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An Equilibrium Model of the Term Structure of Interest Rates: Recursive Preferences at Play

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Abstract

In this paper we analyze the performance of an equilibrium model of the term structure of the interest rate under Epstein-Zin/Weil preferences in which consumption growth and inflation follow a VAR process with logistic stochastic volatility. We find that the model can successfully reproduce the first moment of yields and their persistence, but fails to reproduce their standard deviation. The filtered stochastic volatility is a good indicator of crises and shows high persistence, but it is not enough to generate a slowly decaying volatility of yields with respect to maturity. Preference parameters are estimated to be about 4 for the coefficient of relative risk aversion and infinity for the elasticity of intertemporal substitution.

Keywords: Yield curve; Recursive preferences; Logistic stochastic volatility; Nonlinear Kalman filter; Quadrature-based methods.

JEL Classification Numbers: E43, G12, C32.

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1 Introduction

The literature on equilibrium models of the term structure of interest rates has emphasized matching primarily three stylized facts of the yield curve as one way to evaluate the performance of the models: 1) an upward sloping average term structure, 2) a slowly decaying volatility with respect to maturity, and 3) varying term premia. Examples include, among others, Backus et al. (1989), Donaldson et al. (1990), Pennacchi (1991), Boudoukh (1993), Backus and Zin (1994), Canova and Marrinan (1996), and, more recently, Piazzesi and Schneider (2006), Wachter (2006), Gallmeyer et al. (2007), and Doh (2008). The more recent literature focuses on incorporating alternative preferences with respect to the power utility case. This study examines how good an equilibrium model of the term structure of nominal interest rates is for reproducing the moments of yields under Epstein-Zin/Weil preferences and persistent stochastic volatility in inflation.

We model consumption growth and inflation as a VAR(3) process with stochastic volatility in inflation (SV-VAR hereupon). We specify the lag order at three based on various lag selection criteria, and justify volatility in inflation given the results of heteroscedasticity tests performed on the homoscedastic version of the model. Volatility is modeled as a logistic function of a unit root process like in Lee (2008). We estimate the model by maximum likelihood and, given the nonlinearity relating observables and non-observables, we numerically integrate densities to obtain the likelihood function.

Park (2002) shows that a nonlinear nonstationary heteroscedasticity generated by an asymptotically homogeneous function, like the logistic function, has a long memory. Another property of this type of heteroscedasticity is that it generates sample kurtosis with truncated supports, unlike an exponential stochastic volatility model in which volatility may explode if the process is too persistent. By having this specification for volatility, we can incorporate a persistent factor into the model to generate a slowly decaying volatility of yields with respect to maturity, as pointed out by Gallmeyer, Hollifield, Palomino and Zin (2007) (GHPZ hereupon). Additionally, we can generate varying term premia, as originally evidenced by

Campbell and Shiller (1991). Given the form of the logistic function, volatility is bounded between "high volatility" and "low volatility" regimes, with a transition stage corresponding to a "middle volatility" regime in which agents can not anticipate with certainty where the economy will end up. This setup is compatible with Epstein-Zin/Weil preferences, as noted by Kim et al. (2008), because agents would prefer an early resolution of uncertainty under certain value of preference parameters, as observed by Epstein and Zin (1991).

As noted in Piazzesi and Schneider (2006) (PS hereupon), negative linear relationships between unanticipated inflation and consumption and between anticipated inflation and consumption, along with recursive preferences, are needed to generate an upward sloping yield curve. The SV-VAR is able to incorporate these possibilities, and results show a negative correlation between unexpected inflation and consumption growth, as well as a negative linear relationship between anticipated inflation and future consumption growth.

We feed the pricing kernel corresponding to Epstein-Zin/Weil preferences with estimates from the SV-VAR to price bonds using the recursive pricing equation. Preference parameters, i.e., elasticity of intertemporal substitution and coefficient of relative risk aversion, are estimated by the simulated method of moments where we use the average of historical yields as moment conditions. Unlike previous studies where recursive preferences have been incorporated, we do not assume an affine structure for the pricing kernel like in GHPZ and Doh (2008), or a particular value for a preference parameter like in PS. In order to solve for the conditional expectation to price bonds, we recur to numerically integrate this expression using the approach shown in Tauchen and Hussey (1991).

Results show that the model can successfully reproduce the average term structure, the 10-year bond term premium, as well as the first order autocorrelation of yields, but fails to reproduce their standard deviation. We argue that the reason for which the model does not perform as expected with respect to the volatility of yields resides in the numerical integration technique. Since the integration uses Markov transition probabilities to compute the expectation, and given that inflation and consumption growth follow a stationary VAR pro-

cess, these transition probabilities are dominated by the stationary component even though the stochastic volatility process is highly persistent.

The structure of this document is as follows: Section 2 describes the model to price bonds in equilibrium, Section 3 shows the estimation methodology of the SV-VAR and estimation results, Section 4 explains the methodology to estimate preference parameters along with the numerical integration technique, the way to simulate yields, and discusses results. Finally, Section 5 concludes. Details of derivations are shown in an Appendix.

2 The Model

We consider an endowment economy populated by a representative investor in which endowment (e_t) and inflation (π_t) are exogenously given. This section illustrates the setup for the valuation of real and nominal bonds under recursive preferences and under a stochastic volatility VAR(p) specification for the exogenous processes.

2.1 Preferences

We assume an exchange economy in which a representative agent chooses her consumption level to maximize the recursive utility function proposed by Epstein and Zin (1989). Given a sequence of consumption $\{c_t, c_{t+1}, c_{t+2}, ...\}$ with random realizations of future consumption, the intertemporal utility function, U_t, is the solution to the recursive equation,

$$U_t \;=\; [(1-\beta)c_t^{\rho} + \beta\mu_t(U_{t+1})^{\rho}]^{1/\rho},$$

where $0 < \beta < 1$ is the discount factor, $\rho \leqslant 1$ is a preference parameter measuring the degree of intertemporal substitution (the elasticity of intertemporal substitution, EIS hereupon, for a deterministic flow of consumption is given by $1/(1-\rho)$), and the certainty equivalent of

random future utility is

$$\mu(U_{t+1}) \equiv \mathbb{E}_t[U_{t+1}^{\alpha}]^{\frac{1}{\alpha}},$$

where $\alpha \leqslant 1$ measures the risk aversion in a static framework (the coefficient of relative risk aversion for static lotteries is $1-\alpha$). The intertemporal marginal rate of substitution, M_{t+1}, is given by

$$M_{t+1} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{\rho-1} \left(\frac{U_{t+1}}{\mu_t\left(U_{t+1}\right)}\right)^{\alpha-\rho}. \tag{1}$$

Notice that when the coefficient of relative risk aversion is equal to the inverse of the EIS, i.e., $\alpha = \rho$, the marginal rate of substitution reduces to the one obtained under power utility.

2.2 Exogenous processes setup

We consider a VAR(p), with p finite, as the specification of the stochastic process for the rate of growth of the endowment and inflation. We also introduce stochastic volatility in the innovations of the VAR(p). Following Boudoukh (1993), we assume that there is stochastic volatility in inflation only. This assumption is consistent with the analysis shown in PS about expected and unexpected inflation being a carrier of bad news, and the effect of inflation on bond prices. Denote

$$\mathbf{z}_{t+1} = \begin{bmatrix} g_{t+1} \\ \pi_{t+1} \end{bmatrix}.$$

Then our assumptions imply

$$\mathbf{z}_{t+1} = \mathbf{\Phi}_0 + \sum_{l=1}^{p} \mathbf{\Phi}_l \mathbf{z}_{t+1-l} + \boldsymbol{\varepsilon}^*_{t+1},$$

where $\mathbf{\Phi}_0$ is a 2×1 vector of parameters, $\mathbf{\Phi}_l$ is a 2×2 matrix of parameters for $l = 1, 2, \ldots, p$, and $\boldsymbol{\varepsilon}^*_{t+1} = [\varepsilon^*_{g,t+1}, \varepsilon^*_{\pi,t+1}]'$ are the innovations in endowment growth and inflation, respectively. Here we assume that $var_t(\boldsymbol{\varepsilon}^*_{t+1})$ is not constant and that it depends on a non observable nonstationary factor. More specifically, we assume

$$\mathbf{F}(y_t) = \begin{bmatrix} \sigma_g & 0 \\ 0 & \sqrt{f(y_t)} \end{bmatrix},$$

and follow Lee (2008) to make $f(y) = \theta_0 + \frac{\theta_1}{1+\exp(-\lambda y)}$, with $\theta_0 > 0$, $\theta_1 > 0$, $\lambda > 0$, $y_{t+1} = y_t + u_{t+1}$, and

$$\begin{bmatrix} \varepsilon_{g,t+1} \\ \varepsilon_{\pi,t+1} \\ u_{t+1} \end{bmatrix} | \mathfrak{F}_t \sim \text{iid}\mathbb{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right), \tag{2}$$

where $\boldsymbol{\varepsilon}^*_{t+1} = \mathbf{F}(y_t)\boldsymbol{\varepsilon}_{t+1}$, with $\boldsymbol{\varepsilon}_{t+1} = [\varepsilon_{g,t+1}, \varepsilon_{\pi,t+1}]'$, $\mathfrak{F}_t = \sigma(\{\mathbf{z}_s\}_{s=0}^t)$ is the information available at time t, and $|\nu| \leq 1$.

Under this stochastic volatility setup, we model volatility as a logistic function of a unit root process, which implies that volatility is bounded and has a smooth transition between two regimes: low volatility (θ_0) and high volatility ($\theta_0 + \theta_1$), while the smoothness of the transition is measured by the coefficient λ. Lee (2008) mentions this characteristic as opposed to what happens with an exponential stochastic volatility model, in which volatility would explode if there is enough persistence in it. Also, the smooth transition along with the nonstationary latent factor $\{y_t\}$ allow for volatility clustering. Additionally, Park (2002) shows that this type of nonlinear heteroscedasticity has sample kurtosis with truncated supports. On a related matter, Kim et al. (2008) points out that since agents dislike the uncertainty that arises when volatility is in between of the two regimes, because they would prefer an early resolution of uncertainty (when $\gamma > \frac{1}{\psi}$), the recursive preferences specification in this work is compatible with our setup for volatility.

The SV-VAR has some advantages with respect to other models used in the literature of term structure of interest rates with recursive preferences and/or stochastic volatility. First, we assume that the lag order of the VAR is finite, as opposed to the invertible VARMA(1,1) proposed by PS. Second, we allow for the lag order of SV-VAR to be greater than one, extending the setup in Boudoukh (1993).

2.3 Pricing bonds

In this section we use the intertemporal marginal rate of substitution to price bonds under the SV-VAR specification for endowment growth and inflation rates. It is important to point out that we do not make any assumption about the value of the EIS, as opposed to PS, who set this coefficient to unity.

2.3.1 Real bond pricing

Recalling the pricing equation for bonds, we have that if the price, in consumption units, at t of a bond of maturity n is denoted by $Q_{n,t}$, then

$$Q_{n,t} = \mathbb{E}_t(M_{t+1}Q_{n-1,t+1}) \tag{3}$$

In equilibrium, the real pricing kernel is given by (1) with $c_t = e_t$ and V_t being the value of utility. Consequently, the (log of the) pricing kernel in equilibrium is given by

$$\ln M_{t+1} = \ln \beta + (\rho - 1)\, g_{t+1} + (\alpha - \rho)\left(\ln V_{t+1} - \ln \mu_t\left(V_{t+1}\right)\right), \tag{4}$$

where $g_{t+1} \equiv \ln(e_{t+1}/e_t)$ is the endowment growth rate between t and $t+1$. To obtain a fully parametric (and able to be simulated) expression for the (log of the) pricing kernel, we follow GHPZ. We obtain the following result, whose derivation appears in the Appendix:

$$\ln M_{t+1} \approx \ln \beta - (\alpha - \rho)\, A_t + \left[(\rho - 1) + (\alpha - \rho)\,(1 + v_1)\right] g_{t+1} + (\alpha - \rho)\left[w_1 \pi_{t+1} + v_f f\left(y_{t+1}\right)\right], \tag{5}$$

where the approximation follows because of the linear approximation that we made to the logistic function. A_t is an expression that includes preference parameters, parameters from the SV-VAR and variables at t. v_1, w_1 and v_f are parameters from the (log of the scaled) value function which, in turn, are functions of the preference parameters and parameters from the SV-VAR. Again, if $\alpha = \rho$, we return to the conventional real pricing kernel obtained from a power utility setup.

2.3.2 Nominal bond pricing

Since the pricing equation (3) must hold for real prices of nominal bonds, we have

$$\frac{Q^{\$}_{n,t}}{P_t} = \mathbb{E}_t \left(M_{t+1} \frac{Q^{\$}_{n-1,t+1}}{P_{t+1}} \right),$$

where $Q^{\$}_{n,t}$ denotes the nominal price at t of a bond maturing n periods ahead, and P_t denotes the price of the consumption good at t. Therefore, we can write

$$Q^{\$}_{n,t} = \mathbb{E}_t \left(M^{\$}_{t+1} Q^{\$}_{n-1,t+1} \right), \tag{6}$$

where the (log of the) nominal pricing kernel is given by

$$\ln M^{\$}_{t+1} = \ln M_{t+1} - \pi_{t+1}.$$

From the structures of the real and the nominal pricing kernels, we can see that the properties of inflation influence the price of nominal bonds. Let us analyze the effects of inflation on the real pricing kernel (5). First, under the power utility setup, the effect of inflation on the pricing kernel is weighted by the EIS only (or the coefficient of relative risk aversion). Second, under recursive preferences, inflation not only enters the pricing kernel weighted by the EIS, but also by the difference between the coefficient of relative risk aversion and the EIS. Third, stochastic volatility is also a determinant of the pricing kernel and its weight

is determined by the difference between the coefficient of relative risk aversion and the EIS too.

3 SV-VAR Estimation

In this section we proceed to estimate the parameters of the SV-VAR which will serve as inputs for the estimation of preference parameters which, in turn, will allow us to simulate the term structure of interest rates.

To estimate the model presented in section 2.2 we use annualized quarterly consumption growth and inflation covering the period 1952:1 - 2008:3. We specify consumption as real per capita consumption of nondurables and services as reported by the Bureau of Economic Analysis. Data on the consumer price index (CPI) are from the Bureau of Labor Statistics of the U.S. Department of Labor.

3.1 VAR lag selection and heteroscedasticity

In order to proceed with the estimation of the SV-VAR model we need to offer an adequate specification for the conditional mean vector of consumption growth and inflation. Table 1 reports values of four different lag order selection criteria of a homoscedastic VAR: Sequential Modified LR, Akaike IC, Schwarz IC and Hannan-Quinn IC. The optimal lag order is 3, according to all the criteria used. Additionally, with this lag order we do not reject the null hypothesis of absence of serial correlation in the error term at the 1% level of significance.

With the lag order set to 3 we proceed to estimate a homoscedastic VAR in order to test for varying conditional variance. Table 2 shows results of the estimation as well as the heteroscedasticity tests performed: ARCH LM test in each equation of the VAR, single-equation White test, and joint White test. The first two tests reject the null hypothesis of homoscedasticity in inflation but not in consumption growth at conventional significance levels. The joint White heteroscedasticity test also rejects constancy of the VAR's conditional

variance covariance matrix. All these results support our setup for the SV-VAR model with respect to the assumption of including stochastic volatility in the inflation equation only.

3.2 Maximum likelihood estimation of the SV-VAR

We estimate parameters of the SV-VAR by maximum likelihood, using a nonlinear Kalman filter to obtain the appropriate densities. Tanizaki (1996) shows how to construct the prediction, updating and smoothing steps. First, notice that

$$\begin{aligned} \mathbf{z}_{t+1}|y_t, \mathfrak{F}_t, \Theta &\sim \mathbb{N}\left(\Phi_0 + \sum_{l=1}^{3} \mathbf{\Phi}_l \mathbf{z}_{t+1-l}, \Omega(y_t)\right) \\ y_t|y_{t-1}, \mathfrak{F}_t, \Theta \overset{d}{=} y_t|y_{t-1}, \Theta &\sim \mathbb{N}(y_{t-1}, 1), \end{aligned} \tag{7}$$

where $\Omega(y_t) = \mathbf{F}(y_t)\Gamma\mathbf{F}(y_t)$, Θ is the parameter space, and Γ is the upper left 2×2 block of the variance-covariance matrix in (2).

The maximum likelihood estimator is given by

$$\hat{\Theta} = \arg\max_{\Theta} \, l\left(\mathbf{z}_1, \ldots, \mathbf{z}_n | \, \Theta\right),$$

where $l\left(\mathbf{z}_1, \ldots, \mathbf{z}_n | \, \Theta\right) = \sum_{t=1}^{n} \ln p\left(\mathbf{z}_t | \mathfrak{F}_{t-1}, \Theta\right)$, and

$$\begin{aligned} p\left(\mathbf{z}_t | \, \mathfrak{F}_{t-1}, \Theta\right) &= \int p\left(\mathbf{z}_t, y_{t-1} | \, \mathfrak{F}_{t-1}, \Theta\right) dy_{t-1} \\ &= \int p\left(\mathbf{z}_t | \, y_{t-1}, \mathfrak{F}_{t-1}, \Theta\right) p\left(y_{t-1} | \, \mathfrak{F}_{t-1}, \Theta\right) dy_{t-1}. \end{aligned}$$

In order to integrate out the non observable and non stationary variable y_t, we proceed to numerically integrate the densities in which it appears. To that end, we apply the approach used in Lee (2008) and described in extent in the Appendix along with details about the prediction and updating steps of the filter. Results of the estimation are shown in Table 3.

We can see that the parameters corresponding to the conditional mean of the SV-VAR(3)

are very similar to those obtained from the homoscedastic VAR(3) in Table 2. Regarding parameters of the conditional volatility, all of them are statistically significant (positive) at the 5% level of significance. In particular, given these estimates, the shape of the logistic function is shown in Figure 1. The figure shows the smooth transition from the low volatility regime to the high volatility one. The transition is determined by the parameter λ, which is close to one, implying a transition smooth enough to make agents dislike a scenario of moderate volatility because they do not know where the economy will end up, as mentioned previously.

From the estimation of the SV-VAR, and by making use of a non linear Kalman filter, we obtain the filtered volatility, which appears in Figure 2 along with the short rate and the NBER recession periods. The graph shows that volatility is indeed highly persistent. It also shows a highly volatile period between the second half of the 1970s and during the 1980s. There is also a low volatility regime which corresponds to the period between the early 1990s until the early 2000s. It is also evident an increase in volatility at the end of the present decade due to the effects of the current crisis. In fact, volatility increases every time that a recession affects the economy, except for one crisis episode at the beginning of the sample. When comparing time series of the filtered volatility with those of the short-term nominal interest rate, we can see that there is a positive covariation between them, except for the period between the early 1990s and the mid 2000s. This is a sign that our volatility setup could help capture salient features of interest rates [1] along with the other factors used here, namely, consumption growth and inflation.

4 Implications for bond yields

We describe the methodology for preference parameters estimation as well as comparisons with results in other studies of equilibrium modeling of the term structure in Section 4.1. In Section 4.2 we discuss the term structure implications of the model. In Section 4.3 we show

[1] And it could also explain features of the equity premium, as explored in Kim et al. (2008).

the model performance with respect to reproducing time series characteristics of the short term yield and the yield spread. Here we use yields of maturities 1, 4, 8, 12, 20, 28 and 40 quarters. Data up to 1991 are from McCulloch and Kwon (1993), then we use yields on U.S. Treasury securities at constant maturities reported by the Federal Reserve Bank.

4.1 Preference parameter estimation

Besides parameters involving preferences towards risk (α) and preferences towards intertemporal allocations of consumption (ρ), we have additional parameters involved in the (log of the) pricing kernel. The parameters under discussion include the discount factor, β, and other parameters related to the (log of the scaled) value function, namely, v_1, w_1, and v_f, as well as other that appear in A_t and that are related to the value function too (they are denoted v_2, v_3, w_2, and w_3). Given α and ρ, these eight additional parameters are obtained from a system of nonlinear equations designed to match the mean of the short-term interest rate. The system of equations is shown in detail in the Appendix.

Regarding the preference parameters α and ρ, we estimate them so that the mean of the yields produced by our model match as closely as possible the observed average term structure. To that end, we make use of the simulated method of moments firstly introduced by Lee and Ingram (1991), and extended by Burnside (1993) to the numerical solution to asset pricing models proposed by Tauchen and Hussey (1991).

Before proceeding to the estimation of preference parameters we notice that, given the nature of the logistic function assumed for volatility, we can define the process $\{f_{t+1}\}_{t=-\infty}^{\infty} \equiv \{f(y_{t+1})\}_{t=-\infty}^{\infty}$ with a well defined conditional density function, $p(f_{t+1}|f_t)$, since the Markov property of the process $\{y_t\}_{t=-\infty}^{\infty}$ is inherited by $\{f_{t+1}\}_{t=-\infty}^{\infty}$ because $f(\cdot)$ is bijective. We show the expression for $p(f_{t+1}|f_t)$ in the Appendix. This step is important because it allows us to restrict the range of an integration from $(-\infty, \infty)$ to an integration on $(\theta_0, \theta_0 + \theta_1)$, which is bounded.

Now, by defining $\mathbf{s}_t \equiv \{\mathbf{z}_t, \mathbf{z}_{t-1}, \mathbf{z}_{t-2}\}$, we notice that, since the real pricing kernel in (5)

is a function of $\mathbf{z}_{t+1}$, f_{t+1}, $\mathbf{s}_t$, and f_t, the nominal pricing kernel is also a function of these variables. We write $M^{\$}(\mathbf{z}_{t+1}, f_{t+1}, \mathbf{s}_t, f_t)$ to express the dependence of the nominal pricing kernel on the mentioned variables. Therefore, the pricing equation (6) implies that nominal bond prices are functions of $\mathbf{s}_t$ and f_t only:

$$Q^{\$}_{n,t} = H_n(\mathbf{s}_t, f_t),$$

with $H_0(\mathbf{s}_t, f_t) = 1 \ \forall t$, and

$$\begin{aligned} H_n(\mathbf{s}_t, f_t) &= \mathbb{E}_t M^{\$}(\mathbf{z}_{t+1}, f_{t+1}, \mathbf{s}_t, f_t) H_{n-1}(\mathbf{s}_{t+1}, f_{t+1}) \\ &= \int_{-\infty}^{\infty} \int_{\theta_0}^{\theta_0+\theta 1} M^{\$}(\mathbf{z}_{t+1}, f_{t+1}, \mathbf{s}_t, f_t) H_{n-1}(\mathbf{s}_{t+1}, f_{t+1}) p(\mathbf{z}_{t+1}, f_{t+1}|\mathbf{s}_t, f_t)\, df_{t+1}\, d\mathbf{z}_{t+1}. \end{aligned} \tag{8}$$

Further, we notice that the conditional density involved in the integration can be written as follows:

$$\begin{aligned} p(\mathbf{z}_{t+1}, f_{t+1}|\mathbf{s}_t, f_t) &= p(\mathbf{z}_{t+1}|\mathbf{s}_t, f_t, f_{t+1}) p(f_{t+1}|\mathbf{s}_t, f_t) \\ &= p(\mathbf{z}_{t+1}|\mathbf{s}_t, f_t) p(f_{t+1}|f_t), \end{aligned}$$

where passing from the first to the second identity is done because $\mathbf{z}_{t+1}|\mathfrak{F}_t, f_t$ is independent of f_{t+1}, as shown in (7), and because $\{f_{t+1}\}_{t=-\infty}^{\infty}$ is first-order Markov.

We also point out that there is an integration across three dimensions involved in (8). Tauchen and Hussey (1991) suggests using a Gaussian quadrature method to discretize the space of the state variables in order to write

$$H_n(\mathbf{s}_j, f_l) = \sum_{i=1}^{N_{\mathbf{z}}} \sum_{k=1}^{N_f} M^{\$}(\mathbf{z}_i, f_k, \mathbf{s}_j, f_l) H_{n-1}(\mathbf{s}_i, f_k)\, \Pi_{ji,l}\, \Pi_{lk}, \tag{9}$$

with

$$\begin{aligned}\Pi_{ji,l} &= \text{Prob}\{\mathbf{z}_{t+1} = \mathbf{z}_i | \mathbf{s}_t = \mathbf{s}_j, f_t = f_l\} \\ \Pi_{lk} &= \text{Prob}\{f_{t+1} = f_k | f_t = f_l\},\end{aligned}$$

where $j = 1, 2, \ldots, \hat{N}_{\mathbf{z}}$, $l = 1, 2, \ldots, N_f$, and $\hat{N}_{\mathbf{z}} = {N_{\mathbf{z}}}^3$. Here N_f and $\hat{N}_{\mathbf{z}}$ denote the number of quadrature abscissa points.[2] We assume the same number of abscissa points for each of the variables in the VAR, which is 5, making $N_{\mathbf{z}} = 25$ and $\hat{N}_{\mathbf{z}} = 15,625$. Furthermore, we assume $N_f = 6$, making the total number of abscissa points $\hat{N}_{\mathbf{z}} \times N_f = 93,750$. In the Appendix we discuss the approach of Tauchen and Hussey to obtain the transition probabilities.

In order to obtain an extension from the discrete-space solution (9) to the continuous-space solution, we use step functions. Burnside (1999) points out that any Gaussian quadrature rule divides the real line for each of the variables into non-overlapping segments, therefore we can extend the discrete-space solution to any $\mathbf{s} \in \mathbb{R}^3$ and $f \in (\theta_0, \theta_0 + \theta_1)$. In the Appendix we discuss how to obtain this result.

Once we are able to obtain yields for different maturities from the model, we proceed to estimate preference parameters by the simulated method of moments whose setup is shown in the Appendix. For the moments we choose the means of yields, that is, we have 7 moment conditions. Results of the estimation are shown in Table 4.

These results imply that the coefficient of relative risk aversion is approximately 4, while the EIS is infinite. As mentioned before, β is estimated to match the average short-term interest rate. The value of the coefficient of risk aversion is lower (and significantly lower in some cases) compared to those obtained in the literature of term structure modeling with recursive preferences. Values range from 5-7 in GHPZ, 16-17 in Doh (2008), and 43-85 in PS. The value of the coefficient of EIS in this study is the same as the one obtained in

[2]Notice that the value for the abscissa points as well as the weights of the quadrature depend on the number of abscissa points, but we omit them here from the discretization expressions to avoid excess of notation.

GHPZ, much higher than the unity coefficient assumed in PS, and the estimated coefficient in Doh (2008), which is in the range 1.4-1.6. We notice that, since $\gamma > \frac{1}{\psi}$, meaning that agents prefer an early resolution of uncertainty, our volatility setup is compatible with the preferences used in this study.

PS needs the risk aversion coefficient to be in the mentioned range because, given that the EIS is set to unity, the only way to generate a positive expected excess return is by giving an important weight to the negative covariance between inflation and expected future consumption growth. The mechanism that allows PS and our work to have a positive excess return has to do with the fact that higher inflation rates (that affect real payoffs of nominal bonds, particularly long bonds) bring news about lower expected consumption growth rates and, since the real payoff of long bonds is lower in bad times and agents prefer an early resolution of uncertainty, the required premium on these bonds increases compared to short bonds.

With respect to the EIS parameter, we point out that the higher it is, the less demand for smoothing consumption over time. Increasing the EIS decreases the demand for long-term real bonds to smooth consumption, leading to a higher required real premium on these bonds. The fact that the EIS is high allows us to reproduce more satisfactorily the slope of the term structure. Epstein and Zin (1991) mentions the infinite-elasticity case as the case in which the C-CAPM reduces to the static CAPM, and points out that the static CAPM emerges due to the perfect substitutability of consumption across time.

4.2 Term structure implications

We obtain time series for each of the yields with the same time horizon as the sample data, 1952:1 to 2008:3, using the same step functions that allowed us to extend the discrete-space solution to the continuous-space one. That is, we consider the consumption growth rate, inflation rate and filtered volatility of a particular quarter and choose the yield for which each of the variables lay in one of the intervals given by the step function. In Table 5

we show averages, standard deviations and first order autocorrelations of yields. The model reproduces satisfactorily the average term structure but fails to reproduce the volatility of yields. Particularly, the simulated yields' volatilities decay in an exponential fashion with respect to maturity, as opposed to the slowly decaying volatilities of observed yields. This sharp decay is observed despite the introduction of a persistent factor for explaining yields, which is the stochastic volatility. Figure 3 and Figure 4 show the average term structure and the standard deviation of yields of both the observed and simulated data. The persistence of the different yields, however, is satisfactorily reproduced by the model and here persistence of the stochastic volatility process plays a fundamental role.

The reason for which the model is incapable of reproducing the slowly decaying standard errors of yields lies in the way the pricing equation (8) is computed. Here we use the numerical integration technique suggested by Tauchen and Hussey (1991) and divided the conditional density inside the integration into the product of two conditional densities: the conditional density for the stationary processes, namely consumption growth and inflation, and the conditional density for stochastic volatility, which is highly persistent. When computing the numerical integral, we transform these densities into transition probabilities, and the transition probabilities corresponding to consumption growth and inflation dominate the transition probabilities for stochastic volatility, making the solution look like as if it came from a purely stationary exogenous process. This situation can be more easily seen when looking at equation (9), where the product of the two transition probabilities appears explicitly. Affine models of the term structure of interest rates under Epstein-Zin/Weil preferences do not have this kind of problem because they can be solved explicitly by either assuming that no correlation exists between consumption growth and inflation like in GHPZ, or by assuming an affine functional form for the return on consumption claims, like in Doh (2008).

4.3 Time series implications

Figure 5 shows time series of the nominal yield on the three-month bond obtained from the model and from observed data (both de-meaned). The simulated rate (dashed line) shows more variability than the observed rate (solid line), but the model is able to satisfactorily reproduce many of the movements in the short interest rate, except for a slight deviation at the beginning of the 1990s and also at the end of the considered sample. The overall correlation between the observed and the simulated rate is 0.73.

In Figure 6 we show the term spread of the 10-year bond with respect to the 3-month bond. The simulated yield spread is more variable than the observed spread and the model predicts a higher spread before the 1970s and a lower term spread during the 1970s than what is observed. However, towards the end of the sample the model does a reasonable job describing movements in the term spread. The overall correlation between the observed and the simulated term spread is 0.14.

5 Conclusions

This paper presents an equilibrium model of the term structure of interest rates with Epstein-Zin/Weil preferences and an exogenous process that includes logistic stochastic volatility. The model's performance is very reasonable with respect to matching the first moment of the term structure of interest rates and the persistence of yields. The model, however, is not good at reproducing the slowly decaying standard errors of yields with respect to maturity. This last shortcoming occurs despite of the highly persistent stochastic volatility introduced as an additional factor to explain yields. The reason for the failure of the model at explaining the standard deviation of yields lies in the way the numerical integration to solve for the pricing equation is performed. Since the joint density inside the integration can be split into the product of the conditional density for the stationary processes, namely consumption growth and inflation, and the density for the stochastic volatility, the product

of the two is dominated by the first stationary conditional density, regardless of volatility's persistence.

The estimated preference parameters imply an infinite elasticity of substitution between present and future consumption, whereas the coefficient of relative risk aversion toward static lotteries is about 4. These estimates can be seen as an argument in favor of our volatility specification (given that $\gamma > \frac{1}{\psi}$), since agents prefer an early resolution of uncertainty and our volatility function does not reveal the ending state if the initial volatility state is of medium uncertainty.

One future line of research is to consider a semi-affine specification for the pricing kernel, in the line of Duarte (2004), which is a flexible form that would allow us to incorporate the covariance between consumption growth and inflation and, at the same time, avoid a numerical integration to price bonds.

Another future line of research is to incorporate specifications of monetary or fiscal policy that introduce persistence in the volatility of yields, in the line of GHPZ.

6 Appendix

6.1 Derivation of the pricing kernel

In order to obtain equation (5), we need to make a linear approximation of the logistic function around some value $\bar{f} \in (\theta_0, \theta_0 + \theta_1)$. We obtain

$$f(y_{t+1}) \approx f(y_t) + \bar{\theta} u_{t+1},$$

where $\bar{\theta} = \frac{\lambda\left(\bar{f}-\theta_0\right)}{\theta_1}\left(\theta_0 + \theta_1 - \bar{f}\right) > 0$. Also

$$\sqrt{f(y_t)} \approx \frac{1}{2}\left(\sqrt{\bar{f}} + \frac{f(y_t)}{\sqrt{\bar{f}}}\right).$$

From here on, we follow Gallmeyer et al. (2007) in the derivations. By homogeneity of $\mu_t(\cdot)$ we can write

$$\frac{V_t}{e_t} = \left[(1-\beta) + \beta\mu_t\left(\frac{V_{t+1}}{e_{t+1}} \times \frac{e_{t+1}}{e_t}\right)^{\rho}\right]^{\frac{1}{\rho}}.$$

Taking logs and defining $v_t = \ln(V_t/e_t)$, we have

$$v_t = \frac{1}{\rho}\ln\left[(1-\beta) + \beta\exp\left(\rho\tilde{\mu}_t\right)\right],$$

where $\tilde{\mu}_t \equiv \ln\left(\mu_t\left(\exp\left(v_{t+1} + g_{t+1}\right)\right)\right)$.

Approximating v_t around $\tilde{\mu}_t = \bar{m}$ yields

$$v_t \approx \eta_0 + \eta_1\tilde{\mu}_t,$$

where

$$\begin{aligned}
\eta_0 &= \frac{1}{\rho}\ln\left[(1-\beta) + \beta\exp\left(\rho\bar{m}\right)\right] - \frac{\beta\exp\left(\rho\bar{m}\right)}{1-\beta+\beta\exp\left(\rho\bar{m}\right)}\bar{m}, \\
\eta_1 &= \frac{\beta\exp\left(\rho\bar{m}\right)}{1-\beta+\beta\exp\left(\rho\bar{m}\right)}, \qquad 0 < \eta_1 < 1.
\end{aligned}$$

If we evaluate at $\bar{m} = 0$, these expressions imply $\eta_0 = 0$, $\eta_1 = \beta$.

In order to parameterize the log of the value function, conjecture that

$$v_t = \bar{v} + v_1 g_t + v_2 g_{t-1} + v_3 g_{t-2} + w_1\pi_t + w_2\pi_{t-1} + w_3\pi_{t-2} + v_f f(y_t), \tag{10}$$

which implies that

$$\begin{aligned} v_{t+1}+g_{t+1} &= \bar{v}+(1+v_1)\,g_{t+1}+v_2 g_t+v_3 g_{t-1}+w_1\pi_{t+1}+w_2\pi_t+w_3\pi_{t-1}+v_f f\left(y_{t+1}\right) \\ &\approx \bar{v}+(1+v_1)\,g_{t+1}+v_2 g_t+v_3 g_{t-1}+w_1\pi_{t+1}+w_2\pi_t+w_3\pi_{t-1}+ \\ &\quad +v_f\left(f\left(y_t\right)+\bar{\theta}u_{t+1}\right). \end{aligned} \tag{11}$$

Taking conditional expectation and variance on (11), we obtain

$$\begin{aligned} \mathbb{E}_t\left(v_{t+1}+g_{t+1}\right) &\approx \bar{v}+(1+v_1)\,\mathbb{E}_t g_{t+1}+v_2 g_t+v_3 g_{t-1}+w_1\mathbb{E}_t\pi_{t+1}+w_2\pi_t+w_3\pi_{t-1}+v_f f\left(y_t\right), \\ var_t\left(v_{t+1}+g_{t+1}\right) &\approx (1+v_1)^2\sigma_g^2+w_1^2 f\left(y_t\right)+2\left(1+v_1\right)w_1\nu\sigma_g\sqrt{f\left(y_t\right)}+v_f^2\bar{\theta}^2 \\ &\approx (1+v_1)^2\sigma_g^2+w_1^2 f\left(y_t\right)+\left(1+v_1\right)w_1\nu\sigma_g\left(\sqrt{\bar{f}}+\frac{f\left(y_t\right)}{\sqrt{\bar{f}}}\right)+v_f^2\bar{\theta}^2 \\ &= (1+v_1)^2\sigma_g^2+\left(w_1^2+\frac{\left(1+v_1\right)w_1\nu\sigma_g}{\sqrt{\bar{f}}}\right)f\left(y_t\right)+\left(1+v_1\right)w_1\nu\sigma_g\sqrt{\bar{f}}+v_f^2\bar{\theta}^2. \end{aligned}$$

Since $v_{t+1}+g_{t+1}$ is normally distributed conditional on the information available at t, we have

$$\tilde{\mu}_t=\mathbb{E}_t\left(v_{t+1}+g_{t+1}\right)+\frac{\alpha}{2}var_t\left(v_{t+1}+g_{t+1}\right),$$

then

$$\begin{aligned} \tilde{\mu}_t &\approx \bar{v}+\left(1+v_1\right)\Phi_{01}+w_1\Phi_{02}+\frac{\alpha}{2}\left[(1+v_1)^2\sigma_g^2+\left(1+v_1\right)w_1\nu\sigma_g\sqrt{\bar{f}}+v_f^2\bar{\theta}^2\right]+ \\ &\quad +\left[\left(1+v_1\right)\Phi_{11}^{(1)}+v_2+w_1\Phi_{21}^{(1)}\right]g_t+ \\ &\quad +\left[\left(1+v_1\right)\Phi_{11}^{(2)}+v_3+w_1\Phi_{21}^{(2)}\right]g_{t-1}+ \\ &\quad +\left[\left(1+v_1\right)\Phi_{11}^{(3)}+w_1\Phi_{21}^{(3)}\right]g_{t-2}+ \\ &\quad +\left[\left(1+v_1\right)\Phi_{12}^{(1)}+w_1\Phi_{22}^{(1)}+w_2\right]\pi_t+ \\ &\quad +\left[\left(1+v_1\right)\Phi_{12}^{(2)}+w_1\Phi_{22}^{(2)}+w_3\right]\pi_{t-1}+ \\ &\quad +\left[\left(1+v_1\right)\Phi_{12}^{(3)}+w_1\Phi_{22}^{(3)}\right]\pi_{t-2}+ \\ &\quad +\left[v_f+\frac{\alpha}{2}\left(w_1^2+\frac{\left(1+v_1\right)w_1\nu\sigma_g}{\sqrt{\bar{f}}}\right)\right]f\left(y_t\right), \end{aligned}$$

where $\Phi_{ij}^{(l)}$ is the element (i,j) of $\mathbf{\Phi}_l$ for $i=0,1,2$, $j=1,2$, and $l=0,1,2,3$. Now, by using $v_t\approx\eta_0+\eta_1\tilde{\mu}_t$, (10), and $\eta_0=0$, $\eta_1=\beta$, we can get $\bar{v}$, v_1, v_2, v_3, w_1, w_2, w_3.

Recall that

$$\begin{aligned} \tilde{\mu}_t &\approx \bar{v}+\left(1+v_1\right)\mathbb{E}_t g_{t+1}+v_2 g_t+v_3 g_{t-1}+w_1\mathbb{E}_t\pi_{t+1}+w_2\pi_t+w_3\pi_{t-1}+v_f f\left(y_t\right)+ \\ &\quad +\frac{\alpha}{2}\left[(1+v_1)^2\sigma_g^2+w_1^2 f\left(y_t\right)+2\left(1+v_1\right)w_1\nu\sigma_g\sqrt{f\left(y_t\right)}+v_f^2\bar{\theta}^2\right], \end{aligned}$$

and

$$\ln V_{t+1} - \ln \mu_t (V_{t+1}) = v_{t+1} + g_{t+1} - \tilde{\mu}_t,$$

therefore we have

$$\begin{aligned} \ln V_{t+1} - \ln \mu_t (V_{t+1}) &\approx (1+v_1)(g_{t+1} - \mathbb{E}_t g_{t+1}) + w_1 (\pi_{t+1} - \mathbb{E}_t \pi_{t+1}) + v_f (f(y_{t+1}) - f(y_t)) - \\ &- \frac{\alpha}{2}\left[(1+v_1)^2 \sigma_g^2 + w_1^2 f(y_t) + 2(1+v_1) w_1 \nu \sigma_g \sqrt{f(y_t)} + v_f^2 \bar{\theta}^2\right]. \quad (12) \end{aligned}$$

Replacing (12) into (4) we obtain the real pricing kernel

$$\ln M_{t+1} \approx \ln \beta - (\alpha - \rho) A_t + [(\rho - 1) + (\alpha - \rho)(1 + v_1)] g_{t+1} + (\alpha - \rho)[w_1 \pi_{t+1} + v_f f(y_{t+1})],$$

where

$$\begin{aligned} A_t &= (1+v_1)\mathbb{E}_t g_{t+1} + w_1 \mathbb{E}_t \pi_{t+1} + v_f f(y_t) + \\ &+ \frac{\alpha}{2}\left[(1+v_1)^2 \sigma_g^2 + w_1^2 f(y_t) + 2(1+v_1) w_1 \nu \sigma_g \sqrt{f(y_t)} + v_f^2 \bar{\theta}^2\right], \end{aligned}$$

To obtain the nominal short-term interest rate, we need to get expressions for $\mathbb{E}_t(\ln M_{t+1} - \pi_{t+1})$ and $var_t(\ln M_{t+1} - \pi_{t+1})$ which, by the log-normality of the pricing kernel, allow us to write

$$\begin{aligned} r_{t+1}^{\$} &\approx -\ln \beta + (1-\rho)\mathbb{E}_t g_{t+1} + \mathbb{E}_t \pi_{t+1} - \\ &- \frac{1}{2}\{\tilde{\sigma}_t^2 - (\alpha - \rho)\alpha[(1+v_1)^2 \sigma_g^2 + w_1^2 f(y_t) + 2(1+v_1) w_1 \nu \sigma_g \sqrt{f(y_t)} + v_f^2 \bar{\theta}^2]\}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\sigma}_t^2 &= [(\rho - 1) + (\alpha - \rho)(1 + v_1)]^2 \sigma_g^2 + [(\alpha - \rho) w_1 - 1]^2 f(y_t) + (\alpha - \rho)^2 v_f^2 \tilde{\theta}^2 + \\ &+ 2[(\rho - 1) + (\alpha - \rho)(1 + v_1)][(\alpha - \rho) w_1 - 1] \nu \sigma_g \sqrt{f(y_t)}. \end{aligned}$$

6.2 Filtering procedure for the SV-VAR estimation

For the prediction step, we have[3]

$$\begin{aligned} p(y_t | \mathfrak{F}_t, \Theta) &= \int p(y_t, y_{t-1} | \mathfrak{F}_t, \Theta)\, dy_{t-1} \\ &= \int p(y_t | y_{t-1}, \mathfrak{F}_t, \Theta) p(y_{t-1} | \mathfrak{F}_t, \Theta)|, dy_{t-1} \\ &= \int p(y_t | y_{t-1}, \Theta) p(y_{t-1} | \mathfrak{F}_t, \Theta)\, dy_{t-1}. \end{aligned}$$

[3]Notice that, due to the nature of how the information is revealed, $p(y_t|\mathfrak{F}_t)$ refers to the density of the prediction, while $p(y_t|\mathfrak{F}_{t+1})$ is the density for the updating step, once z_{t+1} (and its variance) has been observed.

For the updating step, we have

$$\begin{aligned} p(y_t|\mathfrak{F}_{t+1},\Theta) &= p(y_t|\mathbf{z}_{t+1},\mathfrak{F}_t,\Theta) \\ &= \frac{p(y_t,\mathbf{z}_{t+1}|\mathfrak{F}_t,\Theta)}{p(\mathbf{z}_{t+1}|\mathfrak{F}_t,\Theta)} \\ &= \frac{p(\mathbf{z}_{t+1}|y_t,\mathfrak{F}_t,\Theta)p(y_t|\mathfrak{F}_t,\Theta)}{p(\mathbf{z}_{t+1}|\mathfrak{F}_t,\Theta)}. \end{aligned}$$

Regarding the estimation strategy, we need to numerically integrate the densities in order to proceed to the maximization of the log-likelihood function. However, since the state variable may show high persistence, we will follow Lee (2008) to make the Gauss-Legendre quadrature rule depend on the previous value of the state. For the prediction step, we assume that $p(y_{t-1}|\mathfrak{F}_t)$ is being integrated over $[-c+y_{t-1|t-1}, c+y_{t-1|t-1}]$ for some $c>0$, where $y_{t-1|t-1}\equiv\mathbb{E}(y_{t-1}|\mathfrak{F}_{t-1})$ is the prediction of y_{t-1}. Then we have,

$$\begin{aligned} p(y_t|\mathfrak{F}_t) &= \int p(y_t|y_{t-1})p(y_{t-1}|\mathfrak{F}_t)\,dy_{t-1} \\ &\approx \int_{-c+y_{t-1|t-1}}^{c+y_{t-1|t-1}} p(y_t|y_{t-1})p(y_{t-1}|\mathfrak{F}_t)\,dy_{t-1} \end{aligned}$$

for $t=1,...,n$, with $y_{0|1}$ to be estimated, and $p(y_0|\mathfrak{F}_1)=1$ at $y_0|\mathfrak{F}_1=y_{0|1}$.

For the updating step we need $p(z_{t+1}|\mathfrak{F}_t)$, which can be approximated as follows:

$$\begin{aligned} p(z_{t+1}|\mathfrak{F}_t) &= \int p(z_{t+1}|y_t,\mathfrak{F}_t)p(y_t|\mathfrak{F}_t)\,dy_t \\ &= \int\int p(z_{t+1}|y_t\mathfrak{F}_t)p(y_t|y_{t-1})p(y_{t-1}|\mathfrak{F}_t)dy_{t-1}dy_t \\ &\approx \int_{-c+y_{t|t}}^{c+y_{t|t}}\int_{-c+y_{t-1|t-1}}^{c+y_{t-1|t-1}} p(z_{t+1}|y_t\mathfrak{F}_t)p(y_t|y_{t-1})p(y_{t-1}|\mathfrak{F}_t)dy_{t-1}dy_t, \end{aligned}$$

where $y_{t|t}=y_{t-1|t}$ is the prediction of y_t conditional on $\mathfrak{F}_t$. Now,

$$y_{t|t+1}\approx\int_{-c+y_{t|t}}^{c+y_{t|t}} y_t p(y_t|\mathfrak{F}_{t+1})\,dy_t,$$

with $p(y_t|\mathfrak{F}_{t+1})=\frac{p(z_{t+1}|y_t,\mathfrak{F}_t)p(y_t|\mathfrak{F}_t)}{p(z_{t+1}|\mathfrak{F}_t)}$.

We obtain the filtered stochastic volatility from

$$f_{t|t}=\int_{-c+y_{t|t}}^{c+y_{t|t}} f(y_t)p(y_t|\mathfrak{F}_t)\,dy_t.$$

6.3 Nonlinear system of equations

The nonlinear system of equations is given by

$$
\begin{aligned}
v_1 &= \frac{\beta}{1-\beta\Phi_{gg}^{(1)}}\left[\Phi_{11}^{(1)}+v_2+w_1\Phi_{21}^{(1)}\right] \\
v_2 &= \beta\left[(1+v_1)\,\Phi_{11}^{(2)}+v_3+w_1\Phi_{21}^{(2)}\right] \\
v_3 &= \beta\left[(1+v_1)\,\Phi_{11}^{(3)}+w_1\Phi_{21}^{(3)}\right] \\
w_1 &= \frac{\beta}{1-\beta\Phi_{22}^{(1)}}\left[(1+v_1)\,\Phi_{12}^{(1)}+w_2\right] \\
w_2 &= \beta\left[(1+v_1)\,\Phi_{12}^{(2)}+w_1\Phi_{22}^{(2)}+w_3\right] \\
w_3 &= \beta\left[(1+v_1)\,\Phi_{12}^{(3)}+w_1\Phi_{22}^{(3)}\right] \\
v_f &= \frac{\beta}{1-\beta}\frac{\alpha}{2}\left(w_1^2+\frac{(1+v_1)\,w_1\nu\sigma_g}{\sqrt{\bar{f}}}\right) \\
\bar{r^{\$}} &= -\ln\beta+(1-\rho)\left[\Phi_{01}+\left(\Phi_{11}^{(1)}+\Phi_{11}^{(2)}+\Phi_{11}^{(3)}\right)\bar{g}+\left(\Phi_{12}^{(1)}+\Phi_{12}^{(2)}+\Phi_{12}^{(3)}\right)\bar{\pi}\right]+ \\
&\quad +\Phi_{02}+\left(\Phi_{21}^{(1)}+\Phi_{21}^{(2)}+\Phi_{21}^{(3)}\right)\bar{g}+\left(\Phi_{22}^{(1)}+\Phi_{22}^{(2)}+\Phi_{22}^{(3)}\right)\bar{\pi}- \\
&\quad -\frac{1}{2}\left\{\begin{array}{c}\left[(\rho-1)+(\alpha-\rho)\,(1+v_1)\right]^2\sigma_g^2+\left[(\alpha-\rho)\,w_1-1\right]^2\bar{f}+(\alpha-\rho)^2\,v_f^2\bar{\theta}^2+ \\ +2\left[(\rho-1)+(\alpha-\rho)\,(1+v_1)\right]\left[(\alpha-\rho)\,w_1-1\right]\nu\sigma_g\sqrt{\bar{f}}- \\ -(\alpha-\rho)\,\alpha\left[(1+v_1)^2\,\sigma_g^2+w_1^2\bar{f}+2\,(1+v_1)\,w_1\nu\sigma_g\sqrt{\bar{f}}+v_f^2\bar{\theta}^2\right],\end{array}\right\}
\end{aligned}
$$

where $\bar{r^{\$}}$, $\bar{g}$ and $\bar{\pi}$ are the sample means of the short-term nominal interest rate, consumption growth and inflation, respectively. For $\bar{f}$ we take the median of the filtered $\{f_t\}$ because of the existence of the two specified volatility regimes.

6.4 Stochastic volatility's transition density

The transition density for the stochastic volatility process is given by

$$
p(f_{t+1}|f_t)=\left\{\begin{array}{ll}\frac{1}{\sqrt{2\pi}\lambda}\left(\frac{1}{f_{t+1}-\theta_0}+\frac{1}{\theta_1+\theta_0-f_{t+1}}\right)\exp\left(-\frac{1}{2\lambda^2}\left(\ln\frac{\frac{f_{t+1}-\theta_0}{\theta_1+\theta_0-f_{t+1}}}{\frac{f_t-\theta_0}{\theta_1+\theta_0-f_t}}\right)^2\right) & \text{if } \theta_0<f_{t+1}<\theta_0+\theta_1 \\ 0 & \text{o.w.}\end{array}\right.
$$

6.5 Transition probabilities

Assume that integration is performed against the density $p(\mathbf{z}_{t+1}|\mathbf{s}_t,f_t)p(f_{t+1}|f_t)$. Tauchen and Hussey (1991) suggests to replace the integral with summation using the quadrature rule, and then normalizing so that the weights add to unity. In our case, since we need to consider

the states for the stochastic volatility, the normalization is performed as follows:

$$\begin{aligned}\Pi_{ji,l} &= \frac{p(\mathbf{z}_i|\mathbf{s}_j, f_l)}{p(\mathbf{z}_i|\mu_{\mathbf{s}}, f_l)} \frac{w^{\mathbf{z}}_{i,l}}{a^{\mathbf{z}}_{j,l}}, \\ \Pi_{lk} &= \frac{p(f_k|f_l)}{a^f_l} w^f_k,\end{aligned}$$

where

$$\begin{aligned}a^{\mathbf{z}}_{j,l} &= \sum_{i=1}^{N_{\mathbf{z}}} \frac{p(\mathbf{z}_i|\mathbf{s}_j, f_l)}{p(\mathbf{z}_i|\mu_{\mathbf{s}}, f_l)} w^{\mathbf{z}}_{i,l}, \\ a^f_l &= \sum_{k=1}^{N_f} p(f_k|f_l) w^f_k,\end{aligned}$$

$w^{\mathbf{z}}_{i,l}$, for $i = 1, 2, \ldots, N_{\mathbf{z}}$, denote the weights given by the Gauss-Hermite quadrature; w^f_k, for $k = 1, 2, \ldots, N_f$, denote the weights given by the Gauss-Lebesgue quadrature, and $\mu_{\mathbf{s}}$ is the unconditional mean of $\{\mathbf{z}_t\}_{t=-\infty}^{\infty}$.

6.6 Continuous-space solution from discrete-space solution

We can extend the discrete-space solution to the continuous-space solution by letting

$$H_n(\mathbf{s}, f) = \sum_{j=1}^{\hat{N}_{\mathbf{z}}} \sum_{l=1}^{N_f} H_n(\mathbf{s}_j, f_l) \mathbb{1}_{j,l}(\mathbf{s}) \mathbb{1}_l(f),$$

where $\mathbb{1}_{j,l}(\mathbf{s}) = \mathbb{1}_l(\mathbf{z}_1)\mathbb{1}_l(\mathbf{z}_2)\mathbb{1}_l(\mathbf{z}_3)$, and

$$\begin{aligned}\mathbb{1}_l(\mathbf{z}_i) &= \begin{cases} 1, & \text{if } \mathbf{z}_i \in (\mathbf{x}^{\mathbf{z}}_{j-1,l}, \mathbf{x}^{\mathbf{z}}_{j,l}) \\ 0, & \text{otherwise,} \end{cases} \\ \mathbb{1}_l(f) &= \begin{cases} 1, & \text{if } f \in (x^f_{l-1}, x^f_l) \\ 0, & \text{otherwise,} \end{cases}\end{aligned}$$

with $\mathbf{x}^{\mathbf{z}}_{j,l} = \mu_{\mathbf{z}} + \text{chol}(\Omega(f_l))\mathbf{x}_{j,l}$, where $\mathbf{x}_{j,l}$ is the solution to $w^{\mathbf{z}}_{j,l} = \int_{\mathbf{x}_{j-1,l}}^{\mathbf{x}_{j,l}} \exp(-\mathbf{v}'\mathbf{v})\, d\mathbf{v}$, $\mathbf{x}_{0,l} = -\infty$, $\mathbf{x}_{N_{\mathbf{z}},l} = \infty$, and $w^{\mathbf{z}}_{j,l}$ are the weights given by the Gauss-Hermite quadrature rule. In a similar reasoning, $x^f_l = x^f_{l-1} + \frac{1}{2}\theta_1 w^f_l$, where $x^f_0 = \theta_0$, $x^f_{N_f} = \theta_0 + \theta_1$, and w^f_l are the weights given by the Gauss-Lebesgue quadrature rule.

6.7 Simulated method of moments estimation

The estimators are defined, for $\Upsilon \equiv \{(-\infty, 1), (-\infty, 1)\}$, as

$$\{\hat{\alpha}, \hat{\rho}\} = \underset{\{\alpha,\rho\}\in\Upsilon}{\arg\min}\ m_T(\alpha, \rho)' D_T m_T(\alpha, \rho),$$

where, for $\mathbf{r}_t$ denoting the vector of yields for the different maturities considered,

$$m_T(\alpha, \rho) = \frac{1}{T} \sum_{t=1}^{T} \psi(\mathbf{r}_t, \alpha_0, \rho_0) - \mu_T(\alpha, \rho),$$

and α_0 and ρ_0 are the true preference parameters. For the estimation we choose the means of the yields, which give us 8 moment conditions, that is, we choose $\psi(\cdot)$ to be the identity function. For D_T we use the inverse of a HAC variance-covariance matrix of the method of moments estimator of the means of the yields. Burnside (1993) provides conditions under which the estimators obtained from this methodology are consistent.

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Table 1: Lag order selection of the VAR

Lag	**LR**[a]	**AIC**[b]	**SC**[c]	**HQ**[d]
1	139.24	-8.96	-8.87	-8.92
2	53.49	-9.18	-9.02	-9.11
3	34.66*	-9.31*	-9.09*	-9.22*
4	2.87	-9.28	-9.00	-9.17
5	9.46	-9.29	-8.95	-9.15
6	5.25	-9.28	-8.88	-9.12
7	4.95	-9.27	-8.80	-9.08

[a] Sequential modified LR test statistic.
[b] Akaike information criterion.
[c] Schwarz information criterion.
[d] Hannan-Quinn information criterion.
* Lag order selected by the criterion.
p-values for the LM test of residual serial correlation of the VAR(3) for lags 1 to 10 are, respectively: 0.876, 0.862, 0.176, 0.521, 0.200, 0.413, 0.349, 0.014, 0.710, 0.883.

Table 2: Consumption Growth and Inflation VAR

Parameter	Estimate
$\mathbf{\Phi}_0$	$\underset{(0.005)}{0.028}$ $\underset{(0.003)}{-0.006}$
$\mathbf{\Phi}_1$	$\underset{(0.069)}{0.117}$ $\underset{(0.084)}{-0.277}$ $\underset{(0.053)}{0.110}$ $\underset{(0.064)}{0.223}$
$\mathbf{\Phi}_2$	$\underset{(0.070)}{0.056}$ $\underset{(0.083)}{0.168}$ $\underset{(0.053)}{0.136}$ $\underset{(0.063)}{0.327}$
$\mathbf{\Phi}_3$	$\underset{(0.067)}{0.063}$ $\underset{(0.087)}{-0.100}$ $\underset{(0.051)}{0.073}$ $\underset{(0.066)}{0.401}$
σ_g	0.026
σ_π	0.020
ν	-0.153
HET1[a]	eq: g_{t+1} [0.868] eq: π_{t+1} [0.001]
HET2[b]	eq: g_{t+1} [0.855] eq: π_{t+1} [0.001]
HET3[c]	[0.008]

Values in parenthesis denote standard errors.
Values in square brakets denote p-values.
[a] ARCH LM test.
[b] White heteroskedasticity test with cross terms for individual equations.
[c] Joint White heteroskedasticity test with cross terms.

$$\mathbf{z}_{t+1} = \mathbf{\Phi}_0 + \sum_{l=1}^{3} \mathbf{\Phi}_l \mathbf{z}_{t+1-l} + \boldsymbol{\varepsilon}_{t+1}$$

$$\begin{bmatrix} \varepsilon_{g,t+1} \\ \varepsilon_{\pi,t+1} \end{bmatrix} | \mathfrak{F}_t \sim \text{iid}\mathbb{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_g & 0 \\ 0 & \sigma_\pi \end{bmatrix} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_g & 0 \\ 0 & \sigma_\pi \end{bmatrix} \right)$$

Table 3: Estimates of the SV-VAR

Parameter	Estimate
$\mathbf{\Phi}_0$	$\underset{(0.004)}{0.030}$ $\underset{(0.002)}{-0.007}$
$\mathbf{\Phi}_1$	$\underset{(0.067)}{0.108}$ $\underset{(0.082)}{-0.302}$ $\underset{(0.040)}{0.105}$ $\underset{(0.069)}{0.186}$
$\mathbf{\Phi}_2$	$\underset{(0.068)}{0.050}$ $\underset{(0.083)}{0.158}$ $\underset{(0.042)}{0.123}$ $\underset{(0.068)}{0.356}$
$\mathbf{\Phi}_3$	$\underset{(0.066)}{0.066}$ $\underset{(0.085)}{-0.078}$ $\underset{(0.039)}{0.089}$ $\underset{(0.065)}{0.402}$
σ_g	$\underset{(0.001)}{0.026}$
θ_0	$\underset{(2.15\times10^{-5})}{6.05\times10^{-5}}$
θ_1	$\underset{(3.6\times10^{-4})}{0.001}$
λ	$\underset{(0.403)}{0.950}$
ν	$\underset{(0.067)}{-0.175}$

Values in parenthesis denote standard errors.

$$\mathbf{z}_{t+1} = \mathbf{\Phi}_0 + \sum_{l=1}^{3} \mathbf{\Phi}_l \mathbf{z}_{t+1-l} + \boldsymbol{\varepsilon}^*_{t+1}$$

$$\begin{bmatrix} \varepsilon^*_{g,t+1} \\ \varepsilon^*_{\pi,t+1} \\ u_{t+1} \end{bmatrix} | \mathfrak{F}_t \sim \mathbb{N}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_g & 0 & 0 \\ 0 & \sqrt{f(y_t)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_g & 0 & 0 \\ 0 & \sqrt{f(y_t)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$\begin{aligned} f(y_t) &= \theta_0 + \frac{\theta_1}{1 + \exp\left(-\lambda y_t\right)} \\ y_{t+1} &= y_t + u_{t+1} \end{aligned}$$

Table 4: Estimates of α, ρ, and β

Parameter	Estimate
α	-3.014 (0.050)
ρ	0.999 (0.016)
β	0.982

Values in parenthesis denote standard errors.
β comes from the solution to the non-linear system of equations.
$\alpha = 1-\gamma$, where γ is the coefficient of relative risk aversion, and $\rho = 1-1/\psi$, where ψ is the elasticity of intertemporal substitution.

Table 5: Moments of the Yield Curve

	$\mathbb{E}(y_t^1)$	$\mathbb{E}(y_t^4)$	$\mathbb{E}(y_t^8)$	$\mathbb{E}(y_t^{12})$	$\mathbb{E}(y_t^{20})$	$\mathbb{E}(y_t^{28})$	$\mathbb{E}(y_t^{40})$
data	5.100	5.514	5.748	5.885	6.085	6.222	6.311
model	4.953	5.412	5.731	5.922	6.133	6.242	6.332
	$\sigma(y_t^1)$	$\sigma(y_t^4)$	$\sigma(y_t^8)$	$\sigma(y_t^{12})$	$\sigma(y_t^{20})$	$\sigma(y_t^{28})$	$\sigma(y_t^{40})$
data	2.877	2.922	2.866	2.812	2.743	2.700	2.653
model	4.203	3.414	2.868	2.512	2.007	1.649	1.279
	$\rho(y_t^1, y_{t-1}^1)$	$\rho(y_t^4, y_{t-1}^4)$	$\rho(y_t^8, y_{t-1}^8)$	$\rho(y_t^{12}, y_{t-1}^{12})$	$\rho(y_t^{20}, y_{t-1}^{20})$	$\rho(y_t^{28}, y_{t-1}^{28})$	$\rho(y_t^{40}, y_{t-1}^{40})$
data	0.926	0.935	0.942	0.948	0.958	0.963	0.968
model	0.882	0.912	0.922	0.925	0.927	0.927	0.927

Numbers are annual percentages, except for autocorrelations.

Figure 1: Implied Logistic Function

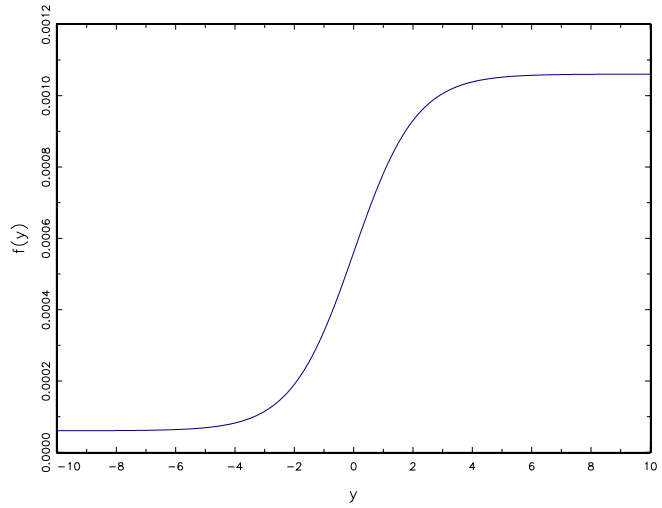

Figure 2: Filtered Logistic Volatility

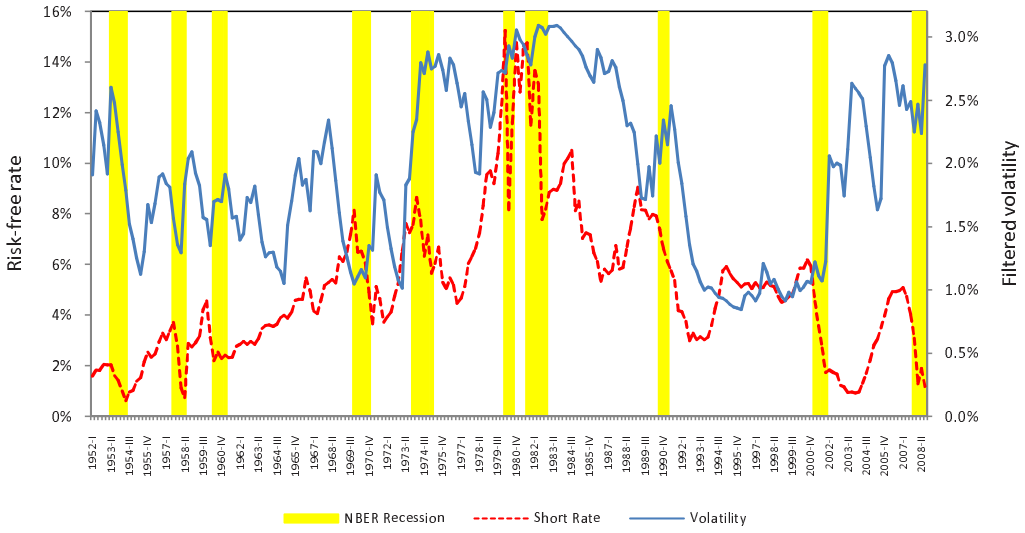

Figure 3: Average Term Structure

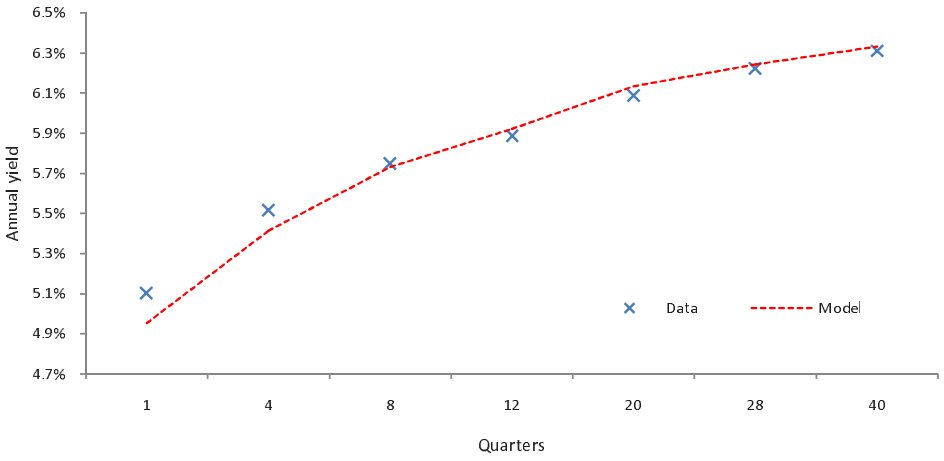

Figure 4: Volatility of Yields

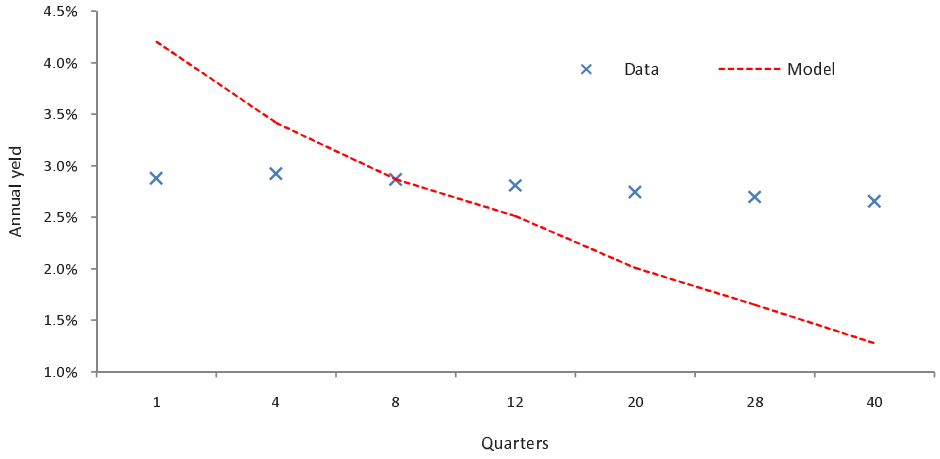

Figure 5: Risk-free Rate

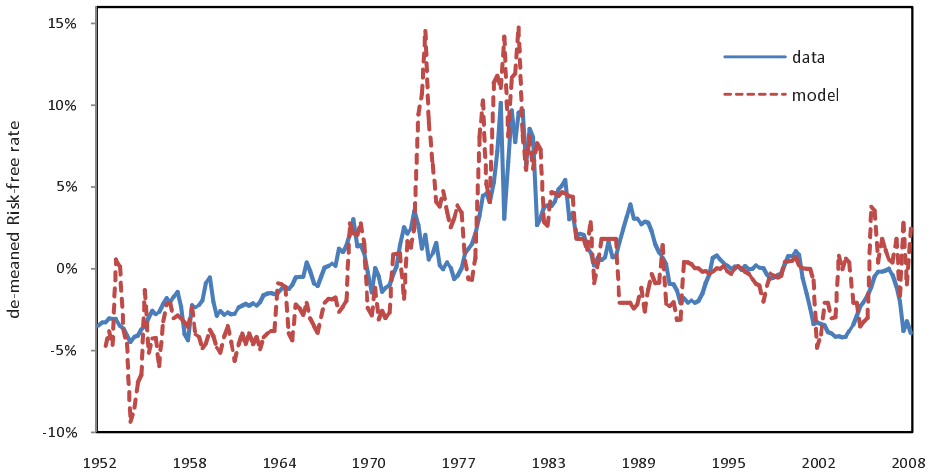
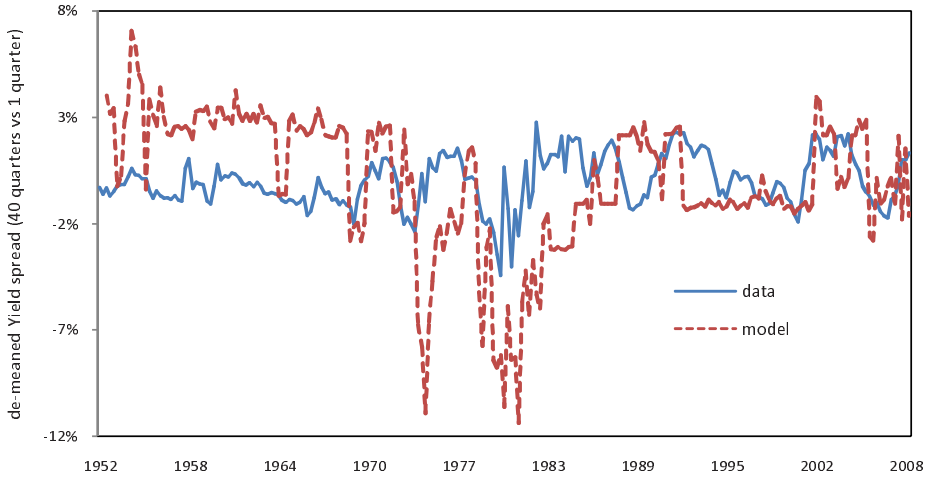
Figure 6: Yield Spread