}{\bar{\pi}_2^T} \frac{(1 - \delta_i)}{(1 - \delta)} \frac{(1 - \delta_{i^{\times\times}} + \bar{\pi}_2^T \frac{1}{2}\delta_{i^{\times\times}})}{(1 - \delta_i + \bar{\pi}_2^T \frac{1}{2}\delta_i)} v_{i^{\times\times}1}^T di.\end{aligned}$$

Since $\int \nu_{i2} di = \nu_2^L$, we know from our discussion in the proof of Lemma 1 that either

$$\text{(a)} \quad \mu_2^L = \nu_2^L < \mu_2^T, \quad \bar{\pi}_2^T = \frac{\nu_2^L}{\mu_2^T} \quad \text{and} \quad \pi_{i2}^L = \frac{\nu_{i2}}{\nu_2^L}$$

or

$$\text{(b)} \quad \mu_2^T = \nu_2^L \leq \mu_2^L, \quad \bar{\pi}_2^T = 1 \quad \text{and} \quad \pi_{i2}^L = \frac{\nu_{i2}}{\mu_2^L}.$$

Under (a) and for small e, the right hand side of (25) is monotonically and continuously decreasing in μ_2^T and approaches $-\infty$ as μ_2^T goes to infinity. Under (b) and for small e, the right hand side of (25) is monotonically and continuously increasing in μ_2^L and approaches

$$\frac{\beta_{i^{\times\times}2}}{\delta} - \frac{1 - \delta_{i^{\times\times}} + \frac{1}{2}\delta_{i^{\times\times}}}{\frac{1}{2}\delta} v_{i^{\times\times}1}^T \geq \frac{\beta_{i^{\times\times}2}}{\delta} - \frac{\max(0, \beta_{i^{\times\times}1} - e\delta)}{\delta} > 0$$

as μ_2^L goes to infinity. Consequently, there must be unique $\mu_2^L > 0$ and $\mu_2^T > 0$ which solve (25). For given $\mu_2^L > 0$ and $\mu_2^T > 0$, on the other hand, we observe that (5), (6),(19), $v_{i^{\times\times}2}^T = v_{i^{\times\times}1}^T$, $\nu_{i^{\times\times}2} = 1$ and (15) uniquely determine μ_{i2}^T and v_{i2}^T :

$$\mu_{i2}^T = \frac{\nu_{i2}}{\bar{\pi}_2^T} \text{ if } \nu_{i2} > 0, \tag{26}$$

$$\mu_{i2}^T = 0 \text{ if } \nu_{i2} = 0, \text{ and} \tag{27}$$

$$v_{i2}^T = \frac{\bar{\pi}_2^T \frac{1}{2} \max(0, \beta_{i2} - v_2^L \delta)}{1 - \delta_i + \bar{\pi}_2^T \frac{1}{2} \delta_i} \text{ with} \tag{28}$$

$$v_2^L = \frac{\beta_{i^{\times\times}2}}{\delta} - \frac{1 - \delta_{i^{\times\times}} + \bar{\pi}_2^T \frac{1}{2} \delta_{i^{\times\times}}}{\bar{\pi}_2^T \frac{1}{2} \delta} v_{i^{\times\times}1}^T. \tag{29}$$

Step 3. So far, we have shown that *some* of our equilibrium conditions uniquely pin down all endogenous variables for $\nu = \nu^{CH}$. To complete the proof of Lemma 6, we now show that with (δ, e) sufficiently close to $(1, 0)$, these endogenous variables do indeed satisfy *all* equilibrium conditions if δ_P is sufficiently close to 1.

With $\nu^L > \nu^T$ and $i^\times = 0$, our definition of ν_{ij}^{CH} obviously reveals that (3) is valid, and that (7) and (11) do not apply.[20] Since $\int \nu_{i2} di = \nu_2^L$ and $\int \nu_{i1} di = \nu^T - \nu_2^L < \nu_1^L$, we also find that (4) is valid. The conditions (1), (5) and (6), on the other hand, hold by construction since we used them to express μ_{i1}^T, μ_{i2}^T, v_{i1}^T and v_{i2}^T explicitly. Formally, (1) is equivalent to (23) and (28), (5) is equivalent to (21) and (26), and (6) is equivalent to (22) and (27).

When we somewhat manipulate (23) and (28), we can show that

$$v_{ij}^T (1 - \delta_i) = \bar{\pi}_j^T \tfrac{1}{2} \max(0, \beta_{ij} - v_j^L \delta - v_{ij}^T \delta_i) \tag{30}$$

which proofs that $\beta_{ij} - v_j^L \delta - v_{ij}^T \delta_i > 0$ iff $v_{ij}^T > 0$. For (δ, e) sufficiently close to $(1, 0)$, (23) and (24) reveal that $v_{i1}^T > 0$ for all i, and in particular, that $v_{i1}^T > 0$ if $\nu_{i1} > 0$. When we plug (29) into (28), we find that $v_{i^{\times\times}2}^T = v_{i^{\times\times}1}^T$. Exploiting (28), $v_{i^{\times\times}2}^T = v_{i^{\times\times}1}^T > 0$ and type monotonicity, we can further show that $\beta_{i2} - v_2^L \delta > 0$ if $i > i^{\times\times}$ which implies that $v_{i2}^T > 0$ if $\nu_{i2} > 0$. The

[20] Equation (7) only matters in "shut-down" equilibria. Equation (11), however, does matter in cases where $\nu^T > \nu^L$. In the omitted part of this proof, for example, analogous arguments as in Step 2 must be used in both markets since (11) and (9) imply that $v_{i^\times 1}^T = e$ if $i^\times > 0$.

comments in this paragraph add up to the conclusion that $\beta_{ij}-v_j^L\delta-v_{ij}^T\delta_i>0$ if $\nu_{ij}>0$, i.e., that (13) must hold. Using (23) and (28), also notice that,

$$\beta_{ij}-v_j^L\delta>0 \text{ if } \nu_{ij}>0. \tag{31}$$

When we insert (29) into (28) and, at the same time, apply (31), we get that

$$v_{i2}^T=\frac{\bar{\pi}_2^T\frac{1}{2}(\beta_{i2}-\beta_{i\times\times 2})}{1-\delta_i+\bar{\pi}_2^T\frac{1}{2}\delta_i}+\frac{1-\delta_{i\times\times}+\bar{\pi}_2^T\frac{1}{2}\delta_{i\times\times}}{1-\delta_i+\bar{\pi}_2^T\frac{1}{2}\delta_i}v_{i\times\times 1}^T \text{ if } \nu_{i2}>0.$$

Using (27), note that π_{i2}^L=0 if $\nu_{i2}=0$. Exploiting (25) and (29), we can, thus, show that $v_2^L(1-\delta)$=$\int\pi_{i2}^L(1-\delta_i)v_{i2}^T/\bar{\pi}_2^T di$. When we finally plug (30) into this equation, we find that

$$v_2^L(1-\delta)=\int\pi_{i2}^L\tfrac{1}{2}\max(0,\beta_{i2}-v_2^L\delta-v_{i2}^T\delta_i)di,$$

i.e., that (2) must hold for $j=2$.

As an immediate consequence of (22), we observe that π_{i1}^L=0 if ν_{i1}=0. Exploiting (20), (23), (24) and (31), we can, thus, show that $v_1^L(1-\delta)$= $\int\pi_{i1}^L(1-\delta_i)v_{i1}^T/\bar{\pi}_1^T di$. Inserting (30) into this expression, we find that

$$v_1^L(1-\delta)=\int\pi_{i1}^L\tfrac{1}{2}\max(0,\beta_{i1}-v_1^L\delta-v_{i1}^T\delta_i)di,$$

i.e., that (2) must hold for $j=1$.

We can exploit (20), (31) and (24) to show that

$$e>\int_I\frac{\pi_{i1}^L\frac{1}{2}(\beta_{i1}-e\delta)}{1-\delta+\bar{\pi}_1^T\frac{1}{2}\delta}di. \tag{32}$$

With $\lim\bar{\pi}_1^T=0$ and (δ,e) sufficiently close to $(1,0)$,[21] this requirement must certainly be violated since $\lim\bar{\pi}_1^T=0$ would imply that $\lim\pi_{i1}^L$=$\nu_{i1}/(\nu^T-\nu_2^L)$ >0 for all $i\in[\nu_P^T,\nu^T]$. Using (23), (24) and $\lim\bar{\pi}_1^T>0$, we can, thus , verify that $\lim(v_{i1}^T-e)=\beta_{i1}>0$ for all i, [22] i.e., that (9) must hold if (δ,e) is sufficiently close to $(1,0)$.[23]

[21] At an abuse of notation and throughout the paper, we use lim as a shorthand for $\lim_{(\delta,e)\to(1,0)}$.

[22] Notice, thereby, that $\delta_i\geq\delta$.

[23] Since $\nu_{i1}+\nu_{i2}=1$ for all i, (9) must be valid if $v_{i1}^T>e$ for all i.

When we manipulate (25), we can easily verify that

$$\frac{\beta_{i^{\times\times}2}}{\delta} > \int_{i^{\times\times}}^{\nu_P^T} \frac{\pi_{i2}^L}{\bar{\pi}_2^T} \frac{(1-\delta_P)}{(1-\delta)} v_{i^{\times\times}1}^T di.$$

With $\lim(1-\delta_P)/\bar{\pi}_2^T > 0$ and (δ, e) sufficiently close to $(1,0)$,[24] this requirement must certainly be violated since $\lim(1-\delta_P)/\bar{\pi}_2^T > 0$ would imply that $\lim \pi_{i2}^L = \nu_{i2}/\nu_2^L > 0$ for all $i \in [i^{\times\times}, \nu_P^T)$.[25] Exploiting $\lim(1-\delta_P)/\bar{\pi}_2^T = 0$, $\lim v_{i^{\times\times}1}^T = \beta_{i^{\times\times}1}$ and (29), we can, thus, show that $\lim v_2^L - e = \beta_{i^{\times\times}2} - \beta_{i^{\times\times}1} > 0$, i.e., that (8) and (10) must hold if (δ, e) is sufficiently close to $(1,0)$.[26]

We can manipulate $\lim \bar{\pi}_1^T > 0$, $\lim(1-\delta_P)/\bar{\pi}_2^T = 0$ and $\lim v_2^L = \beta_{i^{\times\times}2} - \beta_{i^{\times\times}1}$, (23), (24) and (28) to show that

$$\lim v_{i2}^T - v_{i1}^T = \beta_{i2} - \beta_{i^{\times\times}2} - (\beta_{i1} - \beta_{i^{\times\times}1}) \text{ if } i \in [i^{\times}, \nu_P^T).$$

Since $\partial\beta_{i2}/\partial i > \partial\beta_{i1}/\partial i$ by assumption, this implies that $\lim v_{i2}^T - v_{i1}^T \geq 0$ if $i \in [i^{\times\times}, \nu_P^T)$ and $\lim v_{i1}^T - v_{i2}^T > 0$ if $i \in [i^{\times}, i^{\times\times})$, i.e., that (12) must hold for $i \in [i^{\times}, \nu_P^T)$ if (δ, e) is sufficiently close to $(1,0)$.

Finally, we need to focus on the incentive constraint of the impatient tenant types. First, we proof that $\lim_{\delta_P \to 1} \bar{\pi}_2^T = 0$ and that $\lim_{\delta_P \to 1} \bar{\pi}_1^T > 0$ for any given (δ, e) sufficiently close to $(1,0)$. Then, we show that all impatient tenant types do indeed choose a low quality level if relative to δ, δ_P is sufficiently close to 1.

Assume that $\lim_{\delta_P \to 1} \bar{\pi}_2^T > 0$ for given (δ, e) sufficiently close to $(1,0)$. Then, the right hand side of (25) must approach $\beta_{i^{\times\times}2}/\delta - v_{i^{\times\times}1}^T/\delta$ as we let δ_P go to 1. Using (23), however, notice that $\beta_{i^{\times\times}1} \geq v_{i^{\times\times}1}^T$ and that $\beta_{i^{\times\times}2}/\delta - v_{i^{\times\times}1}^T/\delta > 0$. Hence, we observe that $\lim_{\delta_P \to 1} \bar{\pi}_2^T > 0$ is in contradiction with (25). To summarize, $\lim_{\delta_P \to 1} \bar{\pi}_2^T = 0$ for any given (δ, e) sufficiently close to $(1,0)$.

Assume that $\lim_{\delta_P \to 1} \bar{\pi}_1^T = 0$ for given (δ, e) sufficiently close to $(1,0)$. Then, the right hand side of (32) must approach

$$\int_I \frac{\nu_{i1}}{\nu^T - \nu_2^L} \frac{\frac{1}{2}(\beta_{i1} - e\delta)}{1-\delta} di$$

[24] Since $\delta_P > \delta$, notice that $\lim \bar{\pi}_2^T = 0$ if $\lim(1-\delta_P)/\bar{\pi}_2^T > 0$.

[25] Using the result in the previous paragraph, notice that $\lim v_{i^{\times\times}1}^T = \beta_{i^{\times\times}1} > 0$.

[26] Since $0 < \int \nu_{i1} di = \nu^T - \nu_2^L < \nu_1^L$ and $\int \nu_{i2} di = \nu_2^L$, (8) and (10) hold iff $v_1^L = e$ and $v_2^L \geq e$.

as we let δ_P go to 1. For (δ, e) sufficiently close to $(1,0)$, however, this expression is certainly greater than e. Hence, we find that $\lim_{\delta_P \to 1} \bar{\pi}_1^T = 0$ is in contradiction with (32). To summarize, $\lim_{\delta_P \to 1} \bar{\pi}_1^T > 0$ for any given (δ, e) sufficiently close to $(1,0)$.

Using (23), (24), (28) and the results in the two previous paragraphs, we can show that with given (δ, e) sufficiently close to $(1,0)$, $\lim_{\delta_P \to 1} v_{i1}^T > 0$ and $\lim_{\delta_P \to 1} v_{i2}^T = 0$ if $i \in [\nu_P^T, \nu^T]$. Hence, we have proofed that with given (δ, e) sufficiently close to $(1,0)$, $v_{i1}^T > v_{i2}^T$ for all $i \in [\nu_P^T, \nu^T]$ if $\delta_P > \delta$ is sufficiently close to 1. Note that all other equilibrium conditions hold for *all* $\delta_P > \delta$ given that (δ, e) is sufficiently close to $(1,0)$. □

6.2 Crowding Out in Both Markets

Lemma 7 *For $\nu_2^L < \nu_P^T$, and any given* (δ,e) *sufficiently close* (1,0)*, one and at most one equilibrium with $\nu = \nu^{CB}$ does exist if δ_P is sufficiently close to 1.*

Lemma 7 reveals that there is a second strong deviation from the Walrasian outcome which does exist for arbitrarily small frictions. In an equilibrium with $\nu = \nu^{CB}$, only patient tenants receive an apartment, and the impatient tenants whose demand strength is strictly higher do not even enter the market. The crucial feature which makes such an outcome possible in our model is a high tenant per apartment ratio in each of the markets which converges to infinity as the frictions become removed. δ_P is an important variable in an equilibrium with $\nu = \nu^{CB}$ because $\lim_{\delta_P \to 1} \mu_2^T / \mu_2^L = \lim_{\delta_P \to 1} \mu_1^T / \mu_1^L = \infty$ for any given (δ,e). The discount factor of the patient tenant types must be close enough to 1 in order to make sure that all impatient tenant types want to stay out of the market. Since it is just a variation of Lemma 6's proof, we omit Lemma 7's proof.

6.3 Positively Assortative Matching

Lemma 8 *For $\nu_2^L < \nu_P^T$ and any* (δ,δ_P,e) *sufficiently close* (1,1,0)*, one and at most one equilibrium with $\nu = \nu^{PA}$ does exist.*

The fact that our frictional model can generate a Walrasian apartment assignment when we asymptotically remove all frictions is not very surprising since it confirms the widespread intuition that the impact of the match creation costs on the expected payoffs should become negligible when all traders become very patient, i.e., that only prices should determine the expected payoffs in an environment with infinitesimal frictions. From the analysis above, we know, however, that this intuition is based on the assumption that the matching probabilities do not approach zero as frictions become removed. Since it is again just a variation of Lemma 6's proof, we also omit Lemma 8's proof.

7 Discussion

Our result in Proposition 2 shows that lowly ranked but patient players can get highly ranked partners in a marriage model with unboundedly low frictions. In the following, we will briefly discuss the modeling choices under which we obtain this remarkable result.

The special type distribution which we use throughout the paper is certainly not an essential ingredient in our model. With a discrete distribution on one side of the market and a uniform one on the other, the description of an equilibrium matching pattern just becomes especially easy. At the expense of notation, an extension of our result to arbitrary type distributions is a straight forward exercise. Such an extension should also be possible if we assume that the patient players and all others on their side of the market can only choose the quality of their potential partner within given ranges and not exactly. In such a framework, however, the analysis becomes much more involved because meetings need not always end up with partnership formations.

The main finding in this paper should further be independent of the way in which we model the steady state player flows. We should be able to make similar observations in models with a finite measure of infinitely lived players whose matches become randomly destroyed by nature at a constant rate. In such an environment, the steady state population share of some patient player types can still approach 1 as frictions vanish, i.e., as the match creation rate becomes unbounded: Their stocks may simply remain positive while that of all other player types go to zero.

The most unusual aspect of our model is the assumption that the patient players and all others on their side of the market can choose the quality of the partners they meet. We can easily think of situations where such an asymmetry is plausible, [27] but primarily we make this assumption because it tremendously simplifies our steady state description: Meetings must always end up with partnership formations under an endogenous matching technology of this nature.

A small but positive stochastic perturbation of the quality choice in our model should not destroy our main result because the discrete quality and entry decisions will typically remain optimal as long as this perturbation is sufficiently small. Whether our result can survive in frameworks where all players get matched exogenously and at random, is a more complicated question. According to my intuition it does when we assume that lowly ranked but patient players do not only appear on one but on both sides of the market: In a model of this nature, there should be an equilibrium in which lowly ranked but patient players crowd both market sides and, at the same time, do not trade with each other because they can get highly ranked but impatient partners.[28]

With homogenous discount factors and unboundedly low frictions, we never obtain a deviation from the Walrasian outcome in the framework of this paper.[29] But even under these circumstances, it sometimes remains possible that lowly ranked players get highly ranked partners. An example for that is given by Ramsauer (1998) who studies homogenous discount factors and a purely discrete type distribution in a framework which is otherwise identical to the one in this paper. The result in Ramsauer(1998) is driven by a high steady state population share of the worst active players which makes it hard for the other players to meet each other. In such an equilibrium, some marginal players are very choosy because their match value and, hence, their match delay costs are much lower than that of all other players. The existence of such an equilibrium requires that the flow measure of all marginal players is above a certain positive threshold. Obviously, this is violated when the

[27] One example is a housing market with tenants who are familiar with the city and landlords who must advertise the location of their apartment: The tenants in such a market do receive a strong quality signal before they contact a partner, but the landlords do not.

[28] Notice that this assumption would also work in the framework of this paper: The high quality market would be crowded with lowly ranked but patient tenants, the low quality market would be crowded with patient landlords, and the highly ranked but impatient tenants would choose a low quality level.

[29] According to Lemma 1(a), the crowding intensities must always remain finite in such cases.

type distribution is nonatomic.

An extension of our results to nontransferable utility frameworks should also be possible. Again, we can imagine an equilibrium in which lowly ranked but patient players crowd both sides of the market. When atoms appear in the type distribution, however, we may run into problems. An example for that is the nontransferable utility version of the model in this paper: A very patient landlord in a high quality market which is crowded with lowly ranked tenants would only want to form a partnership with the very best of them. In our transferable utility version, this problem does not arise because better tenants also demand a higher compensation. Without atoms, the problem would not arise because no landlord type would have to form partnerships with a fixed variety of tenant types.

8 Related Literature

Since the pioneering work of Diamond (1971), it has been well known that the equilibrium in a search model with infinitesimal search costs need not be close to the equilibrium of the corresponding Walrasian model. Regardless of how small the search frictions are, and regardless of how many buyers and sellers there are, the monopoly price is the only equilibrium price in Diamond's simple model. This surprising result is often referred to as the "Diamond paradox," and the crucial feature which is responsible for its appearance is an extremely asymmetric bargaining procedure in which only the sellers can commit to a price.

Diamond's introduction of an equilibrium approach to search theory initiated the labor market oriented "search and bargaining" literature which is primarily concerned with generating non-Walrasian outcomes, often denoted as "non-degenerate equilibrium price distributions," that are observed in real world markets. The issue of convergence to a Walrasian outcome as search costs become negligible is typically not studied in this literature. The first paper which explicitly raises this question is Rubinstein and Wolinsky (1985). They focus on the steady state of a simple search model with a continuum of traders, indivisible, homogenous goods and a symmetric bargaining procedure. Their main finding is that the equilibrium price for infinitesimal frictions deviates from the Walrasian price if the Walrasian price is defined relative to trader stocks. In a response to this result, Gale (1987) argues that the competitive equilibrium in a model with an infinite measure of traders should be defined relative to the trader flows, and shows that the unique search equilibrium in his model yields the alternatively defined Walrasian equilibrium when the rate of time preference converges to 0. A succession of papers which are concerned with the "game theoretic foundations of Walrasian equilibrium" has emerged since this important debate about the right manner of defining a Walrasian equilibrium. Osborne and Rubinstein (1990) survey this literature through the end of the eighties. An often used phrase in this literature is "decentralized trade" which stands for search and bargaining environments. "Centralized trade," on the other hand, is reserved for Walrasian procedures.

Several approaches have led to search and bargaining models with equilibrium outcomes which do not converge to the Walrasian benchmark as the frictions become asymptotically removed: multistage bargaining procedures with outside options [Bester(1988a), Muthoo (1993), Shaked (1987), Shaked and Sutton (1984)], absence of anonymity [Rubinstein and Wolinsky (1990), Hendon and Tranæs (1991), Hendon and Tranæs (1995)], incomplete information [Bester (1988b), Samuelson (1992)],[30] simultaneous offer bargaining [Serrano and Yosha (1995a)], simultaneous offer bargaining with incomplete information [Serrano and Yosha (1995b)] and simultaneous offer bargaining with incomplete information and common values [Wolinsky (1990)]. The approach which we take in this paper is qualitatively different from all other approaches in the literature. The traders in our model are perfectly anonymous and possess complete information about their partner's type. The bargaining procedure, on the other hand, is symmetric and based on "take-it-or-leave-it" offers. A closely related model which also possesses these feature is the one of Gale (1987), but the unique equilibrium in Gale (1987) converges to the Walrasian outcome as the frictions become asymptotically removed. The crucial element in our model which makes it possible that we arrive at a different conclusion than Gale (1987) is a trader population with heterogeneous discount factors. Notice that we also use the flow concept rather than the stock concept for the definition of our Walrasian benchmark.

[30]Samuelson (1992) explains why disagreement may arise in a market with matching and bargaining. His equilibrium outcomes, however, do not deviate from the Walrasian benchmark if we define this benchmark in terms of trader flows.

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