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Lottery Qualities

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Abstract: The aim of this paper is to propose a model of decision-making for lotteries. Lottery qualities are the key concepts of the theory. Qualities allow the derivation of optimal decision-making processes and are taken explicitly into account for lottery evaluation. Our contribution explains the major violations of the expected utility theory for decisions on two-point lotteries and shows the necessity of giving explicit consideration to lottery qualities. Judged certainty equivalent and choice certainty equivalent concepts are discussed in detail along with the comparison of lotteries. Examples are provided by using different test results in the literature.

Keywords: Lottery choice, common ratio, preference reversal, pricing, lottery test, cognitive process, certainty equivalent

JEL Classification: D81

Résumé: L'objectif de cette recherche est de proposer un nouveau modèle de décision sur les loteries. Les qualités des loteries sont les concepts clés de cette théorie. Les qualités permettent de dériver un processus de décision optimal et sont explicitement prises en compte dans l'évaluation des loteries. Notre contribution explique les lacunes majeures du modèle d'espérance d'utilité en ce qui a trait aux loteries à deux points. L'équivalent certain relié à un jugement et l'équivalent certain relié à un choix sont discutés en détail en relation avec la comparaison des loteries. Des exemples sont donnés à l'aide de résultats de différents tests de la littérature.

Mots clés: Choix de loterie, ratio commun, préférence renversée, tarification, test de loterie, processus cognitif, équivalent certain

Introduction

Over the last fifty years many theories have been proposed to explain the results of lottery tests (for surveys of the main results see Machina, 1987; McFadden, 1999; Luce, 2000). However, even for the simplest two-point lotteries, no theory is able to take into account all the tests together. This is because agents employ many different mental operations to evaluate lotteries. In particular, they use lottery qualities and different processes to evaluate lotteries.

Regarding the existence of lottery qualities, Tversky and Kahneman (1992), among others, have already tested the difference between the positive and negative qualities for monetary amounts x_i. Prelec (1998) has pointed out the qualitative difference between impossibility ($p_i = 0$) and possibility ($p_i \in]0,1[$) for probabilities, while Tversky and Kahneman (1979) have looked at certainty ($p_i = 1$) as another quality. The presence of more than one cognitive process can be illustrated by the preference reversal paradox (Tversky et al., 1990, Lichtenstein and Slovic 1971), where a majority of subjects would prefer lottery A to lottery B in a direct choice but give a higher judged price to lottery B. In this choice, it was always possible for subjects to price each lottery first and then compare the two prices. This test result clearly shows that individuals do not price before making their choices. We must then conclude that there exist at least two different cognitive processes and that individuals have preferences regarding these processes.

The goal of this paper is to use lottery qualities to build up a model that will take into account tests related to both the pricing and comparison of two-point lotteries. The concept of qualities will be shown to be useful in two different ways. First, qualities determine what elements are judged together and, second, they serve explicitly in the lottery judgment (comparison or pricing).

The model can be applied to judge any good. For example a TV set can have a large screen (L) or not (NL) and can be made of wood (W) or not (NW). So for each TV type we have a vector of qualities where the first element is L or NL and the second is W or NW. For example, a TV with a large screen and made of wood has the vector (L, W). Suppose that qualities L and NL in the first position of the vector are more important than qualities W and NW in the second position. The basic concept is that an agent is able to consider the difference between only two qualities and that he will consider the more important difference between qualities. For example, when he compares a TV with a large screen and made of wood (L, W) to a TV without a large screen and not made of wood (NL, NW), the qualities selected are the first ones—L and NL. However, if TV (L, W) is compared to TV (L, NW), the qualities considered in making the choice are then W and NW, since the first two are equivalent. Even if this behavioural hypothesis seems plausible many others can be proposed. The only way to evaluate the hypothesis is to consider tests with subjects. In this paper we shall show how lottery tests are explained with the proposed model.

The paper is organized as follows. Section 1 lists the fourteen more problematic empirical facts culled from the literature on two-point lotteries and Section 2 defines vectors of qualities and shows how to use them in order to evaluate the lotteries. Section 3 presents the results and Section 4 concludes.

1 Facts about two-point lotteries

1.1 Notation

We assume that all lotteries are identified by the vector (p_i, x_i) with the prize x_i (dollars) having probability p_i and the prize 0 (dollars) having probability $1\text{-}p_i$. We consider bundles of one or two lotteries. For example the bundle {(1, 10), (0.3, 5)} contains the sure monetary amount 10 and a lottery where you can win 5 with probability 0.3 or 0 otherwise.

A bundle can take into account lotteries where the two monetary amounts are different from 0, by assuming that the agent uses the segregation proposed by Kahneman and Tversky (1979): the agent splits a lottery where he can win x_i with probability p_i and x_{i-1} with probability $1 - p_i$ into a sure monetary amount $(1, x_{i-1})$ and a lottery $(p_i, x_i - x_{i-1})$. For convenience we assume that $(|x_i| > |x_{i-1}|)$. For example, a two-point lottery where the agent can win 15 with probability 0.3 or 10 otherwise is represented by the bundle {(1, 10), (0.3, 5)}.[1]

The more basic tests for two-point lotteries are the judged certainty equivalent (subjects are asked to select a price), the choice certainty equivalent (subjects choose between a lottery and a sure monetary amount) and the comparison of two lotteries. Almost every test in the literature involves some difficulties. We list below fourteen of the more problematic facts associated with these tests. Table 1 summarizes all facts for bundles $\{(1, x_1), (p_2, x_2)\}$, $x_2 > 0$. There exist three possibilities for the monetary amounts ($x_1 = 0$, $x_1 < 0$ and $x_1 > 0$) combined with high (H) or low (L) probabilities.

(Table 1 here)

[1] For $x_1 < 0 < x_2$, Kahneman and Tversky (1979) do not use segregation. This procedure would not affect the results of this paper. Perhaps it is best not to use segregation when x_1 and x_2 have almost the same size and to use it when they are very different (see Wu and Markle, 2004).

Bundles $\{(p_i, x_i)\}$ with $x_i > 0$

Fact 1: In a lottery choice between lotteries (p_1, x_1) and (p_2, x_2), if p_1 is high and p_2 is low, both yielding the same von Neuman-Morgenstern expected utility, a majority of subjects will select lottery (p_1, x_1). (Tversky et al., 1990).

Fact 2: When two lotteries (p_1, x_1) and (p_2, x_2) with the same expected value are compared and the probabilities p_1, p_2 $(p_1 > p_2)$ are both high, a majority of subjects will choose the first lottery. However, when both probabilities are low and the ratio p_1/p_2 remains the same, a majority of subjects will choose the second lottery. This is the common ratio paradox. (Kahneman and Tversky, 1979; MacCrimon and Larsson, 1979).

Fact 3: When subjects are asked to select a price (judged certainty equivalent, JCE) for the lotteries (p_i, x_i), a lottery with a high probability of winning is underestimated, while a lottery with a low probability of winning is overestimated (Birnbaum et al., 1992). Facts 1 and 3 together lead to the preference reversal paradox.

Fact 4: In comparing two lotteries, it is always possible for subjects to price each lottery first and then compare the two prices. But the test results (Facts 1 and 3) clearly imply that individuals do not price before making their choices. So we have to explain why pricing lotteries is not optimal when subjects face a lottery choice.

Fact 5: If we compare a lottery with a sure monetary amount or with a series of sure monetary amounts, we obtain the choice certainty equivalent CCE. Tversky et al. (1990) found that CCE = JCE for lotteries with high probabilities but CCE < JCE for lotteries with low probabilities. We have to explain these results and why it is not optimal for subjects to price (JCE) first when asked to choose between a lottery with a high p_i and a sure monetary amount (CCE).

Fact 6: One can also note that for the CCE and the common ratio paradox where $x_i > 0$, the observed preferences run counter to the ones for lotteries where $x_i < 0$ (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992).

Bundles $\{(1, x_{i-1}), (p_i, x_i)\}$ with $x_i, x_{i-1} > 0$

This bundle contains a sure monetary amount $(1, x_{i-1})$ and a lottery (p_i, x_i). We can also consider it as the segregated form of a lottery where you can win $x_i + x_{i-1}$ with probability p_i or x_{i-1} otherwise.

Fact 7: When p_i is high, the JCE of the bundle is smaller than the JCE of a lottery (p_i, x_i+x_{i-1}) even if $x_{i-1} + p_i x_i > p_i (x_i + x_{i-1})$ (Birnbaum et al., 1992).

Fact 8: In direct choices, the bundle $\{(1, x_{i-1}), (p_i, x_i)\}$ is preferred to a lottery $(p_i, x_i + x_{i-1})$. This result, opposite to that of Fact 7, yields another reversal of preferences (Birnbaum and Sutton, 1992). Consequently, the agent does not price the two bundles when facing a choice and we have to show again that the pricing of each bundle is not always optimal for this case.

Fact 9: The graph of the JCE for the bundle $\{(1, x_{i-1}), (p_i, x_i)\}$ as a function of p_i has an inverse S-shape like the one for the case where $x_{i-1} = 0$. Moreover, x_{i-1} + JCE of $(p_i, x_i) \neq$ JCE of $\{(1, x_{i-1}), (p_i, x_i)\}$ and the difference between the two JCEs decreases when p_i increases (Birnbaum and Sutton, 1992).

Fact 10: When p_i is high, the CCE of a bundle $\{(1, x_{i-1}), (p_i, x_i)\}$ is smaller than the CCE of a lottery $(p_i, x_i + x_{i-1})$ (Birnbaum, 1992). The use of CCE does not change fact 7 where the JCE is used.

Bundles $\{(1, x_{i-1}), (p_i, x_i)\}$ with $x_{i-1} < 0 < x_i$

Fact 11: When an agent is indifferent to his choice between a bundle $\{(1, x_{i-1}), (0.5, x_i)\}$ and a sure monetary amount 0, the value of $|x_{i-1}|$ is a lot smaller than $|x_i|$. This result is far too extreme to be explained by a wealth effect or by decreasing risk aversion, as Tversky and Kahneman (1992) have pointed out.

Fact 12: For comparisons of bundles with the same expected value as in Bostic et al. (1990), the bundle with the monetary amount x_{i-1} closer to 0 is always chosen.

Fact 13: In two of the four tests used by Lichtenstein and Slovic (1971) and Bostic et al. (1990), there exists a reversal of preferences, while there is no reversal for the other two tests. This situation is more complex than the one for lotteries (p_i, x_i), where reversals are observed for all tests (Tversky et al. 1990).

Fact 14: For this type of bundle with $x_{i-1} < 0 < x_i$, CCE = JCE for high probabilities but CCE < JCE for low probabilities (Bostic et al. 1990). This result is like the ones for lotteries (p_i, x_i) (Fact 5) but differs from bundles with $x_{i-1}, x_i > 0$ (Fact 10).

These fourteen facts strongly suggest that it is impossible to build up a single evaluation function designed to take all of them into account simultaneously. This is why we think that lottery qualities are the basic concept for lottery judgments.

2 Lottery qualities

2.1 Vector of qualities

We describe four collections of sets of qualities for lotteries. In a first step, each group of elements p_i and x_i is naturally split into elements that belong respectively to the set of probabilities P and the set of monetary amounts X. So, the first collection of sets of qualities becomes $\wp_1 = \{P, X\}$. Monetary amounts can be positive or negative. We define the two sets: $X^+ = \{x_i \mid x_i \in]0, +\infty[\}$ and $X^- = \{x_i \mid x_i \in]-\infty, 0[\}$. So $\wp_2 = \{X^+, X^-\}$. These two qualities (X^+ and X^-) are often mentioned in the literature (see Tversky and Kahneman 1992 among others).

Kahneman and Tversky (1979) mention that probabilities have surety (S) and risk (R) qualities. Prelec (1998) obtains a $w(p_i)$ function and mentions that it takes into account the qualitative difference between impossibility (I) and risk (R). We use these qualities and $\wp_3 = \{S, R, I\}$ where $S = \{1\}$, $R = \{p_i \mid p_i \in]0,1[\}$ and $I = \{0\}$. We add to the literature two new qualities for probabilities that indicate whether a lottery has high chances of winning or not. So $\wp_4 = \{H, L\}$ where $L = \{p_i \mid p_i \in [0, p^*[\}$ and $H = \{p_i \mid p_i \in [p^*, 1]\}$. We assume that p*, which can be the fixed point of the inverse S-shape probability weighting function, belongs to [0.3, 0.5]. (See Prelec, 1998, for a discussion)[2]. The existence of H and L is empirically supported by the common ratio paradox (Fact 2), the comparison in the preference reversal (Fact 1) and the pricing of lotteries (Fact 3), where, in each of these tests, one can observe a different way of judging the probabilities that belong to H and L. We then have four collections of sets of qualities:

$$\wp_1 = \{P, X\},\ \wp_2 = \{X^+, X^-\},\ \wp_3 = \{R, S, I\},\ \wp_4 = \{H, L\}.$$

[2] p* is such that for $p \in]0,1[$ $w(p) < p$ if $p > p^*$ and $w(p) > p$ if $p < p^*$.

Each element p_i, x_i and each lottery (p_i, x_i) has a vector of qualities denoted $Q(\bullet) = (q_1, q_2, q_3, q_4)$ where the set $q_j \in \wp_j$, j = 1, ..., 4. The q_j associated with one p_i or x_i is the one that contains the corresponding element. If no set of the collection $\wp_j$ contains the element then the q_j of this element is $\varnothing$. So for a collection defined for probabilities such as $\wp_4 = \{H, L\}$, for example, the q_4 of all monetary amounts is $\varnothing$, because a monetary amount belongs neither to H nor to L. The j^{th} quality of a lottery (p_i, x_i) is the union of the j^{th} qualities of each element p_i and x_i.

We can summarize the previous definitions as follows. For a probability, the vector $Q(p_i)$ is defined in (1.1) while that of a monetary amount $Q(x_i)$ is defined in (1.2) and that of a lottery $Q(p_i, x_i)$ is defined in (1.3).

Definition 1: Vector of Qualities

$Q(p_i) = (P, \varnothing, q_3, q_4)$ where $p_i \in q_3$ and $p_i \in q_4$ (1.1)

$Q(x_i) = (X, q_2, \varnothing, \varnothing)$ where $x_i \in q_2$. (1.2)

$Q(p_i, x_i) = (P \cup X, \varnothing \cup q_2, q_3 \cup \varnothing, q_4 \cup \varnothing) = (P \cup X, q_2, q_3, q_4)$ (1.3)

Example 1

Suppose a bundle contains the lottery (0.2, 1). The vectors of qualities are:

$Q(0.2) = (P, \varnothing, R, L)$ since $0.2 \in P$, $(0.2 \notin X^+, 0.2 \notin X^-)$, $0.2 \in R$, and $0.2 \in L$,

$Q(10) = (X, X^+, \varnothing, \varnothing)$ since $10 \in X$, $10 \in X^+$, $(10 \notin R, 10 \notin S, 10 \notin I)$, and $(10 \notin W, 10 \notin L)$,

$Q(0.2, 0.10) = (P \cup X, \varnothing \cup X^+, R \cup \varnothing, L \cup \varnothing) = (P \cup X, X^+, R, L)$.

We can use these vectors of qualities to form a matrix of qualities for a set that contains elements p_i and x_i. There are two possible kinds of matrix: The matrix of lotteries ML where the lines are the qualities of lotteries (p_i, x_i) and the matrix of elements ME where the lines are the qualities of each element p_i and x_i. These matrices are two different ways of taking into account the qualities of lotteries.

Example 2

The set that contains the components of the bundles {(0.8, 10)} and {(0.2, 40)} is {0.8, 10, 0.2, 40}. The two corresponding matrices are in Figure 1.

(Figure 1 here)

2.2 Three-step process

With the components of the bundles we can form different sets. In this section we propose a way to split a set A of elements p_i and x_i into two subsets A_1 and A_2 by using the qualities of the column j* where j* is the column of qualities selected by the three-step process. The qualities are q^1_{j*} for A_1 and q^2_{j*} for A_2. Payne et al. (1993) pointed out that it is more complex to evaluate two different elements (p_1 and x_1, for example) than to compare two similar elements (p_1 and p_2, for example). We follow the same idea and we assume that the decision- maker will judge similar elements together. The agent will use the largest difference between qualities of elements where the first position in the vector contains a larger qualitative difference than the second one and so on. However, any other order is possible and, for example, some agents may find that H and L (0.9 vs 0.2) are more different than R and S (0.9 vs 1)[3]. This way of doing things is close to a lexicographic order relation used in a more technical paper (Alarie and Dionne 2004). We

[3] The positions of the qualities in the vectors are important and based on tests in the literature. The most natural difference between qualities is the one between probabilities (P) and monetary amounts (X). That is why this difference comes first in the vector of qualities. In lottery tests, Tversky and Kahneman (1992) obtained a very significant effect for the difference between a positive and a negative monetary amount. So this difference becomes the second important one. According to different test results, the difference between the qualities of S, I, and R already pointed out in the literature seems more significant than the difference between the qualities of H and L suggested in this article. So S, I, and R are considered before H and L. In the remainder of this article we will keep this order.

now present the three-step process that determines the q^1_{j*}, q^2_{j*}, A_1, and A_2 the agent will set to make decisions about lotteries. He first selects the relevant matrix and then the relevant column j*. In the third step, the elements associated with the lines that have the same j*th quality are put in the same set.

STEP 1: Select the relevant matrix

The agent must take into account the natural link between a x_i and its corresponding p_i and avoid judgments, as for example the one of p_1 and x_2 together. Another example is when a set A contains a bundle of two lotteries {(p_1, x_1), (p_2, x_2)} where the judgment (addition) of the two probabilities p_1, p_2 together is not typical. In order to rule out these judgments the agent considers the matrix of lotteries (ML) when there are two lotteries of the same bundle in set A. Otherwise he uses the matrix of elements. The set {0.8, 10, 0.2, 40} of Example 2 contains one lottery (0.8, 10) from the first bundle and one lottery (0.2, 40) from the second bundle and the agent must consider the matrix of elements ME as shown in Figure 2. We summarize the previous discussion by introducing the first decision step.

STEP 1: Select matrix ML if set A contains two lotteries from the same bundle and matrix ME if not.

(Figure 2 here)

STEP 2: Select the relevant column

Once the matrix (ME or ML) is built, the agent has to select the column j* that he will consider to obtain the two subsets A_1 and A_2. In order to take into account every line of the matrix, the agent retains the columns where the qualities exist for every line.

Consequently, he rules out the columns where ∅ exists. Remark that the first column never contains ∅. Among the remaining columns, he selects the first relevant column where there are two different qualities. If there is no such column he selects the last of the columns that have no ∅. This procedure is close to a lexicographic order relation, as we have already pointed out, and the agent considers the largest qualitative difference according to his ordering of the columns. For matrix ME of Example 2, the agent rules out columns 2, 3, and 4 that contain ∅. He selects the first column, as indicated by the vertical arrow in Figure 2, that contains two different qualities P and X

STEP 2: Rule out the columns that contain a symbol ∅. Then select the first column where there are two different qualities or, if not, the last one.

STEP 3: Select the relevant lines

In Step 3, the agent builds up the sets A_1 and A_2. If A contains only two elements (#A = 2, where # means cardinality) he puts one element in each set. If A contains more than two elements (#A > 2), he considers the lines associated with the same j^{*th} quality and puts the elements p_i and x_i in the same set. For matrix ML, A_1 and A_2 contain the components of the lotteries that have the same j^{*th} quality. For matrix ME, A_1 and A_2 contain the elements that have the same j^{*th} quality. The j^{*th} qualities of each set are q^1_{j*}, q^2_{j*}. In Example 2, the matrix of elements leads to the qualities P and X and the elements 0.2 and 0.8 belong to P, while the elements 10 and 40 belong to X. The sets are {0.2, 0.8} and {10, 40}, as the horizontal arrows indicate in Figure 2. So the agent, by looking for the largest qualitative difference, builds up sets that contain the most similar elements.

STEP 3: If #A = 2 put one element in each set A_1, A_2. If #A ≥ 2 split the set A into A_1, A_2 according to the two $j*^{th}$ qualities found in STEP 2. q^1_{j*}, q^2_{j*} are the $j*^{th}$ qualities of A_1 and A_2.

2.3 Values of sets

To obtain the value of set A which contains at least two elements, the agent judges the values of the two subsets A_1 and A_2 by considering the $j*^{th}$ qualities of each subset q^1_{j*} and q^2_{j*} found in the three-step process. The value of a set A is v(A) = $V_{q^1_{j*}q^2_{j*}}(v(A_1), v(A_2)) : R \times R \to R$. So for any judgment of one or two bundles, the agent starts with the set that contains all elements of the bundles and uses the three-step process to obtain two sets. He then applies the three-step process for each of the two sets and so on. He stops when each subset contains only one element.

For a set that contains only one element, we assume that $v(x_i) = x_i$ for all x_i and $v(p_i) = p_i$ except for some $p_i \in R$ which can be judged with a boundary[4]. Steps 1, 2, and 3 allow to decide whether a boundary is used or not. Kahneman and Tversky (1979) and Prelec (1998) have pointed out that the shape of the weighting probability function reflects the qualitative difference between the boundaries 0 and 1 and the other probabilities. In the same spirit, judgments of one probability with its corresponding boundary are allowed. The boundary associated with a probability p_i is b_{p_i}. If $p_i \in H$ then $b_{p_i} = 1$ and if $p_i \in L$ then $b_{p_i} = 0$ where H and L are the set of high and low probabilities.

[4] We make this assumption to emphasize the role of probabilities. This is equivalent to assuming a linear utility function. This procedure simplifies the discussion. In other words, a non-linear $u(x_i)$ function (obtained from a judgment with another boundary $x_i = 0$, pointed out in Kahneman and Tversky 1979), would not affect the results in this paper. However, as in Tversky and Kahneman (1992), there is a difference between strictly positive and strictly negative monetary amounts.

The last $V_{q^1_{j*}q^2_{j*}}(\cdot)$ obtained by repeating the three-step process presented above for #A ≥ 2, evaluates two elements. In particular, a p_i is judged with the more similar element allowed by the process. A boundary is used if p_i is more similar to the boundary than to the other element. So the agent just has to add the boundary b_{p_i} to the set and apply Steps 1, 2, and 3 which put similar elements together. The boundary is used if p_i and b_{p_i} are in the same set, since p_i is then more similar to the boundary than to the other element. By doing Steps 1, 2, and 3 for every possible element, we obtain that boundaries are used with p_i for two cases[5]: when the other element is x_i or when it is the remaining boundary different from b_{p_i}. We summarize the preceding discussion by writing the values of sets according to their cardinality.

Definition 2:

Value of a set (# A≥2)

$v(A) = V_{q^1_{j*}q^2_{j*}}(v(A_1), v(A_2))$

Value of a single element

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[5] If $p_1 \in H$ and x_1 are in the same set, the matrix of $\{p_1, x_1, b_{p_1}\}$ is:

$$\begin{bmatrix} Q(p_1) \\ Q(b_{p_1}) \\ Q(x_1) \end{bmatrix} = \begin{bmatrix} P & \varnothing & R & H \\ P & \varnothing & S & H \\ X & X^+ & \varnothing & \varnothing \end{bmatrix}$$

The first different qualities are P and X and the two subsets are $\{p_1, b_{p_1}\}$ and $\{x_1\}$. So p_1 is more similar to the boundary than to x_1 and the boundary is used. We obtain the same result if we use the boundary which differs from b_{p_1} instead of x_1. However if the two-element set is $\{p_1, b_{p_1}\}$, we have the matrix:

$$\begin{bmatrix} Q(p_1) \\ Q(b_{p_1}) \\ Q(b_{p_1}) \end{bmatrix} = \begin{bmatrix} P & \varnothing & R & H \\ P & \varnothing & S & H \\ P & \varnothing & S & H \end{bmatrix}$$

and the two subsets are $\{p_1\}$ and $\{b_{p_1}, b_{p_1}\}$. Then the boundary is not used. We obtain the same result if we use a probability that belongs to R rather than b_{p_1}.

$v(x_i) = x_i$. $v(p_i) = p_i$ except for two cases: $V_{q^1_{j*}q^2_{j*}}(v(p_i), v(\theta)) =$

$V_{q^1_{j*}q^2_{j*}}(V_{q^1_{j*}q^2_{j*}}(p_i, b_{p_i}), \theta)$

when $p_i \in R$ and $\theta = x_i$ or θ is the boundary such that $\theta \neq b_{p_i}$.

Example 3

Suppose an individual faces two bundles of lotteries {(1, 4), (0.2, 10)} and {(0.9, 5)}. The corresponding set is {1, 4, 0.2, 10, 0.9, 5}.

STEP 1: Since the set contains the two lotteries of the first bundle, the agent employs the matrix of lotteries (ML).

$$\begin{bmatrix} Q(1,4) \\ Q(0.2,10) \\ Q(0.9,5) \end{bmatrix} = \begin{bmatrix} P \cup X & X^+ & S & H \\ P \cup X & X^+ & R & L \\ P \cup X & X^+ & R & H \end{bmatrix}$$

STEP 2: There is no $\varnothing$ so he must consider every column. The first two different qualities S and R are in the third column. STEP 3: He selects the set {1, 4} for S and {0.2, 10, 0.9, 5} for R. He obtains $V_{RS}(v(0.9, 5, 0.2, 10), v(1, 4))$. The agent repeats the three-step process for each set {1, 4} and {0.2, 10, 0.9, 5} because the value of each set is unknown so far. For the first set {1, 4} we have.

STEP 1: The matrix for {1, 4} is:

$$\begin{bmatrix} Q(1) \\ Q(4) \end{bmatrix} = \begin{bmatrix} P & \varnothing & S & H \\ X & X^+ & \varnothing & \varnothing \end{bmatrix}.$$

STEP 2: The qualities are P and X. STEP 3: There are only two elements, so he puts each element in one set. We obtain $V_{PX}(v(1), v(4))$. Since there is no probability in R, the boundaries are not used and we obtain $V_{PX}(1, 4)$.

For the second set {0.9, 5, 0.2, 10}, we have:

STEP 1: The agent uses ME since there is one lottery of each bundle in the set, so the matrix is:

$$\begin{bmatrix} Q(0.9) \\ Q(5) \\ Q(0.2) \\ Q(10) \end{bmatrix} = \begin{bmatrix} P & \varnothing & R & H \\ X & X^{+} & \varnothing & \varnothing \\ P & \varnothing & R & L \\ X & X^{+} & \varnothing & \varnothing \end{bmatrix}.$$

STEP 2: The first two different qualities P and X are in the first column. STEP 3: We obtain the two sets {0.9, 0.2} and {10, 5}. We have V_{PX} (v(0.9, 0.2), v(10, 5)). Again the agent repeats the process for each set.

STEP 1: The matrix for {0.9, 0.2} is:

$$\begin{bmatrix} Q(0.9) \\ Q(0.2) \end{bmatrix} = \begin{bmatrix} P & \varnothing & R & H \\ P & \varnothing & R & L \end{bmatrix}.$$

STEP 2: The relevant qualities are H and L. STEP 3: The agent puts one element in each set and obtains V_{HL} (v(0.9), v(0.2)). Since the two probabilities belong to R, the boundaries are not used and we obtain V_{HL} (0.9, 0.2). For the second set we have:

STEP 1: The matrix for {10, 5} is:

$$\begin{bmatrix} Q(10) \\ Q(5) \end{bmatrix} = \begin{bmatrix} X & X^{+} & \varnothing & \varnothing \\ X & X^{+} & \varnothing & \varnothing \end{bmatrix}.$$

STEP 2: The agent rules out the columns with a ∅ and the qualities are the last ones X^+ and X^+.

STEP 3: The agent puts one element in each set and obtain***s*** $V_{X^+X^+}$(v(10), v(5)). There is no boundary for x_i and we obtain $V_{RS}(V_{PX}(1, 4), V_{PX}(V_{HL}\ (0.9, 0.2), V_{X^+X^+}(10, 5)))$. The agent is able to find the two values $V_{HL}(\cdot)$, $V_{X^+X^+}(\cdot)$. He can then evaluate the two values $V_{PX}(\cdot)$ and ends up with $V_{RS}(\cdot)$. So he is able to select the preferred bundle.

3 Results

We now derive, for bundles $\{(1, x_{i-1}), (p_i, x_i)\}$, the judgments associated with three different tests (JCE, CCE, and the comparison of lotteries). x_i is strictly larger than 0 and x_{i-1} is either equal, larger or smaller than 0. We present one test for each case, unless the use of different qualities leads to different judgments (with a boundary or not). Each value of the set that contains all elements considered in the test is obtained by using the three-step process made in details in Appendix I.

3.1 Tests

We start with the bundles where $x_{i-1} = 0$.

Test 1 (Comparison)

Let $\{(p_1, x_1)\}$ be compared to $\{(p_2, x_2)\}$, where $Q(p_1) = (P, \varnothing, R, H)$, $Q(p_2) = (P, \varnothing, R, L)$. Then the value is $V_{PX}(V_{HL}(p_1, p_2), V_{X^+X^+}(x_1, x_2))$.

Test 2 (JCE)

Suppose $\{(p_1, x_1)\}$ has to be evaluated where $Q(p_1) = (P, \varnothing, R, H)$. Then the value is $V_{PX}(V_{RS}(p_1, b_{p_1}), x_1)$.

If in Test 1 we use two probabilities that both belong to H or L, we obtain $V_{HH}(p_1, p_2)$ or $V_{LL}(p_1, p_2)$ rather than $V_{HL}(p_1, p_2)$. For Test 2, the use of a probability in L also leads to similar judgments, but we obtain $V_{RI}(p_1, b_{p_1})$ rather than $V_{RS}(p_1, b_{p_1})$. However, for the

CCE, we present the two tests (3 and 4), where p_1 belong to H or L since the judgments are very different.

Test 3 (CCE)

Let a lottery $\{(p_1, x_1)\}$ be compared to a sure monetary amount $\{(1, x_2)\}$, where $Q(p_1) = (P, \emptyset, R, H)$. Then the value is $V_{PX}(V_{RS}(p_1, 1), V_{X^+X^+}(x_1, x_2))$.

Test 4 (CCE)

Let a lottery $\{(p_1, x_1)\}$ be compared to a sure monetary amount $\{(1, x_2)\}$, where $Q(p_1) = (P, \emptyset, R, L)$. Then the value is $V_{PX}(V_{SR}(V_{RI}(p_1, b_{p_1}), 1), V_{X^+X^+}(x_1, x_2))$.

We now consider the lotteries where $x_{i-1} > 0$.

Test 5 (Comparison)

Let $\{(1, x_1), (p_2, x_2)\}$ be compared to $\{(p_3, x_3)\}$, where $p_2 = p_3$ and $Q(p_2) = (P, \emptyset, R, H) = Q(p_3)$. Then the value is $V_{SR}(V_{PX}(1, x_1), V_{PX}(V_{HH}(p_2, p_3), V_{X^+X^+}(x_3, x_2)))$.

Test 6 (JCE)

Suppose $\{(1, x_1), (p_2, x_2)\}$ has to be evaluated where $Q(p_2) = (P, \emptyset, R, L)$. Then the value is $V_{RS}(V_{PX}(1, x_1), V_{PX}(V_{RI}(p_2, b_{p_2}), x_2))$.

Test 7 (CCE)

Let $\{(1, x_1), (p_2, x_2)\}$ be compared to $\{(1, x_3)\}$, where $Q(p_2) = (P, \varnothing, R, L)$. Then the value is $V_{RS}(V_{PX}(V_{HH}(1, 1), V_{X^+X^+}(x_1, x_3)), V_{PX}(V_{RI}(p_2, b_{p_2}), x_2))$.

The following tests consider lotteries where $x_{i-1} < 0$.

Test 8 (Comparison)

Let $\{(1, -x_1), (p_2, x_2)\}$ be compared to $\{(1, -x_3), (p_4, x_4)\}$, where $Q(p_2) = (P, \varnothing, R, H)$ and $Q(p_4) = (P, \varnothing, R, L)$. Then the value is $V_{X^-X^+}(V_{PX}(V_{HH}(1, 1), V_{X^-X^-}(-x_1, -x_3)), V_{PX}(V_{HL}(p_2, p_4), V_{X^+X^+}(x_2, x_4)))$.

Test 9 (JCE)

Suppose $\{(1, -x_1), (p_2, x_2)\}$ has to be evaluated and $Q(p_2) = (P, \varnothing, R, H)$. Then the value is $V_{X^-X^+}(V_{PX}(1, -x_1), V_{PX}(V_{RS}(p_2, b_{p_2}), x_2))$.

Test 10 (CCE)

Let $\{(1, -x_1), (p_2, x_2)\}$ be compared to $\{(1, x_3)\}$, where $Q(p_2) = (P, \varnothing, R, H)$. Then the value is $V_{X^-X^+}(V_{PX}(1, -x_1), V_{PX}(V_{RS}(p_2, 1), V_{X^+X^+}(x_2, x_3)))$.

Test 11 (CCE)

Let $\{(1, -x_1), (p_2, x_2)\}$ be compared to $\{(1, x_3)\}$ where $Q(p_2) = (P, \varnothing, R, L)$. Then the value is $V_{X^-X^+}(V_{PX}(1, -x_1), V_{PX}(V_{SR}(V_{RI}(p_2, b_{p_2}), 1), V_{X^+X^-}(x_2, x_3))$.

We can already discuss many facts by using the general functions of Tests 1 to 11. For example, the pricing in Test 2, which uses boundaries, is different from the comparison in Test 1 where the two probabilities are judged together, as in Rubinstein (1988). The difference between the CCE and JCE is also straightforward if we look, when $p_i \in L$, at Tests 2 and 4. But in order to be more concrete we will use numerical examples.

3.2 Numerical examples

Many judgments can be used, such as an inverse S-shaped function w (p_i) and the standard utility function u (x_i). Since the form of the evaluation function is beyond the scope of this paper, we select the simple function $\sum_{i=1}^{2} p_i^a x_i^a - \sum_{i=1}^{2} p_i^b x_i^b$, where a bundle a is compared to a bundle b. Examples of functions $V_{q_{j*}^1 q_{j*}^2}$ (v(A_1), v(A_2)) that lead to this evaluation are in Appendix 2. For each function $V_{q_{j*}^1 q_{j*}^2}$ (v(A_1), v(A_2)), the qualities will be taken into account by a simple parameter $\alpha_{q_{j*}^1 q_{j*}^2}$ that multiplies one of the two values v(A_1) or v(A_2). It can be either v(A_1) or v(A_2). The selected value does not matter but the agent must select one. This is achieved in Appendix 2. From Appendix 2, we obtain the following evaluations for each test.

Test 1: $\alpha_{HL}\ p_1x_1 - p_2x_2$

Test 2: $\alpha_{RS}\ p_1x_1$

Test 3: $\alpha_{RS}\ p_1x_1 - x_2$

Test 4: $\alpha_{RI}\alpha_{RS}\ p_1x_1 - x_2$

Test 5: $x_1 + \alpha_{RS}p_2x_2 - \alpha_{RS}\ p_3x_3$

Test 6: $x_1 + \alpha_{RS}\alpha_{RI}p_2x_2$

Test 7: $x_1 + \alpha_{RS}\alpha_{RI}p_2x_2 - x_3$

Test 8: $-\alpha_{X^+X^-}\ x_1 + \alpha_{HL}p_2x_2 + \alpha_{X^+X^-}\ x_3 - p_4x_4$

Test 9: $-\alpha_{X^+X^-}\, x_1 + \alpha_{RS} p_2 x_2$

Test 10: $-\alpha_{X^+X^-}\, x_1 + \alpha_{RS}\, p_2 x_2 - x_3$

Test 11: $-\alpha_{X^+X^-}\, x_1 + \alpha_{RS}\, \alpha_{RI}\, p_2 x_2 - x_3$

We now apply these evaluations to different test results, in order to obtain numerical values for the parameters and explain the fourteen facts presented in Section 2. The parameters α_{RI}, α_{SR} and α_{HL} obtained from test results take into account the qualitative difference between two probabilities. One can expect that these parameters should increase the differences between the probabilities. For example, the difference between $0.9 \in R$ and $1 \in S$ will be larger than 0.1 as noticed by Kahneman and Tversky (1979). We expect the same pattern for the qualities X^+ and X^-.

Weighting probability function: Facts 1, 2, 3, 4, 6

The next example is about the comparison in the preference reversal paradox. In this section, we assume that p* = 4, the middle point in the interval [0.3, 0.5].

> Example 4
>
> In Tversky et al. (1990) we observe that 83% of the subjects choose (0.97, 4) over (0.31, 16). To explain the result we use the evaluation of Test 1 with $p_1 > p^* > p_2$. We obtain $\alpha_{HL} p_1 x_1 - p_2 x_2$. From the test result, α_{HL} $(0.97 \times 4) - (0.31 \times 16) > 0$ and α_{HL} must be greater than 1.28 to obtain the desired result. We observe that the qualitative difference between elements of H and L increases the difference between the two p_i, so Fact 1 is explained. It is important to notice that the way we introduce the parameter α_{HL} does not affect the conclusion. If α_{HL} multiplies the smallest probability 0.31, then α_{HL} must be lower than 1/1.28 to explain the result. The parameter still increases the difference between probabilities.
>
> Example 5

Kahneman and Tversky (1979) test sequentially a choice between (0.45, 6,000) and (0.90, 3,000) and another choice between (0.001, 6,000) and (0.002, 3,000). 86% of the subjects select the second lottery in the first task, but 73% select the first lottery in the second task. This is the common ratio paradox. By using the same evaluation of Example 4 with the parameter α_{HH}, we obtain: $\alpha_{HH}0.90 \times 3{,}000 > 0.45 \times 6{,}000$ and with the parameter α_{LL} we obtain $\alpha_{LL}0.002 \times 3{,}000 < 0.001 \times 6{,}000$ Consequently, we must have $\alpha_{HH} > 1 > \alpha_{LL}$ to solve Fact 2. If the monetary amounts are both negative, we have the opposite result (Fact 6). For example $\alpha_{HH}0.90 \times (-3{,}000) < 0.45 \times (-6{,}000)$.

The next example is about the JCE (Fact 3).

Example 6

Birnbaum et al. (1992) obtained that the JCE of the lottery (0.95, 96) has a value of around 70. The JCE for this type of lottery is obtained from Test 2 and we have α_{RS} p_1x_1. We obtain α_{RS} $0.95 \times 96 = 70$, which implies that $\alpha_{RS} = 0.77$. For a small probability, the lottery is overestimated and then $\alpha_{RI} > 1$. This solves Fact 3. Consequently, the judgment of probabilities with boundaries is similar to the inverse S-shape used by Prelec (1998), Tversky and Kahneman (1992) and Wu and Gonzalez (1996)[6]. If we combine Test 2 and Test 1 (with the qualities of H and L), we explain the existence of the preference reversal paradox (Tversky et al. 1990).

For the comparison of lotteries, we obtain three judgments of probabilities $V_{HH}(p_1, p_2)$, $V_{LL}(p_1, p_2)$ and $V_{HL}(p_1, p_2)$. For Test 2, the agent uses $V_{RS}(p_1, 1)$ to obtain the JCE. If the agent had used a probability with the quality L, he would have obtained $V_{RI}(p_1, 0)$. It is optimal, in our model, to judge a probability with a boundary to obtain the JCE. In contrast, for the comparison of lotteries, the judgment with boundaries was not shown to

[6] As Prelec (1998) has pointed out, the closer the probabilities are to boundaries the greater the effects of RS or RI. We can take this fact into account by setting: $\alpha_{RS}(p^*) = 1$ and $d\alpha_{RI}/dp < 0$ for $p < p^*$ and $d\alpha_{RS}/dp > 0$ for $p > p^*$. The same idea could be applied to other parameters such as α_{HH}, for example.

be optimal (Fact 4), since no boundary satisfies the condition. Consequently, qualities play an important role by allowing both kinds of judgment and by identifying when it is optimal to use one type of judgment (with a boundary) instead of the other (without a boundary).

For the JCE, we obtain either $V_{RI}(p_1, 0)$ or $V_{RS}(p_1, 1)$. The difference between the qualities increases the difference between the probability and the boundary. This is the reason that justifies the underestimation of high p_i and the overestimation of low p_i and explains the existence of the inverse S-shaped probability function for the pricing of lotteries. However this function does not fit the data for the comparison of lotteries. In particular, this function runs counter to the data of the comparison involving H and L which is used in the preference reversal paradox. Furthermore, Alarie and Dionne (2001) show that a one parameter weighting probability function ($w(p_i)$) cannot take into account simultaneously these three comparisons of lotteries. This is seen with this model by the fact that the agent uses three different ways of judging probabilities $V_{HH}(p_1, p_2)$, $V_{LL}(p_1, p_2)$ and $V_{HL}(p_1, p_2)$.

In summary, the above discussion proposes five different judgment functions $V_{RI}(p_1, 0)$, $V_{RS}(p_1, 1)$, $V_{HH}(p_1, p_2)$, $V_{LL}(p_1, p_2)$ and $V_{HL}(p_1, p_2)$ and each of them implies a different way of judging the probabilities. So the evaluation of probabilities is a lot more complicated than what we could have expected at first sight and cannot be taken into account by a one-parameter function.

JCE vs CCE: Facts 5, 7, 10, 14

Another important group of facts concerns the difference between the CCE and the JCE. Tversky et al. (1990) introduced the CCE in order to obtain a lottery price from a comparison with a sure monetary amount. As Bostic et al. (1990) have pointed out, this procedure is closer to the comparison of two lotteries than the JCE and can thus reduce the number of reversals. We will see that this is not necessarily the case.

Example 7

Test 3 and Test 4 are about Fact 5. For Test 3 the evaluation of the lottery is given by $\alpha_{RS}p_1x_1$ as in Test 2. So there is no difference between the JCE and the CCE for high probabilities.

The evaluation of the lottery for Test 4 is given by $\alpha_{RS}\alpha_{RI}\, p_1x_1$ = CCE. If the agent now uses JCE instead of CCE for low probabilities, Test 2 implies that JCE = $\alpha_{RI}\, p_1x_1$ and then CCE = α_{RS}JCE. Since, from Example 6, $\alpha_{RS} < 1$, the JCE is larger than the CCE.

One can note that the reason why the use of the CCE decreases the number of reversals is not, in this paper, because the CCE is closer to the comparison of lotteries than the JCE. This is rather because there are two judgments for the probability $p_1 \in L$. The agent first judges the probability with the boundary 0 as in the JCE, and then compares the result to the sure probability 1 which involves another judgment that considers the qualities R and S (Test 4). So JCE > CCE.

When $x_1 < 0$ and $x_2 > 0$, the JCE for high probabilities is given by Test 9. For low probabilities, 9 becomes 9′. The corresponding JCE and CCE are:

Test 9′: $-\alpha_{X^+X^-}\, x_1 + \alpha_{RI}\, p_2x_2$ = JCE

Test 11: $-\alpha_{X^+X^-}\, x_1 + \alpha_{RS}\, \alpha_{RI}\, p_2x_2$ = CCE

and for high probabilities

Test 9: $-\alpha_{X^+X^-}\, x_1 + \alpha_{RS}\, p_2x_2$ = JCE

Test 10: $-\alpha_{X^+X^-}\, x_1 + \alpha_{RS}\, p_2x_2$ = CCE

The evaluations have two parts. The first ones $-\alpha_{X^+X^-}\, x_1$ are the same for all evaluations and only the second parts differ. These second parts are identical to those where a monetary amount is equal to 0. So, for $p_2 \in H$, the JCE = CCE (Test 9 vs Test 10) and, for $p_2 \in L$, the JCE $\neq$ CCE (Test 9' vs Test 11) for the same reasons that apply for those used to explain Tests 2, 3 and 4. This is why the test results obtained by Bostic et al. (1990) where $x_1 < 0$ (Fact 14) are the same as those in Tversky et al. (1990) where $x_1 = 0$. We now provide a test result where CCE = JCE for all probabilities.

When x_i and $x_{i-1} > 0$ and $p_i \in L$, the evaluation for Test 7 gives CCE = $x_1 + \alpha_{RS}\, \alpha_{RI}\, p_2x_2$. Test 6 gives JCE = $x_1 + \alpha_{RS}\, \alpha_{RI}\, p_2x_2$ and CCE = JCE for low probabilities. We have the same result when $p_i \in H$. Then, for this type of lottery, there is no difference between JCE and CCE for all probabilities, and this explains Fact 10. Since for high probabilities CCE = JCE, the use of CCE does not change the result of fact # 7.

Facts 8, 9, 11, 12, 13

Test 5 gives the way to compare a lottery where the two $x_i > 0$ with a lottery where one $x_i > 0$ and the other $x_i = 0$. Note that, in all tests, $\{(1, x_1), (p_2, x_2)\}$ is preferred to $\{(p_3, x_3)\}$ in a direct choice when $p_2 = p_3$ and $x_3 = x_2 + x_1$. This test result obtained by Birnbaum and Sutton (1992) is foreseeable since, for both cases, one can win $x_2 + x_1$ with probability p_2 and one can win x_1 with probability $1 - p_2$ for the first bundle and 0 with probability $1 - p_2$ for the second one. Test 5 gives $x_1 + \alpha_{RS}\, p_2x_2 - \alpha_{RS}\, p_2(x_2 + x_1)$. So the difference between the two lotteries is $x_1 - \alpha_{RS}\, p_2x_1 > 0$. This is positive because both α_{RS} and p_2 are smaller than 1. This explains Fact 8. When we consider the pricing of these lotteries, Birnbaum and Sutton (1992) obtained the surprising result discussed below.

Example 8

Birnbaum and Sutton (1992) obtained that JCE of $\{(p_3, x_3)\}$ > JCE of $\{(1, x_1), (p_2, x_2)\}$ when $x_3 = 96 = x_2 + x_1$, $x_2 = 72$, $x_1 = 24$ and $p_3 = p_2 = 0.8$.

We now show that this contradictory result can be rationalized. This test result is difficult to accept intuitively because the expected value of the bundle with the higher JCE is lower than the one for the other bundle $p_2 (x_2 + x_1) < p_2x_2 + x_1$. In fact for the JCE of $(p_2, x_2 + x_1)$ we have $\alpha_{RS}p_2(x_2 + x_1)$ and for the JCE of $\{(1, x_1), (p_2, x_2)\}$ he uses Test 6 with a high probability and the evaluation of this bundle is $x_1 + \alpha_{RS}\, \alpha_{RS}\, p_2x_2$. From Birnbaum and Sutton (1992) data, we have (Fact 7) $x_1 + \alpha_{RS}\, \alpha_{RS}\, p_2x_2 < \alpha_{RS}\, p_2(x_2 + x_1)$ when $\alpha_{RS} \in\,]1/2, 5/6[$. This interval contains $\alpha_{RS} = 0.77$ which corresponds to the value found for the test in Example 6 (Birnbaum et al., 1992). This result (Test 6), along with Test 5, involves a second reversal of preferences explained by the model. The first was explained by Tests 1 and 2 together.

Birnbaum and Sutton (1992) pointed out that the JCE of $\{(1, 24), (p_2, 72)\}$ is different from 24 + JCE of $\{(p_2, 72)\}$. Moreover, the spread between the two JCE decreases when the probability p_2 increases (Fact 9). We can explain this result by taking the derivative of (JCE of $\{(1, x_1), (p_2, x_2)\}$ – JCE of $\{(p_2, x_2)\}$) with respect to p_2 where $p_2 \in H$. The difference between the two JCE is equal to $x_1 + \alpha_{RS}\, \alpha_{RS}\, p_2x_2 - \alpha_{RS}\, p_2x_2$ and the derivative with respect to p_2 yields $(\alpha_{RS} - 1)\, \alpha_{RS}\, x_2 < 0$, since $0 < \alpha_{RS} < 1$ and $x_2 > 0$. For $p_2 \in L$ we also obtain $(\alpha_{RS} - 1)\, \alpha_{RI}\, x_2 < 0$ since $\alpha_{RI} > 1$ and $\alpha_{RS} < 1$.

One can note that we have a comparison with the boundaries as in the case where $x_1 = 0$. This is the reason why the evaluation of these lotteries as a function of p_2 still has an inverse S-shaped curve, as discussed in note 6 (Fact 9).

Luce et al. (1993) use four pairs of lotteries taken from Lichtenstein and Slovic (1971). These tests consider lotteries where $x_{i-1} < 0 < x_i$ and are very difficult to explain since, contrary to the cases where $x_1 = 0$ or $x_1 > 0$, the reversal does not occur systematically. So the value of each parameter is important. These tests also provide an opportunity to check if the values of the parameters obtained from all preceding examples are coherent among themselves.

Table 2 shows the results for the four cases where in each case A – B measures the difference between the values of lotteries A and B when they are compared. For these lotteries, the agent can win x_2 with probability p_2 and x_1 with probability $1 - p_2$.

(Table 2 here)

Example 9

For the comparison of pairs of lotteries in Bostic et al. (1990), the agent uses the evaluation function of Test 8. For the parameter that takes into account the difference between a positive and a negative monetary amount, Tversky and Kahneman (1992) obtained a value around 2.25. So we use $\alpha_{X^+X^-} = 2.25$. We also use $\alpha_{HL} = 1.19$, since this is the average of the 14 tests in Tversky et al. (1990). We set $1/\alpha_{RI} = \alpha_{RS} = 0.77$, which corresponds to the values in the preceding examples. For the comparison (Test 8) of two bundles $\{(1, x^a_1), (p^a_2, x^a_2 - x^a_1)\}$ and $\{(1, x^b_1), (p^b_2, x^b_2 - x^b_1)\}$, we have $\alpha_{X^+X^-} (x^a_1 – x^b_1) + p^a_2(x^a_2 – x^a_1) – \alpha_{HL}\, p^b_2(x^b_2 – x^b_1)$ = (A – B in Table 2). The JCE (Test 9) for lotteries with low probabilities and high probabilities are respectively:

$$\alpha_{X^+X^-}\, x_1 + \alpha_{RI}\, p_2(x_2 - x_1)$$

$$\alpha_{X^+X^-}\, x_1 + \alpha_{RS}\, p_2(x_2 - x_1).$$

The corresponding values for the JCE are given in Table 2.

So the lottery with the x_1 ($x_1 < 0$) closer to 0 is always selected in a direct comparison (Fact 12) and reversals occur for lotteries 1 and 4 (Fact 13). The parameters we use fit the data well, in particular $\alpha_{X^+X^-} = 2.25$ taken from Tversky and Kahneman (1992) which is the most significant (Fact 11).

4 Conclusion

The basic block of this paper is used to consider the largest qualitative difference needed to obtain sets with similar elements and then to judge them by considering qualities. This is achieved in step 1, 2, and 3. The same three-step process is employed to decide whether boundaries are to be used or not. Qualities have already been used to evaluate lotteries in the literature, but the major contribution of this paper is to show that the same qualities also decide which elements are to be judged together.

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APPENDIX 1

Test 1 <u>(comparison)</u>

Let $\{(p_1,x_1)\}$ be compared to $\{(p_2, x_2)\}$, where $Q(p_1) = (P, \varnothing, R, H)$, $Q(p_2) = (P, \varnothing, R, L)$. Then the value is $V_{PX}(V_{HL}(p_1, p_2), V_{X^+X^+}(x_1, x_2))$.

Since the first set contains one lottery of each bundle, the agent uses ME and we have:

$$\begin{bmatrix} Q(p_1) \\ Q(p_2) \\ Q(x_1) \\ Q(x_2) \end{bmatrix} = \begin{bmatrix} P & \varnothing & R & H \\ P & \varnothing & R & L \\ X & X^+ & \varnothing & \varnothing \\ X & X^+ & \varnothing & \varnothing \end{bmatrix}$$

The first different qualities are P and X and the two sets are $\{p_1, p_2\}$ and $\{x_1, x_2\}$. He then uses $V_{PX}(\cdot)$. For $\{p_1, p_2\}$, he rules out the second column and the qualities used are the last ones and we have $V_{HL}(p_1, p_2)$.

$$\begin{bmatrix} Q(p_1) \\ Q(p_2) \end{bmatrix} = \begin{bmatrix} P & \varnothing & R & H \\ P & \varnothing & R & L \end{bmatrix}$$

The boundaries are not used since the two probabilities belong to R. For x_i, the agent rules out the last two columns and uses the last qualities that are different from the empty set and we have $V_{X^+X^+}(x_1, x_2)$.

$$\begin{bmatrix} Q(x_1) \\ Q(x_2) \end{bmatrix} = \begin{bmatrix} X & X^+ & \varnothing & \varnothing \\ X & X^+ & \varnothing & \varnothing \end{bmatrix}$$

No boundary is used for x_i. □

Test 2 <u>(JCE)</u>

Suppose $\{(p_1, x_1)\}$ has to be evaluated where $Q(p_1) = (P, \varnothing, R, H)$. Then the value is $V_{PX}(V_{RS}(p_1, b_{p_1}), x_1)$.

We have $Q(p_1) = (P, \varnothing, R, H)$ and $Q(x_1) = (X, X^+, \varnothing, \varnothing)$ and the matrix is:

$$\begin{bmatrix} Q(p_1) \\ Q(x_1) \end{bmatrix} = \begin{bmatrix} P & \varnothing & R & H \\ X & X^+ & \varnothing & \varnothing \end{bmatrix}$$

Since $P \neq X$, he uses $V_{PX}(v(p_1), v(x_1))$. $b_{p_1} = 1$ and he uses boundary 1 since $p_1 \in R$ and the other element is a monetary amount x_1. $Q(1) = (P, \varnothing, S, H)$ and he considers the first different qualities to obtain $V_{RS}(p_1, b_{p_1})$. The value $v(x_1)$ is x_1 and we obtain $V_{PX}(V_{RS}(p_1, b_{p_1}), x_1)$. □

<u>Test 3</u> <u>(CCE)</u>

Let a lottery $\{(p_1, x_1)\}$ be compared to a sure monetary amount $\{(1, x_2)\}$, where $Q(p_1) = (P, \varnothing, R, H)$. Then the value is $V_{PX}(V_{RS}(p_1, 1), V_{X^+X^+}(x_1, x_2))$.

$\{p_1, 1\}$ and $\{x_1, x_2\}$ are selected as in Test 1. He does not use boundaries. The judgment for the probabilities is $V_{RS}(p_1, 1)$ and we obtain $V_{PX}(V_{RS}(p_1, 1), V_{X^+X^+}(x_1, x_2))$. □

<u>Test 4</u> <u>(CCE)</u>

Let a lottery $\{(p_1, x_1)\}$ be compared to a sure monetary amount $\{(1, x_2)\}$, where $Q(p_1) = (P, \varnothing, R, L)$. Then the value is $V_{PX}(V_{SR}(V_{RI}(p_1, b_{p_1}), 1), V_{X^+X^+}(x_1, x_2))$.

As in test 1, the two sets are $\{1, p_1\}$ and $\{x_1, x_2\}$. The qualities for the probabilities are RS and the ones for the monetary amounts are X^+X^+. He uses boundaries for $v(p_1)$. Since $Q(p_1) = (P, \varnothing, R, L)$ and $Q(b_{p_1}) = (P, \varnothing, I, L)$, he uses the qualities I and R. □

Test 5 (comparison)

Let $\{(1, x_1), (p_2, x_2)\}$ be compared to $\{(p_3, x_3)\}$, where $p_2 = p_3$, and $Q(p_2) = (P, \varnothing, R, H) = Q(p_3)$. Then the value is $V_{SR}(V_{PX}(1, x_1), (V_{PX}(V_{HH}(p_2, p_3), V_{X^+X^+}(x_3, x_2))$.

Since $\{1, x_1, p_2, x_2, p_3, x_3\}$ contains the two lotteries of the same bundle $\{(1, x_1), (p_2, x_2)\}$, he uses matrix ML. He considers the three lotteries and we have:

$$\begin{bmatrix} Q(p_2, x_2) \\ Q(1, x_1) \\ Q(p_3, x_3) \end{bmatrix} = \begin{bmatrix} P \cup X & X^+ & R & H \\ P \cup X & X^+ & S & H \\ P \cup X & X^+ & R & H \end{bmatrix}$$

The two first different qualities R and S are in the third column. So we obtain the two sets $\{1, x_1\}$ and $\{p_2, x_2, p_3, x_3\}$ and $V_{RS}(v(1, x_1), v(p_2, x_2, p_3, x_3))$. For the first set he uses ME:

$$\begin{bmatrix} Q(x_1) \\ Q(1) \end{bmatrix} = \begin{bmatrix} X & X^+ & \varnothing & \varnothing \\ P & \varnothing & S & H \end{bmatrix}$$

The two first different qualities are P and X and we obtain $V_{PX}(v(1), v(x_1))$ and $v(x_1) = x_1$. $v(1) = 1$, since $1 \notin R$. For the second set, the agent uses ME and the matrix is:

$$\begin{bmatrix} Q(p_2) \\ Q(p_3) \\ Q(x_2) \\ Q(x_3) \end{bmatrix} = \begin{bmatrix} P & \varnothing & R & H \\ P & \varnothing & R & H \\ X & X^+ & \varnothing & \varnothing \\ X & X^+ & \varnothing & \varnothing \end{bmatrix}$$

The first two different qualities P and X are in the first column. So we obtain $V_{PX}(v(p_2, p_3), v(x_2, x_3))$. The boundaries are not used and we obtain: $V_{SR}(V_{PX}(1, x_1), V_{PX}(V_{HH}(p_2, p_3), V_{X^+X^+}(x_3,x_2))$. □

Test 6 (JCE)

Suppose $\{(1, x_1), (p_2, x_2)\}$ has to be evaluated where $Q(p_2) = (P, \varnothing, R, L)$. Then the value is $V_{RS}(V_{PX}(1, x_1), V_{PX}(V_{RI}(p_2, b_{p_2}), x_2))$.

Since the set contains 2 lotteries from the same bundle, he uses ML.

$$\begin{bmatrix} Q(p_2, x_2) \\ Q(1, x_1) \end{bmatrix} = \begin{bmatrix} P \cup X & X^+ & R & L \\ P \cup X & X^+ & S & H \end{bmatrix}$$

The first different qualities are R and S and we obtain $V_{RS}(v(p_2, x_2), v(1, x_1))$. The judgment of $\{p_2, x_2\}$ is like Test 2 and the other is $V_{PX}(v(1), v(x_1))$. $\square$

Test 7 (CCE)

Let $\{(1, x_1), (p_2, x_2)\}$ be compared to $\{(1, x_3)\}$, where $Q(p_2) = (P, \varnothing, R, L)$. Then the value is $V_{RS}(V_{PX}(V_{HH}(1, 1), V_{X^+X^+}(x_1, x_3)), V_{PX}(V_{RI}(p_2, b_{p_2}), x_2))$.

As in Test 5, the agent uses ML and we have:

$$\begin{bmatrix} Q(p_2, x_2) \\ Q(1, x_1) \\ Q(1, x_3) \end{bmatrix} = \begin{bmatrix} P \cup X & X^+ & R & L \\ P \cup X & X^+ & S & H \\ P \cup X & X^+ & S & H \end{bmatrix}$$

The qualities are R and S and we obtain $V_{RS}(v(1, x_1, 1, x_3), v(p_2, x_2))$. The first set is judged like Test 1 and the second like Test 2. $\square$

Test 8 (comparison)

Let $\{(1, -x_1), (p_2, x_2)\}$ be compared to $\{(1, -x_3), (p_4, x_4)\}$, where $Q(p_2) = (P, \varnothing, R, H)$ and $Q(p_4) = (P, \varnothing, R, L)$. Then the value is $V_{X^-X^+}(V_{PX}(V_{HH}(1, 1), V_{X^-X^-}(-x_1, -x_3)), V_{PX}(V_{HL}(p_2, p_4), V_{X^+X^+}(x_2, x_4)))$.

He uses ML and the matrix is:

$$\begin{bmatrix} Q(p_2, x_2) \\ Q(1, -x_1) \\ Q(p_4, x_4) \\ Q(1, -x_3) \end{bmatrix} = \begin{bmatrix} P \cup X & X^+ & R & H \\ P \cup X & X^- & S & H \\ P \cup X & X^+ & R & L \\ P \cup X & X^- & S & H \end{bmatrix}$$

The first qualities are X^+ and X^- and the judgment is $V_{X-X^+}(v(1, 1, -x_1, -x_3), v(p_2, p_4, x_2, x_4))$. The first set is judged like Test 7 and the second one is judged like Test 1. □

Test 9 (JCE)

Suppose $\{(1, -x_1), (p_2, x_2)\}$ has to be evaluated and $Q(p_2) = (P, \varnothing, R, H)$. Then the value is $V_{X\text{-}X^+}(V_{PX}(1, -x_1), V_{PX}(V_{RS}(p_2, b_{p_2}), x_2))$.

Since the set contains 2 lotteries of the same bundle, the agent uses ML.

$$\begin{bmatrix} Q(p_2, x_2) \\ Q(1, -x_1) \end{bmatrix} = \begin{bmatrix} P \cup X & X^+ & R & H \\ P \cup X & X^- & S & H \end{bmatrix}$$

The first different qualities are X^+ and X^- and we obtain $V_{X^+X^-}(v(p_2, x_2), v(1, -x_1))$. The judgment of $\{p_2, x_2\}$ is like Test 2 and the other is $V_{PX}(1, -x_1)$. □

Test 10 (CCE)

Let $\{(1, -x_1), (p_2, x_2)\}$ be compared to $\{(1, x_3)\}$, where $Q(p_2) =(P, \varnothing, R, H)$. Then the value is $V_{X\text{-}X^+}(V_{PX}(1, -x_1), ((V_{PX}(V_{RS}(p_2, 1), V_{X^+X^+}(x_2, x_3)))$.

He uses ML to obtain:

$$\begin{bmatrix} Q(p_2, x_2) \\ Q(1, -x_1) \\ Q(1, x_3) \end{bmatrix} = \begin{bmatrix} P \cup X & X^+ & R & H \\ P \cup X & X^- & S & H \\ P \cup X & X^+ & S & H \end{bmatrix}$$

The qualities are X^+ and X^- and the judgment is $V_{X^-X^+}(v(1, -x_1), v(p_2, 1, x_2, x_3))$. The first set is like Test 2 and the other one is like Test 3. □

<u>Test 11</u> <u>(CCE)</u>

Let $\{(1, -x_1), (p_2, x_2)\}$ be compared to $\{(1, x_3)\}$ where $Q(p_2) = (P, \varnothing, R, L)$. Then the value is $V_{X^-X^+}(V_{PX}(1, -x_1), V_{RS}(V_{PX}(1, x_3), V_{PX}(V_{RI}(p_2, b_{p_2}), x_2))$.

He uses ML to obtain:

$$\begin{bmatrix} Q(p_2, x_2) \\ Q(1, -x_1) \\ Q(1, x_3) \end{bmatrix} = \begin{bmatrix} P \cup X & X^+ & R & L \\ P \cup X & X^- & S & H \\ P \cup X & X^+ & S & H \end{bmatrix}$$

The qualities are X^+ and X^- and the judgment is $V_{X^-X^+}(v(1, -x_1), v(p_2, 1, x_2, x_3))$. The first set is like Test 2 and the other one is like Test 4. □

APPENDIX 2

The agent judges the two values with the functions below, where a bundle A is compared to a bundle B and θ^1, θ^2 are sets of elements:

$V^1(p_i, x_i)$ = sgn $p_i x_i$. The sgn = + if p_i and $x_i \in$ A and sgn = – if not. (1.1)

$V^2(p_1, p_2) = (p_1 - p_2)(x_1 + x_2)/2$ when $p_1 \in$ A and $p_2 \in$ B. (1.2)

$V^3(x_1, x_2) = (x_1 - x_2)(p_1 + p_2)/2$ when $x_1 \in$ A and $x_2 \in$ B. (1.3)

$V^4(v(\theta^1), v(\theta^2)) = v(\theta^1) + v(\theta^2)$ when $\#(\theta^1), \#(\theta^2) > 1$. (1.4)

$V^5(p_i, b_{p_i}) = p_i$. (1.5)

The first equation is equivalent to the mathematical expectation. When we put together the results from (1.2) and (1.3) by using (1.4), we obtain $p_1x_1 - p_2x_2$. When (p_1, x_1) is compared to (p_2, x_2), we then have:

$$\begin{aligned} p_1x_1 - p_2x_2 &= 2(p_1x_1 - p_2x_2)/2 + (p_1x_2 - p_2x_1)/2 - (p_1x_2 - p_2x_1)/2 \\ &= (p_1 - p_2)(x_1 + x_2)/2 + (x_1 - x_2)(p_1 + p_2)/2 \\ &= V^2(p_1, p_2) + V^3(x_1, x_2) \\ &= V^4(V^2(p_1, p_2), V^3(x_1, x_2)). \end{aligned}$$

So these judgments lead to expressions with terms p_ix_i even though the agent is allowed to compare two p_i or two x_i and the evaluation of a lottery is given by Σp_ix_i. Finally, for the last equation, we obtain a judged value of the probability by considering a boundary.

In order to take into account the qualities, we introduce a parameter $\alpha_{q^1_{j*}q^2_{j*}}$ that multiplies one of the two values in $V_{q^1_{j*}q^2_{j*}}(v(A_1), v(A_2))$ and we obtain for example

$V(\alpha_{q^1_{j*}q^2_{j*}}\, v(A_1), v(A_2))$. The choice of the set A_m, m = 1, 2, does not matter, but the agent must select one. In this paper we focus only on the qualities below.

i) $\alpha_{X^+X^-}$, α_{HL}, α_{RS}, or α_{RI} multiplies $v(A_m)$ when A_m has respectively the qualities X^-, H or R.

ii) α_{HH} or α_{LL}, multiplies $v(A_m)$ when A_m contains the largest p_i.

All other parameters $\alpha_{q^1_{j*}q^2_{j*}}$ are equal to 1. The average of x_i used in (1.2) and that of p_i used in (1.3) are also obtained by considering qualities. For example, in Test 1 we have $V_{PX}(V_{HL}(p_1, p_2), V_{X^+X^+}(x_1, x_2))$ and by using (i) and (ii) the only parameter is α_{HL} which multiplies the largest probability p_1. $V_{HL}(p_1, p_2) = (\alpha_{HL}p_1 - p_2)(x_1 + x_2)/2$ and $V_{X^+X^+}(x_1, x_2)) = (x_1 - x_2)(\alpha_{HL}p_1 + p_2)/2$ by (1.2) and (1.3). Finally, (1.4) gives $(\alpha_{HL}p_1 - p_2)(x_1 + x_2)/2 + (x_1 - x_2)(\alpha_{HL}p_1 + p_2)/2 = \alpha_{HL}\, p_1x_1 - p_2x_2$.

Table 1
Main facts in the literature obtained from tests

	Comparison			**Pricing**			
				JCE		**CCE**	
x_1	H vs H	L vs L	H vs L	H	L	H	L
$x_1 = 0$	Common ratio F2, F4		H > L F1	Under F3	Over F3	= JCE F5	< JCE F5
			Preference Reversal F4				
$x_1 < 0$			Preference Reversal Non-Systematic F12, F13			= JCE F11, F14	< JCE F11, F14
$x_1 > 0$	H vs H			JCE and CCE of (p_2,x_2+x_1) larger than JCE and CCE of $\{(1,x_1),(p_2,x_2)\}$; $p_2 > p^*$ F7, F9, F10			
	F8						

$H = \{p_2 \mid p_2 \geq p^*\}$, $p^* \in [0.3,0.5]$
$L = \{p_2 \mid p_2 < p^*\}$
JCE: Judged Certainty Equivalent: the subject states a price for the lottery
CCE: Choice Certainty Equivalent: the subject compares the lottery with a sure monetary amount or a series of sure monetary amounts
Under: underestimated
Over: overestimated
Fx: Fact x

Table 2
Gambles Used by Luce et al. (1993)

Case	X_2	P_2	X_1	A–B	JCE	Reversal
1A	16	0.3056	–1.5	–1.56	3.58	Yes
1B	4	0.9722	–1.0		1.49	
2A	9	0.1944	–0.5	0.10	1.28	No
2B	2	0.8056	–1.0		–0.38	
3A	6.5	0.5000	–1.0	0.38	0.64	No
3B	3	0.9444	–2.0		–0.86	
4A	8.5	0.3889	–1.5	–1.63	1.68	Yes
4B	2.5	0.9444	–0.5		1.05	

Contrary to the case where $x_1 = 0$ and $x_2 > 0$, the reversals do not occur systematically. The calibration of the parameters obtained from other tests explains the reversals for lottery pairs 1 and 4 and consistent preferences for pairs 2 and 3, where $\alpha_{HH} = \alpha_{HL} = 1.19$.

$$
\text{Matrix of elements} \qquad\qquad\qquad\qquad \text{Matrix of lotteries}
$$

$$
\mathrm{ME} = \begin{bmatrix} Q(.8) \\ Q(.2) \\ Q(10) \\ Q(40) \end{bmatrix} = \begin{bmatrix} P & \varnothing & R & H \\ P & \varnothing & R & L \\ X & X^{+} & \varnothing & \varnothing \\ X & X^{+} & \varnothing & \varnothing \end{bmatrix} \quad \mathrm{ML} = \begin{bmatrix} Q(.8{,}10) \\ Q(.2{,}40) \end{bmatrix} = \begin{bmatrix} P \cup X & X^{+} & R & H \\ P \cup X & X^{+} & R & L \end{bmatrix}
$$

Figure 1. The matrices ME and ML.

$$\begin{array}{ccc} & & \begin{array}{cccc} \Downarrow & & & \end{array} \\ \begin{bmatrix} Q(.8) \\ Q(.2) \\ Q(10) \\ Q(40) \end{bmatrix} & \begin{matrix} \longleftarrow \\ \longleftarrow \\ \Longleftarrow \\ \Longleftarrow \end{matrix} & \begin{bmatrix} P & \varnothing & R & H \\ P & \varnothing & R & L \\ X & X^{+} & \varnothing & \varnothing \\ X & X^{+} & \varnothing & \varnothing \end{bmatrix} \end{array}$$

Figure 2. The three-step process.

In Step 1 the agent selects the matrix of elements ME as the relevant one. In Step 2 he selects the relevant column as indicated by the vertical arrow. In step 3 he chooses the relevant lines corresponding to two different sets as the horizontal arrows indicate.