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Econometric Inference, Cyclical Fluctuations, and Superior Information

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Abstract:
This paper presents and assesses a procedure to estimate conventional parameters characterizing fluctuations at the business cycle frequency, when the economic agents' information set is superior to the econometrician's one. Specifically, we first generalize the conditions under which the econometrician can estimate these 'cyclical fluctuation' parameters from augmented laws of motion for forcing variables that fully recover the agents' superior information. Second, we document the econometric properties of the estimates when the augmented laws of motion are possibly misspecified. Third, we assess the ability of certain information criteria to detect the presence of superior information.

Keywords: Block bootstrap, Hidden variables, Laws of motion for forcing variables, Monte Carlo simulations

JEL Classification: C14, C15, C32, E32

1. Introduction

A vast literature attempts to explain fluctuations of macroeconomic aggregates at the business cycle frequency from dynamic artificial economies. These economies usually rely on calibrated forward-looking rules relating choice variables to expected future forcing variables. These rules capture the decisions formed by economic agents, given for example preferences and technologies. Hence, the rules reflect assumptions that are generally viewed as fundamental since they characterize agents' economic behavior. The artificial economies also typically rely on estimated laws of motion depicting the dynamics of forcing variables. These laws of motion are required to forecast future forcing variables involved in the rules. Hence, the laws of motion reflect assumptions that are frequently perceived as auxiliary since they define information sets. The artificial economies are then used to estimate the parameters summarizing the salient features of cyclical fluctuations for selected variables. The conventional 'cyclical fluctuation' parameters include standard deviations, first-order autocorrelation coefficients, and dynamic cross-correlations. These parameters characterize the volatility, persistence, and comovements of selected variables (relative to a measure of the business cycle).

Arguably, the estimates may differ from the true values of the cyclical fluctuation parameters, not because the rules are inadequate, but simply because the laws of motion are misspecified. In such a case, this occurs not because the fundamental assumptions on agents' economic behavior are incorrect, but rather because the auxiliary assumptions defining information sets are invalid. In most studies, the laws of motion correspond to vector autoregression (VAR) processes involving only forcing variables. These standard laws of motion postulate that the relevant information sets include only the history of forcing variables. However, it is possible that agents exploit additional information to improve their forecasts of future forcing variables. The extra relevant information is embodied in exogenous variables, hidden variables, which are either unknown or omitted by the econometrician. The existence of hidden variables ensure that the agents' relevant information set is superior to the econometrician's one.

Some empirical analyses try to account for superior information by using augmented laws of motion (Campbell and Shiller 1987; Campbell and Deaton 1989; Flavin 1993; Normandin 1999; Boileau and Normandin 2002, 2003). These laws of motion correspond to VAR processes involving forcing variables and a single choice variable. It can be shown that this specification is adequate only when the true law of motion corresponds to a VAR process containing forcing variables and a single hidden variable (Boileau and Normandin 2002). In this context, the choice variable is a surrogate for the hidden variable. This occurs because, under the assumed rules, all the relevant extra information is summarized by agents' optimal decisions. Consequently, the feedbacks from the lagged choice variable to current forcing variables in the augmented law of motion reflect the existence of a hidden variable.

This paper pursues three objectives. The first goal is to generalize the development just described for the case of multiple hidden variables. To do so, we consider an actual economy that is characterized by forward-looking rules for nonpredetermined variables and a true law of motion, which corresponds to a VAR process involving many forcing and hidden variables. For this environment, we derive the sufficient (rank) and necessary (order) conditions under which the actual data generating process can be summarized by an augmented law of motion, which corresponds to a VAR process involving exclusively forcing and nonpredetermined variables. Specifically, the order conditions reveal that there must be as many nonpredetermined variables, or surrogates, as hidden variables to fully recover the agents' relevant information. In principle, the laws of motion with multiple nonpredetermined variables admit richer information sets than the case with a unique nonpredetermined variable. Moreover, these augmented laws of motion offer the considerable advantage to avoid selecting the hidden variables.

The second objective of our paper seeks to document the properties of the estimates of cyclical fluctuation parameters when the law of motion is possibly misspecified. In particular, we pay attention to artificial economies where the standard law of motion correctly (incorrectly) specifies the true law of motion due to the absence (presence) of superior information. We also analyze cases where the augmented law of motion represents an appropriate (inappropriate) specification. More precisely, the augmented law of motion can be misspecified because there is no superior information, or because it includes too many or too few nonpredetermined variables to adequately capture the superior information.

For this purpose, we design a simulation procedure relying on four steps. First, the true values of cyclical fluctuation parameters are obtained from a parametrized version of the actual economy, where the true law of motion may or may not involve hidden variables. Second, the estimates of the parameters are calculated from an artificial economy that correctly specifies the rules for nonpredetermined variables, but that possibly misspecifies the law of motion for forcing variables. Third, the confidence intervals for the estimates are derived from the bootstrap percentile method, where the estimates are generated from a general Block Bootstrap approach (Künsch 1989; Bühlmann 2002; Lahiri 2003). Fourth, a Monte Carlo experiment is conducted to compute the coverage probability of the confidence intervals, the average length of the confidence intervals, and the root mean square error of the estimates.

The simulation results indicate that the estimates of cyclical fluctuation parameters computed from the standard laws of motion yield efficient inference under the absence of superior information, but are severely biased under the existence of superior information. In contrast, the estimates obtained from the augmented laws of motion lead to a sizeable loss of efficiency when there is no superior information, but are never substantially biased whether or not there is superior information. Moreover, the estimates obtained from the augmented laws of motion with a single nonpredetermined variable are efficient and unbiased, whether unique or multiple nonpredetermined variables are required to fully recover the superior information. The latter econometric properties are attractive, given that, in practice, artificial economies with augmented laws of motion are much more tractable when they involve only a single nonpredetermined variable. These econometric properties also suggest the relevance of previous empirical studies, based on augmented laws of motion with a single choice variable.

Finally, the third goal of our paper is to assess the ability of the Bayesian and Akaike information criteria (BIC and AIC) to detect the absence or presence of superior information. To this end, these criteria are first define to select the appropriate number of nonpredetermined variables to include in the law of motion. Then, our simulation procedure is used to compute the proportions of Monte Carlo replications for which the information criteria correctly select the (standard or augmented) law of motion that adquately specifies the true law of motion.

The results reveal that the BIC choses much more often the standard law of motion when there is no superior information, while both the BIC and AIC never wrongly select the standard law of motion when there is superior information. Overall, these findings suggest that the BIC is very useful in empirical analysis to determine whether the standard law of motion is a relevant specification. This is crucial given that the standard law of motion yields efficient estimates of the cyclical fluctuation parameters under the absence of superior information, whereas it leads to biased estimates under the existence of superior information.

This paper is organized as follows. Section 2 presents the actual economy. Section 3 describes the artificial economy. Section 4 elaborates the simulation procedure. Section 5 reports the simulation results. Section 6 concludes by presenting several extensions.

2. The Actual Economy

Throughout our analysis, we consider an actual economy that is governed by forward-looking rules and dynamic laws of motion. In this section, we describe the actual economy, derive the underlying data generating process, and present the associated cyclical fluctuation parameters.

2.1 The Rules

The actual economy is characterized by the relations:

$$\mathbf{m}_t = \mathbf{\Phi} \sum_{j=1}^{\infty} \beta^j E_t \mathbf{f}_{t+j}. \tag{1}$$

The term E_t represents the expectation operator conditional on the information available in period t. The $(n_m \times 1)$ vector $\mathbf{m}_t$ contains (endogenous) nonpredetermined variables. The $(n_f \times 1)$ vector $\mathbf{f}_t$ incorporates (exogenous) forcing variables. The matrix $\mathbf{\Phi}$ includes coefficients, while the scalar $|\beta| < 1$ is a discount factor.

The relations (1) are forward-looking rules: they relate nonpredetermined variables to expected future forcing variables. The rules reflect the decisions formed by economic agents such as consumers, producers, and policy makers. Also, the coefficients represent the structural parameters associated with economic notions such as preferences and technologies. Hence, the rules rely on assumptions that are generally viewed as fundamental since they characterize agents' economic behavior.

The rules (1) are widely used in dynamic analyses, and in particular, in macroeconomics, monetary economics, and financial economics. In macroeconomics, for example, seminal applications of the Permanent Income Hypothesis relate private saving to expected future changes of labor income (Campbell 1987) and the current account (i.e. national saving net of investment) to expected future changes of net output (i.e. output net of investment and government expenditures) (Sheffrin and Woo 1990). In monetary economics, pioneer work on hyperinflation establishes the logarithm of prices from the present value of expected future logarithms of money supply (Cagan 1956), whereas the recent New Keynesian Phillips curve expresses the inflation rate as the sum of discounted expected future firms' real marginal costs (Gali and Gertler 1999). In financial economics, the expectation theory of the term structure links the spread between long- and short-term interest rates to expected future changes of short-term interest rate (Shiller 1979), while the conventional stock pricing model relates the difference between the stock price and a multiple of dividend to expected future changes of dividend (Campbell and Shiller 1987).

Other applications rely on rules that are more complex than (1). These cases are analyzed in our extensions (section 6).

2.2 The True Law of Motion

In the actual economy, the forcing variables are governed by the stationary first-order VAR process:

$$\begin{pmatrix} \mathbf{f}_t \\ \mathbf{h}_t \end{pmatrix} = \begin{pmatrix} \mathbf{\Pi}_{11} & \mathbf{\Pi}_{12} \\ \mathbf{\Pi}_{21} & \mathbf{\Pi}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{f}_{t-1} \\ \mathbf{h}_{t-1} \end{pmatrix} + \begin{pmatrix} \mathbf{u}_{f,t} \\ \mathbf{u}_{h,t} \end{pmatrix},$$

or more compactly

$$\mathbf{w}_t = \mathbf{\Pi}\mathbf{w}_{t-1} + \mathbf{u}_t. \tag{2}$$

The $(n_h \times 1)$ vector $\mathbf{h}_t$ includes additional (exogenous) variables. The vector $\mathbf{w}_t$ is observed by agents. The vector $\mathbf{u}_t$ incorporates the innovations of exogenous variables, with zero means and covariance matrices $\mathbf{\Omega}_{ff} = E(\mathbf{u}_{f,t}\mathbf{u}'_{f,t})$ and $\mathbf{\Omega} = E(\mathbf{u}_t\mathbf{u}'_t)$. The matrix $\mathbf{\Pi}$ contains the feedback coefficients.

The VAR (2) corresponds to the true law of motion: it correctly specifies the dynamics of exogenous variables. The law of motion depicts the stochastic environment affecting the temporal evolution of exogenous variables. Also, the feedback coefficients reveal the composition of the agents' relevant information set to form their expectations of future forcing variables. In particular, $\mathbf{\Pi}_{12} = \mathbf{0}$ implies that the agents' relevant information set incorporates only the history of these variables. In contrast, $\mathbf{\Pi}_{12} \neq \mathbf{0}$ implies that the agents' information set includes the history of all exogenous variables, because the additional variables contain extra information useful to track the temporal movements of forcing variables. Overall, the law of motion reflects assumptions that are frequently perceived as auxiliary since they define the agents' information set.

The VAR processes are widely used in dynamic analyses, and in particular, in data descriptions and linear projections. In data descriptions, the stochastic properties are conveniently documented from impulse responses and from forecast error variance decompositions (Sims 1980). In linear projections, the forecasting power of variable groups is assessed from Granger-causality tests (Granger 1969; Sims 1972) and the expected values of future variables are computed from Wiener-Kolmogorov formula (Hansen and Sargent 1980).

Other applications rely on stationary p-order VAR processes. Our analysis takes into account these cases by rewriting the VAR processes as first-order systems.

2.3 The Actual Data Generating Process

Agents use the true law of motion (2) to form their expectations of future forcing variables involved in the rules (1). The agents' solution is:

$$\mathbf{m}_t = \mathbf{\Phi} \mathbf{e}_f \beta \mathbf{\Pi} \Big[\mathbf{I}_{(n_f+n_h)} - \beta \mathbf{\Pi} \Big]^{-1} \mathbf{w}_t = \mathbf{\Theta} \mathbf{w_t} = \mathbf{\Upsilon}_f \mathbf{f}_t + \mathbf{\Upsilon}_h \mathbf{h}_t. \tag{3}$$

The selection matrix $\mathbf{e}_f$ is defined as $\mathbf{f}_t = \mathbf{e}_f \mathbf{w}_t$ and the $(n_d \times n_d)$ identity matrix is denoted $\mathbf{I}_{n_d}$. Importantly, the solution (3) depends on all variables that contain information useful to forecast future forcing variables. Therefore, (3) involves the additional variables ($\mathbf{\Upsilon}_h \neq \mathbf{0}$) if and only if these variables Granger-cause forcing variables ($\mathbf{\Pi}_{12} \neq \mathbf{0}$).

As the actual economy, the true law of motion (2) and agents' solution (3) yield the data of $\mathbf{f}_t$, $\mathbf{h}_t$, and $\mathbf{m}_t$ for fixed values of β, $\mathbf{\Phi}$, $\mathbf{\Pi}$, and $\mathbf{\Omega}$. However, it will prove useful to summarize this actual data generating process by a representation involving exclusively forcing and nonpredetermined variables. To do so, it is assumed for the moment that the numbers of nonpredetermined and hidden variables are the same ($n_m = n_h$). In this environment, the representation corresponds to:

$$\begin{pmatrix} \mathbf{f}_t \\ \mathbf{m}_t \end{pmatrix} = \begin{pmatrix} \mathbf{\Gamma}_{11} & \mathbf{\Gamma}_{12} \\ \mathbf{\Gamma}_{21} & \mathbf{\Gamma}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{f}_{t-1} \\ \mathbf{m}_{t-1} \end{pmatrix} + \begin{pmatrix} \mathbf{v}_{f,t} \\ \mathbf{v}_{m,t} \end{pmatrix},$$

or

$$\mathbf{x}_t = \mathbf{\Gamma}\mathbf{x}_{t-1} + \mathbf{v}_t. \tag{4}$$

When $\mathbf{\Pi}_{12} = \mathbf{0}$, then $\mathbf{x}_t = \mathbf{\Psi}\mathbf{f}_t$, $\mathbf{v}_t = \mathbf{\Psi}\mathbf{u}_{f,t}$, $\mathbf{\Gamma} = \mathbf{\Psi}\mathbf{\Pi}_{11}\mathbf{e}_f$, $\mathbf{\Lambda} = \mathbf{\Psi}\mathbf{\Omega}_{ff}\mathbf{\Psi}'$, and $\mathbf{\Psi} = (\, \mathbf{I}_{n_f} \quad \mathbf{\Upsilon}_f' \,)'$. Note that $\mathbf{\Gamma}_{12} = \mathbf{0}$ implies that the block of equations for forcing variables in (4) coincides with the true law of motion (2). Also, $\mathbf{v}_t = \mathbf{\Upsilon}\mathbf{u}_{f,t}$ reveals that the innovations of nonpredetermined variables depend exclusively on the innovations of forcing variables.

When $\mathbf{\Pi}_{12} \neq \mathbf{0}$, then $\mathbf{x}_t = \mathbf{\Upsilon}\mathbf{w}_t$, $\mathbf{v}_t = \mathbf{\Upsilon}\mathbf{u}_t$, $\mathbf{\Gamma} = \mathbf{\Upsilon}\mathbf{\Pi}\mathbf{\Upsilon}^{-1}$, $\mathbf{\Lambda} = \mathbf{\Upsilon}\mathbf{\Omega}\mathbf{\Upsilon}'$, and $\mathbf{\Upsilon} = (\, \mathbf{e}_f' \quad \mathbf{\Theta}' \,)'$. Note that $\mathbf{\Gamma}_{12} = \mathbf{\Pi}_{12}\mathbf{\Upsilon}_h^{-1} \neq \mathbf{0}$ reflects the effects of lagged additional variables on current forcing variables highlighted in the true law of motion (2). Moreover, $\mathbf{v}_t = \mathbf{\Upsilon}\mathbf{u}_t$ states that the innovations of nonpredetermined variables are functions of the innovations of all exogenous variables. This accords with the notion that nonpredetermined variables fully summarize the agents' relevant information.

Importantly, for the case $\mathbf{\Pi}_{12} \neq \mathbf{0}$, (4) is a well defined representation of the actual data generating process under certain conditions.

Sufficient condition. The rank condition states that $\mathbf{\Upsilon}_h$ is of full rank. To see this, note that the matrix $\mathbf{\Gamma}$ involves:

$$\mathbf{\Upsilon}^{-1} = \begin{pmatrix} \mathbf{I}_{n_f} & \mathbf{0} \\ -\mathbf{\Upsilon}_h^{-1}\mathbf{\Upsilon}_f & \mathbf{\Upsilon}_h^{-1} \end{pmatrix}, \tag{5}$$

which exists only if $\mathbf{\Upsilon}_h$ is invertible.

Necessary conditions. The order conditions state that $n_f \geq n_h$ and $n_m = n_h$. These are required for $\mathbf{\Upsilon}_h$ to be invertible, given that under these conditions the rank of $\mathbf{\Upsilon}_h$ is $\rho(\mathbf{\Upsilon}_h) = \min(n_f, n_h)$ — where $\mathbf{\Upsilon}_h = \mathbf{\Phi\Xi}$ is a $(n_h \times n_h)$ matrix, $\mathbf{\Phi}$ is a $(n_h \times n_f)$ matrix, and $\mathbf{\Xi} = \beta^2\mathbf{\Pi}_{11}\Big[(\mathbf{I}_{n_f} - \beta\mathbf{\Pi}_{11}) - \beta^2\mathbf{\Pi}_{12}(\mathbf{I}_{n_h} - \beta\mathbf{\Pi}_{22})^{-1}\mathbf{\Pi}_{21}\Big]^{-1}\mathbf{\Pi}_{12}(\mathbf{I}_{n_h} - \beta\mathbf{\Pi}_{22})^{-1} + \beta\mathbf{\Pi}_{12}(\mathbf{I}_{n_h} - \beta\mathbf{\Pi}_{22})^{-1}\Big[\mathbf{I}_{n_h} + \beta^2\Pi_{21}\Big[(\mathbf{I}_{n_f} - \beta\mathbf{\Pi}_{11}) - \beta^2\mathbf{\Pi}_{12}(\mathbf{I}_{n_h} - \beta\mathbf{\Pi}_{22})^{-1}\mathbf{\Pi}_{21}\Big]^{-1}\mathbf{\Pi}_{12}(\mathbf{I}_{n_h} - \beta\mathbf{\Pi}_{22})^{-1}\Big]$ is a $(n_f \times n_h)$ matrix. In addition, these conditions reveal that there must be as many nonpredetermined variables as additional exogenous variables to recover adequately the agents' relevant information.

Few applications rely on the representation (4), with $n_f = 2$ and $n_m = n_h = 1$ (Boileau and Normandin 2002, 2003). Our derivations formalize and generalize the conditions under which the representation (4) is valid. Recall, however, that these derivations hold when the numbers of nonpredetermined and additional exogenous variables are identical. This condition is relaxed in our extensions (section 6).

2.4 The Cyclical Fluctuation Parameters

The actual data generating process (4) yields the population moments:

$$vec(\mathbf{\Sigma}_0) = \left[\mathbf{I}_{(n_f+n_m)^2} - \mathbf{\Gamma} \otimes \mathbf{\Gamma}\right]^{-1} vec(\mathbf{\Lambda}), \tag{6.1}$$

and

$$vec(\mathbf{\Sigma}_k) = \left[\mathbf{I}_{(n_f+n_m)} \otimes \mathbf{\Gamma}^k\right] vec(\mathbf{\Sigma}_0). \tag{6.2}$$

The term *vec* represents the vectorization operator (stacking by columns), $\otimes$ denotes the Kronecker product, and $\mathbf{\Sigma}_k = E(\mathbf{x}_t \mathbf{x}'_{t-k})$.

In addition, the population moments (6) lead to the following parameters.

Volatility. The volatility of a selected nonpredetermined variable m_t is measured by its standard deviation σ_m. The variable m_t is more (less) volatile than the measure of business cycle y_t when σ_m is larger (smaller) than σ_y. Note that y_t can correspond in some contexts to another nonpredetermined variable and in other environments to a forcing variable.

Persistence. The persistences of m_t and y_t are measured by their first-order autocorrelation coefficients ρ_m and ρ_y. The selected variable is more (less) persistent than the business cycle when ρ_m is larger (smaller) than ρ_y.

Comovements. The comovements between m_t and y_t are measured by the dynamic cross-correlations $corr(m_t, y_{t+k})$. The selected variable is procyclical (acyclical) [countercyclical] when $corr(m_t, y_t) > 0$ ($corr(m_t, y_t) = 0$) [$corr(m_t, y_t) < 0$]. The selected variable is a leading (coincident) [lagging] indicator of the business cycle when $corr(m_t, y_{t+k})$ reaches a maximum, in absolute value, for $k > 0$ ($k = 0$) $[k < 0]$.

These parameters are frequently invoked to characterize the salient features of the actual fluctuations at the business cycle frequency. In particular, these cyclical fluctuation parameters are extensively used in the real business cycle literature (Kydland and Prescott 1982; King, Plosser, and Rebelo 1988).

3. The Artificial Economy

In what follows, we analyze an artificial economy that posits the correct fundamental assumptions on agents' economic behavior, but may impose invalid auxiliary assumptions defining information sets. Specifically, the artificial economy is characterized by the rules (1) and two laws of motion, that may differ from the true law of motion (2). In this section, we specify the alternative laws of motion, derive the artificial data generating process, and obtain the associated estimates of cyclical fluctuation parameters.

3.1 The Alternative Laws of Motion

In the artificial economy, the forcing variables are assumed to be governed by the stationary VAR process:

$$\begin{pmatrix} \mathbf{f}_t \\ \mathbf{m}_t \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{\Pi}}_{11} & \tilde{\mathbf{\Pi}}_{12} \\ \tilde{\mathbf{\Pi}}_{21} & \tilde{\mathbf{\Pi}}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{f}_{t-1} \\ \mathbf{m}_{t-1} \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{u}}_{f,t} \\ \tilde{\mathbf{u}}_{m,t} \end{pmatrix},$$

or

$$\mathbf{x}_t = \tilde{\mathbf{\Pi}} \mathbf{x}_{t-1} + \tilde{\mathbf{u}}_t. \tag{7}$$

The vector $\mathbf{x}_t$ contains the variables observed by the econometrician. In contrast, the vector $\mathbf{h}_t$ captures variables that are hidden to (unknown or omitted by) the econometrician. The vector $\tilde{\mathbf{u}}_t$ includes error terms, with covariance matrices $\tilde{\mathbf{\Omega}}_{ff} = E(\tilde{\mathbf{u}}_{f,t}\tilde{\mathbf{u}}'_{f,t})$ and $\tilde{\mathbf{\Omega}} = E(\tilde{\mathbf{u}}_t\tilde{\mathbf{u}}'_t)$. The matrix $\tilde{\mathbf{\Pi}}$ contains the feedback coefficients.

The VAR (7) embodies two laws of motion: the standard and augmented ones. The standard law of motion imposes that $\tilde{\mathbf{\Pi}}_{12} = \mathbf{0}$. In this case, the law of motion involves only forcing variables. This means that the econometrician's information set used to construct agents' expectations of future forcing variables incorporates only the history of these variables.

When $\mathbf{\Pi}_{12} = \mathbf{0}$, it is assumed that the standard law of motion coincides with the true one, that is $\tilde{\mathbf{\Pi}}_{11} = \mathbf{\Pi}_{11}$ and $\tilde{\mathbf{\Omega}}_{ff} = \mathbf{\Omega}_{ff}$. This ensures that economic agents and the econometrician use identical information, since forcing variables are observed by everyone. When $\mathbf{\Pi}_{12} \neq \mathbf{0}$, the standard law of motion is misspecified. This is because agents exploit the additional information contained in hidden variables to improve their forecasts of future forcing variables. In this context, agents have superior (richer) information relative to the econometrician, given that hidden variables are only observed by agents.

The augmented law of motion relaxes the restrictions $\tilde{\mathbf{\Pi}}_{12} = \mathbf{0}$. Thus, it augments the standard law of motion by including nonpredetermined variables. This means that the econometrician constructs agents' expectations of future forcing variables from the information contained in the history of both forcing and nonpredetermined variables.

When $\mathbf{\Pi}_{12} = \mathbf{0}$, the augmented law of motion is misspecified. This occurs because the econometrician's information set includes redundant nonpredetermined variables. When $\mathbf{\Pi}_{12} \neq \mathbf{0}$, it is postulated that the augmented law of motion coincides with the data generating process, that is $\tilde{\mathbf{\Pi}} = \mathbf{\Gamma}$ and $\tilde{\mathbf{\Omega}} = \mathbf{\Lambda}$. In this environment, the feedbacks from lagged nonpredetermined variables to current forcing variables reflect the notion that agents exploit the information contained in hidden variables.

Importantly, these feedbacks do not indicate economic causalities, but rather Granger-causalities. Hence, the nonpredetermined variables do not cause forcing variables in an economic sense, but rather are surrogates for the hidden variables that cause forcing variables. Interestingly, the Granger-causalities allow the verification of the existence of superior information from (7), which exclusively contains variables that are in the econometrician's information set, instead of (2), which also contains variables out of that information set. Finally, the surrogates permit the econometrician to recover the agents' extra information, and avoids the difficult task of identifying and measuring all hidden variables.

In practice, standard laws of motion are almost always used in empirical analyses. Few exceptions use augmented laws of motion, but with a single nonpredetermined variable (Campbell and Shiller 1987; Campbell and Deaton 1989; Flavin 1993; Normandin 1999; Boileau and Normandin 2002, 2003).

3.2 The Artificial Data Generating Process

The econometrician uses the law of motion (7) to construct the agents' expectations involved in the rules (1). This econometrician's solution is:

$$\tilde{\mathbf{m}}_t = \mathbf{\Phi}\mathbf{e}_f\beta\tilde{\mathbf{\Pi}}\left[\mathbf{I}_{(n_f+n_h)} - \beta\tilde{\mathbf{\Pi}}\right]^{-1}\mathbf{x}_t = \tilde{\mathbf{\Theta}}\mathbf{x}_t = \tilde{\mathbf{\Upsilon}}_f\mathbf{f}_t + \tilde{\mathbf{\Upsilon}}_m\mathbf{m}_t, \tag{8}$$

The solution (8) depends on all variables included in the econometrician's information set. In particular, (8) involves nonpredetermined variables ($\tilde{\boldsymbol{\Upsilon}}_m \neq \mathbf{0}$) if and only if these variables Granger-cause forcing variables ($\tilde{\boldsymbol{\Pi}}_{12} \neq \mathbf{0}$).

Furthermore, the econometrician's solution (8) and the agents' solution (3) are identical when the law of motion (7) correctly specifies the true law of motion (2). For example, the artificial series $\tilde{\mathbf{m}}_t$ exactly match the actual data $\mathbf{m}_t$, as long as $\tilde{\boldsymbol{\Pi}}_{12} = \mathbf{0}$ when agents do not possess superior information $\boldsymbol{\Pi}_{12} = \mathbf{0}$. In this case, it is easy to show that $\tilde{\mathbf{m}}_t = \mathbf{m}_t$ since $\tilde{\boldsymbol{\Upsilon}}_f = \boldsymbol{\Upsilon}_f$ and $\tilde{\boldsymbol{\Upsilon}}_m = \boldsymbol{\Upsilon}_m = \mathbf{0}$. Likewise, $\tilde{\mathbf{m}}_t = \mathbf{m}_t$, as long as $\tilde{\boldsymbol{\Pi}}_{12} \neq \mathbf{0}$ when agents have superior information $\boldsymbol{\Pi}_{12} \neq \mathbf{0}$. This occurs because $\tilde{\boldsymbol{\Upsilon}}_f = \mathbf{0}$ and $\tilde{\boldsymbol{\Upsilon}}_m = \mathbf{I}_{n_m}$, given that $\tilde{\boldsymbol{\Theta}} = \boldsymbol{\Theta}\boldsymbol{\Upsilon}^{-1} = \mathbf{e}_m$ — where $\mathbf{e}_m$ is a selection matrix defined as $\mathbf{m}_t = \mathbf{e}_m \mathbf{x}_t$, $\tilde{\boldsymbol{\Pi}} = \boldsymbol{\Gamma} = \boldsymbol{\Upsilon}\boldsymbol{\Pi}\boldsymbol{\Upsilon}^{-1}$, and $\mathbf{e}_f \boldsymbol{\Upsilon} = \mathbf{e}_f$.

In contrast, the econometrician's solution (8) and the agents' solution (3) are different when the law of motion (7) incorrectly specifies the true law of motion (2). Importantly, the artificial series deviate from the actual data, not because the rules are inadequate, but simply because the law of motion is misspecified. Put differently, the artificial economy do not match the actual one, not because the fundamental assumptions on agents' economic behavior are incorrect, but rather because the auxiliary assumptions defining information sets are invalid.

Finally, the artificial data generating process is summarized as:

$$\begin{pmatrix} \mathbf{f}_t \\ \tilde{\mathbf{m}}_t \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{\Gamma}}_{11} & \tilde{\boldsymbol{\Gamma}}_{12} \\ \tilde{\boldsymbol{\Gamma}}_{21} & \tilde{\boldsymbol{\Gamma}}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{f}_{t-1} \\ \tilde{\mathbf{m}}_{t-1} \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{v}}_{f,t} \\ \tilde{\mathbf{v}}_{m,t} \end{pmatrix},$$

or

$$\tilde{\mathbf{x}}_t = \tilde{\mathbf{\Gamma}}\tilde{\mathbf{x}}_{t-1} + \tilde{\mathbf{v}}_t. \tag{9}$$

When $\tilde{\mathbf{\Pi}}_{12} = \mathbf{0}$, then $\tilde{\mathbf{x}}_t = \tilde{\mathbf{\Psi}}\mathbf{f}_t$, $\tilde{\mathbf{v}}_t = \tilde{\mathbf{\Psi}}\tilde{\mathbf{u}}_{f,t}$, $\tilde{\mathbf{\Gamma}} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Pi}}_{11}\mathbf{e}_f$, $\tilde{\mathbf{\Lambda}} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Omega}}_{ff}\tilde{\mathbf{\Psi}}'$, and $\tilde{\mathbf{\Psi}} = (\,\mathbf{I}_{n_f} \quad \tilde{\mathbf{\Upsilon}}'_f\,)'$. This artificial data generating process is identical to the data generating process (4) under the absence of superior information ($\mathbf{\Pi}_{12} = \mathbf{0}$), since $\tilde{\mathbf{\Pi}}_{11} = \mathbf{\Pi}_{11}$, $\tilde{\mathbf{\Omega}}_{ff} = \mathbf{\Omega}_{ff}$, and $\tilde{\mathbf{\Upsilon}}_f = \mathbf{\Upsilon}_f$. In contrast, it differs from (4) under the existence of superior information ($\mathbf{\Pi}_{12} \neq \mathbf{0}$).

When $\tilde{\mathbf{\Pi}}_{12} \neq \mathbf{0}$, then $\tilde{\mathbf{x}}_t = \tilde{\mathbf{\Upsilon}}\mathbf{x}_t$, $\tilde{\mathbf{v}}_t = \tilde{\mathbf{\Upsilon}}\tilde{\mathbf{u}}_t$, $\tilde{\mathbf{\Gamma}} = \tilde{\mathbf{\Upsilon}}\tilde{\mathbf{\Pi}}\tilde{\mathbf{\Upsilon}}^{-1}$, $\tilde{\mathbf{\Lambda}} = \tilde{\mathbf{\Upsilon}}\tilde{\mathbf{\Omega}}\tilde{\mathbf{\Upsilon}}'$, and $\tilde{\mathbf{\Upsilon}} = (\,\mathbf{e}'_f \quad \tilde{\mathbf{\Theta}}'\,)'$. This formulation deviates from the data generating process (4) when $\mathbf{\Pi}_{12} = \mathbf{0}$. However, it is the same as (4) when $\mathbf{\Pi}_{12} \neq \mathbf{0}$, given that $\tilde{\mathbf{\Pi}} = \mathbf{\Gamma}$, $\tilde{\mathbf{\Omega}} = \mathbf{\Lambda}$, and $\tilde{\mathbf{\Theta}} = \mathbf{e}_m$.

3.3 The Cyclical Fluctuation Estimates

The artificial data generating process (9) produces:

$$vec(\tilde{\mathbf{\Sigma}}_0) = \left[\mathbf{I}_{(n_f+n_m)^2} - \tilde{\mathbf{\Gamma}} \otimes \tilde{\mathbf{\Gamma}}\right]^{-1} vec(\tilde{\mathbf{\Lambda}}), \tag{10.1}$$

and

$$vec(\tilde{\mathbf{\Sigma}}_k) = \left[\mathbf{I}_{(n_f+n_m)} \otimes \tilde{\mathbf{\Gamma}}^k\right] vec(\tilde{\mathbf{\Sigma}}_0), \tag{10.2}$$

where $\tilde{\mathbf{\Sigma}}_k = E(\tilde{\mathbf{x}}_t\tilde{\mathbf{x}}'_{t-k})$.

Expressions (10) yield estimates of the volatility and persistence of selected nonpredetermined variables, as well as the comovements between these variables and the business cycle. In principle, these estimates are identical to the true values of the cyclical fluctuation parameters obtained from (6) when the law of motion (7) correctly specifies the true law of motion (2). In practice, however, the estimates may slightly differ because they are computed from the estimated coefficients of (7), while the true values are constructed from fixed (known) coefficients of (2). Finally, the estimates may substantially deviate when the law of motion (7) misspecifies the true law of motion (2).

4. Simulation Procedure

In this section, we design a simulation procedure to document the properties of the inference of cyclical fluctuations under various specifications of the alternative law of motion (7). For these evaluations, we rely on some parameters of interest, such as σ_m, ρ_m, and $corr(m_t, f_{t+k})$ for $k = -4$, -2, 0, 2, and 4 — where m_t and f_t correspond to the first elements of the vectors $\mathbf{m}_t$ and $\mathbf{f}_t$. These cyclical fluctuation parameters summarize salient features of the economy, such as the volatility and the persistence of m_t, as well as the comovements between m_t and f_t. In practice, estimates of these parameters are often used to describe fluctuations of macroeconomic aggregates at the business cycle frequency.

For the alternative law of motion (7), we pay attention to the cases where the standard law of motion correctly (incorrectly) specifies the true law of motion due to the absence (presence) of superior information. We also analyze artificial economies where the augmented law of motion represents an appropriate (inappropriate) specification.

Specifically, the procedure involves the following steps.

Step 1. The true values of the cyclical fluctuation parameters are computed from expressions (6) and given values for $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ of the actual data generating process (4). These matrices are calculated from a specific parametrization of the coefficients of the rules and the true law of motion. For the rules (1), we impose $n_m = n_f$ and set $\beta = 0.99$ to a standard value of the discount factor for a quarterly frequency, as well as $\mathbf{\Phi} = [\phi_{ij}]$ with $\phi_{ii} = 1$ and $\phi_{ij} = 0.5$ (for $i \neq j$) to ensure that each nonpredetermined variable is affected by all forcing variables. For the true law of motion, we amend (2) as:

$$\begin{pmatrix} \mathbf{f}_t \\ \mathbf{h}_t \\ \mathbf{h}_t^+ \end{pmatrix} = \begin{pmatrix} \mathbf{\Pi}_{11} & \mathbf{\Pi}_{12} & \mathbf{\Pi}_{13} \\ \mathbf{\Pi}_{21} & \mathbf{\Pi}_{22} & \mathbf{\Pi}_{23} \\ \mathbf{\Pi}_{31} & \mathbf{\Pi}_{32} & \mathbf{\Pi}_{33} \end{pmatrix} \begin{pmatrix} \mathbf{f}_{t-1} \\ \mathbf{h}_{t-1} \\ \mathbf{h}_{t-1}^+ \end{pmatrix} + \begin{pmatrix} \mathbf{u}_{f,t} \\ \mathbf{u}_{h,t} \\ \mathbf{u}_{h,t}^+ \end{pmatrix},$$

or

$$\mathbf{w}_t = \mathbf{\Pi}\mathbf{w}_{t-1} + \mathbf{u}_t, \tag{2'}$$

where the $(n_{h^+} \times 1)$ vector $\mathbf{h}_t^+$ contains 'redundant' hidden variables. For (2′), we set $n_f = n_h + n_{h^+}$, which will prove computationally convenient to analyze cases where the number of nonpredetermined variables ($n_m = n_f$) is greater or equal to that of 'relevant' hidden variables (n_h). Also, we fix $\mathbf{\Pi}_{11} = \begin{pmatrix} \mathbf{\Pi}_{22} & \mathbf{\Pi}_{23} \\ \mathbf{\Pi}_{32} & \mathbf{\Pi}_{33} \end{pmatrix}$, $\mathbf{\Pi}_{21} = \mathbf{\Pi}_{31} = \mathbf{0}$, $\mathbf{\Pi}_{11} = [\pi_{11,ij}]$ with $\pi_{11,ii} = 0.5$ and $\pi_{11,ij} = 0.1$ (for $i \neq j$), $\mathbf{\Pi}_{12} = [\pi_{12,ij}]$ with $\pi_{12,ii} = 0.5$ and $\pi_{12,ij} = 0.1$ (for $i \neq j$), $\mathbf{\Pi}_{13} = [\pi_{13,ij}]$ with $\pi_{13,ii} = 0.001$ and $\pi_{13,ij} = 0$ (for $i \neq j$), and $\mathbf{\Omega} = E(\mathbf{u}_t\mathbf{u}_t') = \mathbf{I}_{2n_f}$. This ensures that the hidden variables included in $\mathbf{h}_t$ are 'relevant', since they contain useful information to forecast future forcing variables ($\mathbf{\Pi}_{12} \neq \mathbf{0}$). In contrast, the hidden variables incorporated in $\mathbf{h}_t^+$ are 'redundant', since they do not Granger-cause forcing variables ($\mathbf{\Pi}_{13} \approx \mathbf{0}$). In this context, $n_h = 0$ ($n_h > 0$) reflects the absence (presence) of superior information. Finally, note that the parametrization yields real positive eigenvalues between 0.4 and 0.9. This guarantees that the process (2′) is stationary and that exogenous (forcing and hidden) variables exhibit smooth and persistent dynamics.

Step 2. The estimates of the cyclical fluctuation parameters are calculated from equations (10) and given values for $\tilde{\mathbf{\Gamma}}$ and $\tilde{\mathbf{\Lambda}}$ of the artificial data generating process (9). These matrices are calculated from the parametrized version of the rules and an estimated version of the alternative law of motion. For this law of motion, we amend (7) as:

$$\begin{pmatrix} \mathbf{f}_t \\ \mathbf{m}_t^- \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{\Pi}}_{11} & \tilde{\mathbf{\Pi}}_{12} \\ \tilde{\mathbf{\Pi}}_{21} & \tilde{\mathbf{\Pi}}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{f}_{t-1} \\ \mathbf{m}_{t-1}^- \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{u}}_{f,t} \\ \tilde{\mathbf{u}}_{m,t}^- \end{pmatrix},$$

or

$$\mathbf{x}_t = \tilde{\mathbf{\Pi}}\mathbf{x}_{t-1} + \tilde{\mathbf{u}}_t, \tag{7'}$$

where the $(n_{m^-} \times 1)$ vector $\mathbf{m}_t^-$ selects the first n_{m^-} nonpredetermined variables included in $\mathbf{m}_t$. For (7′), we set $n_m \geq n_m^- \geq 0$, which will prove useful to enrich our analysis to cases where the number of nonpredetermined variables involved in the rules (n_m) is greater or equal to that of nonpredetermined variables considered by the econometrician (n_{m^-}). In this environment, $n_{m^-} = 0$ ($n_{m^-} > 0$) recovers the standard (augmented) law of motion, while $n_{m^-} = n_h$ ($n_{m^-} \neq n_h$) yields correctly (incorrectly) specified laws of motion. Also, the law of motion (7′) is estimated by applying Ordinary Least Squares (OLS) on simulated data. These data are computed recursively from the actual data generating process (4), intial conditions $\mathbf{x}_0 = \mathbf{0}$, and innovations $\mathbf{v}_t = \mathbf{\Upsilon}\mathbf{u}_t$ where $\mathbf{u}_t$ is drawn from a normal distribution $[N(\mathbf{0}, \mathbf{\Omega})]$ for $t = 1, \ldots, T$. We set $T = 250$ to a traditional sample size in business cycle analyses, which usually focus on the quarterly data covering the post World War II period.

Step 3. The 95% confidence intervals for the estimates of the cyclical fluctuation parameters are computed from the bootstrap percentile method. Specifically, we use this method to construct a typical interval by selecting the middle 95% of the 500 ordered estimates of a cyclical fluctuation parameter, generated from a general Block Bootstrap approach. This approach estimates the parameters from symmetric functions of bootstrap samples containing vectors of consecutive observations for forcing and nonpredetermined variables (Bühlmann 2002; Lahiri 2003). The bootstrap samples are constructed by forming blocks of the consecutive vectorized observations, selecting some blocks at random with replacement and joining them together (Künsch 1989). In our application, we fix the size of the vectors to 2, which is a natural selection given that the actual data generating process (4) corresponds to a first-order VAR process. (Using a size of 4 yields similar results.) Note that these vectors are important because some cyclical fluctuation parameters (such as the first-order autocorrelation and dynamic cross-correlations) are defined through the joint distribution of 2 or more consecutive observations. Thus, a basic block bootstrap (i.e. on individual observations) does not work because 2 consecutive observations at the junction point of 2 blocks in the bootstrap sample are not necessarily true consecutive observations. In addition, we set the size of blocks to 7, which accords with the optimal theoretical block length of order $T^{1/3}$ (Lahiri 2003). Note that these blocks are important to preserve the dependence structure of the data, given that they are not independently distributed. Interestingly, this is achieved without resorting to any parametric nor distributional assumptions.

Step 4. Some statistics used to assess the econometric properties of the estimates are computed from a Monte Carlo experiment with 1000 replications. These statistics are the coverage probability CP of the confidence intervals, the average length of the confidence intervals $ALCI$, and the root mean square error $RMSE$ of the estimates. CP corresponds to the fraction of the replications for which the true values of the parameters fall in the confidence intervals. $ALCI$ is calculated by averaging over all replications the differences between the upper and lower bounds of the confidence intervals. $RMSE$ is computed by taking the square root of the sum over all replications of the square deviations between the estimates and the true values of the parameters, divided by 1000. For each replication, steps 2 and 3 are done.

Steps 1 to 4 are performed for all combinations $n_f \geq n_{m^-}$ and $n_f \geq n_h$ (to satisfy the necesary conditions derived in section 2), with $n_f = n_m = 1$, 3, and 5, $n_{m^-} = 0$, 1, 3, and 5, as well as $n_h = 0$, 1, 3, and 5. The combinations $n_h = n_{m^-}$ ($n_h \neq n_{m^-}$) capture the cases where the law of motion ($7'$) is well specified (misspecified). In particular, $n_h = n_{m^-} = 0$ ($n_h > n_{m^-} = 0$) corresponds to environments where the standard law of motion is well specified (misspecified) due to the absence (presence) of superior information. Moreover, $n_h = n_{m^-} > 0$ ($n_h \neq n_{m^-} > 0$) represents artificial economies where the augmented law of motion is well specified (misspecified). Interestingly, the augmented law of motion can be misspecified because there is no superior information ($n_h = 0$), or because there is too many or too few nonpredetermined variables ($n_{m^-} > n_h$ or $n_{m^-} < n_h$) included in ($7'$) to adequately recover the superior information.

5. Simulation Results

In this section, we first present the results obtained by applying the simulation procedure just explained. We then verify the robustness of our findings to different sample sizes and various parametrizations of the true law of motion. We finally assess whether some information criteria are successful in detecting the absence or presence of superior information.

5.1 Basic Results

Table 1 reports the statistics CP, $ALCI$, and $RMSE$ for actual economies with no superior information ($n_h = 0$). First, the artificial economies with standard laws of motion ($n_{m^-} = 0$) always yield values of CP close to 95% for the 95% confidence intervals of the various cyclical fluctuation estimates. As expected, this occurs because the standard laws of motion correspond to adequate specifications of the true laws of motion. Interestingly, the artificial economies with augmented laws of motion involving single and multiple hidden variables ($n_{m^-} = 1$, 3, and 5) also systematically produce CP-values approaching 95%. This arises although the augmented laws of motion are misspecified, because they encompass the standard laws of motion.

Second, the standard laws of motion ($n_{m^-} = 0$) always generate values of $ALCI$ that are smaller than those obtained from the various augmented laws of motion ($n_{m^-} > 0$). Intuitively, this is because the augmented laws of motion include redundant nonpredetermined variables, and as such capture noise that reduces the accuracy of the inference of the cyclical fluctuation parameters. Empirically, the discrepancies between the $ALCI$-values obtained from the augmented and standard laws of motion are substantial: the deviations exceed 10% for almost half of the cases and reach a maximum of 21%. It is precisely because the augmented laws of motion tend to produce wider confidence intervals that they generate large CP-values.

Third, the standard laws of motion ($n_{m^-} = 0$) most frequently produce values of $RMSE$ that are smaller than those derived from the augmented laws of motion ($n_{m^-} > 0$). The differences between the $RMSE$-values derived from the augmented and standard laws of motion are numerically nonnegligible: the deviations exceed 10% for rouhgly one fifth of the cases and reach a maximum of 24%. These differences are mainly due to the large standard errors of the estimates of the cyclical fluctuation parameters associated with the augmented laws of motion, rather than to large mean biases of these estimates. (These results are not reported, but are available upon request.) This accords with the fact that the augmented laws of motion generally produce large coverage probabilities and wide confidence intervals.

Table 2 presents the statistics for an environment where the superior information is summarized by a single hidden variable ($n_h = 1$). First, the results indicate that the standard laws of motion ($n_{m^-} = 0$) frequently produce CP-values that are substantially smaller than 95%: these values are smaller than 70% for almost half of the cases and attain a minimum of 0.1%. This occurs because the standard laws of motion represent inadequate specifications of the true laws of motion. In contrast, the augmented laws of motion including only one nonpredetermined variable ($n_{m^-} = 1$) always yield CP-values of about 95%. This arises because these augmented laws of motion exactly incorporate the appropriate number of nonpredetermined variable to adequately recover the superior information. Interestingly, this finding also holds for the other augmented laws of motion ($n_{m^-} > 1$), even if they include too many nonpredetermined variables.

Second, the augmented laws of motion with one nonpredetermined variable ($n_{m^-} = 1$) generate $ALCI$-values that are sometimes smaller than those obtained from the standard laws of motion ($n_{m^-} = 0$), and almost always smaller than those induced by the augmented laws of motion with multiple nonpredetermined variables ($n_{m^-} > 1$). The discrepancies between the $ALCI$-values obtained from the well specified augmented and misspecified standard laws of motion are important: the deviations exceed 10% for all, but one, cases and reach a maximum of 6500%. In contrast, the differences between the $ALCI$-values obtained from the well specified and misspecified augmented laws of motion are mild: the deviations are less than 5% for almost all cases and reach a maximum of 8%.

Third, the augmented laws of motion with a single nonpredetermined variables ($n_{m^-} = 1$) produce $RMSE$-values that are almost always smaller than those derived from the standard laws of motion ($n_{m^-} = 0$) and the augmented laws of motion with multiple nonpredetermined variables ($n_{m^-} > 1$). The differences between the $RMSE$-values associated with the well specified augmented and misspecified standard laws of motion are pronounced: the deviations exceed 50% for more than half of the cases and reach a maximum of 362%. These differences are primarily explained by the large mean biases of the estimates associated with the standard laws of motion ($n_{m^-} = 0$). This is consistent with the observation that these laws of motion usually generate small coverage probabilities. Conversely, the differences between the $RMSE$-values computed from the well specified and misspecified augmented laws of motion are modest: the deviations are less than 5% for four fifth of the cases and attain a maximum of 9%. These differences are essentially due to the large standard errors of the estimates obtained from the augmented laws of motion with too many nonpredetermined variables ($n_{m^-} > 1$). This accords with the fact that these laws of motion tend to generate wide confidence intervals.

Tables 3 and 4 report the statistics for the cases where the superior information is captured by multiple hidden variables ($n_h = 3$ and $n_h = 5$). First, the standard laws of motion ($n_{m^-} = 0$) often generate CP-values that are much smaller than 95%: these values are smaller than 50% for half of the cases and attain a minimum of 3%. In contrast, the augmented laws of motion involving at least as many nonpretermined variables as hidden variables ($n_{m^-} \geq n_h$) yield CP-values that are around 95%. So far, this corroborates the results presented above. In addition, the augmented laws of motion containing less nonpredetermined variables than hidden variables ($n_{m^-} < n_h$) also produce CP-values of about 95%. This finding is surprising, given that these laws of motion are mispecified since they include too few nonpredetermined variables to completely recover the superior information.

Second, the well specified augmented laws of motion ($n_{m^-} = n_h$) generate $ALCI$-values that are sometimes smaller than those obtained from the standard laws of motion ($n_{m^-} = 0$), and almost always smaller than those induced by the augmented laws of motion with too many nonpredetermined variables ($n_{m^-} > n_h$). Also, these discrepancies are always more severe for the standard laws of motion. This confirms the findings already presented. In addition, the well specified augmented laws of motion ($n_{m^-} = n_h$) yield $ALCI$-values that are systematically larger than the augmented laws of motion with too few nonpredetermined variables ($n_{m^-} < n_h$). In fact, the various confidence intervals associated with the augmented laws of motion with one nonpredetermined variable ($n_{m^-} = 1$) are always the narrowest.

Third, the well specified augmented laws of motion ($n_{m^-} = n_h$) produce $RMSE$-values that are always substantially smaller than those derived from the standard laws of motion ($n_{m^-} = 0$), and sometimes slightly smaller than those computed from the augmented laws of motion with too many nonpredetermined variables ($n_{m^-} > n_h$). Again, this is in line with our previous findings. Furthermore, the well specified augmented laws of motion ($n_{m^-} = n_h$) yield $RMSE$-values that are similar to those calculated from the augmented laws of motion with too few nonpredetermined variables ($n_{m^-} < n_h$).

Overall, the results indicate that the estimates of the cyclical fluctuation parameters computed from the standard laws of motion yield efficient inference under the absence of superior information, but are severely biased under the existence of superior information. In the latter case, the estimates are often significantly different from the true values of the cyclical fluctuation parameters not because the fundamental assumptions on agents' economic behavior reflected in the rules are incorrect, but rather because the auxiliary assumptions defining information sets from the standard laws of motion are invalid. In contrast, the estimates obtained from the augmented laws of motion lead to a sizeable loss of efficiency when there is no superior information, but are never substantially biased whether or not there is superior information. Finally, the estimates obtained from the augmented laws of motion with a single nonpredetermined variable are efficient and unbiased, whether unique or multiple nonpredetermined variables are required to fully recover the superior information. These last econometric properties are attractive, given that, in practice, artificial economies with augmented laws of motion are much more tractable when they involve only single nonpredetermined variables.

5.2 Robustness

The robustness of the results is verified in two ways. First, we consider different sample sizes. Specifically, we set $T = 100$ to a typical small sample size in international business cycle studies, which often rely on quarterly data covering the post Bretton Woods period. Also, we fix $T = 1000$ to a large size that tends to an asymptotic sample. For the samples with $T = 100$ and $T = 1000$, the bootstrap block sizes are set to 5 and 10, and the simulation procedure is redone for $n_f \geq n_{m^-}$ and $n_f \geq n_h$ with $n_f = n_m = 1$, 3, and 5, $n_{m^-} = 0$, 1, 3, and 5, as well as $n_h = 0$, 1, 3, and 5. These exercises provide 6 (12) cases where the standard law of motion is appropriately (inappropriately) specified. They also yield 12 environements where the augmented law of motion is correctly specified, 12 cases where it is misspecified because there is no superior information, and 8 (8) artificial economies where there are too many (too few) nonpredetermined variables incorporated in the law of motion ($7'$) to adequately recover the superior information.

Second, we analyze alternative parametrizations of the true law of motion ($2'$). In particular, we alter the baseline parametrization presented above to consider the combinations $\pi_{11,ii} = 0.5$ and $\pi_{12,ii} = -0.5$, $\pi_{11,ii} = -0.5$ and $\pi_{12,ii} = 0.5$, as well as $\pi_{11,ii} = -0.5$ and $\pi_{12,ii} = -0.5$. The parametrization with $\pi_{11,ii} = 0.5$ implies real positive eigenvalues between 0.4 and 0.9, which guaranty that the process ($2'$) is stationary and displays smooth dynamics. The parametrizations with $\pi_{11,ii} = -0.5$ yield real negative eigenvalues between -0.6 and -0.1, which ensure that ($2'$) is stationary but exhibits oscillating dynamics. The parametrization with $\pi_{12,ii} = 0.5$ indicates that future forcing variables are positively linked to current hidden variables, while those with $\pi_{12,ii} = -0.5$ imply the opposite relation. For each parametrization, the simulation procedure is redone for $T = 250$, $n_f \geq n_{m^-}$ and $n_f \geq n_h$ with $n_f = n_m = 1$, 3, and 5, $n_{m^-} = 0$, 1, 3, and 5, as well as $n_h = 0$, 1, 3, and 5. These exercises provide 3 (15) cases where the standard law of motion is appropriately (inappropriately) specified. They also yield 18 environments where the augmented law of motion is correctly specified, 18 cases where it is misspecified because there is no superior information, and 12 (12) artificial economies where there are too many (too few) nonpredetermined variables included in the law of motion ($7'$) to adequately recover the superior information.

Importantly, the exercises involving exclusively alternative sample sizes or parametrizations yield similar results to those reported previously. Likewise, the exercises implying jointly alternative sample sizes and parametrizations produce similar findings. (All results are available upon request.)

5.3 Information Criteria

The simulation results reported so far highlight that the estimates of cyclical fluctuation parameters obtained from standard (augmented) laws of motion tend to be biased (inefficient) under the existence (absence) of superior information. In this context, it becomes crucial to detect the presence (if any) of superior information. As explained above, this is done from the alternative law of motion ($7'$) by verifying the Granger-causalities (if any) of the nonpredetermined variables on forcing variables, where these variables are all in the econometrician's information set.

For this purpose, we propose the modified Bayesian and Akaike information criteria (BIC and AIC):

$$BIC = \log|\tilde{\mathbf{\Omega}}_{ff}| + \frac{\log T}{T}(n_f \times n_{m^-}), \tag{11}$$

$$AIC = \log|\tilde{\mathbf{\Omega}}_{ff}| + \frac{2}{T}(n_f \times n_{m^-}), \tag{12}$$

where $\tilde{\mathbf{\Omega}}_{ff}$ are evaluated at their OLS estimates and $(n_f \times n_{m^-})$ is the number of estimated feedback coefficients in the matrix $\tilde{\mathbf{\Pi}}_{12}$. Expressions (11) and (12) select the appropriate number of nonpredetermined variables (n_{m^-}) to include in (7′). This contrasts with conventional applications, which use information criteria to chose the adequate number of lags. Also, the formulations (11) and (12) preserve the considerable advantage of being easy to implement, as in standard applications.

Table 5 presents statistics assessing the performance of the different information criteria. These statistics are computed from our baseline simulation procedure. They measure the proportions of Monte Carlo replications for which the BIC and AIC correctly select the alternative (standard or augmented) law of motion (7′) that adquately specifies the true law of motion (2′) $(n_h = n_{m^-})$, and incorrectly chose the standard law of motion $(n_h > n_{m^-} = 0)$.

The results reveal that the BIC strikingly overperforms the AIC in identifying the appropriate specification of (7′). In particular, the BIC choses much more often the standard law of motion when there is no superior information $(n_h = n_{m^-} = 0)$, especially when there are multiple forcing variables $(n_f > 1)$. Consequently, the BIC tends to identify a number of nonpredetermined variables that is smaller than that find from the AIC. This finding occurs because the BIC penalizes extra parameters more heavily, for typical sample sizes T. Also, this result parallels the well-documented notion establishing that the BIC selects a number of lags (which converges asymptotically to the true lag length) that is smaller than that obtained from the AIC (Hannan and Quinn 1979; Shibata 1980). In addition, our findings show that both the BIC and AIC never wrongly select the standard law of motion when there is superior information $(n_h > n_{m^-} = 0)$. (These results are robust to the alternative sample sizes and parametrizations, and are available from the authors.)

Overall, these findings suggest that the BIC is useful in empirical analysis to determine whether the standard law of motion is a relevant specification. This is important given that this law of motion yields efficient estimates of the cyclical fluctuation parameters under the absence of superior information, whereas it leads to biased estimates under the existence of superior information. In contrast, the information criteria are not so useful to determine whether the augmented law of motion should contain a single or multiple nonpredetermined variables, given that these specifications produce estimates of the cyclical fluctuation parameters which exhibit similar properties. Moreover, recall that the augmented law of motion with a single nonpredetermined variable offers, in practice, the important advantage of being more tractable.

6. Extensions

This paper has presented and assessed a procedure to estimate conventional parameters characterizing fluctuations at the business cycle frequency, when the economic agents' information set is superior to the econometrician's one. First, we showed that, under certain conditions, augmented laws of motion involving forcing and nonpredetermined variables capture the agents' superior information. Second, we found from our simulation procedure that the estimates of cyclical fluctuation parameters obtained from standard (augmented) laws of motion tend to be biased (inefficient) under the existence (absence) of superior information. In addition, the estimates obtained from the augmented laws of motion with a single nonpredetermined variable are efficient and unbiased, whether unique or multiple nonpredetermined variables are required to fully recover the superior information. Third, we concluded that a variant of the traditional Bayesian information criterion successfully detect the absence or presence of superior information.

Recall that our procedure relies on the rules (1). Admittedly, many macroeconomic environments lead to rules that are more complex than (1). For example, the rules frequently involve not only nonpredetermined variables as in (1), but also predetermined variables. Specifically, the rules often take the form:

$$\hat{\mathbf{n}}_t = \mathbf{\Phi}_{m1}\mathbf{f}_t + \mathbf{\Phi}_{m2}\sum_{j=1}^{\infty}\beta^j E_t\mathbf{f}_{t+j}, \tag{13.1}$$

$$\hat{\mathbf{p}}_{t+1} = \mathbf{\Phi}_{p1}\mathbf{f}_t + \mathbf{\Phi}_{p2}E_t\mathbf{f}_{t+1} + \mathbf{\Phi}_{p3}\sum_{j=1}^{\infty}\beta^j E_t\mathbf{f}_{t+j}, \tag{13.2}$$

where

$$\hat{\mathbf{n}}_t \equiv \mathbf{n}_t - \mathbf{\Theta}_m\mathbf{p}_t, \tag{14.1}$$

$$\hat{\mathbf{p}}_{t+1} \equiv \mathbf{p}_{t+1} - \mathbf{\Theta}_p\mathbf{p}_t. \tag{14.2}$$

Here, the $(n_n \times 1)$ vector $\mathbf{n}_t$ now denotes nonpredetermined variables and the $(n_p \times 1)$ vector $\mathbf{p}_{t+1}$ contains predetermined variables. Also, the vectors $\hat{\mathbf{n}}_t$ and $\hat{\mathbf{p}}_{t+1}$ refer to adjusted nonpredetermined and predetermined variables. The definitions (14) allow the econometrician to measure the adjusted variables from actual data of $\mathbf{n}_t$ and $\mathbf{p}_{t+1}$ and calibrated values of $\mathbf{\Theta}_m$ and $\mathbf{\Theta}_p$. The rules (13) are forward-looking, as adjusted variables are exclusively related to current and expected future forcing variables. Importantly, this implies that all adjusted variables $\mathbf{m}_t = \begin{pmatrix} \hat{\mathbf{n}}_t' & \hat{\mathbf{p}}_{t+1}' \end{pmatrix}'$

represent appropriate surrogates for hidden variables. Hence, augmented laws of motion similar to (7), involving both forcing and adjusted variables, adequately recover the superior information. As a result, these augmented laws of motion with the rules (13) and the definitions (14) can be used to estimate the parameters characterizing the cyclical fluctuations of nonpredetermined and predetermined variables (Boileau and Normandin 2003).

Also, many macroeconomic models yield rules that, in contrast to (1), are nonlinear in forcing variables. Interestingly, a wide variety of these nonlinear environments can be numerically solved by approximation methods that generate linear forward-looking rules similar to (13) and (14) (King, Plosser, and Rebelo 2002). It then become possible to estimate the cyclical fluctuation parameters as above.

Finally, recall that our procedure relies on necessary conditions stating that the numbers of nonpredetermined and hidden variables are identical ($n_m = n_h$). In principle, if there are less nonpredetermined variables than hidden variables ($n_m < n_h$), then augmented laws of motion cannot recover the superior information. In practice, however, our simulation results highlight that the inclusion of a single nonpredetermined variable ($n_m = 1$) is enough to yield adequate estimates of the cyclical fluctuation parameters.

Conversely, if there are more nonpredetermined variables than hidden variables ($n_m > n_h$), then a collection of augmented laws of motion can be stacked as:

$$\begin{pmatrix} \mathbf{x}_{1,t} \\ \vdots \\ \mathbf{x}_{\kappa,t} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{\Pi}}_1 & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \tilde{\mathbf{\Pi}}_\kappa \end{pmatrix} \begin{pmatrix} \mathbf{x}_{1,t-1} \\ \vdots \\ \mathbf{x}_{\kappa,t-1} \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{u}}_{1,t} \\ \vdots \\ \tilde{\mathbf{u}}_{\kappa,t} \end{pmatrix}$$

or more compactly

$$\mathbf{x}_t = \tilde{\mathbf{\Pi}}\mathbf{x}_{t-1} + \tilde{\mathbf{u}}_t \tag{15}$$

Here, $\mathbf{x}_{i,t} = \begin{pmatrix} \mathbf{f}'_t & \mathbf{m}'_{i,t} \end{pmatrix}'$, where the $\left(\frac{n_m}{\kappa} \times 1\right)$ vectors $\mathbf{m}_{i,t}$ contain the ith block of selected nonpredetermined variables (for $i = 1, \ldots, \kappa$). In principle, when $\frac{n_m}{\kappa} = n_h$ then each augmented law of motion in (15) fully captures the superior information. (If n_m is not a multiple of n_h, then some nonpredetermined variables must be omitted.) In practice, however, our simulation results suggest that setting $\left(\frac{n_m}{\kappa}\right) = 1$ yield appropriate estimates of the cyclical fluctuation parameters.

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Table 1. Statistics: No Superior Information ($n_h = 0$)

		$n_{m^-}=0$			$n_{m^-}=1$			$n_{m^-}=3$			$n_{m^-}=5$		
		CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$
	σ_m	0.927	1.147	0.292	0.944	1.188	0.296						
	ρ_m	0.929	0.209	0.054	0.958	0.214	0.054						
	$corr(m_t, f_{t-4})$	0.929	0.103	0.027	0.957	0.112	0.028						
$n_f = 1$	$corr(m_t, f_{t-2})$	0.929	0.202	0.053	0.944	0.204	0.053						
	$corr(m_t, f_t)$	—	—	—	—	—	—						
	$corr(m_t, f_{t+2})$	0.929	0.202	0.053	0.937	0.228	0.055						
	$corr(m_t, f_{t+4})$	0.929	0.103	0.027	0.958	0.121	0.028						
	σ_m	0.908	4.091	1.042	0.929	4.296	1.051	0.939	4.332	1.056			
	ρ_m	0.921	0.176	0.048	0.959	0.179	0.045	0.957	0.178	0.045			
	$corr(m_t, f_{t-4})$	0.933	0.202	0.053	0.955	0.212	0.054	0.964	0.215	0.054			
$n_f = 3$	$corr(m_t, f_{t-2})$	0.937	0.259	0.069	0.952	0.262	0.069	0.952	0.262	0.068			
	$corr(m_t, f_t)$	0.953	0.351	0.089	0.946	0.366	0.093	0.930	0.369	0.095			
	$corr(m_t, f_{t+2})$	0.933	0.287	0.076	0.932	0.304	0.078	0.926	0.310	0.079			
	$corr(m_t, f_{t+4})$	0.919	0.207	0.055	0.946	0.227	0.056	0.937	0.233	0.057			
	σ_m	0.919	56.13	10.54	0.955	67.28	10.67	0.960	66.61	10.60	0.959	65.25	10.54
	ρ_m	0.923	0.113	0.031	0.977	0.114	0.028	0.979	0.115	0.028	0.985	0.115	0.028
	$corr(m_t, f_{t-4})$	0.927	0.354	0.095	0.946	0.369	0.093	0.944	0.368	0.093	0.950	0.367	0.093
$n_f = 5$	$corr(m_t, f_{t-2})$	0.933	0.313	0.084	0.934	0.338	0.087	0.935	0.336	0.086	0.939	0.335	0.086
	$corr(m_t, f_t)$	0.945	0.277	0.071	0.925	0.334	0.087	0.920	0.334	0.087	0.913	0.335	0.088
	$corr(m_t, f_{t+2})$	0.926	0.336	0.088	0.920	0.381	0.099	0.917	0.382	0.100	0.905	0.383	0.101
	$corr(m_t, f_{t+4})$	0.923	0.365	0.096	0.927	0.410	0.104	0.927	0.412	0.105	0.917	0.412	0.106

Notes: CP represents the coverage probability of the confidence intervals, $ALCI$ is the average length of the confidence intervals, and $RMSE$ is the root mean square error of the estimates of cyclical fluctuation parameters. σ_m and ρ_m are the standard deviation and the first-order autocorrelation coefficient of m_t, while $corr(m_t, f_{t+k})$ are the dynamic cross-correlations between m_t and f_t (for $k = -4$, -2, 0, 2, and 4). m_t and f_t are the first elements of the vectors $\mathbf{m}_t$ and $\mathbf{f}_t$ containing the nonpredetermined and forcing variables. n_m=n_f is the number of nonpredetermined and forcing variables in the rules (1) and in the true law of motion (2′), n_h is the number of 'relevant' hidden variables in the true law of motion (2′), and n_{m^-} is the number of nonpredetermined variables in the alternative (standard or augmented) law of motion (7′). $n_{m^-} = 0$ ($n_{m^-} > 0$) corresponds to the standard (augmented) law of motion. $n_{m-} = n_h$ ($n_{m-} \neq n_h$) implies that the alternative law of motion (7′) is an adequate (inadequate) specification of the true law of motion (2′). — indicates omitted cases where by construction $|corr(m_t, f_t)| = 1$ in the actual economy, so that the parameter value is on the boundary of the set of admissible values.

Table 2. Statistics: Superior Information ($n_h = 1$)

		$n_{m^-} = 0$			$n_{m^-} = 1$			$n_{m^-} = 3$			$n_{m^-} = 5$		
		CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$
$n_f = 1$	σ_m	0.673	2.247	0.943	0.932	2.640	0.694						
	ρ_m	0.665	0.190	0.083	0.933	0.166	0.044						
	$corr(m_t, f_{t-4})$	0.559	0.165	0.093	0.933	0.210	0.057						
	$corr(m_t, f_{t-2})$	0.228	0.225	0.170	0.925	0.205	0.056						
	$corr(m_t, f_t)$	0.001	0.004	0.319	0.939	0.264	0.069						
	$corr(m_t, f_{t+2})$	0.320	0.225	0.158	0.934	0.325	0.086						
	$corr(m_t, f_{t+4})$	0.560	0.165	0.084	0.938	0.212	0.056						
$n_f = 3$	σ_m	0.859	6.732	1.804	0.931	7.312	1.705	0.935	7.503	1.726			
	ρ_m	0.815	0.164	0.058	0.944	0.140	0.037	0.941	0.150	0.040			
	$corr(m_t, f_{t-4})$	0.892	0.264	0.075	0.935	0.292	0.075	0.948	0.306	0.077			
	$corr(m_t, f_{t-2})$	0.483	0.266	0.153	0.938	0.233	0.060	0.944	0.233	0.059			
	$corr(m_t, f_t)$	0.501	0.242	0.158	0.956	0.247	0.060	0.948	0.257	0.063			
	$corr(m_t, f_{t+2})$	0.827	0.279	0.109	0.940	0.326	0.082	0.944	0.331	0.083			
	$corr(m_t, f_{t+4})$	0.707	0.251	0.112	0.936	0.267	0.070	0.934	0.277	0.071			
$n_f = 5$	σ_m	0.922	167.3	24.32	0.924	161.1	24.62	0.917	167.5	24.66	0.922	160.6	24.52
	ρ_m	0.889	0.096	0.030	0.930	0.075	0.021	0.937	0.080	0.022	0.936	0.081	0.022
	$corr(m_t, f_{t-4})$	0.828	0.389	0.147	0.933	0.317	0.082	0.930	0.334	0.089	0.931	0.332	0.088
	$corr(m_t, f_{t-2})$	0.820	0.308	0.121	0.933	0.240	0.060	0.934	0.243	0.062	0.932	0.242	0.062
	$corr(m_t, f_t)$	0.823	0.207	0.069	0.939	0.221	0.055	0.939	0.230	0.057	0.934	0.232	0.058
	$corr(m_t, f_{t+2})$	0.907	0.291	0.081	0.932	0.319	0.083	0.928	0.327	0.084	0.928	0.329	0.085
	$corr(m_t, f_{t+4})$	0.926	0.362	0.096	0.927	0.395	0.104	0.924	0.401	0.104	0.923	0.403	0.105

Notes: See Table 1.

Table 3. Statistics: Superior Information ($n_h = 3$)

		$n_{m^-} = 0$			$n_{m^-} = 1$			$n_{m^-} = 3$			$n_{m^-} = 5$		
		CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$
	σ_m	0.962	38.51	7.872	0.913	21.55	5.044	0.915	21.68	5.046			
	ρ_m	0.903	0.120	0.038	0.919	0.113	0.030	0.940	0.112	0.030			
	$corr(m_t, f_{t-4})$	0.900	0.331	0.094	0.924	0.274	0.074	0.933	0.274	0.073			
$n_f = 3$	$corr(m_t, f_{t-2})$	0.790	0.269	0.101	0.928	0.194	0.052	0.942	0.197	0.052			
	$corr(m_t, f_t)$	0.230	0.202	0.183	0.934	0.286	0.074	0.933	0.289	0.074			
	$corr(m_t, f_{t+2})$	0.215	0.295	0.231	0.939	0.393	0.103	0.943	0.397	0.103			
	$corr(m_t, f_{t+4})$	0.209	0.344	0.248	0.940	0.404	0.106	0.939	0.409	0.106			
	σ_m	0.973	2278.	615.8	0.938	773.6	89.75	0.944	787.6	90.28	0.942	788.5	90.81
	ρ_m	0.879	0.057	0.021	0.940	0.054	0.015	0.943	0.054	0.015	0.937	0.054	0.015
	$corr(m_t, f_{t-4})$	0.989	0.269	0.058	0.943	0.158	0.041	0.951	0.158	0.041	0.942	0.163	0.043
$n_f = 5$	$corr(m_t, f_{t-2})$	0.997	0.183	0.036	0.947	0.145	0.036	0.951	0.147	0.036	0.947	0.151	0.038
	$corr(m_t, f_t)$	0.154	0.099	0.103	0.944	0.234	0.060	0.948	0.235	0.060	0.957	0.239	0.061
	$corr(m_t, f_{t+2})$	0.276	0.192	0.150	0.945	0.356	0.094	0.945	0.357	0.094	0.951	0.361	0.094
	$corr(m_t, f_{t+4})$	0.355	0.280	0.196	0.944	0.467	0.124	0.944	0.468	0.124	0.950	0.471	0.124

Notes: See Table 1.

Table 4. Statistics: Superior Information ($n_h = 5$)

		$n_{m^-} = 0$			$n_{m^-} = 1$			$n_{m^-} = 3$			$n_{m^-} = 5$		
		CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$	CP	$ALCI$	$RMSE$
	σ_m	0.984	6081.	2239.	0.933	1634.	178.8	0.941	1654.	179.4	0.946	1664.	180.2
	ρ_m	0.445	0.039	0.033	0.940	0.052	0.015	0.941	0.052	0.015	0.948	0.052	0.014
	$corr(m_t, f_{t-4})$	1.000	0.186	0.026	0.945	0.093	0.024	0.945	0.094	0.024	0.947	0.095	0.024
$n_f = 5$	$corr(m_t, f_{t-2})$	0.991	0.123	0.041	0.957	0.133	0.033	0.953	0.135	0.033	0.956	0.137	0.034
	$corr(m_t, f_t)$	0.025	0.072	0.121	0.940	0.245	0.064	0.954	0.247	0.064	0.961	0.249	0.065
	$corr(m_t, f_{t+2})$	0.031	0.141	0.195	0.942	0.373	0.100	0.947	0.376	0.100	0.958	0.378	0.100
	$corr(m_t, f_{t+4})$	0.036	0.206	0.267	0.939	0.490	0.132	0.945	0.493	0.132	0.955	0.495	0.132

Notes: See Table 1.

Table 5. Information Criteria

Akaike Information Criterion (AIC)

	$n_h = 0$	$n_h = 1$		$n_h = 3$		$n_h = 5$	
	$n_{m^-} = 0$	$n_{m^-} = 0$	$n_{m^-} = 1$	$n_{m^-} = 0$	$n_{m^-} = 3$	$n_{m^-} = 0$	$n_{m^-} = 5$
$n_f = 1$	0.840	0.000	1.000				
$n_f = 3$	0.371	0.000	0.180	0.000	1.000		
$n_f = 5$	0.094	0.000	0.000	0.000	0.009	0.000	0.996

Bayesian Information Criterion (BIC)

	$n_h = 0$	$n_h = 1$		$n_h = 3$		$n_h = 5$	
	$n_{m^-} = 0$	$n_{m^-} = 0$	$n_{m^-} = 1$	$n_{m^-} = 0$	$n_{m^-} = 3$	$n_{m^-} = 0$	$n_{m^-} = 5$
$n_f = 1$	0.980	0.000	1.000				
$n_f = 3$	0.856	0.000	0.793	0.000	0.939		
$n_f = 5$	0.588	0.000	0.010	0.000	0.635	0.000	0.362

Notes: Entries are the proportions of times that the AIC and BIC correctly select the alternative (standard or augmented) law of motion ($7'$) that adequately specifies the true law of motion ($2'$) ($n_h = n_{m-}$), and incorrectly chose the standard law of motion ($n_h > n_{m-} = 0$). n_m=n_f is the number of nonpredetermined and forcing variables in the rules (1) and in the true law of motion ($2'$), n_h is the number of 'relevant' hidden variables in the true law of motion ($2'$), and n_{m^-} is the number of nonpredetermined variables in the alternative (standard or augmented) law of motion ($7'$). $n_h = 0$ ($n_h > 0$) indicates the absence (presence) of superior information. $n_{m^-} = 0$ ($n_{m^-} > 0$) corresponds to the standard (augmented) law of motion.