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On the Necessity of Using Lottery Qualities

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Abstract: The aim of this paper is to propose a model of decision-making for lotteries. The key element of the theory is the use of lottery qualities. Qualities allow the derivation of optimal decision-making processes and are taken explicitly into account for lottery evaluation. Our contribution explains the major violations of the expected utility theory for decisions on two-point lotteries and shows the necessity of giving explicit consideration to the lottery qualities.

Résumé: L'objet de cette recherche est de proposer un modèle de décision pour les loteries. L'élément clé de la théorie est l'utilisation des caractéristiques des loteries. Les qualités permettent de dériver les processus optimaux de décision et sont explicitement prises en compte dans l'évaluation des loteries. Notre contribution explique les principales violations de la théorie de l'espérance d'utilité pour les décisions sur les loteries à deux points et montre la nécessité d'utiliser les qualités des loteries.

Keywords: Lottery choice, common ratio, preference reversal, pricing, lottery test, cognitive process, certainty equivalent, lottery quality

Mots clés: Choix de loterie, ratio commun, renversement des préférences, tarification, test de choix de loterie, processus cognitif, équivalent certain, qualité d'une loterie

JEL Classification: D81

1. Introduction

Over the last fifty years many theories have been proposed to explain the results of lottery tests (for a survey of the main results see Machina, 1987; McFadden, 1999; Luce, 2000). However, even for the simplest two–point lotteries, no theory is able to take into account all tests together. The goal of this paper is to use lottery qualities to build up a model that will take into account all possible tests related to both the pricing and comparison of two–point lotteries and, second, explain why it is optimal for an agent to act according to the test results.

There exist two important features about lottery tests: (1) the existence of lottery qualities and (2) the presence of more than one cognitive process. Regarding the existence of lottery qualities, Tversky and Kahneman (1992), among others, have already tested the difference between the positive and negative qualities for monetary amounts x_i. Prelec (1998) has pointed out the qualitative difference between impossibility ($p_i = 0$) and possibility ($p_i \in]0,1[$) for probabilities, while Tversky and Kahneman (1979) have looked at certainty ($p_i = 1$) as another quality. The presence of more than one cognitive process can be illustrated by the preference reversal paradox (Tversky et al., 1990, Lichtenstein and Slovic 1971), where a majority of subjects would prefer lottery A to lottery B in a direct choice but give a higher judged price to lottery B. In this choice, it was always possible for subjects to price each lottery first and then compare the two prices. The test result obtained clearly shows that individuals do not price before making their choices. We must then conclude that there exist at least two different cognitive processes and that individuals have preferences regarding these processes. In this paper, we shall condense the choices of all processes into one simple principle which consists in splitting of sets into more homogeneous subsets.

The concept of qualities will be shown to be useful in two different ways. First, qualities determine the decision process and, second, they serve as explicit elements in the lottery judgment. So, the role played by qualities seems strong enough to justify the necessity of using them.

In non–expected utility (NEU) models such as $\Sigma w(p_i)u(x_i)$ (Kahneman and Tversky, 1979, Edwards, 1955), p_i and x_i are always first evaluated with the functions w and u and then the summation of the different products are used to evaluate the lotteries. In expected utility (EU), $w(p_i) = p_i$ for all i, and the same type of evaluation process is used for all lotteries. In this paper, we shall show that considering qualities in the choice process makes it possible to extend the most common models in two directions. First, as in the current literature, it allows individuals to make a primary judgment of p_i and x_i whenever it seems optimal. Second, individuals can use the product of judged p_i and x_i (as in NEU or EU) but again only when it is optimal. For example, they may also, as in Rubinstein (1988) and Leland (1994), compare the two p_i and the two x_i in some cases.

The paper is organized as follows. Section 2 lists the fourteen more problematic empirical facts culled from the literature on two–point lotteries and Section 3 defines vectors of qualities and a relation that orders these vectors. In Section 4, we use these two definitions to explain how the agent selects optimal processes (Definition 4) and evaluates lotteries (Definitions 5.1 and 5.2). Section 5 presents numerical examples and discusses the fourteen facts in relation with these examples. Section 6 concludes.

2. Facts about two–point lotteries

Notation

The notation {a,b} is for a set and the notation (a,b) is for an ordered pair. A monetary amount with values in]–∝,∝[is denoted x_i, and $x_i \in X$, the set of monetary amounts. A probability with values in [0,1] is denoted p_i, and $p_i \in P$, the set of probabilities. A lottery where the agent can win x_i with probability p_i and 0 otherwise is denoted (p_i,x_i).The set of the two elements of this lottery is $l_i = \{p_i,x_i\}$. A lottery where the agent can win x_i with probability p_i and x_{i-1} with probability $1-p_i$ is denoted $(1,x_{i-1};p_i,x_i-x_{i-1})$ and $l_i = \{1,x_{i-1},p_i,x_i-x_{i-1}\}$. For convenience we assume that $|x_{i-1}| < |x_i|$. This notation puts the emphasis on the fact that, when $x_{i-1} \neq 0$, the agent first considers a sure monetary amount x_{i-1} and then a lottery (p_i,x_i-x_{i-1}), which is almost equivalent to the concept of segregation in Kahneman and Tversky (1979) as discussed in Luce (2000).[1]

The more basic tests for two–point lotteries are the judged certainty equivalent (subjects are asked to select a price), the choice certainty equivalent (subjects choose between a lottery and a sure monetary amount) and the comparison involving two lotteries. Almost every test involves some difficulties for theoreticians. We list below fourteen of the more problematic facts associated with these tests.

Lotteries: (p_i,x_i), $x_i > 0$.

Fact #1:

In a lottery choice between lotteries A and B, if p_i^A is high and p_i^B is low, both yielding the same von Neuman–Morgenstern expected utility, a majority of subjects will select lottery A. (Tversky et al., 1990).

Fact #2:

When two lotteries A and B with the same expected value are compared and the probabilities $p_i^A > p_i^B$ are both high, a majority of subjects will choose lottery A. However, when both probabilities are low and the ratio p_i^A/p_i^B remains the same, a majority of subjects will choose lottery B. This is the common ratio paradox. (Kahneman and Tversky, 1979; MacCrimon and Larsson, 1979).

Fact # 3:

When subjects are asked to select a price (judged certainty equivalent JCE) for the lotteries (p_i,x_i), lottery A with high probability of winning is underestimated, while lottery B with a low probability of winning is overestimated (Birnbaum et al., 1992).

Facts #1 and #3 together lead to the preference reversal paradox. Moreover, Alarie and Dionne (2001) have shown that a one–parameter weighting probability function $w(p_i)$ is not able to explain the choices observed for these three facts involving the need for a function $w(p_i^A; p_i^B)$.

A very important point often neglected in the literature is the next one.

Fact # 4:

In comparing two lotteries, it is always possible for subjects to price each lottery first and then compare the two prices. But the test results (Facts #1 and #3) clearly imply that individuals do not price before making their choices. So we have to explain why pricing lotteries is not optimal when subjects face a lottery choice.

Fact # 5:

If we compare a lottery with a sure monetary amount or with a series of sure monetary amounts, we obtain the choice certainty equivalent CCE. Tversky et al. (1990) found that CCE = JCE for lotteries with high probabilities but CCE < JCE for lotteries with low probabilities. We have to explain these results and why it is not optimal for subjects to price (JCE) first when asked to choose between a lottery with a high p_i and a sure monetary amount (CCE).

Lotteries: (p_i,x_i), $x_i < 0$.

Fact # 6:

One can also note that for the CCE and the common ratio paradox where $x_i > 0$, the observed preferences run counter to the ones for lotteries where $x_i < 0$ (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992).

Lotteries: $(1,x_{i-1};p_i,x_i-x_{i-1})$ $x_i,x_{i-1} > 0$.

Fact # 7:

When p_i is high, the JCE of a lottery $(1,x_{i-1};p_i,x_i-x_{i-1})$ where an agent can win x_i with probability p_i or x_{i-1} otherwise is smaller than the JCE of a lottery (p_i,x_i) where the agent can still win x_i with probability p_i but 0 otherwise (Birnbaum et al., 1992).

Fact # 8:

In direct choices, the lottery $(1,x_{i-1};p_i,x_i-x_{i-1})$ where an agent can win x_i with probability p_i or x_{i-1} otherwise is preferred to a lottery (p_i,x_i) where the agent can still win x_i with probability p_i but 0 otherwise. This result, opposite to that of Fact #7, yields another reversal of preferences (Birnbaum and Sutton, 1992).

Consequently, the agent does not price the two lotteries when facing a lottery choice and we have to show again that the pricing of each lottery is not always optimal for this case.

Fact #9:

The graph of the JCE for the lottery $(1,x_{i-1};p_i,x_i-x_{i-1})$ as a function of p_i has an inverse S–shape like the one for the case where $x_{i-1}= 0$. Moreover, x_{i-1} + JCE of

$(p_i,x_i) \neq$ JCE of $(1,x_{i-1};p_i,x_i)$ and the difference between the two JCE decreases when p_i increases (Birnbaum and Sutton, 1992).

Fact # 10:

For lotteries (x_i, $x_{i-1} > 0$) the JCE is equal to the CCE for high and low probabilities of gain, contrary to Fact #5 where $x_{i-1} = 0$ and $x_i > 0$ (Birnbaum, 1992) (see Alarie and Dionne, 2004, for a discussion).

Lotteries: $(1,x_{i-1};p_i,x_i-x_{i-1})$, $x_{i-1} < 0 < x_i$.

This subsection introduces an additional complexity, namely the presence of negative and positive outcomes in the same lottery. Kahneman and Tversky (1979) have already pointed out the asymmetry between these outcomes and its consequence for expected utility theory. Below, we present additional facts that show the significance of this asymmetry for lottery choices and pricing.

Fact # 11:

When an agent is indifferent to a choice between a lottery $(1,x_{i-1}; p_i,x_i-x_{i-1})$ and a sure monetary amount 0, the value of $|x_{i-1}|$ is a lot smaller than $|x_i|$. This result is far too extreme to be explained by a wealth effect or by decreasing risk aversion as Tversky and Kahneman (1992) have pointed out.

Fact # 12:

For comparisons of lotteries with the same expected value as in Bostic et al. (1990), the lottery with the monetary amount x_{i-1} closer to 0 is always chosen.

Fact # 13:

In two of the four tests in Bostic et al. (1990) there exists a reversal of preferences, while there is no reversal for the other two tests. This situation is more complex than the one for lotteries (p_i,x_i) where reversals are observed for all tests (Tversky et al. 1990).

Fact # 14:
For this type of lottery ($x_{i-1} < 0 < x_i$), CCE = JCE for high probabilities but CCE < JCE for low probabilities (Bostic et al. 1990). This result is like the ones for lotteries (p_i,x_i) (Fact #5) and runs counter to the ones where x_{i-1},$x_i > 0$ (Fact #10).

These fourteen facts strongly suggest that an evaluation function designed to take all of them into account simultaneously would be different from those already documented in the literature. This is why we analyse the processes behind the preferences in order to construct a unified explanation of the fourteen facts above listed. Then an evaluation function is derived from this analysis in Section 5.

3. Qualities of lotteries

We now present a model which yields optimal decision processes and takes into account the preceding facts. We first define the vector of qualities associated with any set of elements p_i and x_i, and then the lexicographic order relation $\prec_L$ used to compare different vectors.

3.1 *Vector of Qualities*

We described four collections of sets of qualities $\wp_j$, j=1,...,4. They may contain the two sets P and X, their union and their intersection or any set of other lottery qualities described below, along with their unions and their intersections. In a first step, each group of elements p_i and x_i is naturally split into elements that belong respectively to the sets P and X defined in Section 2. So, the first collection of sets of qualities becomes $\wp_1 = \{P,X,P \cup X,\varnothing\}$ where $\varnothing = P \cap X$. Monetary amounts can be positive or negative and these qualities are already mentioned in the literature (Tversky and Kahneman 1992). So, we define two other sets: $X^+ = \{x_i / x_i \in]0, \propto[\}$ and $X^- = \{x_i / x_i \in]-\propto,0\ [\}$. By considering the union and the intersection of these two sets we obtain $\wp_2 = \{X^+,X^-,X^- \cup X^+,\varnothing\}$. Note that $x_i = 0$ does not belong to any set in $\wp_2$ since this monetary value is considered neutral (Kahneman and Tversky, 1992).

As in Kahneman and Tversky (1979), we assume that probabilities have surety (S) and risk (R) qualities. Prelec (1998) obtained a $w(p_i)$ function that takes into account the qualitative difference between impossibility (I) and risk (R). So, we define a partition of P as {S,R,I}: $S = \{1\}$, $R = \{p_i/p_i \in]0,1[\}$ and $I = \{0\}$. The third collection of sets of qualities becomes $\wp_3 = \{R,S,I,R \cup S,R \cup I,S \cup I,P,\varnothing\}$ where $P = R \cup S \cup I$ and $\varnothing = R \cap S \cap I = R \cap S = S \cap I = R \cap I$. We also add to the literature two new qualities for probabilities that indicate whether a lottery has high chances of winning or not. In the partition {H,L} of P, the elements of H have the high quality and those of L have the low quality: $L = \{p_i/p_i \in [0, p^*[\}$ and $H = \{p_i/p_i \in [p^*,1]\}$. A value of p*, the fixed point of the inverse S–shape probability weighting function, larger than 0.3 but smaller than 0.5, is observed in many tests (see Prelec, 1998, for a discussion)[2]. In this paper we assume that p* belongs to [0.3,0.5]. So the fourth collection is $\wp_4 = \{H,L,P,\varnothing\}$. The existence of H and L is empirically supported by the common ratio paradox (Fact # 2), the comparison in the preference reversal (Fact # 1) and the pricing of lotteries (Fact # 3), where, in each of these tests, one can observe a different way of judging the probabilities that belong to H and L.

Each element p_i or x_i or each set of these elements has a vector of qualities denoted $Q(\bullet) = (q_1,q_2,q_3,q_4)$ where the set $q_j \in \wp_j$. The set q_j associated with one p_i or x_i is the intersection of all sets of $\wp_j$ that contains this p_i or x_i. For example, the probability $p_i = 1$ belongs to the next four sets of $\wp_3$ that are S, R∪S, S∪I, and P. Then S ∩ (R∪S) ∩ (S∪I) ∩ P = S and the q_3 of $p_i = 1$ is S. S is the smallest set included in all other sets. When there is no such a set, then $q_j = \varnothing$. For example, the q_2 of any probability is ∅, because a probability cannot have monetary values. Any set of probabilities or monetary amounts is denoted Θ^n where the superscript n = 1,2,… identifies the different sets used in the Optimal Process (to be defined). The vector of qualities of the set Θ^n is denoted $Q(\Theta^n)$. Sometimes when it is pertinent we write each element of the set rather than Θ^n. The j^{th} quality of $Q(\Theta^n)$ is the union of the q_j of each element of Θ^n.

Definition 1: Vector of Qualities

1.1) $Q(p_i) = (P,\varnothing,q_3,q_4)$ where $p_i \in q_3 \in \{R, S, I\}$ and $p_i \in q_4 \in \{H, L\}$.

1.2) $Q(x_i) = (X,q_2,\varnothing,\varnothing)$ where $x_i \in q_2 \in \{X^+, X^-\}$ or $q_2 = \varnothing$ if $x_i = 0$.

1.3) The j^{th} quality of $Q(\Theta^n)$ is the union of the j^{th} quality of elements p_i and x_i.

Example 3.1

Suppose we have three probabilities .2, .6, and 1. The vectors of qualities of these probabilities are Q(.2) = (P,∅, R, L), Q(.6) = (P,∅, R, H), and Q(1) = (P,∅, S, H), respectively. For the set {.2,.6}, Q(.2,.6) = (P∪P, ∅∪∅, R∪R, H∪L) = (P,∅,R,P) and Q(.6,1) = (P∪P, ∅∪∅, R∪S, H∪H) = (P,∅,R∪S,H).

A partition of a set Θ^n is a collection of disjoint subsets of Θ^n whose union is all of Θ^n. In this paper, the partitions of Θ^n are always composed of two sets denoted Θ^{2n}, $\Theta^{2n+1} \neq \varnothing$. Figure 1 shows the first partition considered in Example 3.2.

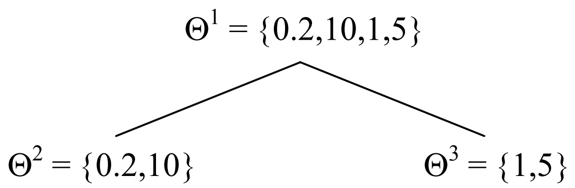

Figure 1: The partition of Θ^n, n=1, is $\Theta^{2n} = \Theta^2$ and $\Theta^{2n+1} = \Theta^3$

Example 3.2

Suppose an individual faces a lottery (0.2,10) and a sure monetary amount (1,5). We then have the set $\Theta^1 = \{0.2,10,1,5\}$. Q(0.2) and Q(1) are defined as in Example 3.1 and Q(10) = Q(5) = $(X,X^+,\varnothing,\varnothing)$. Then the vector associated with the first lottery $Q(\Theta^2) = Q(0.2,10) = (P\cup X,X^+,R,L)$ and the one associated with the sure monetary amount $Q(\Theta^3) = Q(1,5) = (P\cup X,X^+,S,W)$. These vectors take into account every quality of each lottery. For example, they indicate that the first lottery (0.2,10) has the quality $P\cup X$ and this lottery has a positive monetary amount X^+ and is risky (R) while the second lottery is sure (S). The individual can also form other partitions from the set $\Theta^1 = \{0.2,10,1,5\}$ by grouping the two probabilities together and the two monetary amounts together which yields $Q(\Theta^2) = Q(0.2,1) = (P,\varnothing,R\cup S,P)$ and $Q(\Theta^3) = Q(10,5) = (X,X^+,\varnothing,\varnothing)$.

Two vectors of qualities $Q(\Theta^{2n})$ and $Q(\Theta^{2n+1})$ can be more or less different. We use the intersection of the two vectors of qualities $Q(\Theta^{2n}) \cap Q(\Theta^{2n+1}) = (q_1^{2n} \cap q_1^{2n+1}, q_2^{2n} \cap q_2^{2n+1}, q_3^{2n} \cap q_3^{2n+1}, q_4^{2n} \cap q_4^{2n+1})$ in order to judge how different the vectors of qualities are where the j^{th} quality of Θ^n is denoted q_j^n. The lexicographic order relation $\prec_L$ allows the comparison of these vectors of qualities in $\wp_1$ x $\wp_2$ x $\wp_3$ x $\wp_4$. In the next definition, $A \subset B$ means that A is a proper subset of B (i.e. $A \neq B$).

Definition 2: Lexicographic order relation

A relation $\prec_L$ on $\wp_1 \times \wp_2 \times \wp_3 \times \wp_4$ is defined as $Q(\Theta^{2n}) \cap Q(\Theta^{2n+1}) \prec_L Q(\Theta^{2m}) \cap Q(\Theta^{2m+1})$, if there exists a $\hat{j}$ such that $(q_{\hat{j}}^{2n} \cap q_{\hat{j}}^{2n+1}) \subset (q_{\hat{j}}^{2m} \cap q_{\hat{j}}^{2m+1})$ and, for all $j < \hat{j}$, $(q_j^{2n} \cap q_j^{2n+1}) = (q_j^{2m} \cap q_j^{2m+1})$.

So when $Q(\Theta^{2n}) \cap Q(\Theta^{2n+1}) \prec_L Q(\Theta^{2m}) \cap Q(\Theta^{2m+1})$, we observe that the vectors $Q(\Theta^{2n})$ and $Q(\Theta^{2n+1})$ are more different than the vectors $Q(\Theta^{2m})$ and $Q(\Theta^{2m+1})$. We will conclude that the sets Θ^{2n} and Θ^{2n+1} are also more different than the sets Θ^{2m} and Θ^{2m+1}. For a partition of Θ^n into two sets, the more homogeneous their elements, the more the two sets will differ. $\prec_L$ is nonreflexive and transitive, but it is a partial order relation because comparability fails (Munkres, 1975).[3]

> Example 3.3
>
> For the probabilities of Example 3.1, we have $Q(0.2) \cap Q(0.6) = (P\cap P, \varnothing\cap\varnothing, R\cap R, L\cap H) = (P,\varnothing,R,\varnothing)$ and $Q(0.6) \cap Q(1) = (P,\varnothing,\varnothing,H)$. Then $(P,\varnothing,\varnothing,H) \prec_L (P,\varnothing,R,\varnothing)$ since P=P, $\varnothing = \varnothing$ and $\varnothing \subset R$. So 0.6 and 1 are more different than 0.6 and 0.2 because in {0.6,1} one probability is risky (R) and the other probability is sure (S) while in {0.2,0.6} the two probabilities are risky (R) and the set contains more similar elements.

Because we use the lexicographic order relation, the positions of the qualities in the vectors are important. The most natural difference between qualities is the one between probabilities (P) and monetary amounts (X). This is why this difference is the first one, $q_1^{2n} \cap q_1^{2n+1}$, considered in the vector of qualities $(q_1^{2n} \cap q_1^{2n+1}, q_2^{2n} \cap q_2^{2n+1}, q_3^{2n} \cap q_3^{2n+1}, q_4^{2n} \cap q_4^{2n+1})$. Moreover, in lottery tests, Tversky and Kahneman (1992) have reported a very significant effect for the difference between a positive quality (X^+) and a negative (X^-) one for monetary amounts. So this difference becomes the second important one, $q_2^{2n} \cap q_2^{2n+1}$. Finally, according to different test results, the difference between the qualities of S, I and R already pointed out in the literature seems more significant than the one between the qualities of H and L, suggested in this article. In the remainder of this article, we will keep this order based on empirical tests. However, any other order is possible and, for example, some agents may find that H and L (0.9 vs 0.2) are more different than R and S (0.9 vs 1).

4. Optimal process and evaluation of lottery

With the help of the two preceding definitions of vector of qualities and lexicographic order relation $\prec_L$, we are now able to define an optimal process and to evaluate lotteries.

4.1 *Optimal Process*

An optimal process (OP) is obtained by comparing partitions of sets of elements p_i and x_i in a recursive manner. We obtain an OP by first ranking partitions of the first set Θ^1. Then each set of the partition is split into other partitions and so on. We use the lexicographic order relation (Definition 2) to compare the intersection of the vector of qualities of each pair of sets ($Q(\Theta^{2n}) \cap Q(\Theta^{2n+1})$) since the more the two sets differ the more similar the elements contained in each. Payne et al. (1993) pointed out that it is more complex to evaluate two different elements (p_1 and x_1 for example) than to compare two similar elements (p_1 and p_2, for example).

However, all partitions are not admissible in an OP. As we have seen, each set must be different from the empty set. We must also take into account the natural link between an x_i and its corresponding probability p_i to prevent non–natural judgments such as comparing p_1 and x_2, for example. Because these problems occur when there are more than two elements of the same lottery in the set Θ^n, we then constrain the elements p_i and x_i to the same set of the partition of Θ^n. So the constraint is used when $\#(\Theta^n \cap l_i) > 2$ (where # means cardinality and l_i, as defined in Section 2, is the set of the elements of a lottery)

Definition 3: Admissible Partition

An admissible partition of Θ^n is a two–set partition $\{\Theta^{2n}, \Theta^{2n+1}\}$ such that $\Theta^{2n}, \Theta^{2n+1} \neq \emptyset$. If there is an l_i such that $\#(\Theta^n \cap l_i) > 2$ then all couples $p_i, x_i \in \Theta^n$ will belong to the same set of the partition of Θ^n.

An Optimal Process is a series of partitions where each partition is ranked in the sense of Definition 2 and the sets of admissible partitions is constrained as in Definition 3. The first set considered is the union of elements of the two lotteries.

Definition 4: Optimal Process (OP)

An OP for a set $\Theta^{1*} = l_1 \cup l_2$ is a collection of sets $\{\Theta^{1*},\ldots,\Theta^{n*},\ldots\}$. The sets Θ^{2n*}, Θ^{2n+1*} belong to the OP if and only if $\{\Theta^{2n*}, \Theta^{2n+1*}\}$ is an admissible partition of Θ^{n*} such that $Q(\Theta^{2n*}) \cap Q(\Theta^{2n+1*}) \prec_L Q(\Theta^{2n}) \cap Q(\Theta^{2n+1})$ where $\{\Theta^{2n},\Theta^{2n+1}\}$ is any other admissible partition of Θ^{n*}.

Example 4.1

Consider the two lotteries (0.2,35) and (0.6,10). By definition $\Theta^{1*} = \{0.2,35,0.6,10\}$. Since $\#(\Theta^{1*} \cap l_i) \leq 2$ for i = 1,2, we use the set $\{0.2,35,0.6,10\}$ without the restriction in Definition 3. We have $Q(0.2) = (P,\emptyset,R,L)$, $Q(0.6) = (P,\emptyset,R,H)$ and $Q(10) = Q(35) = (X,X^+,\emptyset,\emptyset)$. Moreover $Q(0.6,0.2) = (P,\emptyset,R,P)$ and $Q(10,35) = (X,X^+,\emptyset,\emptyset)$.

If we consider different admissible sets of two elements, the partition $\{\{0.6,0.2\},\{10,35\}\}$ is optimal since $Q(0.6,0.2) \cap Q(10,35) = (\varnothing,\varnothing,\varnothing,\varnothing) \prec_L (X\cup P,X^{+},R,\varnothing) = Q(0.6,10) \cap Q(0.2,35) = Q(0.2,10) \cap Q(0.6,35)$. If we now consider any admissible partition where one set contains one element and the other three elements, the partition $\{\{0.6,0.2\}, \{10,35\}\}$ is still optimal because there is no other partition that yields $(\varnothing,\varnothing,\varnothing,\varnothing)$. For example $Q(0.2) \cap Q(0.8,10,35) = (P,\varnothing,R,\varnothing)$ and then for the first quality $\varnothing \subset P$. Finally, for the set $\{0.6,0.2\}$ the optimal partition is $\{\{0.6\},\{0.2\}\}$and for $\{10,35\}$ the partition is $\{\{10\},\{35\}\}$.

As we shall see in Section 5, for all lottery tests considered in this paper, an OP exists and is unique (see note 4).

4.2 *Lottery evaluation*

In this section we emphasize the role of qualities in the evaluation of lotteries. From an OP the agent already knows the collection of sets $\{\Theta^{1*},...,\Theta^{n*},...\}$. He must now evaluate the set Θ^{n*}, taking the values of its two subsets $\Theta^{2n*},\Theta^{2n+1*}$while considering the qualities.

The optimal partition $\{\Theta^{2n*}, \Theta^{2n+1*}\}$ of a set Θ^{n*} ($\#\Theta^{n*} > 2$) selected from a set of admissible partitions is the one that has the first j such that $q_j^{2n*} \neq q_j^{2n+1*}$ (see note 4). In fact, the agent selects the partition that has the most important qualitative difference. For the evaluation of a set, we will also consider this qualitative difference.

Definition 5.1: Value of sets with more than one element

$v(\Theta^{n^*}) = J_{q_j^{2n^*} q_j^{2n+1^*}}(v(\Theta^{2n^*}), v(\Theta^{2n+1^*}))$: $R \times R \rightarrow R$ where for all j such that $q_j^{2n^*} q_j^{2n+1^*} \neq \varnothing$, J = max j such that for all j < J, $q_j^{2n^*} = q_j^{2n+1^*}$.

When an agent evaluates a set Θ^{n^*}, knowing the values of its two subsets Θ^{2n^*} and Θ^{2n+1^*}, he will select the first two qualities of each of the subsets Θ^{2n^*}, Θ^{2n+1^*} that are different when such qualities do exist. In the above definition, the value of Θ^{n^*} is $v(\Theta^{n^*}) = J_{q_j^{2n^*} q_j^{2n+1^*}}(v(\Theta^{2n^*}), v(\Theta^{2n+1^*}))$ where the value of the two subsets are judged by considering the optimal qualities $q_j^{2n^*}$ and $q_j^{2n+1^*}$.

> Example 4.2
>
> For the comparison of lotteries (0.2,35) and (0.6,10) in Example 4.1, we obtained the partition $\Theta^{2*} = \{0.2, 0.6\}$ and $\Theta^{3*} = \{10, 35\}$. By Definition 1, Q(0.2) = (P,∅,R,L), Q(0.6) = (P,∅,R,H) and since P = P, ∅=∅ and R = R, the first different qualities are H and L. Then, from Definition 5.1, we have $J_{HL}(0.6,0.2)$ and the agent considers these qualities while judging 0.6 and 0.2. The agent can also evaluate the monetary amounts. Since Q(10) = Q(35) = $(X, X^+, \varnothing, \varnothing)$, all qualities are equal and by Definition 5.1, the agent rules out the ∅ and selects the last ones X^+. So we have $J_{X^+X^+}(10,35)$. Finally, to obtain $v(\Theta^{1^*})$ the agent considers Q(0.6,0.2) = (P,∅,R,P) and Q(10,35) = $(X, X^+, \varnothing, \varnothing)$. Since P and X are the first different qualities we have $v(\Theta^{1^*}) = J_{PX}(J_{HL}(0.6,0.2), J_{X^+X^+}(10,35))$. This means that the agent judges the two monetary amounts together and the two probabilities together. Then he makes a third judgment considering the qualities P and X.

It remains to discuss the values of a single element p_i or x_i. Like Tversky and Kahneman (1992), we consider the particular role of the next two boundaries for probabilities 0 and 1. The corresponding boundary of a p_i noted b_{p_i} is 0 if $p_i < p^*$ or 1 if not. As the monetary amount 0 has no quality in $\wp_2$ (i.e. $q_2 = \emptyset$) there is no need for a boundary for x_i and $v(x_i) = x_i$.[5]

Some theories use an evaluation function like $w(p_i)$ in NEU or use directly the value of the elements like p_i in EU. So we must include this possibility of a single–element judgment in our model. In particular Kahneman and Tversky (1979) and Prelec (1998) have pointed out that the shape of the w(p) function reflects the qualitative difference between the boundaries and the other probabilities. In the same spirit, judgments of one probability with its corresponding boundary are allowed. However, these judgments with a boundary will be used only when they are optimal for the agent. The lexicographic order relation $\prec_L$ of Definition 2 sets whether the boundaries are used or not.

A boundary is optimal for judging a probability if there is an element $p_i \in S$ of a set $\{p_i,\theta\}$ that is less different from its corresponding boundary b_{p_i} than from the other element θ, where θ is either a probability or a monetary amount. In fact, θ is the element of Θ^1* which is most similar to p_i, among those that can be judged with p_i in the OP. Then, like the OP that leads to judgments with a similar element, the use of boundaries also leads to judgments with a more similar element for p_i. When a boundary b_{p_i} is used, the judgment of (p_i,b_{p_i}) is as in Definition 5.1 and the value of p_i is $v(p_i) = J_{q_J^p q_J^b}(p_i,b_{p_i})$: RxR→R, where q_J^p and q_J^b are the first different qualities of p_i and b_{p_i}.

Definition 5.2: Value of elements p_i

$v(p_i) = J_{q_J^p q_J^b}(p_i , b_{p_i})$ if and only if there is a set $\Theta^{n^*} = \{p_i,\theta\} \in OP$ and $p_i \in R$, such that $Q(p_i) \cap Q(\theta) \prec_L Q(p_i) \cap Q(b_{p_i})$. $v(p_i) = p_i$ otherwise.

> Example 4.3
> Suppose the lottery (0.8,10) and $b_{0.8}$ is 1 since $0.8 > p^*$. Then $Q(0.8) \cap Q(10) = (\emptyset,\emptyset,\emptyset,\emptyset) \prec_L (P,\emptyset,R\cap S,H) = Q(0.8) \cap Q(1)$. So the boundary 1 is used and the value of 0.8 is $J_{RS}(0.8,1)$.

5. Results

We now present in Table 1 results that derive the judgments of the optimal processes associated with eleven tests found in the literature. As we shall see, they correspond to the fourteen facts listed in Section 2. The other possible tests are a combination of these eleven tests. The proofs are in the Appendix and their implications are discussed in Section 5.1.

(Table 1 here)

5.1 *Examples and discussion*

We now discuss the facts (JCE, CCE, common ratio, preference reversal …) along with the judgments obtained in Table 1 with numerical examples. Without loss of generality, a simple way to take qualities into account when evaluating a lottery is to modify the judgment function by introducing a parameter $\alpha_{q_J^{2n*} q_J^{2n+1*}}$ that multiplies the value of one set $v(\Theta^{2n^*})$, while letting the value of the other $v(\Theta^{2n+1^*})$ unchanged. The judgment $J_{q_J^{2n*} q_J^{2n+1*}}(v(\Theta^{2n^*}),v(\Theta^{2n+1^*}))$ becomes $J(\alpha_{q_J^{2n*} q_J^{2n+1*}} v(\Theta^{2n^*}),$

$v(\Theta^{2n+1*})$). The choice of the set does not matter, so the parameter could multiply $v(\Theta^{2n+1*})$ instead of $v(\Theta^{2n*})$. In this paper, we focus on the qualities in (i) and (ii) below. For all other qualities $\alpha_{q_J^{2n*} q_J^{2n+1*}}$ is equal to 1. From Table 1, we observe, in the last column, the qualities associated with each judgment function. The corresponding $\alpha_{q_J^{2n*} q_J^{2n+1*}}$ and Θ^{2n*} are either:

i) $\alpha_{X^+X^-}$, α_{HL}, or α_{RS}, α_{RI} when Θ^{2n*} has respectively the qualities X^-, H or R.
ii) α_{HH}, α_{LL}, when Θ^{2n*} contains the largest p_i.

In order to emphasize the role of qualities and remain close to the other models in the literature, the evaluation of a lottery $(1,x_1;\ p_2,x_2–x_1)$ is represented by:

$$\Pi_1\ x_1 + \Pi_2\ p_2\,(x_2–x_1) \qquad (1)$$

where Π_1 is the product of the parameter $\alpha_{q_J^{2n*} q_J^{2n+1*}}$ that multiplies the values $v(\Theta^{2n*})$ of the OP, when 1 or x_1 belong to Θ^{2n*}, and Π_2 is the product of the parameter $\alpha_{q_J^{2n*} q_J^{2n+1*}}$ that multiplies the values $v(\Theta^{2n*})$ of the OP, when p_2 or $(x_2 – x_1)$ belong to Θ^{2n*}. The role of each pertinent quality is then taken into account by Equation 1. Note that if qualities play no role all parameters $\alpha_{q_J^{2n*} q_J^{2n+1*}} = 1$ and the model is reduced to expected utility with risk neutrality. Many functions $J_{q_J^{2n*} q_J^{2n+1*}}$ $(v(\Theta^{2n*}), v(\Theta^{2n+1*}))$ can be used to obtain Equation (1) but we do not discuss all of them in this article since it is beyond its scope. An example of such function is presented in note 6. We now apply Equation 1 to the different tests.

Tests 1, 2

The next example is about the comparison in the preference reversal paradox. In this section, we assume that p* = 4, the middle point in the interval [0.3,0.5] discussed in Section 3.1.

Example 5.1

In Tversky et al. (1990) we observe that 83% of the subjects choose (0.97,4) over (0.31,16). To explain the result we use the process of Test 1, so the judgments of the optimal process are $J_{PX}(J_{HL}(p_1, p_2), J_{X^+X^+}(x_1,x_2))$. For the comparison of the two probabilities, the parameter α_{HL} multiplies the highest probability 0.97 as we fixed above in (i) and $\Pi_2 = \alpha_{HL}$. There is no other parameter, since $\alpha_{X^+X^+}$ and α_{PX} are set equal to 1. For the evaluation of the first lottery, Equation 1 gives α_{HL} (0.97×4) and for the second lottery we obtain (0.31×16). From the test result α_{HL} $(0.97 \times 4) - (0.31\times16) > 0$ and α_{HL} must be greater than 1.28 to obtain the desired result. We observe that the qualitative difference between elements of H and L increases the difference between the two p_i, and Fact # 1 is explained. It is important to notice that the way we introduce the parameters does not affect the conclusion. If α_{HL} would multiply the smallest probability 0.31 then α_{HL} must be lower than 1/1.28 to explain the result. The parameter still increases the difference between probabilities.

Example 5.2

Kahneman and Tversky (1979) test sequentially a choice between (0.45, 6000) and (0.90, 3000) and another choice between (0.001, 6000) and (0.002, 3000). 86% of the subjects select the second lottery in the first task but 73% select the first lottery in the second task. This is the common ratio paradox. By using the same process of Example 5.1 with the parameter α_{HH}, we obtain: $\alpha_{HH}0.90 \times 3000 > 0.45 \times 6000$ and with the parameter α_{LL} we obtain $\alpha_{LL}0.002 \times 3000 < 0.001 \times 6000$ Consequently, we must have $\alpha_{HH} > 1 > \alpha_{LL}$ to solve Fact # 2.

The next example is about the JCE (Fact # 3).

Example 5.3

Birnbaum et al. (1992) obtained that the JCE of the lottery (0.95,96) has a value around 70. From Table 1, the JCE for this type of lottery is obtained from Test 2, where the judgments of the OP are $J_{PX}(J_{RS}(p_1, b_{p_1}), x_1)$.There is only one relevant parameter, since P and X are not taken into account by (i) and (ii). From $J_{RS}(p_1, b_{p_1})$, we have that α_{RS} multiplies p_1x_1. We obtain α_{RS} $0.95 \times 96 = 70$, which implies that $\alpha_{RS} = 0.77$. For a small probability, the lottery is overestimated and then the $\alpha_{RI} > 1$. This solves Fact #3. Consequently, the judgment of probabilities with boundaries is similar to the inversed S–shape used by Prelec (1998), Tversky and Kahneman (1992) and Wu and Gonzalez (1996). See note 7. If we combine Test 2 and Test 1 (with the qualities of H and L), we explain the existence of the preference reversal paradox (Tversky et al. 1990).

The use of qualities to obtain an OP for Test 2 leads to almost the same judgment of probabilities as the one in NEU. However, for this test, we use judgments with boundaries instead of the function $w(p_i)$. As the proof of Test 2 shows in Appendix, it is optimal, in our model, to judge a probability with a boundary to obtain the JCE. In contrast, in Test 1 the judgment with boundaries was not shown to be optimal, since no boundary satisfies the condition. An example of judgment of two p_i is to be found in Ranyard (1995), where the subject says: “I’ve chosen option 2 because there’s more chance of winning a smaller amount…” Then the agent clearly compares the two chances of winning or the two probabilities according to Test 1. Consequently, qualities play an important role in defining an OP by allowing both kinds of judgment and by identifying when it is optimal to use one type of judgment (with a boundary) instead of the other (without a boundary).

The inverse S–shaped function $w(p_i)$ does not fit the data for the comparison of two probabilities with the risky quality. In fact, Alarie and Dionne (2001) show that a one parameter $w(p_i)$ function cannot take into account simultaneously the next three comparisons of lotteries: $J_{LL}(p_1, p_2)$, $J_{HH}(p_1, p_2)$ and $J_{HL}(p_1, p_2)$. Examples 5.1 and 5.2 clearly show that the model presented in this paper can solve these three comparisons of lotteries. The judgments that consider HL along with both SR and IR are also very difficult to explain with the inverse S–shaped function $w(p_i)$. In summary, the above discussion uses five different judgment functions $J_{RI}(p_1,0)$, $J_{RS}(p_1,1)$, $J_{HH}(p_1,p_2)$, $J_{LL}(p_1,p_2)$ and $J_{HL}(p_1,p_2)$ to explain Fact #1 to Fact#3.

Since the main tool for explaining paradoxes with positive monetary amounts is the judgment of probabilities, inverse results will then be obtained for negative monetary amounts. This explains Fact #6.

Tests 3, 4

Another important group of facts concerns the difference between the CCE and the JCE (Fact #3). Tversky et al. (1990) introduced the CCE in order to obtain a lottery price from a comparison with a sure monetary amount. As Bostic et al. (1990) have pointed out, this procedure is closer to the comparison of two lotteries than the JCE and can thus reduce the number of reversals. We will see that this is not necessarily the case.

> Example 5.4
>
> Test 3 and Test 4 are about Fact #5. The judgments of the optimal process are $J_{PX}(J_{RS}(p_1,1), J_{X^+X^+}(x_1,x_2))$ for Test 3 and the evaluation of the lottery is given by $\alpha_{RS}p_1x_1$ as in Test 2. So there is no difference between the JCE and the CCE for high probabilities.

The optimal process of Test 4 is $J_{PX}((J_{RS}(J_{RI}(p_1,b_{p_1}),1),\ J_{X^+X^+}(x_1,x_2))$ and the evaluation of the lottery is given by $\alpha_{RS}\ \alpha_{RI}\ p_1x_1$ = CCE. The first parameter α_{RS} comes from $J_{RS}(\bullet)$ and the second α_{RI} from $J_{RI}(\bullet)$. If the agent now uses JCE instead of CCE for low probabilities, Test 2 implies that JCE = $\alpha_{RI}\ p_1x_1$ and then CCE = α_{RS}JCE. Since $\alpha_{RS} < 1$ the JCE is larger than the CCE.

One can note that the reason why the use of the CCE decreases the number of reversals is not, in this paper, because the CCE is closer to the comparison of lotteries than the JCE. It is because there is an additional RxS effect: For $p_1 \in H$, the judgment of p_1 is with the probability 1 in both cases (Tests 2 and 3). However, for $p_1 \in L$, when we test the CCE, the agent first judges the probability, as in the JCE, and compares the result to the sure probability which involves another judgment that considers the qualities R and S (Test 4). So JCE > CCE.

Tests 5, 6, 7

Test 5 gives an optimal way to compare a lottery where $x_1 > 0$ with a lottery where $x_1 = 0$. Note that, in all tests, $(1,x_1;p_2,x_2-x_1)$ is preferred to (p_2,x_2) in a direct choice. This test result obtained by Birnbaum and Sutton (1992) is foreseeable since, for both lotteries, you can win x_2 with probability p_2 and you can win x_1 with probability $1 - p_2$ for the first lottery and 0 with probability $1-p_2$ for the second one. Test 5 gives $x_1 + \alpha_{RS}\, p_2(x_2-x_1)$ for the first lottery and, for the second one, $\alpha_{RS}\, p_2x_2$. So the difference in evaluation between the two lotteries is $x_1 - \alpha_{RS}\, p_2x_1 > 0$. This is positive because both α_{RS} and p_2 are smaller than 1. This explains Fact #8. When we consider the pricing of these lotteries, Birnbaum and Sutton (1992) obtained the surprising result discussed below.

Example 5.5

Birnbaum and Sutton (1992) obtained that JCE of (p_2,x_2) > JCE of $(1,x_1;p_2,x_2-x_1)$ for the lotteries $(1,x_1;p_2,x_2-x_1)$ and (p_2,x_2) where $x_2 = 96$, $x_1 = 24$ and $p_2 = 0.8$.

We now show that this contradictory result can be rationalized. This test result is difficult to accept intuitively because the expected value of the lottery with the higher JCE is lower than the one for the other lottery $p_2x_2 < p_2x_2 + (1-p_2)x_1$. In fact for the JCE of (p_2,x_2) we have $\alpha_{RS}\, p_2\, x_2$, and for the JCE of $(1,x_1;p_2,x_2-x_1)$ we use Test 6 and obtain $J_{RS}(J_{PX}(1,x_1),J_{PX}(J_{RS}(p_2,b_{p_2}),x_2-x_1))$. So the evaluation of this lottery is $x_1 + \alpha_{RS}\, \alpha_{RS}\, p_2(x_2-x_1)$. From Birnbaum and Sutton (1992) data, $x_1 = 24$, $x_2 = 96$ and $p_2 = 0.8$. So we have (Fact #7) $x_1 + \alpha_{RS}\, \alpha_{RS}\, p_2(x_2-x_1) < \alpha_{RS}\, p_2x_2$ when $\alpha_{RS} \in]1/2,5/6[$. This interval contains $\alpha_{RS} = 0.77$ which corresponds to the value found for the test in Example 5.3 (Birnbaum et al., 1992). This result (Test 6), along with Test 5, involves a second reversal of preferences explained by the model. The first was explained by Tests 1 and 2 together.

For the lottery $(1,24;p_2,72)$ where the agent can win 96 with probability p_2 and 24 otherwise, Birnbaum and Sutton (1992)[8] pointed out that the JCE of $(1,24;p_2,72)$ is different from 24 + JCE of $(p_2,72)$. Moreover, the spread between the two JCE decreases when the probability increases (Fact # 9). We can explain this result by taking the derivative of (JCE of $(1,x_1;p_2,x_2-x_1)$ – JCE of (p_2,x_2-x_1)) with respect to p_2 where $p_2 \in H$. The difference between the two JCE is equal to $x_1 + \alpha_{RS}\, \alpha_{RS}\, p_2(x_2-x_1) - \alpha_{RS}\, p_2(x_2-x_1)$ and the derivative with respect to p_2 yields $(\alpha_{RS} - 1)\, \alpha_{RS}\, (x_2-x_1) < 0$, since $\alpha_{RS} < 1$ and $x_2 > x_1 > 0$. For $p_2 \in L$ we obtain $(\alpha_{RS} - 1)\, \alpha_{RI}\, (x_2-x_1) < 0$ since $\alpha_{RI} > 1$ and $\alpha_{RS} < 1$.

One can note that we have a comparison with the boundaries as in the cases where $x_1 = 0$. This is the reason why the evaluation of these lotteries as a function of p_2 still has an inverse S–shaped curve, as discussed in footnote 7 (Fact # 9).

When x_i, $x_{i-1} > 0$ and $p_i \in L$, the process for Test 7 gives CCE= $x_1 + \alpha_{RS}\ \alpha_{RI}\ p_2(x_2-x_1)$. Test 6 gives JCE = $x_1 + \alpha_{RS}\ \alpha_{RI}\ p_2(x_2-x_1)$ and CCE = JCE for low probabilities. We have the same result when $p_i \in W$. Then, for this type of lottery, there is no difference between JCE and CCE for all probabilities (Fact #10).

Tests 8, 9, 10, 11

These tests consider lotteries where $x_{i-1} < 0 < x_i$. We first discuss the JCE and CCE. The judgments of Tests 8, 10 and 11 are:

Test 8) $J_{X^-X^+}(J_{PX}(1,-x_1), ((J_{PX}(J_{RS}(p_2,b_{p_2}), x_2+x_1))$

Test 10) $J_{X^-X^+}(J_{PX}(1,-x_1), ((J_{PX}(J_{RS}(p_2,1), J_{X^+X^+}(x_2+x_1,x_3)))$

Test 11) $J_{X^-X^+}(J_{PX}(1,-x_1), J_{PX}(J_{RS}(J_{RI}(p_2,b_{p_2}), 1), J_{X^+X^+}(x_3, x_2+x_1)))$

For these three OP, the judgment $J_{X^-X^+}$ has two parts. The first ones $J_{PxX}(1,-x_1)$ are the same for all processes and only the second parts differ. These second parts are identical to those of Tests 2, 3, and 4 respectively, where a monetary amount is equal to 0. So, for $p_2 \in W$, the JCE = CCE (Test 8 vs Test 10) and, for $p_2 \in L$, the JCE ≠ CCE (Test 8 with the boundary 0 vs Test 11) for the same reasons that apply for those used to explain Tests 2, 3 and 4. This explains why the test results obtained by Bostic et al. (1990) are the same as those in Fact #14 (Tversky et al., 1990).

Bostic et al. (1990) and Luce et al. (1993) use four pairs of lotteries taken from Lichtenstein and Slovic (1971). These tests are very difficult to explain since, contrary to the cases where $x_1 = 0$ or $x_1 > 0$ (Facts #7 and #8), the reversal does not occur systematically. So the value of each parameter is important. These tests also provide an opportunity to check if the values of the parameters obtained from all preceding examples are correct.

Example 5.6

For the comparison of pairs of lotteries in Bostic et al. (1990), we use Test 9 $J_{X^+X^-}(J_{PX}(J_{HH}(1,1),J_{X^-X^-}(-x_1,-x_3)), J_{PX}(J_{HL}(p_2,p_4), J_{X^+X^+}(x_2+x_1, x_4+x_3)))$. For the parameter that takes into account the difference between a positive and a negative monetary amount, Tversky and Kahneman obtained a value around 2.25. So we use $\alpha_{X^+X^-} = 2.25$. We also use $\alpha_{HL} = 1.19$, since this is the average of the 14 tests in Tversky et al. (1990). We set $1/\alpha_{RI} = \alpha_{RS} = 0.77$, which corresponds to the values in the preceding examples. For the comparison (Test 9), we have $\alpha_{X^+X^-}(x_2-x_3) + p_2(x_2-x_1) - \alpha_{HL}\, p_4(x_4-x_3)$. The JCE (Test 8) for lotteries with low probabilities and high probabilities are respectively:

$$\alpha_{X^+X^-}\, x_{i-1} + \alpha_{RI}\, p_i(x_i-x_{i-1})$$

$$\alpha_{X^+X^-}\, x_{i-1} + \alpha_{RS}\, p_i(x_i-x_{i-1})$$

Table 2 shows the results for the four cases where in each case A–B measures the difference between the lotteries A and B when they are compared. For these lotteries you can win x1 with probability p1 and x2 with probability 1–p1.

(Table 2 here)

So the lottery with the x_1 ($x_1 < 0$) closer to 0 is always selected in a direct comparison (Fact #12) and the reversals occur for lotteries 1 and 4 (Fact #13). The parameters we use fit the data well, in particular $\alpha_{X^+X^-} = 2.25$ taken from Tversky and Kahneman (1992) which is the most significant (Fact #11).

6. CONCLUSION

We have seen that the concept of qualities is useful in two different ways. First the qualities settle the optimal process and, second, they are taken explicitly into account in the judgments. So the role played by qualities seems strong enough to justify the necessity of using them. As shown in Section 5, they serve as a powerful instrument in solving the fourteen facts in the literature (Section 2), which are the most significant for two–point lotteries.

This model can be extended to n point lotteries. Another way to continue the research is to try to explain the difference between the buying and selling prices. Ambiguity is another interesting problem and the explanations of these last two problems seem to be closely related.

Notes

1. For $x_1 < 0 < x_2$ Kahneman and Tversky (1979) do not use segregation. This procedure of doing would not affect the result of this paper. Perhaps the best way is not to use segregation when x_1 and x_2 have almost the same size and to use it when they are very different.

2. p* is such that for $p \in]0,1[$ $w(p) < p$ if $p > p^*$ and $w(p) > p$ if $p < p^*$.

3. For the partition $\Theta^1 = \{p_1,x_1,x_2\}$ and $\Theta^2 = \{p_2\}$, the first quality of $Q(\Theta^1) \cap Q(\Theta^2)$ is P and for the partition $\Theta^3 = \{x_1,p_1,p_2\}$ and $\Theta^4 = \{x_2\}$ the first quality of $Q(\Theta^3) \cap Q(\Theta^4)$ is X. So $X \not\subset P$, $P \not\subset X$ and $P \neq X$ and comparability fails.

4. There exists a unique OP, because for each set considered there is only one optimal partition. When $\#\Theta^n = 2$ the result is obvious. When $\#(\Theta^n \cap l_i) > 2$ each set contains at least one element p_i and one x_i and the first quality is $P \cup X$ for each set. So if some elements have the quality X^- and others have X^+ then putting each x_i in one set according to these qualities is optimal. If all x_i have the same quality then splitting them into two sets, according to R and S qualities, is optimal. This last splitting is always possible, since when a lottery has more than two elements there is always one p_i with the quality R and another one with the quality S. So only one optimal partition is obtained by considering the first different qualities.
   When $\#(\Theta^n \cap l_i) \leq 2$ and $\#\Theta^n > 2$ there is always at least one p_i and one x_i. Since there is no restriction, splitting by using the first different qualities P and X will be optimal.

5. We make this assumption to emphasize the role of probabilities. This is equivalent to assuming a linear utility function. This procedure simplifies the discussion. In other words, a non–linear $u(x_i)$ function (obtained from a judgment with another boundary $x_i = 0$, pointed out in Kahneman and Tversky (1979)) would not affect the results in this paper. However, as in Tversky and Kahneman (1992), there is a difference between strictly positive and strictly negative monetary amounts.

6. Many other judgment functions can lead to Equation 1. For example, in the absence of qualities, when a lottery A is compared to a lottery B we can have:

   1) $J^1(p_i,x_i) = \text{sgn}\, p_i x_i$. sgn = + if p_i and $x_i \in A$ and sgn = – if not.
   2) $J^2(p_1,p_2) = (p_1-p_2)\,(x_1 + x_2)/2$ when $p_1 \in A$ and $p_2 \in B$.
   3) $J^3(x_1,x_2) = (x_1-x_2)(p_1 + p_2)/2$ when $x_i \in A$ and $x_2 \in B$.
   4) $J^4(v(\theta^1), v(\theta^2)) = v(\theta^1) + v(\theta^2)$ when $\#(\theta^1), \#(\theta^2) > 1$.
   5) $J^5(p_i, b_{p_i}) = p_i$

   The first equation is standard, while when we put together the result of (2) and (3) by using (4) we obtain $p_1x_1-p_2x_2$. So these judgments lead to expressions with terms p_ix_i even though the agent is allowed to compare two p_i or two x_i and the evaluation of a lottery is given by $\Sigma p_i x_i$. In Equation 5, the p_i is judged by considering the boundary, in order to obtain a judged value of p_i. Other functions could be used, such as $J^2(p_1,p_2) = p_1/p_2$, for example. To obtain Equation 1, we consider qualities by taking into account (i) and (ii). The average of x_i used in (2) and the one of p_i used in (3) is also obtained by considering qualities.

7. As Prelec (1998) has pointed out, the closer the probabilities are to boundaries the greater the effect of RS or RI. We can take this fact into account by setting: $\alpha_{RS}(p^*)=1$ and $d\alpha_{RI}/dp < 0$ for $p < p^*$ $d\alpha_{RS}/dp > 0$ for $p > p^*$. The same idea could be applied to other parameters such as α_{HH} for example.
8. They use the seller's point of view and not the neutral's one, but the result remains valid.

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APPENDIX

Test 1

Let (p_1,x_1) be compared to (p_2,x_2), where $Q(p_1) = (P,\varnothing,R,H) = Q(p_2)$. Then the judgments of the optimal process are $J_{PX}(J_{HH}(p_1, p_2), J_{X^+X^+}(x_1,x_2))$.

Proof: Since $\#(\Theta^1 \cap l_k) = 2$ there is no restriction, except the one where a set must not be equal to the empty set. Let the partition $\{\{p_1,p_2\}, \{x_1,x_2\}\}$ then $Q(p_1,p_2) \cap Q(x_1,x_2) = (P,\varnothing,R,H) \cap (X,X^+,\varnothing,\varnothing) = (\varnothing,\varnothing,\varnothing,\varnothing)$. For all other partitions that consider sets of two elements, we have that the intersection is equal to $(X\cup P,X^+,R,H)$ and $(\varnothing,\varnothing,\varnothing,\varnothing)\prec_L (X\cup P,X^+,R,H)$. For partitions where one set contains one element, the q_1 is different from $\varnothing$ and none of these partitions are chosen. We have $Q(p_1,p_2) = (P,\varnothing,R,H)$ and $Q(x_1,x_2) = (X,X^+,\varnothing,\varnothing)$ and since $P\neq X$, we use $J_{PX}(\cdot)$.Since $Q(p_1,p_2) \prec_L Q(p_i,b_{p_i})$ for all i, we do not use the boundaries. Since $Q(p_1) = Q(p_2) = (P,\varnothing,R,H)$, we use $J_{HH}(p_1,p_2)$. Since $Q(x_1) = Q(x_2) = (X,X^+,\varnothing,\varnothing)$, we use $J_{X^+X^+}(x_1,x_2)$. $\square$

Test 2

Suppose (p_1,x_1) has to be evaluated where $Q(p_1) = (P,\varnothing,R,H)$. Then the judgments of the optimal process are $J_{PX}(J_{RS}(p_1,b_{p_1}), x_1)$.

Proof: There is only one partition $\{\{p_1\},\{x_1\}\}$. We have $Q(p_1) = (P,\varnothing,R,H)$ and $Q(x_1) = (X,X^+,\varnothing,\varnothing)$ and since $P\neq X$, we use $J_{PX}(\cdot)$. Since $Q(p_1,b_{p_1}) = (P,\varnothing,P,H) \not\prec_L (P\cup X,X^+,R,H) = Q(p_1,x_1)$ we use boundaries. Since $Q(p_1) = (P,\varnothing,R,H)$ and $Q(1) = (P,\varnothing,S,H)$ we use $J_{RS}(p_1,b_{p_1})$. The value of x_1 is x_1 and we obtain $J_{PX}(J_{RS}(p_1,b_{p_1}),x_1)$. $\square$

Test 3

Let a lottery (p_1,x_1) be compared to a sure monetary amount $(1,x_2)$, where $Q(p_1) = (P,\varnothing,R,H)$. Then the judgments of the optimal process are $J_{PX}(J_{RS}(p_1,1), J_{X^+X^+}(x_1,x_2)$.

Proof: The partition $\{\{p_1,1\}, \{x_1,x_2\}\}$ is optimal as in Test 1. Since $Q(p_1,1) = Q(p_1,b_{p_1})$ we do not use boundaries. The judgments are the same as the ones in Test 2 except for $J_{X^+X^+}(x_1,x_2)$. $\square$

Test 4

Let a lottery (p_1,x_1) be compared to a sure monetary amount $(1,x_2)$, where $Q(p_1) = (P,\varnothing,R,L)$. Then the judgments of the optimal process are $J_{PX}(J_{X^+X^+}(x_1,x_2), (J_{SR}(J_{RI}(p_1,b_{p_1}), 1)$.

Proof: As in test 1, the optimal partition is $\{\{1,p_1\},\{x_1,x_2\}\}$. The optimal qualities for the probabilities are RS and X^+X^+ for the monetary amounts. Since $Q(p_1) \cap Q(1) = (P,\varnothing,\varnothing,\varnothing) \prec_L (P,\varnothing,\varnothing,L) = Q(p_1) \cap Q(b_{p_1})$, we use the boundary. We have $Q(p_1) = (P,\varnothing,R,L)$ and $Q(b_{p_1}) = (P,\varnothing,I,L)$ so the qualities used are I and R. □

Test 5

Let $(1,x_1;p_2,x_2–x_1)$ be compared to (p_3,x_3), where $Q(p_2) = (P,\varnothing,R,H) = Q(p_3)$. Then the judgments of the optimal process are $J_{SR}(J_{PX}(1, x_1), (J_{PX}(J_{HH}(p_2, p_3), J_{X^+X^+}(x_3,x_2–x_1))$.

Proof: Since the first lottery has four elements, each p_i and x_i belong to the same set. Since $Q(1,x_1) \cap Q(p_2,x_2–x_1,p_3,x_3) = (X\cup P,X^+,\varnothing,H) \prec_L (X\cup P,X^+,R,H) = Q(1,x_1,p_3,x_3) \cap Q(p_2,x_2–x_1) = Q(1,x_1,p_2,x_2–x_1) \cap Q(p_3,x_3)$ the first partition is optimal. The partition of $\{p_2,x_2–x_1,p_3,x_3\}$ is as in Test 1 and there is only one partition for $\{1,x_1\}$. We use the qualities R and S for the judgment of these two sets together. □

Test 6

Suppose $(1,x_1;p_2,x_2–x_1)$ has to be evaluated where $Q(p_2) = (P,\varnothing,R,L)$. Then the judgments of the optimal process are $J_{RS}(J_{PX}(1, x_1), ((J_{PX}(J_{RI}(p_2, b_{p_2}), x_2–x_1))$.

Proof: Since the first lottery has four elements, p_i and x_i belong to the same set and we have only one admissible partition $\{\{1,x_1\},\{p_2,x_2–x_1\}\}$. Since $Q(1,x_1) = (P\cup X,X^+,S,H)$ and $Q(p_2,x_2–x_1) = (P\cup X,X^+,R,L)$, we use the qualities R and S. The boundary is used to obtain the value of p_2. □

Test 7

Let $(1,x_1;p_2,x_2–x_1)$ be compared to $(1,x_3)$, where $Q(p_2) = (P,\varnothing,R,L)$. Then the judgments of the optimal process are $J_{RS}(J_{PX}(J_{HH}(1,1),J_{X^+X^+}(x_1,x_3)), J_{PX}(J_{RI}(p_2,b_{p_2}), x_2–x_1)$.

Proof: Since the first lottery has four elements, p_i and x_i belong to the same set. $Q(1,1,x_1,x_3) \cap Q(p_2,x_2–x_1) = (X\cup P,X^+,\varnothing,\varnothing) \prec_L (X\cup P,X^+,S,H) = Q(1,x_1) \cap Q(1,x_3,p_2,x_2–x_1) = Q(1,x_3) \cap Q(1,x_1,p_2,x_2–x_1)$.The partition of the first set is as in Test 1 and that of the second set is as in Test 2. □

Test 8

Suppose $(1,-x_1;p_2,x_2+x_1)$ has to be evaluated and $Q(p_2) = (P,\emptyset,R,H)$. Then the judgments of the optimal process are $J_{X^-X^+}(J_{PX}(1,-x_1), ((J_{PX}(J_{RS}(p_2, b_{p_2}), x_2+x_1))$.

Proof: Since the first lottery has four elements, p_i and x_i belong to the same set and there is only one partition. Since $Q(1,-x_1) = (P\cup X,X^-,S,H)$ and $Q(p_2,x_2+x_1) = (P\cup X,X^+,R,H)$ we use the qualities of X^+ and X^-. The boundaries are used for p_2. □

Test 9

Let $(1,-x_1;p_2,x_2+x_1)$ be compared to $(1,-x_3;p_4,x_4+x_3)$, where $Q(p_2) = (P,\emptyset,R,L)$ and $Q(p_4) = (P,\emptyset,R,H)$. Then the judgments of the optimal process are $J_{X^-X^+}(J_{PX}(J_{HH}(1, 1),J_{X^-X^-}(-x_1,-x_3)), (J_{PX}(J_{LH}(p_2, p_4), J_{X^+X^+}(x_2+x_1, x_4+x_3)))$.

Proof: Since the first lottery has four elements, p_i and x_i belong to the same set. We have that the partition $\{\{p_4, p_2,x_4+x_3,x_2+x_1\}, \{1,1,-x_1,-x_2\}\}$ is optimal since this is the only partition where the intersection implies $q_2 = \emptyset$. The partition of each set is as in Test 1. Since $Q(1,1,-x_1,-x_2) = (P\cup X,X^-,S,H)$, and $Q(p_4, p_2,x_4+x_3,x_2+x_1) = (P\cup X,X^+,R,L)$ we use the qualities of X^+ and X^-. □

Test 10

Let $(1,-x_1;p_2,x_2+x_1)$ be compared to $(1,x_3)$, where $Q(p_2) =(P,\emptyset, R,H)$. Then the judgments of the optimal process are $J_{X^-X^+}(J_{PX}(1,-x_1), ((J_{PX}(J_{RS}(p_2,1), J_{X^+X^+}(x_2+x_1,x_3)))$.

Proof: Since the first lottery has four elements, p_i and x_i belong to the same set and the partition $\{1,-x_1\}$ and $\{p_2,x_2+x_1, 1,x_3\}$ is optimal. The partition of the second set is as in Test 3. □

Test 11

Let $(1,-x_1;p_2,x_2+x_1)$ be compared to $(1,x_3)$ where $Q(p_2) = (P,\emptyset,R,L)$. Then the judgments of the optimal process are $J_{X^-X^+}(J_{PX}(1,-x_1), J_{RS}(J_{PX}(1,x_3), J_{PX}(J_{RI}(p_2,b_{p_2}), x_2+x_1))$.

Proof: Since the first lottery has four elements, p_i and x_i belong to the same set. We have that$\{1,-x_1\}$ and $\{p_2,x_2+x_1, 1,x_3\}$ is optimal since $(P\cup X,\emptyset,S,H) \prec_L (P\cup X,X^+,\emptyset,\emptyset)$ which is the one of $\{1,-x_1,1,x_3\}$ and $\{p_2,x_2+x_1\}$. The partition of the second set is as in Test 4. □

Table 1
This table lists the 11 tests used to explain the 14 facts documented in Section 2.
Lottery 1 is compared to Lottery 2 in eight tests or is priced in Tests 2,6,8 (neutral case in Birnbaum et al. 1992).

Tests	Facts	Lottery 1	Lottery 2	Judgments of Optimal Processes
1) Tversky et al. (1990)	1,2,4,6	(p_1,x_1), $p_1 \in L$	(p_2,x_2), $p_2 \in H$	$J_{PX}(J_{HL}(p_1,p_2), J_{X^+X^+}(x_1,x_2))$
2) Birnbaum et al. (1992)	3,6,7	(p_1,x_1), $p_1 \in H$		$J_{PX}(J_{RS}(p_1,b_{p_1}),x_1)$
3) Tversky et al. (1992)	5,6	(p_1,x_1), $p_1 \in H$	$(1,x_2)$	$J_{PX}(J_{RS}(p_1,1), J_{X^+X^+}(x_1,x_2))$
4) Tversky et al. (1992)	5,6	(p_1,x_1), $p_1 \in L$	$(1,x_2)$	$J_{PX}(J_{RS}(J_{RI}(p_1,b_{p_1}),1), J_{X^+X^+}(x_1,x_2))$
5) Birnbaum and Sutton (1992)	8	$(1,x_1;p_2,x_2-x_1)$, $p_2 \in H$	(p_3,x_3), $p_3 \in H$	$J_{RS}(J_{PX}(1,x_1),J_{PX}(J_{HH}(p_2,p_3), J_{X^+X^+}(x_3,x_2-x_1))$
6) Birnbaum et al. (1992)	7,9,10	$(1,x_1;p_2,x_2-x_1)$, $p_2 \in L$		$J_{RS}(J_{PX}(1,x_1), J_{PX}(J_{RI}(p_2,b_{p_2}), x_2-x_1))$
7) Birnbaum (1992)	10	$(1,x_1;p_2,x_2-x_1)$, $p_2 \in L$	$(1,x_3)$	$J_{RS}(J_{PX}(J_{HH}(1,1),J_{X^+X^+}(x_1,x_3), J_{PX}(J_{RI}(p_2,b_{p_2}), x_2-x_1))$
8) Lichtenstein and Slovic (1971)	11,13, 14	$(1,-x_1;p_2,x_2+x_1)$, $p_2 \in H$		$J_{X^-X^+}(J_{PX}(1,-x_1), J_{PX}(J_{RS}(p_2,b_{p_2}), (x_2+x_1))$
9) Lichtenstein and Slovic (1971)	12,13	$(1,-x_1;p_2,x_2+x_1)$, $p_2 \in L$	$(1,-x_3;p_4,x_4+x_3)$, $p_4 \in H$	$J_{X^-X^+}(J_{PX}(J_{HH}(1,1),J_{X^-X^-}(-x_1,-x_3)), J_{PX}(J_{LH}(p_2, p_4), J_{X^+X^+}(x_2+x_1, x_4+x_3)))$
10) Bostic et al. (1990)	14	$(1,-x_1;p_2,x_2+x_1)$, $p_2 \in H$	$(1,x_3)$	$J_{X^-X^+}(J_{PX}(1, -x_1), J_{PX}(J_{RS}(p_2,1), J_{X^+X^+}(x_2+x_1,x_3)))$
11) Bostic et al (1990)	14	$(1,-x_1;p_2,x_2+x_1)$, $p_2 \in L$	$(1,x_3)$	$J_{X^-X^+}(J_{PX}(1,-x_1), J_{PX}(J_{RS}(J_{RI}(p_2,b_{p_2}),1) J_{X^+X^+}(x_3, x_2+x_1))$

Table 2
Gambles used by Luce et al. (1993)

Case	x_1	p_1	x_2	A–B	JCE	Reversal
1A	16	0.3056	–1.5	–1.56	3.58	Yes
1B	4	0.9722	–1.0		1.49	
2A	9	0.1944	–0.5	0.10	1.28	No
2B	2	0.8056	–1.0		–0.38	
3A	6.5	0.5000	–1.0	0.38	0.64	No
3B	3	0.9444	–2.0		–0.86	
4A	8.5	0.3889	–1.5	–1.63	1.68	Yes
4B	2.5	0.9444	–0.5		1.05	

Contrary to the case where $x_1 = 0$ and $x_2 > 0$, the reversals do not occur systematically. The calibration of the parameters obtained from other tests explains the reversals for lottery pairs 1 and 4 and consistent preferences for pairs 2 and 3, where $\alpha_{HH} = \alpha_{HL} = 1.19$.